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IMPROVED SUDAKOV-TYPE BOUNDS FOR
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OF SERIES SYSTEMS

by

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Improved Sudakov-Type Bounds for
Optimal Confidence Limits on the Reliability
of Series Systems

Bernard Harris* and Andrew P. Soms**

Abstract

A sharper Sudakov-type lower bound for the lower confidence limit on the reliability of a series system than the one given in Harris and Soms (1980) is obtained. Numerical examples, coverage probabilities and the listings of the short FORTRAN programs used are also provided.

Key words: Lindstrom-Madden approximation; Optimal confidence bounds; reliability; Series system.

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1. The Improved Sudakov-Type Bound

We will adhere to the notation of Harris and Soms (1980) and we refer the reader to this report for background material and additional references. Let $Y_i, 1 \leq i \leq k$, be independent binomial random variables with parameters n_i and p_i (the success probability) and let $X_i = n_i - Y_i, q_i = 1 - p_i, 1 \leq i \leq k$, with $n_1 \leq n_2 \leq \dots \leq n_k$ here and throughout. It is desired to obtain a "good" lower confidence limit on $\prod_{i=1}^k p_i$, the reliability of a series system with independent components. Let $g(\bar{x}) = g(x_1, x_2, \dots, x_k)$ be a monotonically decreasing or increasing (in each component) ordering function for real (not necessarily integral) $x_i, 0 \leq x_i \leq n_i$, with large or small values, respectively, of $g(\bar{x})$ being best. Let $r_1 > r_2 > \dots > r_s$ be the ordered values of $g(\bar{x})$ in the decreasing case and $r_1 < r_2 < \dots < r_s$ in the increasing and let $A_i = \{\bar{x} | g(\bar{x}) = r_i\}, i = 1, 2, \dots, s$. Then (A_1, A_2, \dots, A_s) is a monotonic partition, i.e., $(0, 0, \dots, 0) \in A_1, (n_1, n_2, \dots, n_k) \in A_s$ and if $\bar{x}_1 = (x_{11}, \dots, x_{1k}), \bar{x}_2 = (x_{21}, \dots, x_{2k})$ with $x_{1j} \leq x_{2j}, j = 1, 2, \dots, k$, then $\bar{x}_1 \in A_j$ implies $\bar{x}_2 \in A_j, j \geq 1$.

Let

$$f(\bar{x}; \bar{p}) = P_{\bar{p}}(\bar{X} = \bar{x}) = \prod_{i=1}^k \binom{n_i}{p_i} p_i^{x_i} q_i^{n_i - x_i} = \prod_{i=1}^k \binom{n_i}{p_i} p_i^{x_i} q_i^{n_i - x_i} \quad (2.1)$$

and for $1 \leq n \leq s-1$, let

$$a_n = \inf \left\{ \prod_{i=1}^k p_i \mid \sum_{\bar{x} \in A_j, j \leq n} f(\bar{x}; \bar{p}) = \alpha \right\} \quad (2.2)$$

and $a_s = 0$. Each such partition may be identified with a function defined on the set of sample outcomes by defining the ordering

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function $g(\bar{x})$, where

$$g(\bar{x}) = n \quad \text{if} \quad \bar{x} \in A_n, \quad 1 \leq n \leq s; \quad (2.3)$$

obviously $g(\bar{x})$ inherits the monotonicity properties of the partition.

For defining Buehler's (1957) method of optimal lower confidence intervals on $\Pi \Pi_k$ we assume that $g(\bar{x})$ has been redefined, if necessary, as in (2.3). Then we have, from Harris and Soms (1980),

Theorem 1. Let \bar{x} be distributed by (2.1). Then $g(\bar{x})$ is a (1- α) lower confidence bound for $\Pi_k \Pi_k$. If $g(\bar{x})$ is also a (1- α) lower confidence bound for $\Pi \Pi_k$ which is monotonically increasing in $g(\bar{x})$, then $b_1 \leq a_1, \dots, 1 \leq t \leq s$.

We now let $g(\bar{x})$ denote the original ordering function, since this is necessary for the applications below. In order to obtain bounds for $g(\bar{x}_0)$ we must assume that \bar{x}_0 is such that for each $t = 1, 2, \dots, k$, the equation

$$g(\{y_1^*, y_2^*, \dots, y_{t-1}^*, y_t, 0, \dots, 0\}) = g(\bar{x}_0) \quad (2.4)$$

has a unique solution y_t^* , $y_t^* < n_t$, where $\{r, r = 1, 2, \dots, t-1\}$, are integers, $0 \leq r \leq y_r^*$. Define y_1^* by $g(0, 0, \dots, 0, y_1^*, 0, \dots, 0) = g(\bar{x}_0)$, where y_1^* is in the 1th position, the rest of the arguments being 0. Note that $y_1 = y_1^*$. A sufficient condition for (2.4) to hold is that

$g(\bar{x})$ is strictly monotonic for $0 \leq y_j \leq y_j^*$, $j=1, 2, \dots, k$

and

$$\lim_{x_r \rightarrow n_r} g(\bar{x}) < g(\bar{x}_0) \quad \text{for } g(\bar{x}) \text{ decreasing} \quad (2.5)$$

or

$$\lim_{x_r \rightarrow n_r} g(\bar{x}) > g(\bar{x}_0) \quad \text{for } g(\bar{x}) \text{ increasing, } r=1, 2, \dots, k.$$

Assume now that (2.5) is satisfied and that in addition

$$\frac{y_r - 1}{n_r - 1} \geq \frac{y_{r+1}}{n_{r+1}}, \quad r=1, 2, \dots, k-1, \quad (2.6)$$

where $\{r$ integral, $0 \leq r \leq y_r$. Let

$$I_p(r, s) = \frac{1}{B(r, s)} \int_0^p t^{r-1} (1-t)^{s-1} dt,$$

and for $0 \leq y < n$, real, define $u(n, y, \alpha)$ by $\alpha = I_{u(n, y, \alpha)}(n-y, y+1)$.

Then it was shown in Harris and Soms (1980) that

$$u(n_1, y_1, \alpha) \leq g(\bar{x}_0) \leq \min_{1 \leq t \leq k} u(n_t, \{y_j^*\}_t, \alpha). \quad (2.7)$$

and thus if y_1 is an integer, $g(\bar{x}_0) = u(n_1, y_1, \alpha)$.

Thus a conservative procedure is to use $u(n_1, y_1, \alpha)$ for

$g(\bar{x}_0)$. It is generally believed that this is quite conservative (see, e.g., Mann, Schafer and Singpurwalla, 1974). It is thus of practical interest to inquire whether the lower bound in (2.7) can be tightened. An examination of the proof of (2.7) in Harris and Soms (1980) shows this to be the case, with the new proof being coincident with the previous one, except for the omission of the final step. We have

Theorem 2. Under the same assumptions as for (2.7),

$$u'(n_1, n_2, \gamma_1, \gamma_2, \alpha) \leq g(\bar{x}_0) \leq \min_{1 \leq i \leq k} u(n_i, \{y_i^*\}, \alpha),$$

where $u'(n_1, n_2, \gamma_1, \gamma_2, \alpha)$ is the solution in α of the equation

$$\sum_{i=1}^k \sum_{j=1}^k p_{ij}^{n_1-1} q_j^{n_2-1} \prod_{t=2}^k p_t^{n_2-\gamma_2+\gamma_1+1} = \alpha.$$

It follows from the proof in Harris and Soms (1980) that

$$u(n_1, \gamma_1, \alpha) \leq u'(n_1, n_2, \gamma_1, \gamma_2, \alpha).$$

Thus the only question is how great the improvement will be in using u' . The numerical examples in 2. show that this can be substantial.

Remarks. u' can be calculated quickly and efficiently by using a short FORTRAN program. The listing is given in the Appendix along with the listing of the program used to calculate coverage probabilities. Two ordering functions that satisfy (2.5) and (2.6) are

$$g(\bar{x}) = \prod_{i=1}^k ((n_1 - x_i)/n_1) \quad \text{if } g(\bar{x}_0) > 0 \text{ and } g(\bar{x}) = \prod_{i=1}^k x_i/n_1 \text{ for sufficiently large } n_1, 1 \leq i \leq k \text{ (see Harris and Soms, 1980, for details).}$$

2. Coverage Probabilities and Numerical Examples

The ordering function used here throughout is

$g(\bar{x}) = \prod_{i=1}^k ((n_1 - x_i)/n_1)$. Table 1 gives the coverage probabilities for $k=3, \bar{n}=(5,7,10), \alpha=.10$ and selected $\bar{p}=(p_1, p_2, p_3)$, for both the lower LB and upper UB bounds of (2.7). While the optimality property of Theorem 1 implies that there are \bar{p} for which the coverage probability is less than .9, it does not seem that there

are very many such \bar{p} .

[Insert Table 1 here.]

Table 2 gives some comparisons of LB, UB, the improved lower bound LBI and the true value TV for $k=2, \alpha=.05$ and selected $\bar{n}=(n_1, n_2)$ and $\bar{x}_0=(x_{01}, x_{02})$. LBI gives substantial improvement when n_1 is small compared to n_2 and little or none if n_1 is approximately the same as n_2 . The TV for $\bar{n}=(5,5)$ and $\bar{x}_0=(1,1)$ agrees with the value in Lifow and Riley (1959).

[Insert Table 2 here.]

Table 3 gives LB, UB and LBI for $k=5, \alpha=.05$ and selected \bar{n} and \bar{x} .

[Insert Table 3 here.]

2. Comparison of Bounds and True Value for $k=2$ and $\alpha=.05$

\bar{n}	\bar{x}_0	LB	LBI	TV	UB
(5,5)	(1,1)	.2166	.2166	.2776	.3426
(5,10)	(1,1)	.2761	.3317	.3317	.3426
(5,10)	(2,3)	.0859	.1521	.1529	.1893
(10,20)	(1,2)	.5037	.5675	.5691	.6058
(10,20)	(4,5)	.1851	.2137	.2137	.2224

3. Comparison of Bounds for $k=5$ and $\alpha=.05$

\bar{n}	\bar{x}_0	LB	LBI	UB
(10,10,10,10,10)	(1,2,0,1,1)	.2893	.2893	.3035
(10,15,20,25,30)	(1,2,1,1,2)	.3599	.3871	.3934
(10,15,20,25,30)	(1,1,2,3,4)	.2838	.3017	.3035
(10,15,20,25,30)	(2,3,4,4,4)	.1319	.1478	.1500

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