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Let $W^{(N,d)}$ be the N-parameter Wiener process with values in R^d. Then we prove that the dimension of the level sets is $N - d/2$ with positive probability if $d < 2N$. The dimension of the graph is a.s. $\min\{2N, N + d/2\}$. The level sets have zero $(N - d/2)$ - measures a.s. and the graph has zero $\min\{2N, N + d/2\}$ - measures a.s.

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The Hausdorff α-Dimensional Measures of the Level Sets and the Graph of the N-Parameter Wiener Process

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Let $W^{(N,d)}$ be the N-parameter Wiener process with values in R^d. Then the dimension of the level sets is $N-d/2$ with positive probability if $d < 2N$. The dimension of the graph is a.s. $\min\{2N, N+d/2\}$. The level sets have zero $(N-d/2)$-measure a.s. and the graph has zero $\min\{2N, N+d/2\}$-measure a.s.

1. Introduction and Preliminaries

Let $W^{(N)}$ be the N-parameter Wiener process, that is a real-valued separable Gaussian process with zero means and covariance $\prod_{i=1}^{N} \min(s_i, t_i)$ where $x = \langle s_i \rangle$, $t = \langle t_i \rangle$, $x_i \geq 0$, $t_i \geq 0$, $i = 1,\ldots,N$. Then $W^{(N,d)}$ is to

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be the process with values in R^d such that each component is an N-parameter Wiener process, and the components are independent. For the case $N = d = 1$, Taylor (1953) showed that a.s. the Hausdorff dimensions of the level sets and the graph of the Brownian motion are $1/2$ and $3/2$ respectively: furthermore, the $1/2$ - measure of the level sets and $3/2$ - measure of the graph are zero. The purpose of this paper is to extend Taylor's results to $W^{(N,d)}$. Our investigation on the level sets is motivated by a question raised by Pyke (1972) as to what topological or dimensional properties are possessed by the zero-sets of $W^{(N,1)}$.

Wherever possible, we shall use the notation of Orey and Pruitt (1973). The parameter space is the set $t \in R_+^N$ with all components non-negative. We sometimes write t as $\langle t_1, \ldots, t_N \rangle$ or simply $\langle t_i \rangle$. When all $t_i = a$, $\langle t_i \rangle$ is written as $\langle a \rangle$. By $t > s$, we mean $t_i > s_i$ for all $1 \le i \le N$. For $s = \langle s_i \rangle$ and $t = \langle t_i \rangle$ with $s_i < t_i$, $X_{i=1}^N [s_i, t_i]$ is denoted by $\Delta(s, t)$ or $\Delta(t)$ in case $s = \langle 0 \rangle$. The interval $\Delta(\langle 1/2 \rangle, \langle 1 \rangle)$ referred to often will be denoted by I for brevity. For simplicity we shall write $W = W^{(N,d)}$. Denote the i^{th} component of W by W^i where $1 \le i \le d$. Let $s, t \in R_+^N$. Then the variance of $W^i(t) - W^i(s)$ can be verified to be $|S(s,t)|$ where $S(s,t)$ is the symmetric difference between $\Delta(s)$ and $\Delta(t)$ and $| \quad |$ denotes the N-dimensional Lebesgue **measure.**

We will often state that W has scaling property and stationary independent increments. For an account of these properties and further information on W, the reader is referred to Kitagawa (1951), Chentsov (1956), Yeh (1960, 1963a, 1963b), Delporte (1966), C. Park (1967), W. J. Park (1970), Zimmerman (1972), Orey and Pruitt (1973) and Tran (1976, 1977). Occasionally we will use c to denote constants whose values are unimportant and may be different from line to line.

2. Dimensions of the level sets and graph

Let $u \in R^d$. Define $W^{-1}(u) = \{t \in \Delta(<0>,<1>) : W(t) = u\}$. when u is not the origin of R^d, and when u is the origin of R^d, define $W^{-1}(u) = \{t \in \Delta(<0>, <1>) : 0 < t$ and $W(t) = 0\}$. When $u = 0$, we require that $t > 0$ to avoid trivialities which arise due to the fact that $W(t) = 0$ if any coordinate of t is zero. We will refer to $W^{-1}(u)$ as the u-level set of W and to $\{(t,W(t)) : t \in R_+^N\}$ as the graph of W.

Theorem 2.1 (i) If $d \geq 2N$ then the dimension of the u-level set is a.s. zero for any u. If $d < 2N$ then the dimension of the u-level set is $N - d/2$ with positive probability for any u. This probability is 1 if $u = 0$.

Proof (i) When $d \geq 2N$, Orey and Pruitt (1973) have shown that W does not hit points. Consequently the u-level set is empty and the result follows trivially.

Assume $d < 2N$. Then computations similar to those of Kahane (1968, p. 147) or Adler (1977) can be carried out for W to show that with positive probability, $W^{-1}(u)$

has positive β capacity for $\beta < N - d/2$, implying that the dimension of $W^{-1}(u)$ is greater than or equal to $N - d/2$ with positive probability. The dimension of $W^{-1}(u)$ is a.s. not greater than $N - d/2$ by Theorem 3.1 to follow. Thus for any u, the dimension of the u-level set is $N - d/2$ with positive probability. We now show that this probability is 1 if $u = 0$. By the scaling property of W, with the same probability the set $\{t \in \Delta(\langle 0\rangle, \langle 1/n\rangle) : 0 < t$ and $W(t) = 0\}$ has dimension $N - d/2$ for all $n \geq 1$. Since this probability is positive, by the Zero-One Law of Orey and Pruitt (1973, p. 143) for almost all ω there is an n such that the set $\{t \in \Delta(\langle 0\rangle, \langle 1/n\rangle) : 0 < t$ and $W(t,\omega) = 0\}$ has dimension $N - d/2$. Thus $W^{-1}(0)$ has dimension $N - d/2$ a.s.

ii) The proof of ii) is similar to the proof of Theorem 2.1 of Tran (1977) and hence is omitted. We now turn to the evaluation of the $N - d/2$ measure of the level sets and the $\{\min 2N, N + d/2\}$-measure of the graph which is a more complicated and interesting problem since Taylor's arguments for $N = 1$ rely on the Strong Markov Property of $W^{(1,d)}$ and thus do not extend easily to $W^{(N,d)}$.

3. The $N - d/2$ measure of the level sets is zero a.s.

First, we partition I into a grid of cubicles as done in Tran (1977). Let m_n be a sequence of positive integers with $m_{n+1} = 2m_n$. For each m_n, partition I into m_n^N cubicles with sides parallel to coordinate axes and equal to $2^{-1}m_n^{-1}$. Let $G(m_n)$ be the collection of these cubicles. Order the cubicles of $\bigcup_{k=1}^{\infty} G(m_k)$ in such a way that the cubicles of $G(m_k)$ preceed the cubicles of $G(m_{k+1})$ for all $k \geq 1$. Denote the ordered collection of cubicles by $\{C_n\}$. Let C_k be a cubicle of $\{C_n\}$ with sides equal to a_k and let t^k be the least vertex of C_k, i.e. closest to $\langle 0 \rangle$. Pick μ to be any positive number.

Define

$$(3.1) \qquad D_k = \left[\omega : \sup_{t \in C_k} \left| W(t, \omega) - W(t^k, \omega) \right| < \mu\, a_k^{1/2} \right]$$

We will need the following lemma, the proof of which can be found in Tran (1977, Lemma 3.1).

<u>Lemma 3.1</u> <u>Let</u> $\{B_n\}$ <u>be a subsequence of the sequence of cubicles</u> $\{C_n\}$ <u>with</u> $B_{n+1} \subset B_n \in G(m_n)$ <u>for all</u> $n \geq 1$ <u>and</u> $\{D_n\}$ <u>be the sequence of events defined in</u> (3.1) <u>corresponding to this subsequence</u>. <u>Then</u> $P(D_n$ infinitely often$) = 1$.

<u>Theorem 3.1</u> <u>Let</u> u <u>be an arbitrary point of</u> R^d. <u>Then</u> $W^{-1}(u)$ <u>has zero</u> $N - d/2$ - <u>measure with probability one</u>.

<u>Proof</u> It is enough to show that $\{t \in I : W(t) = u\}$ has zero $N - d/2$ - measure almost surely. Consider the collection of cubicles $\{C_n\}$ ordered as above. Let t^k be the least vertex of C_k. Pick $\mu > 0$. Corresponding to each cubicle C_k, let S_k be the rectangle in R^{N+d} with sides parallel to the coordinate axes and with least and largest vertices at respectively

$$(t_1^k,\ldots,t_N^k \;,\; W^1(t^k) - \mu a_k^{\frac{1}{2}},\ldots,W^d(t^k) - \mu a_k^{\frac{1}{2}})$$

and

$$(t_1^k + a_k,\ldots,t_N^k + a_k \;,\; W^1(t^k) + \mu a_k^{\frac{1}{2}},\ldots,W^d(t^k) + \mu a_k^{\frac{1}{2}}) \;.$$

Define

$$\begin{aligned} V_1 &= S_1 \quad \text{if} \quad \{(t\,,W(t) : t \in C_1\} \subset S_1 \\ &= \phi \quad \text{otherwise} \end{aligned}$$

with ϕ being the empty set.

For $k > 1$, define

$$\begin{aligned} V_k = S_k \quad \text{if} \quad & \{(t\,,W(t)) : t \in C_k\} \subset S_k \quad \text{and} \\ & \{(t\,,W(t)) : t \in C_k\} \not\subset \bigcup_{j=1}^{k-1} V_j \\ = \phi \quad & \text{otherwise} \end{aligned}$$

Choose K to be a positive integer and write $\sum_{k=1}^{K} (m_k)^N$ as γ_K for brevity. Define $\Psi_K = \bigcup_{k=1}^{\gamma_K} V_k$ and

$$r_K = (m_K^N)^{-1}\,(\#\ \text{of}\ \ C_k \in G(m_K)\ \ \text{with}\ \ \{(t\,,W(t)) : t \in C_k\} \not\subset \Psi_K) \;.$$

We now claim that $Er_K \to o$ as $K \to \infty$. If the claim is not true then by following the same line of argument presented in the proof of Theorem 3.1 in Tran (1977) , we can construct a subsequence $\{B_n\}$ of $\{C_n\}$ with $B_{n+1} \subset B_n \in G(m_n)$ and with $P\Big(\{(t\,,W(t)) : t \in B_n\} \subset \Psi_n\Big) \geq \varepsilon$ for all $n \geq 1$. This can easily be seen to contradict Lemma 3.1 .

For each cubicle $C_k \in G(m_K)$, consider the class ξ_k of rectangles in R^{N+d} with sides parallel to coordinate axes, least and largest vertices of the form

$$(t_1^k,\ldots,t_N^k \;,\; k_1 m_K^{-\frac{1}{2}},\ldots,k_d m_K^{-\frac{1}{2}})$$

and

$$(t_1^k + a_k,\ldots,t_N^k + a_k \;,\; (k_1 + 1)m_K^{-\frac{1}{2}},\ldots,(k_d + 1)m_K^{-\frac{1}{2}}) \;,$$

where $k_1,\ldots,k_d$ are integers.

Let C_k be any cubicle of $G(m_K)$ with

$$\{(t, W(t)) : t \in C_k\} \not\subset \Psi_K .$$

Let M be a large positive integer to be specified later. We shall follow the method used in the proof of Theorem 3.4 of Orey and Pruitt (1973) and count in a central block of $(2M+1)^d$ rectangles centered at E_k where E_k is the element of ξ_k which contains $(t_1^k,\ldots,t_N^k, W^1(t^k),\ldots,W^d(t^k))$. Then, for all cubicles C_k of $G(m_K)$, without taking into account whether $\{(t, W(t)) : t \in C_k\}$ has been covered by Ψ_K or not; we add any rectangles outside this central block which are intersected by $\{(t, W(t)) : t \in C_k\}$. Denote the number of rectangles added by N_k. Now, $\{(t, W(t)) : t \in I\}$ is totally covered by three collections of rectangles.

Let λ_N denote Lebesgue measure in R^N and $L_1(u, m_i)$ denote the union of cubicles $C_k \in G(m_i)$ with $V_k \neq \phi$ and also with $u \in \{W(t) : t \in C_k\}$. Then $\sum_{i=1}^{K} \int_{R^d} \lambda_N(L_1(u, m_i))\, du$ is no larger than the volume of Ψ_K. Evidently, $\sum_{i=1}^{K} \mu^{-d} m_i^{d/2} \int \lambda_N L_1((u, m_i))\, du$ is not larger than the volume of I which is 2^{-N}. Therefore

$$(3.2) \qquad \sum_{i=1}^{K} m_i^{d/2} \int_{R^d} \lambda_N(L_1(u, m_i))\, du < c\mu^d$$

Let $L_2(u, m_K)$ denote the union of cubicles $C_k \in G(m_K)$ with $\{(t, W(t)) : t \in C_k\} \not\subset \Psi_K$ and with $u \in \{W(t) : t \in C_k\}$. Pick $\varepsilon > 0$. By a computation of Orey and Pruitt (1973, Theorem 3.4) , we can pick M large enough such that $EN_k < \varepsilon$, independent of k .

Now we obtain

$$E\int_{R^d} \lambda_N (L_2(u, m_K))\, du \le [Er_K]\, m_K^N\, [(2M+1)\, m_K^{-\frac{1}{2}}]^d\, m_K^{-N} +$$

$$\varepsilon\, m_K^N\, (m_K^{-\frac{1}{2}})^d\, m_K^{-N}$$

$$\le [Er_K]\, (2M+1)^d\, m_K^{-d/2} + \varepsilon\, m_K^{-d/2}$$

which implies that

$$(3.3)\quad m_K^{d/2} \int_{R^d} E\lambda_N(L_2(u, m_K))\, du < c[Er_K] + \varepsilon$$

Let b be a positive number and define a random variable Y as follows

$$Y(\omega) = 1 \quad \text{if} \quad \omega \in [\sup_{t \in I} |\, W(t) - W(<\tfrac{1}{2}>)\,| < b] .$$

$$= 0 \quad \text{otherwise.}$$

Then $E[Y\lambda_N(L_1(u, m_i))]$ is equal to

$$\sum \lambda_N(C_k)\, P([u \in \{W(t) : t \in C_k\}]\, [V_k \ne \phi]\, [Y = 1]) ,$$

where the summation is over all k between $\gamma_{i-1} + 1$ and γ_i . Now the σ field generated by $W(<\frac{1}{2}>)$ is independent of $\bigvee_{t \in I} F(W(t) - W(<\frac{1}{2}>))$ with the latter being the smallest σ field containing all σ fields generated by $W(t) - W(<\frac{1}{2}>)$ with $t \in I$. Observe that the event $[V_k \ne \phi]$ is independent of $W(<\frac{1}{2}>)$. Then $E[Y\lambda_N(L_1(u, m_i))]$ is equal to

$$\sum \lambda_N (C_k) \int P(u - W(<\tfrac{1}{2}>) \in \{W(t, \omega) - W(<\tfrac{1}{2}>, \omega) : t \in C_k\}) \; P(d\omega)$$

where the integration is taken over all ω in

$$[V_k \neq \phi] \cap [Y = 1] .$$

Let a be a positive number and define $U = X_{i=1}^{d} [-a, a]$. The density of $-W(<\tfrac{1}{2}>)$ is bounded below by a positive constant on the set $X_{i=1}^{d} [-a-b, a+b]$ thus for any two points u^1 and u^2 in U and any ω in

$$[V_k \neq \phi] \cup [Y = 1] ,$$

there exists a constant c which depends only on a, b such that

$$P(u^1 - W(<\tfrac{1}{2}>) \in \{W(t, \omega) - W(<\tfrac{1}{2}>, \omega) : t \in C_k\})$$

is bounded above by

$$cP(u^2 - W(<\tfrac{1}{2}>) \in \{W(t, \omega) - W(<\tfrac{1}{2}>, \omega) : t \in C_k\}) .$$

Therefore

$$\text{(3.4)} \quad E[Y\lambda_N(L_1(u^1, m_i))] \leq cE[Y\lambda_N(L_1(u^2, m_i))] .$$

By a similar argument we can show that

$$\text{(3.5)} \quad E[Y\lambda_N(L_2(u^1, m_i))] \leq cE[Y\lambda_N(L_2(u^2, m_i))] .$$

Also, (3.2) and (3.3) imply that

$$(3.6) \qquad \sum_{i=1}^{K} \int_U m_i^{d/2} \, E[Y\lambda_N(L_1(u,m_i))]\, du + \int_U m_K^{d/2} \, E[Y\lambda_N(L_2(u,m_K))]\, du$$

can be made arbitrarily small by choosing μ , ϵ small and K large enough. From (3.4) , (3.5) and (3.6) , we deduce that

$$\sum_{i=1}^{K} m_i^{d/2} \, E[Y\lambda_N(L_1(u\ ,\ m_i))] + m_K^{d/2} \, E[Y\lambda_N(L_2(u\ ,\ m_K)]$$

can be made arbitrarily small for $u \in U$. However

$$\bigcup_{i=1}^{K} L_1(u\ ,\ m_i) \cup L_2(u\ ,\ m_K)$$

forms a covering of

$$\{t \in I : W(t) = u\} \ .$$

Thus clearly for almost all ω with $y(\omega) = 1$

$$\Lambda^{N-d/2}\{t \in I : W(t,\omega) = u\} = 0 \quad \text{for any} \quad u \in U \ .$$

Also, $P(Y=1) \to 1$ as $b \to \infty$ and a can be chosen arbitrarily large. Therefore the $N - d/2$ - measure of $\{t \in I : W(t) = u\}$ is almost surely zero for all $u \in R^d$ including 0 .

4. <u>The $\min\{2N\ ,\ N = d/2\}$ - measure of the graph is zero almost surely.</u>

<u>Theorem 4.1</u> With probability one, the $\min\{2N, N + d/2\}$ - measure of the graph is zero.

<u>Proof</u> (i) $d < 2N$. Choose m_1 so that $m_1 > \mu^{-2}$. Each rectangle V_k in Ψ_K can be contained in n_0 cubes with sides equal to those of C_k where

$$n_0 \le (2\mu a_k^{-1/2} + 1)^d \le (3\mu a_k^{-1/2})^d .$$

Next

$$\sum_{k=1}^{\gamma_K} (\text{diam } V_k)^{N+d/2} \text{ is bounded by}$$

$$\sum_{k=1}^{\gamma_K} (3\mu a_k^{-1/2})^d \, [(N+d)^{1/2} a_k]^{N+d/2} \, g(k)$$

where $g(k) = 1$ if $V_k \ne \phi$
$= 0$ otherwise.

Observe that

$$\sum_{k=1}^{\gamma_K} a_k^N \le 2^{-N} . \quad \text{Thus}$$

(4.1) $$\sum_{k=1}^{\gamma_K} (\text{diam } V_k)^{N-d/2} < c\mu^d$$

The number of rectangles added in the central blocks is

$$r_K m_K^N (2M+1)^d .$$

Each rectangle can be contained in no more than $(m_K^{1/2} + 1)^d$

cubes of side m_K^{-1} . Now

$$(m_K^{1/2} + 1)^d < (2m_K^{1/2})^d .$$

Thus for this collection of rectangles

$$(4.2) \quad E\sum (\text{diam})^{N+d/2} \le [Er_K]\,(2M+1)^d\, m_K^N\, (2m_K^{1/2})^d \times [(N+d)^{1/2}\,(m_K^{-1})]^{N+d/2}$$

$$\le c[Er_K]$$

With respect to the rectangles added outside the central blocks

$$(4.3) \quad E\left[\sum (\text{diam})^{N+d/2}\right] \le \sum_{\gamma_{K-1}+1}^{\gamma_K} [EN_k]\,(2m_K^{1/2})^d\,[N+d)^{1/2}\, m_K^{-1}]^{N+d/2}$$

$$\le c\varepsilon$$

Since (4.1) , (4.2) and (4.3) can be made arbitrarily small by choosing μ , ε small and K large, we can conclude that the expectation of the $N + d/2$ - measure of $\{(t , W(t)) : t \in I\}$ is zero. Therefore the graph has zero $N + d/2$ - measure a.s.

ii) $d \ge 2N$.

Cover the set $\{(t , W(t)) : t \in I\}$ as in part (i) .
Now (4.1) becomes

$$(4.4) \quad \sum (\text{diam } V_k)^{2N} \le \sum_{k=1}^{\gamma_K} [Na_k^2 + d(2\mu a_k^{1/2})^2]^N$$

$$\le \sum_{k=1}^{\gamma_K} [N\mu^2 a_k + d4\mu^2 a_k]^N$$

$$\le c\mu^{2N}$$

Next, (4.2) becomes

$$(4.5) \quad E\left[\sum (\mathrm{diam})^{2N}\right] = [Er_K]\,(2M+1)^d\, m_K^N\,[N(m_K^{-1})^2 + d(m_K^{-1/2})^2]^N$$

$$< [Er_K]\, c$$

Similarly, (4.3) is replaced by

$$(4.6) \quad E\sum (\mathrm{diam})^{2N} < c\varepsilon$$

As above, (4.4), (4.5) and (4.6) imply that the 2N - measure of the graph is zero a.s.

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