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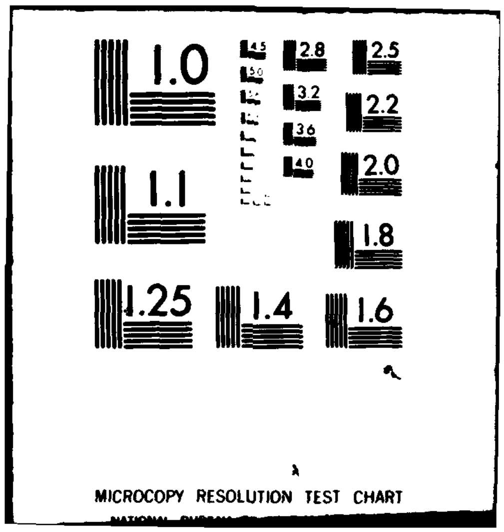
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THE FISHER-BINGHAM DISTRIBUTION ON THE SPHERE

by

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The Fisher-Bingham distribution on the sphere

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Abstract

The Fisher distribution is the analogue on the sphere of the isotropic bivariate normal distribution in the plane. The purpose of this paper is to propose and analyze a spherical analogue of the general bivariate normal distribution. Estimation, hypothesis testing and confidence regions are also discussed.

Key words and phrases: directional data, Fisher distribution, constrained eigenvectors.

AMS 1979 subject classification. primary 62F10;
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Introduction.

The Fisher distribution on the sphere is the analogue of the isotropic bivariate normal, that is the Fisher distribution has circular contours of constant probability. However, in some problems it is desirable to have a more general distribution on the sphere with oval contours in order to provide an analogue of the general bivariate normal distribution. The purpose of this paper is to construct a suitable spherical analogue of the general bivariate normal distribution (denoted as the FB_5 distribution below).

After setting up our notation in Section 2, we define the 8-parameter Fisher-Bingham distribution (FB_8) in Section 3. The FB_5 distribution will appear as a 5-parameter sub-family of FB_8. The limiting normal behaviour of FB_5 for large concentration and other motivating properties of FB_5 are discussed in Sections 4 and 6.

Sufficient statistics for the FB_5 distribution are described in Section 5, estimation of the parameters in Section 8, and a confidence region for the mean direction in Section 10. Several hypothesis tests of interest are discussed in Section 9.

An example to illustrate the use of the use of the FB_5 distribution is given in Section 11. Analogues of the Fisher-Bingham distribution in other dimensions are briefly mentioned in Section 12.

Although the primary emphasis in this paper is on the FB_5 distribution, properties which applying to other sub-families of FB_8 will also be mentioned where relevant.

2. Notation

Let $\Omega_3 = \{\underset{\sim}{x} \in R^3 : \underset{\sim}{x}'\underset{\sim}{x} = x_1^2 + x_2^2 + x_3^2 = 1\}$ denote the unit sphere in R^3. We can write $\underset{\sim}{x}$ in polar coordinates (θ,ϕ) defined by

$$x_1 = \cos\theta, \quad x_2 = \sin\theta\cos\phi, \quad x_3 = \sin\theta\sin\phi \qquad (2.1)$$

where $0\leq\theta\leq\pi$, $0\leq\phi<2\pi$. If $d\underset{\sim}{x}$ denotes Lebesgue measure on Ω_3, then in polar coordinates $d\underset{\sim}{x} = \sin\theta\, d\theta\, d\phi$. Throughout this paper we shall define distributions on Ω_3 in terms of densities with respect to $d\underset{\sim}{x}$.

A useful way to plot spherical data is given by Lambert's equal area projection (see e.g. Mardia, 1972, p.215) defined by

$$z_2 = \rho\cos\phi\ , \quad z_3 = \rho\sin\phi \qquad (2.2)$$

where $\rho = 2\sin(\theta/2)$, $0<\rho<2$.

For any matrix $\underset{\sim}{A}(n\times p)$, let $\underset{\sim}{A}'$ denote the transpose of $\underset{\sim}{A}$, $\underset{\sim}{a}_{(j)}(n\times 1)$ the jth column of $\underset{\sim}{A}$, $j=1,\ldots,p$, and $\underset{\sim}{a}_i(p\times 1)$ the ith row of $\underset{\sim}{A}$ (written as a column vector), $i=1,\ldots,n$.

An orthogonal matrix $\underset{\sim}{\Gamma}(3\times 3)$ of positive determinant depends on 3 polar coordinates. Let us denote by $\underset{\sim}{\Gamma} = \Gamma(\psi,\eta,\xi)$ the matrix defined by

$$\underset{\sim}{\Gamma} = [\underset{\sim}{g}_{(1)},\ \cos\xi\, \underset{\sim}{g}_{(2)} + \sin\xi\, \underset{\sim}{g}_{(3)},\ -\sin\xi\, \underset{\sim}{g}_{(2)} + \cos\xi\, \underset{\sim}{g}_{(3)}] \qquad (2.3)$$

where

$$\underset{\sim}{G} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi\cos\eta & \cos\psi\cos\eta & -\sin\eta \\ \sin\psi\sin\eta & \cos\psi\sin\eta & \cos\eta \end{bmatrix} \qquad (2.4)$$

and $0\leq\psi\leq\pi$, $0\leq\eta,\xi<2\pi$.

We next define a concept we shall need later in the paper. Given a non-zero vector $\underset{\sim}{u}(3\times 1)$ and a symmetric matrix $\underset{\sim}{A}$, let

$\underset{\sim}{E} = \underset{\sim}{E}(\underset{\sim}{u},\underset{\sim}{A})$ be a (3×3) orthogonal matrix such that if $\underset{\sim}{v} = \underset{\sim}{E}'\underset{\sim}{u}$ and $\underset{\sim}{B} = \underset{\sim}{E}'\underset{\sim}{A}\underset{\sim}{E}$, then

$$v_2 = v_3 = 0, \quad b_{23} = 0, \tag{2.5}$$

$$v_1 > 0 \ , \ b_{22} \geq b_{33} \ . \tag{2.6}$$

Call the columns of $\underset{\sim}{E}$ the <u>constrained eigenvectors</u> of $(\underset{\sim}{u},\underset{\sim}{A})$. Note that $\underset{\sim}{e}_{(1)}$ is proportional to $\underset{\sim}{u}$ whereas $\underset{\sim}{e}_{(2)}$ and $\underset{\sim}{e}_{(3)}$ diagonalize $\underset{\sim}{A}$ "as much as possible" subject to being constrained by $\underset{\sim}{e}_{(1)}$. Also note that $\underset{\sim}{e}_{(1)}$ defines a <u>vector</u> (whose sign is determined by (2.6)), whereas $\underset{\sim}{e}_{(2)}$ and $\underset{\sim}{e}_{(3)}$ only define <u>axes</u> (whose order is determined by (2.6)). The constrained eigenvectors can also be viewed as the eigenvectors after projecting $\underset{\sim}{A}$ onto the subspace orthogonal to $\underset{\sim}{u}$ (see Kato, 1966, pp 61-62).

It is also convenient to summarize $(\underset{\sim}{u},\underset{\sim}{A})$ in terms of the <u>size</u> and <u>shape</u> quantities,

$$r_1 = v_1 = \|\underset{\sim}{u}\| \quad \text{and} \quad r_2 = b_{22} - b_{33}, \tag{2.7}$$

respectively, which are invariant under orthogonal changes of the coordinate system.

For computational purposes, the matrix $\underset{\sim}{E}$ is most easily obtained by the following two-step procedure. First choose an orthogonal matrix $\underset{\sim}{H}$ to rotate $\underset{\sim}{u}$ to the north pole $(1,0,0)'$. (In the polar coordinates of (2.1) with $\underset{\sim}{H}$ for $\underset{\sim}{\Gamma}$, choose ψ and η so that $\underset{\sim}{h}_{(1)} = \underset{\sim}{u}$; here ξ is arbitrary, so for simplicity we can take $\xi = 0$.) Then set $\underset{\sim}{w} = \underset{\sim}{H}'\underset{\sim}{u}$, and $\underset{\sim}{C} = \underset{\sim}{H}'\underset{\sim}{A}\underset{\sim}{H}$. Secondly, choose a rotation $\underset{\sim}{K}$ about the north pole to diagonalize $\underset{\sim}{C}_L$, where

$$\underset{\sim}{C}_L = \begin{bmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{bmatrix}$$

is the lower (2×2) submatrix of $\underset{\sim}{C}$. (In the polar coordinates of

(2.1) with $\underset{\sim}{K}$ for $\underset{\sim}{\Gamma}$, take $\psi = 0$, $\eta = 0$ and choose ξ to satisfy

$$\tan 2\xi = 2c_{23}/(c_{22}-c_{33}),$$

ensuring that (2.6) also holds.) Then $\underset{\sim}{\Gamma} = \underset{\sim}{H}\underset{\sim}{K}$.

Note that even after the first stage the size and shape have simple interpretations in terms of $\underset{\sim}{w}$ and $\underset{\sim}{C}$. Letting ℓ_1 and ℓ_2 denote the eigenvalues of $\underset{\sim}{C}_L$ we have

$$r_1 = w_1, \quad r_2 = \ell_1 - \ell_2 \quad . \tag{2.8}$$

3. The Fisher-Bingham distribution.

Define a distribution on Ω_3 by the density

$$f(\underset{\sim}{x}) \propto \exp\{\kappa \underset{\sim}{\nu}'\underset{\sim}{x} + \sum_{j=2}^{3} \beta_j (\underset{\sim}{\gamma}'_{(j)}\underset{\sim}{x})^2\}, \quad \underset{\sim}{x}'\underset{\sim}{x} = 1. \tag{3.1}$$

We shall call (3.1) the Fisher-Bingham distribution since the first factor is proportional to a Fisher density and the second to a Bingham density. The 8 parameters of (3.1) are $\kappa \geq 0$, real-valued $\beta_2 \geq \beta_3$, a unit vector $\underset{\sim}{\nu}$, and an orthogonal matrix $\underset{\sim}{\Gamma} = [\underset{\sim}{\gamma}_{(1)}, \underset{\sim}{\gamma}_{(2)}, \underset{\sim}{\gamma}_{(3)}]$.

We shall also use the name FB_8 for the full family (3.1). Note that the constraint $\sum_1^3 x_i^2 = \sum_1^3 (\underset{\sim}{\gamma}'_{(i)}\underset{\sim}{x})^2 = 1$ implies that a term $\beta_1(\underset{\sim}{\gamma}'_{(1)}\underset{\sim}{x})^2$ in the exponent of (3.1) would be redundant in the specification of the density.

The family of FB_8 distributions is closed under orthogonal transformations. If $\underset{\sim}{x}$ is a random vector from $FB_8(\kappa, \beta_2, \beta_3, \underset{\sim}{\nu}, \underset{\sim}{\Gamma})$ and $\underset{\sim}{H}$ is orthogonal, then $\underset{\sim}{H}'\underset{\sim}{x}$ comes from $FB_8(\kappa, \beta_2, \beta_3, \underset{\sim}{H}'\underset{\sim}{\nu}, \underset{\sim}{H}'\underset{\sim}{\Gamma})$. Note that the transformation $\underset{\sim}{x} \mapsto \underset{\sim}{H}'\underset{\sim}{x}$ can be thought of as changing the frame of reference, with the coordinate axes in the new frame given by the columns of $\underset{\sim}{H}$.

Several interesting distributions appear as special cases of FB_8. Besides the uniform, Fisher and Bingham distributions themselves, we also have the following families, all of which are also closed under orthogonal transformations.

(a) FB_6 $(\kappa,\beta_2,\beta_3,\Gamma)$. Put $\underset{\sim}{\nu} = \underset{\sim}{\gamma}_{(1)}$ so that the Fisher axis lines up with one of the Bingham axes. Then the number of parameters is reduced by 2 to 6 parameters.

(b) FB_5 (κ,β,Γ). Put $\underset{\sim}{\nu} = \underset{\sim}{\gamma}_{(1)}$ and set $\beta_2 = -\beta_3 = \beta$ say, with $\beta \geq 0$. This distribution is a 5-parameter sub-family of FB_6 and is proposed here as a spherical analogue of the bivariate normal distribution. The justification for this proposal will be given in Section 6.

(c) FB_4 $(\kappa,\beta,\underset{\sim}{\gamma}_{(1)})$. Put $\underset{\sim}{\nu} = \underset{\sim}{\gamma}_{(1)}$ and set $\beta_2 = \beta_3 = \beta$, say. Again, FB_4 is a sub-family of FB_6 but with very different behaviour from FB_5. Note that since we cannot distinguish between $\underset{\sim}{\gamma}_{(2)}$ and $\underset{\sim}{\gamma}_{(3)}$ here, FB_4 is only a 4-parameter family. If $\beta > \frac{1}{2}\kappa$, then the mode of the FB_4 density is a small circle whose center lies along the $\underset{\sim}{\gamma}_{(1)}$ axis. This distribution was introduced and studied by Bingham and Mardia (1978).

All of the above families of distributions are closed under arbitrary rotations of the coordinate system. The inclusion relationships between them are summarized in Figure 1.

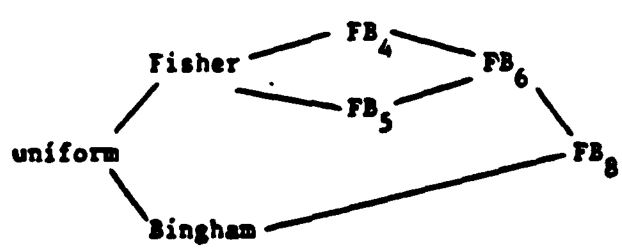

Figure 1.

The Fisher-Bingham distribution was first introduced in Mardia (1975, p.352) on the sphere $\Omega_p, p\geq 2$. It also forms one of a hierarchy of distributions considered in Beran (1979). From Beran's point of view, the Fisher distribution contains an arbitrary linear function of $\underset{\sim}{x}$ in the exponent of the density, and FB_8 contains an arbitrary linear and quadratic function of $\underset{\sim}{x}$. The other distributions in the hierarchy include higher order polynomials in the exponent of the density. An extension of FB_8 to a Stiefel manifold was proposed in Mardia and Khatri (1977).

As noted in Mardia and Khatri (1975), the FB_8 distribution can be obtained by conditioning a trivariate normal distribution with arbitrary mean vector and covariance matrix (a 9-parameter family) to lie on the unit sphere.

Unfortunately, statistical work with the full FB_8 distribution has been hampered by difficulties in estimating and interpreting the parameters (but see de Waal, 1979). However, as we show in this paper, these difficulties do not apply to the FB_5 distribution. For FB_5 the parameters have important and natural interpretations, and estimation is quite feasible.

4. Limiting behaviour of FB_6 for large concentration.

When the FB_6 distribution is highly concentrated about a point, it is well-approximated by a bivariate normal distribution. This result generalizes the well-known property of the Fisher distribution (see e.g. Mardia, 1972, p.246), where an isotropic bivariate normal appears. Details about the closeness of this approximation in the Fisher case can be found in Kent (1978).

For convenience suppose that the orientation matrix $\underset{\sim}{\Gamma}$ equals

$\underset{\sim}{I}$, the identify matrix. Then in the polar coordinates (2.1), the FB_6 density in (3.1) takes the form

$$g(\theta,\phi) = \exp\{\kappa \cos\theta + \beta_2 \sin^2\theta \cos^2\phi + \beta_3 \sin^2\theta \sin^2\phi\} . \tag{4.1}$$

Theorem 4.1. Let $\underset{\sim}{x} \sim FB_6\,(\kappa, \beta_1, \beta_2, \underset{\sim}{I})$ and let the parameters κ, β_1, β_2 vary in such a way that

$$\kappa\to\infty,\ \beta_2/\kappa \to d_2,\ \beta_3/\kappa \to d_3 \ \text{ with } \ -\infty<d_3\le d_2<\tfrac{1}{2} . \tag{4.2}$$

Then as $\kappa\to\infty$,

$$\kappa^{\frac{1}{2}}\begin{bmatrix}\theta\cos\phi\\ \theta\sin\phi\end{bmatrix} \doteq \kappa^{\frac{1}{2}}\begin{bmatrix}x_2\\ x_3\end{bmatrix} \xrightarrow{D} N_2\left(\begin{bmatrix}0\\0\end{bmatrix},\ \begin{bmatrix}(1-2d_2)^{-1} & \\ 0 & (1-2d_3)^{-1}\end{bmatrix}\right), \tag{4.3}$$

where $\doteq$ denotes asymptotically equal in distribution and $\xrightarrow{D}$ denotes convergence in distribution.

Proof. Using the Taylor series expansions

$$\cos\theta = 1-\theta^2/2 + \cdots,\quad \sin\theta = \theta + \cdots \tag{4.4}$$

for θ small and using (4.2) we see that (4.1) is approximately proportional to

$$\exp\{-\tfrac{1}{2}\kappa[\theta^2 - 2d_2\theta^2\cos^2\phi - 2d_3\theta^2\sin^2\phi]\} , \tag{4.5}$$

which is the form of the limiting density in (4.3). To make this argument rigorous, it is merely necessary to show that

(a) the approximation (4.5) is adequate for $|\theta|<\kappa^{-\frac{1}{2}}$ and

(b) the probability mass associated with $|\theta|>\kappa^{-\frac{1}{2}}$ is negligible. The details are straightforward.

Similarly, it is straightforward to show that the two expressions on the left-hand side of (4.3) are asymptotically equal in distribution. $\square$

Corollary 4.1. Let $\underset{\sim}{x} \sim FB_5(\kappa,\beta,\underset{\sim}{I})$. If κ and β vary in such a way that

$$\kappa \to \infty, \quad \beta/\kappa \to d \text{ with } 0 \le d < \tfrac{1}{2},$$

then

$$\kappa^{\frac{1}{2}} \begin{bmatrix} \theta \cos \phi \\ \theta \sin \phi \end{bmatrix} \doteq \kappa^{\frac{1}{2}} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \xrightarrow{D} N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} (1-2d)^{-1} & 0 \\ 0 & (1+2d)^{-1} \end{bmatrix} \right).$$

5. Sufficient statistics

The FB_8 distribution forms a canonical exponential family in its 8 parameters. If we write $\underset{\sim}{\alpha} = \kappa \underset{\sim}{\nu}$ and $\underset{\sim}{\Xi} = \underset{\sim}{\Gamma}\, \mathrm{diag}(0,\xi_2,\xi_3)\underset{\sim}{\Gamma}'$ then

$$f(\underset{\sim}{x}) \propto \exp\{\underset{\sim}{\alpha}'\underset{\sim}{x} + \underset{\sim}{x}'\underset{\sim}{\Xi}\underset{\sim}{x}\} \tag{5.1}$$

and a possible choice for the natural parameter vector is

$$(\alpha_1,\alpha_2,\alpha_3,\ \pi_{12},\pi_{13},\pi_{22},\pi_{23},\pi_{33})' . \tag{5.2}$$

Given an $(n\times 3)$ data matrix $\underset{\sim}{X}$ from the FB_8 distribution, the corresponding sufficient statistic is

$$\underset{\sim}{t} = \frac{1}{n} \sum_{i=1}^{n} (x_{i1},x_{i2},x_{i3},\ 2x_{i1}x_{i2},\ 2x_{i1}x_{i3},\ x_{i2}^2,\ 2x_{i2}x_{i3},\ x_{i3}^2)' \tag{5.3}$$

Note that $\underset{\sim}{t}$ holds the information contained in the sample mean vector and the sample dispersion matrix about $\underset{\sim}{0}$,

$$\bar{\underset{\sim}{x}} = \frac{1}{n}\Sigma\, \underset{\sim}{x}_i, \quad \underset{\sim}{S} = \frac{1}{n}\Sigma\, \underset{\sim}{x}_i\underset{\sim}{x}_i' . \tag{5.4}$$

Unfortunately, the other families described above (FB_4, FB_5 and FB_6) are not canonical exponential families, but instead from curved exponential families. In each case the parameter space has fewer than 8 dimensions, but the minimal sufficient statistic is still given by (5.3).

These remarks about FB_4, FB_5 and FB_6 apply only when all the parameters are unknown. A neater situation arises if we suppose that the mean direction $\underset{\sim}{\nu}$ (and also possibly the concentration κ) is <u>known</u>. With this knowledge FB_4, FB_5 and FB_6 (and also FB_8) now become <u>canonical</u> exponential families.

For definiteness we consider FB_5. After rotating the coordinate system so that $\underset{\sim}{\nu} = \underset{\sim}{\gamma}_{(1)}$ becomes equal to $(1,0,0)'$ the density is proportional to

$$f(\underset{\sim}{x}) = \exp\{\kappa x_1 + \delta_1(x_2-x_3)^2 + 2\delta_2 \; x_{23}^2\} \; , \tag{5.5}$$

or in polar coordinates,

$$g(\theta,\phi) = \exp\{\kappa \cos\theta + \beta \sin^2\theta \; \cos 2(\phi-\chi)\} \; , \tag{5.6}$$

where

$$\delta_1 = \beta \cos 2\chi \; , \qquad \delta_2 = \beta \sin 2\chi \; . \tag{5.7}$$

The parameter $\chi \epsilon [0,2\pi)$ describes the direction of the major axis - see Section 6. Then the natural parameter and sufficient statistic are given by

$$(\kappa,\delta_1,\delta_2)' \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n}(x_{i1}, x_{i2}^2 - x_{i3}^2, \; 2x_{i2}x_{i3})' \; . \tag{5.8}$$

If κ is also known, the natural parameter and sufficient statistic become slightly simpler; namely

$$(\delta_1,\delta_2)' \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n}(x_{i2}^2 - x_{i3}^2, 2x_{i2}x_{i3})' \; . \tag{5.9}$$

6. <u>Properties of the FB_5 distribution</u>.

The FB_5 density was defined in Section 3 by

$$f(\underset{\sim}{x}) = \exp\{\kappa \underset{\sim}{\gamma}'_{(2)}\underset{\sim}{x} + \beta[(\underset{\sim}{\gamma}'_{(2)}\underset{\sim}{x})^2 - (\underset{\sim}{\gamma}'_{(3)}\underset{\sim}{x})^2]\} .$$

As we shall see below the parameters can be described as follows: $\kappa \geq 0$ is the <u>concentration</u>, $\beta \geq 0$ describe the <u>ovalness</u>,

$\underset{\sim}{\gamma}_{(1)}$ is the <u>mean direction</u> or <u>pole</u>, $\underset{\sim}{\gamma}_{(2)}$ is the <u>major axis</u>, and $\underset{\sim}{\gamma}_{(3)}$ is the <u>minor axis</u>. Note that $\underset{\sim}{\gamma}_{(2)}$ and $\underset{\sim}{\gamma}_{(3)}$ are determined only up to sign, so that they do indeed define <u>axes</u> rather than <u>directions</u>.

If we rotate to the frame of reference defined by the columns of $\underset{\sim}{\Gamma}$, the density $f(\underset{\sim}{x})$ takes a particularly simple form. For this reason we shall call this transformation, $\underset{\sim}{x} \mapsto \underset{\sim}{x}^* = \underset{\sim}{\Gamma}'\underset{\sim}{x}$, the transformation to the <u>population standard frame of reference</u>. The density for $\underset{\sim}{x}^*$ takes the form

$$f(\underset{\sim}{x}^*) = \exp\{\kappa x_1^* + \beta(x_2^{*2} - x_3^{*2})\}$$

or in polar coordinates

$$g(\theta,\phi) = \exp\{\kappa \cos\theta + \beta \sin^2\theta \cos 2\phi\} \quad .$$

A sample analogue to the population standard frame will be defined in Section 8.

As stated in the introduction, the FB_5 distribution is proposed here as a spherical analogue of the bivariate normal distribution. The following properties show why this is a sensible proposal. We need to suppose here that $2\beta<\kappa$ to ensure the correct behaviour.

(a) FB_5 and the bivariate normal are both 5-parameter families.

(b) The contours of constant probability near the pole $\underset{\sim}{\gamma}_{(1)}$ are approximately ellipses with major and minor axes $\underset{\sim}{\gamma}_{(2)}$ and $\underset{\sim}{\gamma}_{(3)}$, respectively. (This property follows easily from (4.4).)

(c) The geometric average of $g(\theta,\phi)$ over circles of constant latitude is proportional to a Fisher density, that is,

$$\int_0^{2\pi} \log g(\theta,\phi)d\phi = \kappa \cos\theta + \text{constant.}$$

Thus, in this sense FB_5 is a natural extension of the Fisher distribution.

(d) As θ goes from 0 to π for fixed ϕ, $g(\theta,\phi)$ decreases monotonically. Thus $g(\theta,\phi)$ is unimodal on all great circles through the pole.

(e) For large values of the concentration parameter κ, FB_5 is approximately the same as a bivariate normal distribution with mean $\underset{\sim}{\gamma}_{(1)}$ and major and minor axes $\underset{\sim}{\gamma}_{(2)}$ and $\underset{\sim}{\gamma}_{(3)}$ respectively. (See Corollary 4.1.)

Note that the larger FB_6 family is not a suitable spherical analogue of the general bivariate normal distribution because it has one too many parameters. In Theorem 4.1 we saw that this "extra" parameter is "asymptotically unidentifiable" for large concentration.

Hence we have introduced a further constraint $(\beta_2 = -\beta_3)$ to define the FB_5 family. To some extent this constraint is arbitrary (in fact a different constraint was proposed in Kent, 1980). However, the constraint used here does have some theoretically attractive properties ((c) and (d) above). Moreover, as we shall see in the next section, with this definition of FB_5 the normalization constant takes a reasonably tractable form.

7. Moments of FB_5.

So far we have not dealt with the normalization constant of the FB_5 distribution,

$$c(\kappa,\beta) = \int_0^{\pi} \int_0^{2\pi} \exp\{\kappa \cos\theta + \beta \sin^2\theta \cos 2\phi\} \sin\theta \, d\phi \, d\theta. \quad (7.1)$$

Using the results

$$\int_0^{\pi/2} (\sin\theta)^a(\cos\theta)^b \, d\theta = \tfrac{1}{2} B(\tfrac{a+1}{2}, \tfrac{b+1}{2}), \qquad (7.2)$$

(Abramowitz and Stegun, 1972, (6.2.1), p.258) where $B(\cdot,\cdot)$ is the beta function, and

$$\int_0^{\pi} e^{\kappa\cos\theta} \sin^{2\nu}\theta \, d\theta = \pi^{\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2})(\tfrac{1}{2}\kappa)^{-\nu} I_\nu(\kappa) \qquad (7.3)$$

(Abramowitz and Stegun, 1972, (9.6.18), p.376) where $I_\nu(\kappa)$ is the modified Bessel function, we can expand $c(\kappa,\beta)$ in a series

$$c(\kappa,\beta) = 2\pi \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} \beta^{2k}(\tfrac{1}{2}\kappa)^{-2k-\frac{1}{2}} I_{2k+\frac{1}{2}}(\kappa) . \qquad (7.4)$$

Since sequences of Bessel functions can be quickly and easily computed by the method of Amos (1974), formula (7.4) provides a quick and simple method of calculating $c(\kappa,\beta)$. Note that $c(o,o) = 4\pi$, the surface area of the sphere, and

$$c(\kappa,o) = 4\pi\kappa^{-1} \sinh\kappa ,$$

the normalizing constant for the Fisher distribution.

For large κ (with $2\beta/\kappa<1$ fixed) we have from Corollary 4.1 the asymptotic formula

$$c(\kappa,\beta) \doteq 2\pi e^{\kappa}[(\kappa-2\beta)(\kappa+2\beta)]^{-1/2} . \qquad (7.5)$$

Consider a random vector $\underset{\sim}{x} \sim FB_5(\kappa,\beta,\underset{\sim}{I})$. Differentiating (7.1) under the integral sign and writing $c = c(\kappa,\beta)$, $c_\kappa = \partial c(\kappa,\beta)/\partial\kappa$, etc., we find

$$\begin{aligned} &E x_1 = c_\kappa/c , \quad E x_1^2 = c_{\kappa\kappa}/c \\ &E(x_2^2-x_3^2) = c_\beta/c , \end{aligned} \qquad (7.6)$$

and hence since $x_1^2+x_2^2+x_3^2 = 1$,

$$E x_2^2 = (c - c_{\kappa\kappa} + c_\beta)/2c$$
$$E x_3^2 = (c - c_{\kappa\kappa} - c_\beta)/2c. \qquad (7.7)$$

For later use write

$$\mu = E x_1 \quad \text{and} \quad \sigma_j^2 = E x_j^2, \quad j = 1,2,3. \qquad (7.8)$$

By symmetry, most of the other moments of interest equal 0,

$$E(x_2) = E(x_3) = 0, \qquad (7.9)$$

$$E(x_2 x_3) = 0, \qquad (7.10)$$

$$E(x_1 x_2) = E(x_1 x_3) = 0. \qquad (7.11)$$

8. Moment Estimation for FB_5.

Let $\underline{x}_1, \ldots, \underline{x}_n$ be a sample from $FB_5(\kappa, \beta, \underline{\Gamma})$. The standard way to estimate the parameters of FB_5 is to use maximum likelihood estimates $\hat{\kappa}, \hat{\beta}, \hat{\underline{\Gamma}}$. However, it does not seem possible to obtain explicit expressions for the m.l.e.s, so iterative methods must be used to find them.

In this section we propose simpler estimates which we call the moment estimates $\tilde{\kappa}, \tilde{\beta}, \tilde{\underline{\Gamma}}$ for the parameters of FB_5. They have the following properties.

(a) The moment estimates are consistent estimates of the true parameters and hence provide suitable starting values for maximum likelihood iteration.

(b) The orientation matrix $\tilde{\underline{\Gamma}}$ can be calculated explicitly.

(c) If either the eccentricity $2\beta/\kappa$ is small (the usual case in practice) or if κ is large, then the moment estimates are close to the m.l.e.s.

(d) If the data is highly concentrated, the concentration parameters $\tilde{\kappa}, \tilde{\beta}$ can also be calculated explicitly.

More specifically the moment estimates are defined as follows. Let $\bar{\underset{\sim}{x}}$ and $\underset{\sim}{S}$ be the sample mean vector and the sample dispersion matrix about $\underset{\sim}{0}$ as in (5.4). Then, in the terminology of Section 2, $\bar{\underset{\sim}{\Gamma}}$ is defined to be the matrix of constrained eigenvectors for $(\bar{\underset{\sim}{x}},\underset{\sim}{S})$. Further, letting $r_1 = \|\bar{\underset{\sim}{x}}\|$ and r_2 denote the size and shape quantities for $(\bar{\underset{\sim}{x}},\underset{\sim}{S})$, the estimates $\bar{\kappa},\bar{\beta}$ of concentration parameters are determined implicitly by the equations

$$r_1 - c_\kappa/c = 0, \quad r_2 - c_\beta/c = 0. \tag{8.1}$$

For large κ, the use of (7.5) leads to the explicit solution

$$\begin{aligned}\bar{\kappa} &= (2-2r_1-r_2)^{-1} + (2-2r_1+r_2)^{-1} \\ \bar{\beta} &= \tfrac{1}{2}\{(2-2r_1-r_2)^{-1} - (2-2r_1+r_2)^{-1}\} .\end{aligned} \tag{8.2}$$

The orientation matrix $\bar{\underset{\sim}{\Gamma}}$ has been chosen so that for $\underset{\sim}{x}^*$ the sample analogues of (7.9)-(7.10) (but not (7.11)) will hold. For this reason we shall term the transformation $\underset{\sim}{x} \mapsto \underset{\sim}{x}^* = \bar{\underset{\sim}{\Gamma}}'\underset{\sim}{x}$ the transformation to the <u>sample standard frame of reference</u>.

The rationale behind the moment estimate $\bar{\underset{\sim}{\Gamma}}$ is as follows. The first column $\bar{\underset{\sim}{\gamma}}_{(1)}$ is the mean direction of the sample, which is also the m.l.e. of the mean direction under a Fisher distribution. If the eccentricity $2\beta/\kappa$ is not too large (which is the most important case in practice), then $\bar{\underset{\sim}{\gamma}}_{(1)}$ will also be close to the m.l.e. of the mean direction for FB_5. Further, if the true mean direction $\underset{\sim}{\gamma}_{(1)} = \bar{\underset{\sim}{\gamma}}_{(1)}$ were known, then (7.3)-(7.4) would ensure that $\bar{\underset{\sim}{\gamma}}_{(2)}$ and $\bar{\underset{\sim}{\gamma}}_{(3)}$ would be the m.l.e.s of the major and minor axes, respectively. Hence if $2\beta/\kappa$ is not too large, the moment estimates should be nearly as efficient as the maximum likelihood estimates.

A similar situation arises for large κ when FB_5 is close to a bivariate normal. Then the moment estimates and m.l.e.s for FB_5 will both be close to the corresponding m.l.e.s for the bivariate normal.

9. Some hypothesis tests.

In this section we describe several large-sample hypothesis tests of interest. All of these tests can be carried out using the following general result.

Theorem 9.1. Consider n independent identically distributed observations from a model H_1 with parameters $(\underset{\sim}{\pi},\underset{\sim}{\lambda})$ of dimensions p and q respectively, and consider a null hypothesis $H_0 : \underset{\sim}{\lambda} = \underset{\sim}{0}$. Suppose that under H_0, the model forms a canonical exponential family for $\underset{\sim}{\pi}$ with minimal sufficient statistic $\underset{\sim}{u}$, and that under H_1 (with $\underset{\sim}{\pi}$ known) the model forms a canonical exponential family for $\underset{\sim}{\lambda}$ with minimal sufficient statistic $\underset{\sim}{v}$. Define Rao's score statistic by

$$W_u = [\underset{\sim}{v} - E(\underset{\sim}{v}|\underset{\sim}{u})]' \ \mathrm{Var}(\underset{\sim}{v}|\underset{\sim}{u})^{-1} \ [\underset{\sim}{v} - E(\underset{\sim}{v}|\underset{\sim}{u})] \ , \tag{9.1}$$

where all moments are calculated under H_0 with $\underset{\sim}{\pi} = \hat{\underset{\sim}{\pi}}$ ($\hat{\underset{\sim}{\pi}}$ being the m.l.e. of π under H_0).

Then asymptotically as the sample size $n \to \infty$

$$W_u \sim \chi_q^2 \ , \tag{9.2}$$

and further W_u is asymptotically equivalent to the likelihood ratio statistic $-2\log L$ for H_0 vs H_1. We reject H_0 if W_u is too large.

Proof. This result is a special case of a general result in Cox and Hinkley (1974), p.324, equation after (5.6). See also Rao (1973), p.418. □

Note that the score statistic W_u is usually simpler to calculate than the likelihood ratio statistic because only a parametric model under H_0 need be fitted. Some hypothesis tests which fit into this framework will now be described.

(a) H_0 : Fisher vs H_1 : FB_5

A Fisher distribution with concentration parameter κ and mean direction $\underset{\sim}{\nu}$ forms a canonical exponential family with natural parameters $(\kappa\nu_1, \kappa\nu_2, \kappa\nu_3)$ and sufficient statistic $\underset{\sim}{u} = n^{-1}\sum \underset{\sim}{x}_i$. The m.l.e.s satisfy

$$\hat{\underset{\sim}{\nu}} = n^{-1}\sum \underset{\sim}{x}_i / r_1 \ , \quad I_{3/2}(\hat{\kappa})/I_{1/2}(\hat{\kappa}) = r_1 \tag{9.3}$$

where $r_1 = \|n^{-1}\sum \underset{\sim}{x}_i\|$ is the resultant length, also used in (8.1). Now let $\hat{H}$ be an orthogonal matrix which depends on the data only through $\hat{\nu}$, and whose first column is given by $\underset{\sim}{h}_{(1)} = \hat{\underset{\sim}{\nu}}$. Let $\underset{\sim}{y}_i = \underset{\sim}{H}'\underset{\sim}{x}_i$, $i = 1,\ldots,n$. Then from (5.9), with $\kappa = \hat{\kappa}$ and $\underset{\sim}{\gamma}_{(1)} = \hat{\underset{\sim}{\nu}}$ assumed known, the model under H_1 forms a canonical exponential family with sufficient statistics

$$w_1 = \frac{1}{n}\sum (y_{i2}^2 - y_{i3}^2) \ , \quad w_2 = \frac{2}{n}\sum y_{i2}y_{i3} \ . \tag{9.4}$$

Now the assumption that the $\underset{\sim}{x}_i$ come from a Fisher distribution $F(\hat{\kappa}, \hat{\underset{\sim}{\nu}})$ is equivalent to the assumption that the $\underset{\sim}{y}_i$ come from $F(\hat{\kappa}, (1,0,0)')$. Further, for a random vector $\underset{\sim}{y} \sim F(\hat{\kappa}, (1,0,0)')$, we have by symmetry

$$E(y_2^2 - y_3^2) = 0 \ , \quad E(y_2y_3) = 0$$

$$E(y_2^3y_3) = E(y_2y_3^3) = 0$$

$$E(y_jy_2^2 - y_jy_3^2) = 0 \ , \ E(y_jy_2y_3) = 0 \ , \quad j = 1,2,3,$$

and we have by (7.2) and (7.3),

$$E(y_2^2 - y_3^2) = E(4y_2^2y_3^2) = (\hat{\kappa}/2)^{-2} \quad I_{5/2}(\hat{\kappa})/I_{1/2}(\hat{\kappa}) \ .$$

Hence the test statistic (9.1) takes the form

$$n \frac{(\hat{\kappa}/2)^2 I_{1/2}(\hat{\kappa})}{I_{5/2}(\hat{\kappa})} (w_1^2 + w_2^2) \sim \chi_2^2 \quad . \tag{9.5}$$

Using the notation of Section 2, it is easy to check that $w_1^2 + w_2^2$ equals the squared shape quantity r_2^2 for $(\bar{x}, S)$. In particular it is clear that the statistic W_u does not depend on the arbitrariness in the choice of the second and third columns of H.

In practice this test might be used in the following situation. Given a set of spherical data, an experimenter might first look for directionality by testing H_o: uniform vs H_1: Fisher (the Rayleigh test). If this null hypothesis is rejected he might assess the circular symmetry of the data about the pole by using the test described here.

For large concentration, this test reduces to a test of sphericity for the bivariate normal distribution; see for example Mardia, Kent, and Bibby (1979) p. 134.

(b) H_o: Fisher vs H_1: FB_8.

As in (a), this is a goodness-of-fit test for the Fisher distribution, but here with a more general alternative in mind. The calculations are similar to those in (a) but somewhat more involved. Details are given in Mardia and Holmes (1980).

The analogous test on the circle was given by Cox (1975); see also Cox and Barndorff-Nielsen (1979), p.291.

(c) H_o: Bingham vs H_1: FB_8.

This test is included here because it fits the assumptions of Theorem 9.1, but the application of this test is somewhat different from (a) and (b). Suppose an experimenter has data to which he would like to fit a Bingham distribution. The data are suspected to be but not known to be antipodally symmetric. There are two ways to proceed in this situation.

(i) First use a Rayleigh test ($3n\, \bar{x}'\bar{x} \sim \chi_3^2$) to test uniform vs

Fisher. If uniformity is accepted, then antipodal symmetry can be presumed and a Bingham distribution can be fitted.

(ii) First fit a Bingham distribution, and then assess the goodness-of-fit by using the test of this section. This latter approach seems more suitable when the data is known at the outset not to be uniformly distributed.

Under H_0, $\underset{\sim}{S}$, the sample dispersion matrix about $\underset{\sim}{0}$, is sufficient for the parameters. If $\underset{\sim}{x}$ comes from a Bingham distribution, we have by symmetry

$$E(x_i) = 0\ , \quad E(x_i x_j x_k) = 0, \quad i,j,k = 1,2,3$$

so that

$$E(\bar{\underset{\sim}{x}}) = \underset{\sim}{0}\ , \quad E(\bar{\underset{\sim}{x}} \mid \underset{\sim}{S}) = \underset{\sim}{0}\ .$$

Further, since the m.l.e.s of the Bingham parameters are chosen so that $E(\underset{\sim}{x}\underset{\sim}{x}') = \underset{\sim}{S}$, we have

$\mathrm{Var}(\bar{\underset{\sim}{x}} \mid \underset{\sim}{S}) = \mathrm{Var}(\bar{\underset{\sim}{x}}) = \frac{1}{n}\underset{\sim}{S}$. Hence the score statistic takes the form

$$n\, \bar{\underset{\sim}{x}}' \underset{\sim}{S}^{-1} \bar{\underset{\sim}{x}} \sim \chi_3^2\ . \tag{9.6}$$

Note that when the data is uniformly distributed, $\underset{\sim}{S} \doteq \frac{1}{3}\underset{\sim}{I}$, so that (9.6) reduces to the Rayleigh test statistic.

10. <u>A confidence interval for the mean direction</u>.

Let $\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n$ be a sample from $FB_5(\kappa,\beta,\underset{\sim}{\Gamma})$. First, transform to the <u>population standard frame</u> (see Section 6) $\underset{\sim}{y}_i = \underset{\sim}{\Gamma}'\underset{\sim}{x}_i, i = 1,\ldots,n$, so that the $\underset{\sim}{y}_i$ form a sample from $FB_5(\kappa,\beta,\underset{\sim}{I})$. Then by the central limit theorem and (7.6)-(7.11) we see that the sample mean $\bar{\underset{\sim}{y}} = n^{-1}\sum \underset{\sim}{y}_i$ is asymptotically normally distributed with mean vector $(\mu,0,0)'$ and covariance matrix $n^{-1}\ \mathrm{diag}(\sigma_1^2,\sigma_2^2,\sigma_3^2)$.

The sample mean direction for the $\underline{y}_i$ (which is also the moment estimate of the true mean direction of the $\underline{y}_i$) is defined by $\bar{\underline{\gamma}}_{(1)} = \bar{\underline{y}}/\|\bar{\underline{y}}\|$. Consider the two coordinates $(\bar{\gamma}_{21}, \bar{\gamma}_{31})'$ of $\bar{\underline{\gamma}}_{(1)}$. Then by a general result on transformations (see, e.g. Rao, 1973, p. 387), $(\bar{\gamma}_{21}, \bar{\gamma}_{31})$ is also asymptotically normally distributed,

$$\begin{bmatrix} \bar{\gamma}_{21} \\ \bar{\gamma}_{31} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} , \frac{1}{n a^2} \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_3^2 \end{bmatrix} \right) .$$

Hence, an asymptotic $100(1-\alpha)\%$ probability region about the mean direction of $FB_5(\kappa,\beta,\Gamma)$ is given in population standard coordinates by the ellipse-like region

$$\{\bar{\underline{\gamma}}_{(1)} \in \Omega_3 : \bar{\gamma}_{11} > 0, \quad n\,\mu^2(\bar{\gamma}_{21}^2/\sigma_2^2 + \bar{\gamma}_{31}^2/\sigma_3^2) \le \chi^2_{2;\alpha}\} , \tag{10.1}$$

where $\chi^2_{2;\alpha}$ denotes the upper α critical value of the χ^2_2 distribution.

Inverting (10.1) provides a confidence region for the true mean direction about the sample mean direction. More specifically, suppose $\underline{x}_1,\ldots,\underline{x}_n$ come from $FB_5(\kappa,\beta,\underline{\Gamma})$ with true mean direction $\underline{\nu} = \underline{\gamma}_{(1)}$. Let $\bar{\Gamma}$ be the moment estimate of $\underline{\Gamma}$ (see Section 7) and transform to the sample standard frame, $\underline{x}^*_i = \bar{\underline{\Gamma}}'\,\underline{x}_i$, $i = 1,\ldots,n$, and let $\underline{\nu}^* = \bar{\underline{\Gamma}}'\underline{\nu}$. Then an asymptotic confidence region for $\underline{\nu}^*$ is defined by the ellipse-like region on the sphere

$$\{\underline{\nu}^* \in \Omega_3 : \nu_1^* > 0, \quad n\,\mu^2(\nu_2^{*2}/\sigma_2^2 + \nu_3^{*2}/\sigma_3^2) < \chi^2_{2;\alpha}\} \tag{10.2}$$

Of course, in practice μ, σ_2^2 and σ_3^2 will not be known, but must be replaced by any consistent estimates. One possibility is to estimate κ and β and then use (7.6)-(7.7). However, a simpler technique is to just use the sample moments

$$\tilde{\mu} = n^{-1} \sum x_{i1}^{*} , \quad \tilde{\sigma}_2^2 = n^{-1} \sum x_{i2}^{*2} \quad \text{and} \quad \tilde{\sigma}_3^2 = n^{-1} \sum x_{i3}^{*2} , \tag{10.3}$$

respectively.

Note that in moving from (10.1) to (10.2) we have switched from a frame of reference about the true mean direction to a frame about the sample mean direction. However, since these two points lie within $O(n^{-1})$ of one another in probability, the complications arising from this switch are negligible and (10.2) remains asymptotically valid.

When using the equal-area projection (2.2) (in the sample standard frame) and when using the estimates (10.3), a confidence region asymptotically equivalent to (10.2) is given by

$$\{(z_2, z_3) : n\, \tilde{\mu}^2 (z_2^2/\tilde{\sigma}_2^2 + z_3^2/\tilde{\sigma}_3^2) < \chi^2_{2;\alpha}\}. \tag{10.4}$$

11. Example.

Creer, Irving and Nairn (1959) measured directions of magnetism at $n = 34$ sites in the Great Whin Sill. Their data is summarized in Table 1, column (b) of that paper, pp.311-312 (excluding sites 32 and 34). Their use of declination (D) and inclination (I) is related to our use of polar coordinates in (2.1) by $\theta = 90^\circ + I$, $\phi = 360^\circ - D$. [Change program accordinly).

The summary statistics are given by

$$\bar{x} = \begin{bmatrix} 0.083 \\ -0.959 \\ -0.131 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.045 & -0.075 & 0.014 \\ -0.075 & 0.921 & -0.122 \\ 0.014 & -0.122 & 0.034 \end{bmatrix},$$

from which we find $\tau_1 = 0.971$, $\tau_2 = 0.0229$.

The moment and maximum likelihood estimates of location are both given to three decimal places by

$$\hat{\Gamma} = \tilde{\Gamma} = \begin{bmatrix} 0.085 & -0.979 & 0.185 \\ -0.987 & -0.108 & -0.117 \\ 0.134 & -0.172 & -0.976 \end{bmatrix},$$

and the estimates of concentration by

$\hat{\kappa} = 42.16$ $\hat{\beta} = 9.27$ (exact moment)

$\hat{\kappa} = 41.76$ $\hat{\beta} = 8.37$ (asymptotic moment)

$\hat{\kappa} = 42.16$ $\hat{\beta} = 9.28$ (maximum likelihood).

The data and a 95% confidence region for the mean direction based on (10.2) and (10.3) are given in Figure 2, plotted using the equal-area projection of (2.2) in the sample standard frame.

The hypothesis test of Section 9(b) and the corresponding likelihood ratio test yield the values

$$W_u = 5.96 \quad \text{and} \quad -2 \log L = 6.55.$$

Since the upper 5% critical value of χ_2^2 is 5.99, both statistics show moderate evidence of a departure from a Fisher distribution.

In this example an alternative approach was used by Creer et.al. They managed to transform the data to approximate circular symmetry and then to use statistical techniques applicable to the Fisher distribution.

12. Analogues in other dimensions.

Much of the theory in this paper extends, at least in principle, to other dimensions. The analogue of the full Fisher Bingham family on the unit sphere in R^p, $p>2$, can be written in the general form (2.1) with the summation from 2 to 3 replaced by a summation from 2 to p (as in Beran, 1979).

An analogue of FB_5 can be obtained by introducing the constraint

$$\sum_{j=2}^{p} \beta_j = 0.$$

On the circle, $p = 2$, this analogue of FB_5 is no more general than the von Mises distribution itself. For general $p \geq 3$, provided $|\beta_j| < \kappa/2$, $j = 2,\ldots,p$, properties analogous to those in Sections 4-6 are valid. When $p>3$, moment estimation still can be carried out, although the normalization constant seems to become more complicated as the dimension increases.

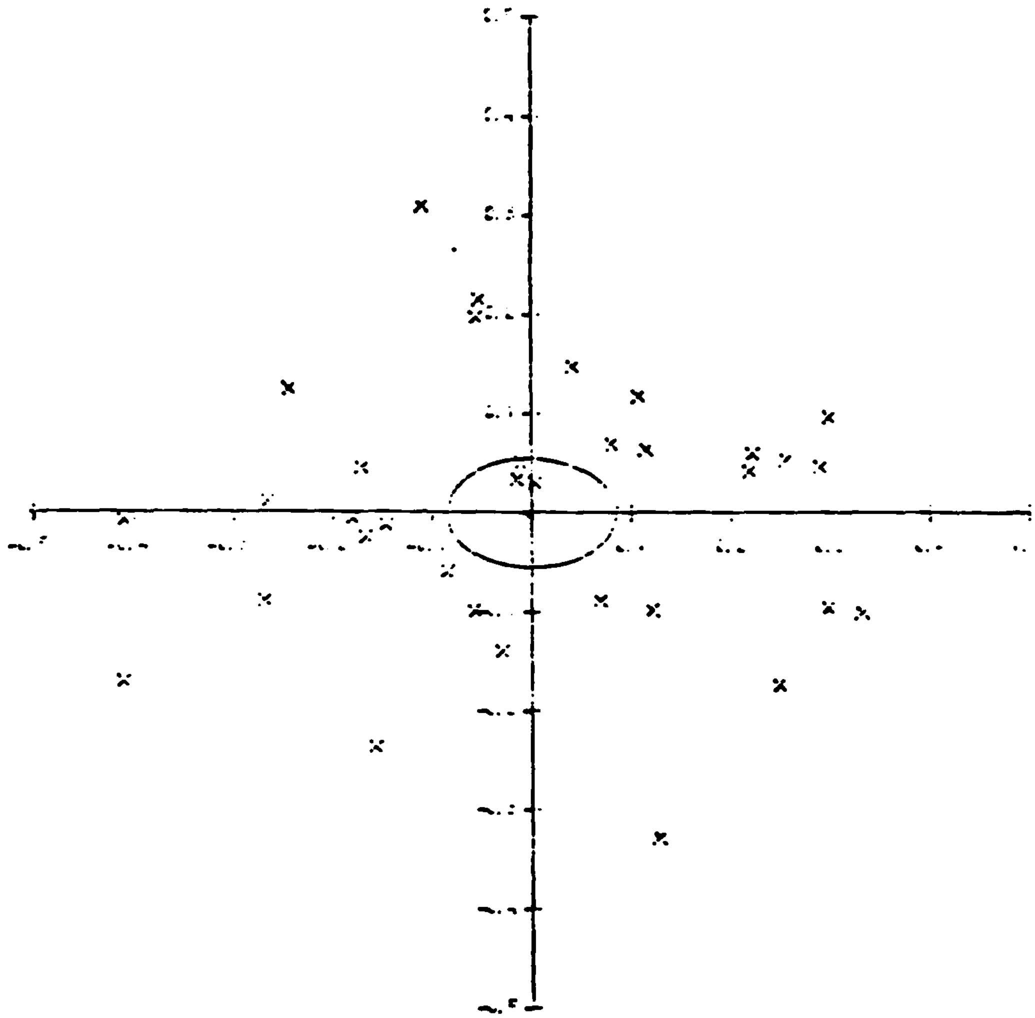

Figure 2: Plot of the Great Whin Sill data using an equal-area projection in the sample standard frame, and a 95% confidence ellipse for the mean direction.

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The Fisher distribution is the analogue on the sphere of the isotropic bivariate normal distribution in the plane. The purpose of this paper is to propose and analyze a spherical analogue of the general bivariate normal distribution. Estimation, hypothesis testing and confidence regions are also discussed.

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