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COMPUTATION OF UNSTEADY TURBULENT BOUNDARY LAYERS WITH FLOW REV--ETC(U)
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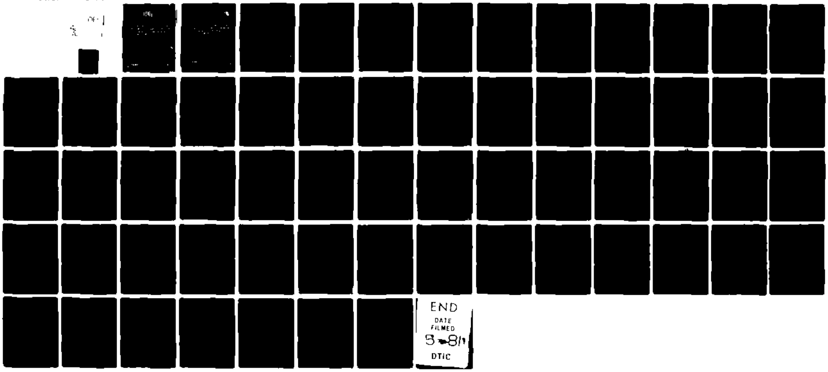

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COMPUTATION OF UNSTEADY TURBULENT BOUNDARY LAYERS WITH FLOW REVERSAL AND EVALUATION OF TWO SEPARATE TURBULENCE MODELS

by

Tuncer Cebeci* and Lawrence W. Carr

SUMMARY

Recently a new procedure, which solves the governing boundary-layer equations with Keller's box method, has been developed for calculating unsteady laminar flows with flow reversal [1]. In regions where the streamwise velocity contains flow reversal, the solution scheme was modified by a procedure which accounted for the downstream influence. With this modification, the unsteady flow over a circular cylinder started impulsively from rest was successfully calculated to values of time and space greater than in any previous solutions. An examination of unsteady separation for laminar flow was made and revealed that the unsteady boundary layer for that flow, even at large times, was free of singularities.

In this report we extend the method of ref. [1] to turbulent boundary layers with flow reversal. Using the algebraic eddy viscosity formulation of Cebeci and Smith [2], we consider several test cases to investigate the proposition that unsteady turbulent boundary layers also remain free of singularities.

Since the solution of turbulent boundary layers requires a closure assumption for the Reynolds shear-stress term and the accuracy of the solutions depend on this assumption, we also perform turbulent flow calculations by using the turbulence model of Bradshaw, Ferriss and Atwell [3]; we solve the governing equations for both models by using the same numerical scheme and compare the predictions with each other, restricting the comparisons to cases in which wall shear is positive.

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The study reveals that, as in laminar flows, the unsteady turbulent boundary layers are free from singularities but there is a clear indication of rapid thickening of the boundary layer with increasing flow reversal. The study also reveals that the predictions of both turbulence models are the same for all practical purposes.

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I. INTRODUCTION

The prediction of unsteady turbulent boundary layers with flow reversal is of importance in a number of aerodynamic problems, notably in dynamic stall, buffeting and gust studies. However, some of the more popular turbulence models implicitly assume that the wall shear is positive and their extension to unsteady flows with flow reversal is not easy. It requires modifications to the functional form of the law of the wall and to the manner in which the wall shear is determined. Two near-wall assumptions are considered here. In the first, the near wall grid point is located in the logarithmic region and the law of the wall is used to link the flow properties at this grid point to the wall. In the second, a Van Driest formulation due to Cebeci and Smith [2] is used; this implies that the grid point closest to the wall will occur in the viscous sublayer.

A further aspect of these flows of current interest is the possibility of a singularity occurring in the reversed-flow region. Examples of this phenomenon have also been reported in laminar flows but, in earlier studies Cebeci [1,4] and Bradshaw [5] have shown that the occurrence is not a feature of the governing equations but is due to the limitations of the numerical procedure used. We shall demonstrate that, for the examples we study, there is no indication of such a singularity in turbulent flow either but there is a clear indication of rapid thickening of the boundary layer.

In addition to the examination of wall functions, we have also considered two turbulence models for unsteady flows without flow reversal. The algebraic eddy-viscosity formulation of Cebeci and Smith (CS) is compared with the transport model of Bradshaw, Ferriss and Atwell [3] (BF). Calculations were performed to determine whether the representation of unsteady flows with strong pressure gradients requires that account be taken of transport of turbulence quantities. As will be shown, the predictions with both models are nearly identical for both steady and unsteady flows with and without strong pressure gradient.

The report has been prepared with six main sections describing, respectively, the governing equations, the numerical procedure, the results, concluding remarks, references and the computer program which uses only the CS model.

II. GOVERNING EQUATIONS

The continuity and momentum equations can be written for two-dimensional unsteady incompressible laminar or turbulent thin shear layers as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} + \frac{\partial \tau}{\partial y} \tag{2}$$

Here

$$\tau = \nu\frac{\partial u}{\partial y} - \overline{u'v'} \tag{3}$$

and we recall that u' and v' denote fluctuations about the ensemble-average velocity; u' and v' are zero in unsteady laminar flow, and $\nu\,\partial u/\partial y$ is negligible outside the viscous sublayer in a turbulent flow. These equations are subject to the usual boundary conditions, which in the case of boundary layers are

$$y = 0, \qquad u = v = 0; \qquad y \to \delta \qquad u \to u_e(x,t) \tag{4}$$

The presence of the Reynolds stress term, $-\overline{u'v'}$ introduces an additional unknown to the system given by Eqs. (2) to (4). In this report we present calculations using two different turbulence models. One is an algebraic eddy-viscosity formulation developed (for steady flows) by Cebeci and Smith and the other is a transport-equation model developed by Bradshaw, Ferriss and Atwell. In the CS model, we write Eq. (3) as

$$\tau = (\nu + \varepsilon_m)\frac{\partial u}{\partial y} \tag{5}$$

with two separate formulas for ε_m. In the so-called inner region of the boundary layer $(\varepsilon_m)_i$ is defined by the following formula:

$$(\varepsilon_m)_i = \{0.4y[1 - \exp(-y/A)]\}^2 \left|\frac{\partial u}{\partial y}\right| \tag{6}$$

where

$$A = 26\nu u_\tau^{-1}[1 - 11.8(p_t^+ + p_x^+)]^{-\frac{1}{2}} \tag{7a}$$

$$u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2}, \qquad p_t^+ = \frac{\nu}{u_\tau^3}\frac{\partial u_e}{\partial t}, \qquad p_x^+ = \frac{\nu u_e}{u_\tau^3}\frac{\partial u_e}{\partial x} \tag{7b}$$

In the outer region ε_m is defined by the following formula

$$(\varepsilon_m)_o = 0.0168 \int_0^\infty (u_e - u)dy \tag{8}$$

The boundary between the inner and outer regions is established by the continuity of the eddy-viscosity formulas.

In the BF model, which is used <u>only</u> outside the viscous sublayer, we assume $\tau = -\overline{u'v'}$ and write a single first-order partial-differential equation for it; the equation was originally developed from the turbulent energy equation but can be equally well regarded as an empirical closure of the exact shear-stress transport equation. **This reads**

$$\frac{D\tau}{Dt} \equiv \frac{\partial\tau}{\partial t} + u\frac{\partial\tau}{\partial x} + v\frac{\partial\tau}{\partial y} = 2a_1\tau\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}(\tau V_\tau) - 2a_1\frac{\tau^{3/2}}{L} \tag{9}$$

Here a_1 is a dimensionless quantity, V_τ is a velocity and L is the dissipation length parameter, specified algebraically by $L/\delta = f(\eta)$ with $\eta = y/\delta$ and $f(\eta)$ given from an analytic fit to an empirical curve by

$$f(\eta) = \begin{cases} 0.4\eta & \eta < 0.18 \\ 0.095 - 0.055(2\eta - 1)^2 & 0.18 \le \eta < 1.1 \\ 0.016\exp[-10(\eta - 1.1)] & \eta \ge 1.1 \end{cases} \tag{10}$$

In a more advanced version of this turbulence model [6] L itself is determined from a transport equation.

The turbulent transport velocity V_τ, nominally $(\overline{p'u'} + \overline{u'v'^2})/\overline{u'v'}$ is proportional to a velocity scale of the large eddies and is chosen to be

$$V_\tau = 2a_1\frac{\tau_{max}}{u_e}g(\eta) \tag{11}$$

where $g(\eta)$ is given by

$$
g(\eta) = \begin{cases} 33.3\eta^2(0.184 + 0.832\eta) & \eta < 0.5 \\ 33.3\eta^3(0.368 + 2.496\eta^3) & 0.5 \le \eta < 1.0 \\ 18.7\eta + 14.60 & \eta \ge 1.0 \end{cases} \tag{12}
$$

In the BF model equations, the inner boundary conditions for (1), (2) and (9) are applied outside the viscous sublayer, usually at $y_1 = 50\nu/u_\tau$. In the steady-flow study reported in [7], these boundary conditions are:

$$
u_1 = u_\tau\left(\frac{1}{\kappa}\ln\frac{y_1 u_\tau}{\nu} + 5.2\right) \tag{13}
$$

$$
v_1 = -\frac{u_1 y_1}{u_\tau}\frac{\partial u_\tau}{\partial x} \tag{14}
$$

$$
\tau_1 = \tau_w + \frac{1}{\rho}\frac{\partial p}{\partial x}y_1 + \alpha^*\frac{\partial \tau_w}{\partial x}y_1 \tag{15}
$$

Here v_1 is evaluated from the continuity equation (1), and α^* is evaluated from (1) and (2) on the assumption that the velocity u is given by

$$
\frac{u}{u_\tau} = \phi\left(\frac{u_\tau y}{\nu}\right) \tag{16}
$$

for $0 < y < y_1$; Eq. (13) is, of course, a special case of (16). The evaluation of α^* is discussed in Ref. [7]; the last term in (15) can be as large as half the second (pressure-gradient) term. In unsteady flow without flow reversal, we use the same inner "boundary" conditions at $y_1 = 50\nu/u_\tau$, but because of the presence of the time-dependent term in (2), α^* becomes more complicated. If we again assume that (16) holds - remember that the turbulence structure of the inner layer is unlikely to be affected unless the external-stream frequency is very high - then (1) and (2) give

$$
\tau = \tau_w + \int_0^{y_1}\frac{\partial u}{\partial t}dy + \frac{\partial p}{\partial x}y_1 + \int_0^{y_1}\frac{\partial}{\partial x}(u^2)dy + uv\Big|_{y=y_1} \tag{17}
$$

Integrating we can write

$$\tau = \tau_w + \frac{\partial}{\partial t}\int_0^{y_1} u dy - u(y_1)\frac{\partial y_1}{\partial t} + \frac{\partial}{\partial x}\int_0^{y_1} u^2 dy - \cancel{u^2(y_1)\frac{\partial y_1}{\partial x}}$$

$$- u(y_1)\frac{\partial}{\partial x}\int_0^{y_1} u dy + u^2(y_1)\frac{\partial y_1}{\partial x} \qquad (18a)$$

because

$$\frac{\partial}{\partial x}\int_0^{y_1^+} \frac{u}{u_\tau} dy^+ \equiv 0$$

We can also write (18a) as

$$\tau = \tau_w + \frac{\partial p}{\partial x} y_1 + \nu\frac{\partial}{\partial t}\int_0^{y_1^+} \frac{u}{u_\tau} dy^+ + u_\tau F(y_1^+)\frac{\nu y_1^+}{u_\tau^2}\frac{\partial u_\tau}{\partial t} + \nu\frac{\partial u_\tau}{\partial x}\int_0^{y_1^+}\left(\frac{u}{u_\tau}\right)^2 dy^+ \quad (18b)$$

or as

$$\tau_1 = \tau_w + y_1 F\frac{\partial u_\tau}{\partial t} + \alpha^* y_1\frac{\partial \tau_w}{\partial x} + \frac{\partial p}{\partial x} y_1 \qquad (18c)$$

where $F = u/u_\tau$ at $y = y_1$ and α^* comes from the last term in (18b) and is the same as in steady flow.

Equation (18c) now replaces (15).

III. SOLUTION PROCEDURE

We use Keller's two-point finite-difference method (called the Box method) to solve the system of equations described in the previous section. The application of this method to unsteady flows with no flow reversal using the CS model has been described in Ref. [8]. Its application to steady two-dimensional flows using the BF model is described in Ref. [7]. Here we present a description of the extension of the CS model to unsteady two-dimensional turbulent flows <u>with</u> flow reversal as well as a description of the extension of the BF model to unsteady turbulent flows with <u>no</u> flow reversal.

3.1 CS Method with and without Flow Reversal

As in previous studies (see, for example [8]), we transform the equations with

$$\bar{x} = x/L, \qquad \bar{t} = tu_0/L, \qquad \eta = (u_0/\nu\bar{x})^{1/2}y \tag{19a}$$

and a dimensionless stream function $f(\bar{x},\eta,\bar{t})$, where

$$\psi = (u_0\nu x)^{1/2}f(\bar{x},\eta,\bar{t}) \tag{19b}$$

Here u_0 is a reference velocity, L a reference length, and ψ is the usual definition of the stream function corresponding to the continuity equation (1). With the relations defined by (19) and with the definition of eddy viscosity, equations (1) to (3) and the boundary conditions can be written as

$$(bf'')' + \frac{1}{2}ff'' + m_3 = \bar{x}\left(f'\frac{\partial f'}{\partial \bar{x}} - f''\frac{\partial f}{\partial \bar{x}} + \frac{\partial f'}{\partial \bar{t}}\right) \tag{20}$$

$$\eta = 0, \quad f = f' = 0; \quad \eta \to \eta_\infty, \quad f' = u_e/u_0 \equiv \bar{u}_e \tag{21}$$

Primes denote differentiation with respect to η and

$$f' = u/u_0, \qquad p_3 = \bar{x}\left(\bar{u}_e\frac{\partial \bar{u}_e}{\partial \bar{x}} + \frac{\partial \bar{u}_e}{\partial \bar{t}}\right)$$

$$b = 1 + \varepsilon_m^+, \qquad \varepsilon_m^+ = \frac{\varepsilon_m}{\nu} \tag{22}$$

For simplicity, we shall now drop the bars on x and t.

We use two separate solution procedures to solve the system given by Eqs. (20) and (21). When there is no flow reversal across the layer, we use the standard Box. On the other hand, when there is flow reversal, then we use the so-called zig-zag Box as described below.

To solve Eqs. (20) and (21) by the standard Box method, we first write Eq. (20) in terms of three first-order equations by introducing new dependent variables $u(x,\eta,t)$, $v(x,\eta,t)$, that is,

$$f' = u \tag{23a}$$

$$u' = v \tag{23b}$$

$$(bv)' + \frac{1}{2} fv + P_3 = x\left(u \frac{\partial u}{\partial x} - v \frac{\partial f}{\partial x} + \frac{\partial u}{\partial t}\right) \tag{23c}$$

We next consider the net cube shown in Fig. 1 and write difference approximations to Eqs. (23). Equations (23a,b) are approximated using centered difference quotients and averaged about the midpoint $(x_i, t_n, \eta_{j-\frac{1}{2}})$. The difference quotients which are to approximate (23c) are written about the midpoint $(x_{i-\frac{1}{2}}, t_{n-\frac{1}{2}}, \eta_{j-\frac{1}{2}})$ of the cube whose mesh widths are r_i, k_n, and h_j. This procedure yields the following equations:

$$f_j^{i,n} - f_{j-1}^{i,n} - h_j u_{j-\frac{1}{2}}^{i,n} = 0 \tag{24a}$$

$$u_j^{i,n} - u_{j-1}^{i,n} - h_j v_{j-\frac{1}{2}}^{i,n} = 0 \tag{24b}$$

$$\frac{(bv)_j^{i,n} - (bv)_{j-1}^{i,n}}{h_j} + \frac{1}{2}(fv)_{j-\frac{1}{2}}^{i,n} - \alpha_i (u^2)_{j-\frac{1}{2}}^{i,n} + \frac{\alpha_i}{2}\left(v_{j-\frac{1}{2}}^{i,n} f_{j-\frac{1}{2}}^{i,n} + m_3 f_{j-\frac{1}{2}}^{i,n} + m_4 v_{j-\frac{1}{2}}^{i,n}\right)$$
$$- 2\beta_n u_{j-\frac{1}{2}}^{i,n} = n_3 \tag{24c}$$

where

$$\alpha_i = \frac{x_{i-\frac{1}{2}}}{x_i - x_{i-1}}, \quad \beta_n = \frac{x_{i-\frac{1}{2}}}{t_n - t_{n-1}}, \quad m_3 = v_{j-\frac{1}{2}}^{234}, \quad m_4 = f_{j-\frac{1}{2}}^{(4)} - 2\overline{f}_{i-1}$$

$$n_3 = \alpha_i[(u^2)_{j-\frac{1}{2}}^{(4)} - 2(\overline{u^2})_{i-1}] - \frac{\alpha_i}{2} m_3 m_4 + 2\beta_n[u_{j-\frac{1}{2}}^{(2)} - 2\overline{u}_{n-1}] - h_j^{-1}[(bv)_j^{234}$$
$$- (bv)_{j-1}^{234}] - \frac{1}{2}(fv)_{j-\frac{1}{2}}^{234} - 4(P_3)_{n-\frac{1}{2}}^{i-\frac{1}{2}}$$

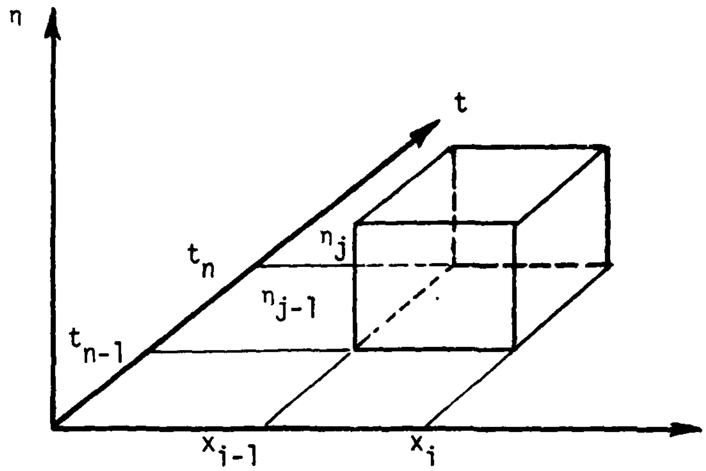

Fig. 1 Net cube for the standard Box method.

Here by v_j^{234} we mean $v_j^{i-1,n} + v_j^{i-1,n-1} + v_j^{i,n-1}$, the sum of the values of v_j at three of the four corners of the face of the box. Also

$$\bar{f}_{i-1} = \frac{1}{2}(f_{j-\frac{1}{2}}^{n,i-1} + f_{j-\frac{1}{2}}^{n-1,i-1})$$

$$\bar{u}_{n-1} = \frac{1}{2}(u_{j-\frac{1}{2}}^{n-1,i} + u_{j-\frac{1}{2}}^{n-1,i-1})$$

$$f_{j-\frac{1}{2}}^{i,n} = \frac{1}{2}(f_j^{i,n} + f_{j-1}^{i,n})$$

The resulting algebraic system given by Eqs. (24) together with the boundary conditions, which now become

$$f_o = u_o = 0, \qquad u_J = \bar{u}_e \tag{25}$$

are nonlinear. We use Newton's method to linearize the system and solve the linear system by the block elimination method discussed in ref. [9].

When there is flow reversal across the boundary layer at some x and t, we modify the standard Box method used for Eq. (23c) but retain that for (23a,b) and still center them at $(x_i, t_n, \eta_{j-\frac{1}{2}})$. To write the difference approximations for the Box centered at $(x_{i-\frac{1}{2}}, t_{n-\frac{1}{2}}, \eta_{j-\frac{1}{2}})$ we examine previously computed values of $u^{i,n}_{j-\frac{1}{2}}$. If $u^{i,n}_{j-\frac{1}{2}} \geq 0$, then we use the standard Box method: if $u^{i,n}_{j-\frac{1}{2}} < 0$, then we write (23c) for the Box centered at P (see Fig. 2) using quantities centered at P, Q, and R, where

$$P \equiv (x_i, t_{n-\frac{1}{2}}, \eta_{j-\frac{1}{2}}), \qquad Q \equiv (x_{i-\frac{1}{2}}, t_n, \eta_{j-\frac{1}{2}})$$

$$R \equiv (x_{i+\frac{1}{2}}, t_{n-1}, \eta_{j-\frac{1}{2}}) \tag{26}$$

Equation (23c) can then be written as

$$(bv)'(P) + \frac{1}{2}(fv)(P) = x(P)\,[\theta u(Q)\frac{\partial u}{\partial x}(Q) + \phi u(R)\frac{\partial u}{\partial x}(R) - \theta v(Q)\frac{\partial f}{\partial x}(Q)$$

$$- \phi v(R)\frac{\partial f}{\partial x}(R) + \frac{\partial u}{\partial t}(P)] \tag{27}$$

Here

$$\theta \equiv \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}, \qquad \phi \equiv \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \tag{28}$$

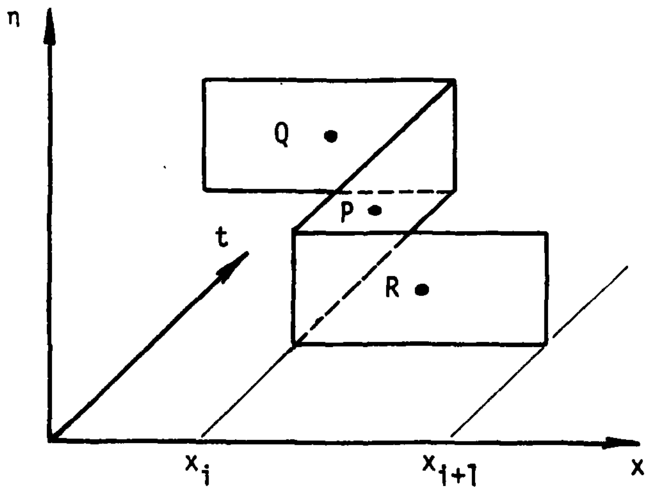

Fig. 2 Finite difference molecule for the Zig-Zag differencing.

The resulting algebraic system is again nonlinear and its solution is obtained by using the procedure followed in the standard Box method.

3.2 BF Method with no Flow Reversal

The solution of the governing equations for unsteady flows with the BF model, even with no flow reversal across the boundary layer, is much more difficult than with the CS model. This is because of the hyperbolic nature of the governing equations, together with the nonlinear boundary conditions, which play an important role in the solution procedure. As is common in most (if not all) methods that use boundary conditions away from the "wall," the wall shear stress is also an unknown parameter; it can be treated as an eigenvalue or as a mechul as described in Ref. [7]. The latter procedure is much more efficient than the former procedure and is used here.

To solve the BF model equations, we first introduce the stream function $\psi(x,y)$ as in Ref. [7] in order to satisfy the continuity equation. With $\sqrt{\tau_w} \equiv w$ treated as mechul, the resulting system can be written as a system of four first-order equations:

$$w' = 0 \tag{29a}$$

$$\psi' = u \tag{29b}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - u'\frac{\partial \psi}{\partial x} = P_3 + \tau' \tag{29c}$$

$$\frac{\partial \tau}{\partial t} + u\frac{\partial \tau}{\partial x} - \tau'\frac{\partial \psi}{\partial x} = 2a_1\left[\tau u' - \frac{\tau^{3/2}}{L} - \tau_{max}^{1/2}(G\tau)'\right] \tag{29d}$$

We again center Eqs. (29a,b) about the midpoint $(x_i, t_n, \eta_{j-1/2})$ and Eqs. (29c,d) about the midpoint $(x_{i-1/2}, t_{n-1/2}, \eta_{j-1/2})$ of the cube shown in Fig. 1. This procedure yields the following nonlinear algebraic equations:

$$w_j^{i,n} - w_{j-1}^{i,n} = 0 \tag{30a}$$

$$\psi_j^{i,n} - \psi_{j-1}^{i,n} - h_j u_{j-1/2}^{i,n} = 0 \tag{30b}$$

$$\frac{\tau_j^{i,n} - \tau_{j-1}^{i,n}}{h_j} + \alpha_i\left[\frac{u_j^{i,n} - u_{j-1}^{i,n}}{h_j}(\psi_{j-1/2}^{i,n} - \psi_{j-1/2}^{i-1,n}) + \psi_{j-1/2}^{i,n}\frac{u_j^{i-1,n} - u_{j-1}^{i,n}}{h_j} - (u^2)_{j-1/2}^{i,n}\right] - 2\beta_n u_{j-1/2}^{i,n} = n_3 \tag{30c}$$

$$\tilde{\beta}_n\tau_{j-\frac{1}{2}}^{i,n} + \tilde{\alpha}_i\left[u_{j-\frac{1}{2}}^{i,n}(\tau_{j-\frac{1}{2}}^{i,n} - \tau_{j-\frac{1}{2}}^{i-1,n}) + u_{j-\frac{1}{2}}^{i-1,n}\tau_{j-\frac{1}{2}}^{i,n}\right] - \tilde{\alpha}_i\left[\frac{\tau_j^{i,n} - \tau_{j-1}^{i,n}}{h_j}(\psi_{j-\frac{1}{2}}^{i,n} - \psi_{j-\frac{1}{2}}^{i-1,n})\right.$$

$$\left. + \psi_{j-\frac{1}{2}}^{i,n}\left(\frac{\tau_j^{i-1,n} - \tau_{j-1}^{i-1,n}}{h_j}\right)\right] - \tau_{j-\frac{1}{2}}^{i,n}\frac{u_j^{i,n} - u_{j-1}^{i,n}}{h_j} + 2\left[\frac{(\tau^{3/2})_{j-\frac{1}{2}}^{i,n} + (\tau^{3/2})_{j-\frac{1}{2}}^{i-1,n}}{L_{j-\frac{1}{2}}^{i,n} + L_{j-\frac{1}{2}}^{i-1,n}}\right]$$

$$+ (G')_{j-\frac{1}{2}}^{i,n}\tau_{j-\frac{1}{2}}^{i,n} + G_{j-\frac{1}{2}}^{i,n}\frac{\tau_j^{i,n} - \tau_{j-1}^{i,n}}{h_j} = n_4 \tag{30d}$$

where now

$$\alpha_i = \frac{1}{x_i - x_{i-1}}, \qquad \beta_i = \frac{1}{t_n - t_{n-1}}, \qquad \tilde{\alpha}_i = \frac{\alpha_i}{2a_1}, \qquad \tilde{\beta}_n = \frac{\beta_n}{2a_1}$$

$$n_3 = -4(P_3)_{n-\frac{1}{2}}^{i-\frac{1}{2}} - (\tau')_{j-\frac{1}{2}}^{234} + 2\beta_n(u_{j-\frac{1}{2}}^{i-1,n} - u_{j-\frac{1}{2}}^{i-1,n-1} - u_{j-\frac{1}{2}}^{i,n-1}) + \alpha_i[(u^2)_{j-\frac{1}{2}}^{i,n-1}$$

$$- (u^2)_{j-\frac{1}{2}}^{i-1,n-1} - (u^2)_{j-\frac{1}{2}}^{i-1,n}] - \alpha_i\left[\{(u')_{j-\frac{1}{2}}^{i,n-1} + (u')_{j-\frac{1}{2}}^{i-1,n-1}\}\right.$$

$$\left.(\psi_{j-\frac{1}{2}}^{i,n-1} - \psi_{j-\frac{1}{2}}^{i-1,n-1}) - (u')_{j-\frac{1}{2}}^{i-1,n}\psi_{j-\frac{1}{2}}^{i-1,n}\right]$$

$$n_4 = \tilde{\alpha}_i(A_3 - A_2) - 2\beta_n A_1 + (\tau u')_{j-\frac{1}{2}}^{234} - 2A_4 - (G'\tau)_{j-\frac{1}{2}}^{234} - (G\tau')_{j-\frac{1}{2}}^{234}$$

$$A_1 = \tau_{j-\frac{1}{2}}^{i-1,n} - \tau_{j-\frac{1}{2}}^{i-1,n-1} - \tau_{j-\frac{1}{2}}^{i-1},$$

$$A_2 = 2u_{j-\frac{1}{2}}^{i-\frac{1}{2},n-1}(\tau_{j-\frac{1}{2}}^{i,n-1} - \tau_{j-\frac{1}{2}}^{i-1,n-1}) - u_{j-\frac{1}{2}}^{i-1,n}\tau_{j-\frac{1}{2}}^{i-1,n}$$

$$A_3 = [(\tau')_{j-\frac{1}{2}}^{i,n-1} + (\tau')_{j-\frac{1}{2}}^{i-1,n-1}][\psi_{j-\frac{1}{2}}^{i,n-1} - \psi_{j-\frac{1}{2}}^{i-1,n-1}] - (\tau')_{j-\frac{1}{2}}^{i-1,n}\,\psi_{j-\frac{1}{2}}^{i-1,n}$$

$$A_4 = \frac{(\tau^{3/2})_{j-\frac{1}{2}}^{i,n-1} + (\tau^{3/2})_{j-\frac{1}{2}}^{i-1,n-1}}{L_{j-\frac{1}{2}}^{i,n-1} + L_{j-\frac{1}{2}}^{i-1,n-1}}$$

Again the system given by Eqs. (30) is nonlinear and is linearized by using Newton's method. This procedure gives rise to the following form $(2 < j < J)$

$$\delta w_j - \delta w_{j-1} = (r_3)_j \tag{31a}$$

$$\delta\psi_j - \delta\psi_{j-1} - \frac{h_j}{2}(\delta u_j + \delta u_{j-1}) = (r_4)_{j-1} \tag{31b}$$

$$(s_1)_j\delta u_j + (s_2)_j\delta u_{j-1} + (s_3)_j\delta\psi_j + (s_4)_j\delta\psi_{j-1} + (s_5)_j\delta\tau_j + (s_6)_j\delta\tau_{j-1} = (r_1)_j \tag{31c}$$

$$(\beta_1)_j\delta u_j + (\beta_2)_j\delta u_{j-1} + (\beta_3)\delta\psi_j + (\beta_4)_j\delta\psi_{j-1} + (\beta_5)_j\delta\tau_j + (\beta_6)_j\delta\tau_{j-1} = (r_2)_j \tag{31d}$$

Here for convenience we have dropped the superscripts i,n and have defined $(s_k)_j$ $(k = 1, 2, \ldots, 6)$

$$(s_1)_j = -\beta_n - \alpha_i u_j + \alpha_i/h_j\,(\psi_{j-\frac{1}{2}} - \psi_{j-\frac{1}{2}}^{i-1,n})$$

$$(s_2)_j = -\beta_n - \alpha_i u_j - \alpha_i/h_j\,(\psi_{j-\frac{1}{2}} - \psi_{j-\frac{1}{2}}^{i-1,n})$$

$$(s_3)_j = \alpha_i/2\,[(u')_{j-\frac{1}{2}} + (u')_{j-\frac{1}{2}}^{i-1,n}],$$

$$(s_4)_j = (s_3)_j$$

$$(s_5)_j = 1/h_j, \qquad (s_6)_j = -1/h_j$$

and $(\beta_k)_j$ $(k = 1,2, \ldots, 6)$

$$(\beta_1)_j = \frac{\tilde{\alpha}_i}{2}\,(\tau_{j-\frac{1}{2}} - \tau_{j-\frac{1}{2}}^{i-1,n}) - \frac{1}{h_j}\,\tau_{j-\frac{1}{2}}$$

$$(\beta_2)_j = \frac{\tilde{\alpha}_i}{2}\,(\tau_{j-\frac{1}{2}} - \tau_{j-\frac{1}{2}}^{i-1,n}) + \frac{1}{h_j}\,\tau_{j-\frac{1}{2}}$$

$$(\beta_3)_j = -\frac{\tilde{\alpha}_i}{2}\,[(\tau')_{j-\frac{1}{2}} + (\tau')_{j-\frac{1}{2}}^{i-1,n}], \qquad (\beta_4)_j = (\beta_3)_j$$

$$(\beta_5)_j = \tilde{\beta}_n + \frac{\tilde{\alpha}_i}{2}\,(u_{j-\frac{1}{2}} - u_{j-\frac{1}{2}}^{i-1,n}) + \frac{\tilde{\alpha}_i}{h_j}\,(\psi_{j-\frac{1}{2}}^{i-1,n} - \psi_{j-\frac{1}{2}})$$
$$+ \frac{1}{2}[(G')_{j-\frac{1}{2}} - (u')_{j-\frac{1}{2}}] + \frac{3}{2}\,\frac{\sqrt{|\tau_j|}}{L_{j-\frac{1}{2}}^{i-1,n} + L_{j-\frac{1}{2}}} + \frac{G_{j-\frac{1}{2}}}{h_j}$$

$$(\beta_6)_j = \tilde{\beta}_n + \frac{\tilde{\alpha}_i}{2}\,(u_{j-\frac{1}{2}} - u_{j-\frac{1}{2}}^{i-1,n}) - \frac{\tilde{\alpha}_i}{h_j}\,(\psi_{j-\frac{1}{2}}^{i-1,n} - \psi_{j-\frac{1}{2}})$$
$$+ \frac{1}{2}\,[(G')_{j-\frac{1}{2}} - (u')_{j-\frac{1}{2}}] + \frac{3}{2}\,\frac{\sqrt{|\tau_j|}}{L_{j-\frac{1}{2}}^{i-1,n} + L_{j-\frac{1}{2}}} - \frac{G_{j-\frac{1}{2}}}{h_j}$$

The terms denoted by $(r_k)_j$ $(k = 1, 2, 3, 4)$ are defined by:

$$(r_3)_j = 0$$

$$(r_4)_{j-1} = \psi_{j-1} - \psi_j + h_j u_{j-\frac{1}{2}}$$

$$(r_1)_j = n_3 - [2\beta_n u_{j-\frac{1}{2}} + \alpha_i (u^2)_{j-\frac{1}{2}} - \alpha_i \{(u')_{j-\frac{1}{2}}(\psi_{j-\frac{1}{2}} - \psi_{j-\frac{1}{2}}^{i-1,n})$$

$$+ (u')_{j-\frac{1}{2}}^{i-1,n}\psi_{j-\frac{1}{2}}\} - (\tau')_{j-\frac{1}{2}}]$$

$$(r_2)_j = n_4 - \Bigg[2\tilde{\beta}_n \tau_{j-\frac{1}{2}} + \tilde{\alpha}_i \{u_{j-\frac{1}{2}}(\tau_{j-\frac{1}{2}} - \tau_{j-\frac{1}{2}}^{i-1,n}) + u_{j-\frac{1}{2}}^{i-1,n}\tau_{j-\frac{1}{2}}\}$$

$$- \tilde{\alpha}_i\{(\tau')_{j-\frac{1}{2}}(\psi_{j-\frac{1}{2}} - \psi_{j-\frac{1}{2}}^{i-1,n}) + (\tau')_{j-\frac{1}{2}}^{i-1,n}\psi_{j-\frac{1}{2}}\} - \tau_{j-\frac{1}{2}}(u')_{j-\frac{1}{2}}$$

$$+ 2\left\{\frac{\tau_{j-\frac{1}{2}}^{3/2} + (\tau^{3/2})_{j-\frac{1}{2}}^{i-1,n}}{L_{j-\frac{1}{2}} + L_{j-\frac{1}{2}}^{i-1,n}}\right\} + G'_{j-\frac{1}{2}}\tau_{j-\frac{1}{2}} + G_{j-\frac{1}{2}}(\tau')_{j-\frac{1}{2}}\Bigg]$$

For $j = 1$, we use the boundary conditions given by Eqs. (13), (14) and (15) and first write them as:

$$u_1 = w_1(2.5 \ln \frac{y_1 w_1}{\nu} + 5.2)$$

$$\frac{\partial \psi_1}{\partial x} - \frac{u_1 y_1}{w_1}\frac{\partial w_1}{\partial x} = 0$$

$$\tau_1 = w_1^2 + y_1 \frac{u_1}{w_1}\frac{\partial w_1}{\partial t} + \alpha^* y_1 \frac{\partial}{\partial x} w_1^2 - P_3 y_1$$

After we write the difference equations and linearize the resulting nonlinear expressions we get

$$\delta u_1 - \left[2.5(\ln \frac{y_1 w_1}{\nu} + \frac{\nu}{y_1})\right]\delta w_1 = (r_1)_1 \tag{32a}$$

$$y_1(w_1 - E_2)\delta u_1 + (w_1 + w_1^{234})\delta\psi_1 + [(\psi_1 - E_1) - y_1(u_1 - u_1^{234})]\delta w_1 = (r_2)_1 \tag{32b}$$

$$\delta\tau_1 + g_7 \delta w_1 = (r_3)_1 \tag{32c}$$

where

$$E_1 = \psi_1^{i-1,n} - \psi_1^{i,n-1} + \psi_1^{i-1,n-1}$$

$$E_2 = (w_1^2)^{i-1,n} - (w_1^2)^{i,n-1} + (w_1^2)^{i-1,n-1}$$

$$E_4 = -w_1^{i,n-1} + w_1^{i-1,n} - w_1^{i-1,n-1}$$

$$E_5 = \frac{1}{2y_1^+} \int_0^{y_1^+} [2.5 \ln(1.0 + y_1^+) + 5.1 - (3.39y_1^+ + 5.1) \exp(-0.37y_1^+)]^2 dy_1^+$$

$$g_7 = -2w_1 \, [1 + \frac{1}{2} y_1 \alpha_i E_5^{1234}] - \frac{1}{2} y_1 \beta_n \left(\frac{u_1}{w_1}\right)^{1234}$$

$$(r_1)_1 = w_1[2.5 \ln \frac{y_1 w_1}{\nu} + 5.2] - u_1$$

$$(r_2)_1 = y_1 u_1^{1234}(w_1 - E_2) - w^{1234}(\psi_1 - E_1)$$

$$(r_3)_1 = (w_1^2)^{234} - \tau_1^{234} - 4(p_3)_{i-\frac{1}{2}}^{n-\frac{1}{2}} y_1 + \frac{1}{2} y_1 \beta_n \left(\frac{u_1}{w_1}\right)^{1234} E_4 + \frac{1}{2} y_1 \alpha_i E_5^{1234} E_3$$
$$- [\tau_1 - w_1^2 - \frac{1}{2} y_1 \beta_n \left(\frac{u_1}{w_1}\right)^{1234} w_1 - \frac{1}{2} y_1 \alpha_i E_5^{1234} w_1^2]$$

For $j = J$, we use the usual boundary condition,

$$u_J = u_e$$

which in its linearized form is

$$\delta u_J = (r_4)_1 = 0 \tag{33}$$

The equations (31) for $2 \leq j < J$ and the boundary conditions given by Eqs. (32) and (33) form a linear system which is solved by the block-elimination method discussed in Ref. [9].

IV. RESULTS AND DISCUSSION

To study the calculation of unsteady turbulent boundary layers with and without flow reversal we have considered three separate test cases. The first one has an external velocity distribution of the form

$$\bar{u}_e = 1 - \alpha(x - x^2)(t^2 - t^3) \qquad 0 < x < 1, \qquad t > 0 \tag{34}$$

where α is a positive constant. The same velocity distribution was recently used by Cebeci [1] for laminar flows in order to study the computation of unsteady laminar flows with flow reversal using the solution procedure described in the previous section and to see whether there is a singularity associated with such flows.

In performing calculations for this case and for the others considered here, care must be taken in generating the initial conditions in the (t,y) and (x,y) planes at some distance, say $x = x_0$. For a laminar flow if $x_0 = 0$, the initial velocity profile for the velocity distribution given by Eq. (34) can be taken as Blasius and there is no difficulty about computing the solution in $x > 0$ since the initial boundary layer is of zero thickness. If $x_0 \neq 0$, we can take

$$\bar{u}_e = 1 - \alpha(x_0 - x_0^2)(t^2 - t^3) \qquad 0 < x < x_0$$

but then at $x = x_0$ there is a discontinuity in the pressure gradient. Since it acts on an already-established boundary layer, the initial response is inviscid leading formally to a velocity slip and hence a subboundary layer at the wall. The treatment of the boundary layer is then rather subtle (see Ref. [10]) but if we are not too concerned with the details of the solution near $x = x_0$, which is the case here, a convenient procedure would be to write Eq. (34) as

$$\bar{u}_e = 1 - \alpha F[(x - x_0)/a](x - x^2)(t^2 - t^3) \tag{35}$$

where F is a smooth function which vanishes if $x < x_0$ and is unity if $x - x_0 > a$. For example, we can take $F(s) = \sin(\pi s/2)$ $0 < s < 1$, and $a = 0.06$ with ten stations between $x = x_0$ and $x = x_0 + a$. A similar difficulty would occur at $t = 0$ if t^2 in Eq. (34) were replaced by t since the boundary layer is well established at $t = 0$.

Figures 3 and 4 show the results for the turbulent flow calculations with the CS model for the test case given by Eq. (35) with $\alpha = 40$ and a unit Reynolds number $u_0/\nu = 2.2 \times 10^6$/m. The results shown in Fig. 3 were obtained by using different expressions for A; those shown by circles were obtained with Eq. (7), and those shown by solid lines with Eq. (7) written as

$$A = 26\left(\frac{\tau}{\rho}\right)_{max}^{-1/2} \tag{36}$$

As can be seen, both expressions give nearly the same results.

The results in Fig. 4, as in laminar flows, exhibit no signs of singularity for all calculated values of t. This is in contrast to the findings of Patel and Nash [11]. Again, as in laminar flows [3], we see the familiar rapid thickening of the boundary layer in the reversed flow region. If it had not been for this, the calculations would have been computed for greater values of t than those considered here.

The two other test cases considered here correspond to Cases 4 and 5, as reported by Carr [12]. Case 4 is for unsteady Howarth flow. It starts from a well-established steady flat-plate flow, on which a linear deceleration of u_e is imposed at $t = 0$. The external velocity distribution is given by

$$\bar{u}_e = 1 - \bar{\alpha}(x - 1.24)t \qquad 1.24 \le x \le 4.69 \tag{37}$$

where $\bar{\alpha}$ is a constant equal to 2.4/3.45 $sec^{-1}m^{-1}$. The flow was assumed to be steady up to $x = 1.24$m; the velocity distribution Eq. (37) was then imposed as a function of x and t. This test case differs from the previous one in that, once the flow separates, it does not reattach. For this reason, the calculations can only be continued as far as the station where the flow reversal first occurs. The initial velocity profile at $x = 1.24$ and for all time correspond to a flat-plate profile with a momentum thickness Reynolds number (R_θ) of 4860, and local skin-friction coefficient c_f of 2.8×10^{-3}.

As in the previous test case, we introduce a function F_1 so that at $x = 1.24$, $du_e/dx = 0$. Since we also want the solutions at $t = 0$ to correspond to steady-state solutions, we introduce another function F_2 in order to set $\partial u_e/\partial t = 0$. With these functions, Eq. (37) then becomes

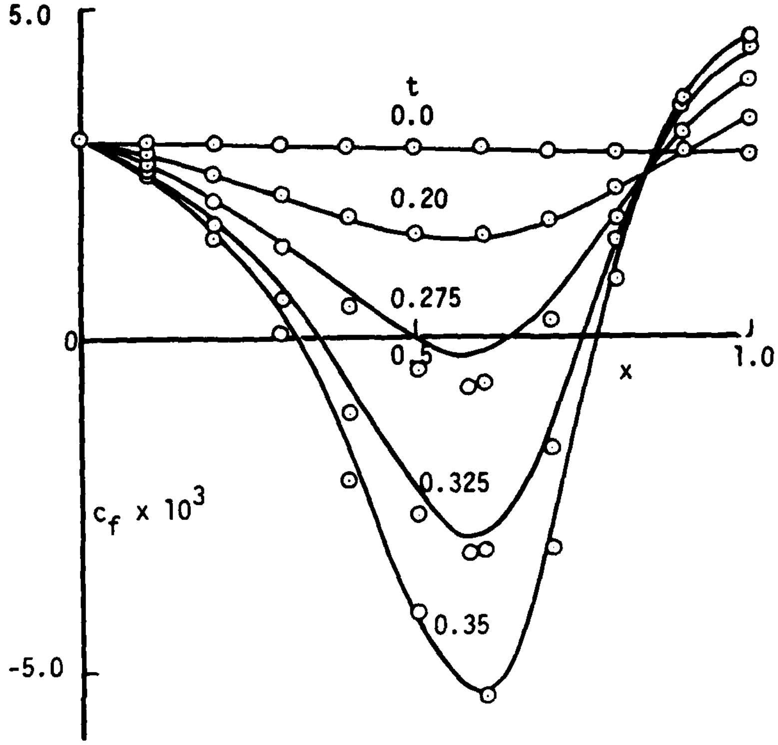

Figure 3. Local skin-friction variation with x for various values of z. Solid lines denote the calculations made by (29) and circles by (8).

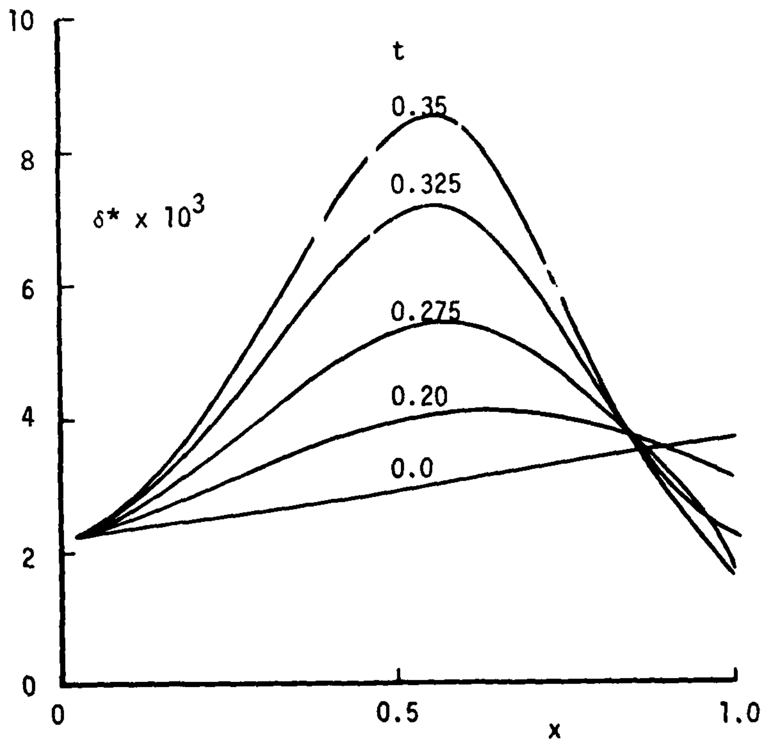

Figure 4. Variation of displacement thickness with x for various values of t.

$$\bar{u}_e = 1 - \bar{\alpha} F_1 F_2 (x - 1.24) t \qquad (38)$$

where

$$F_1 = \sin \frac{\pi}{2} \left(\frac{x - 1.24}{0.1}\right), \qquad F_2 = \sin \frac{\pi}{2} \frac{t}{1.98}$$

Figures 5, 6 and 7 show the calculated local skin-friction coefficient c_f, the shape factor H and the momentum thickness Reynolds number R_θ for this test case. The calculations were done by using both CS and BF models; the results shown by solid lines refer to the predictions of the CS model and those shown by circles refer to the predictions of the BF model.

As seen from these three figures, there is essentially no difference between the predictions of both models. Although there is some discrepancy in the shape factor predictions, this does not seem to be too significant.

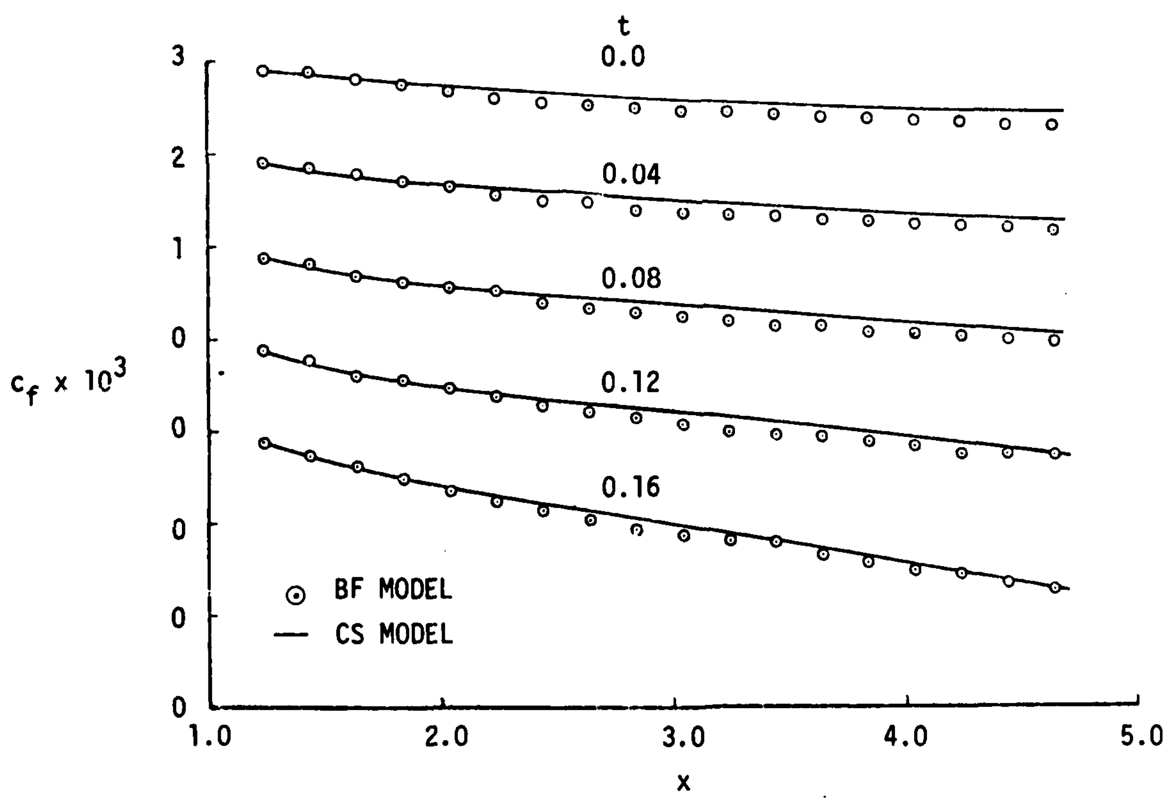

Figure 5. Computed local skin-friction distribution for test case 4.

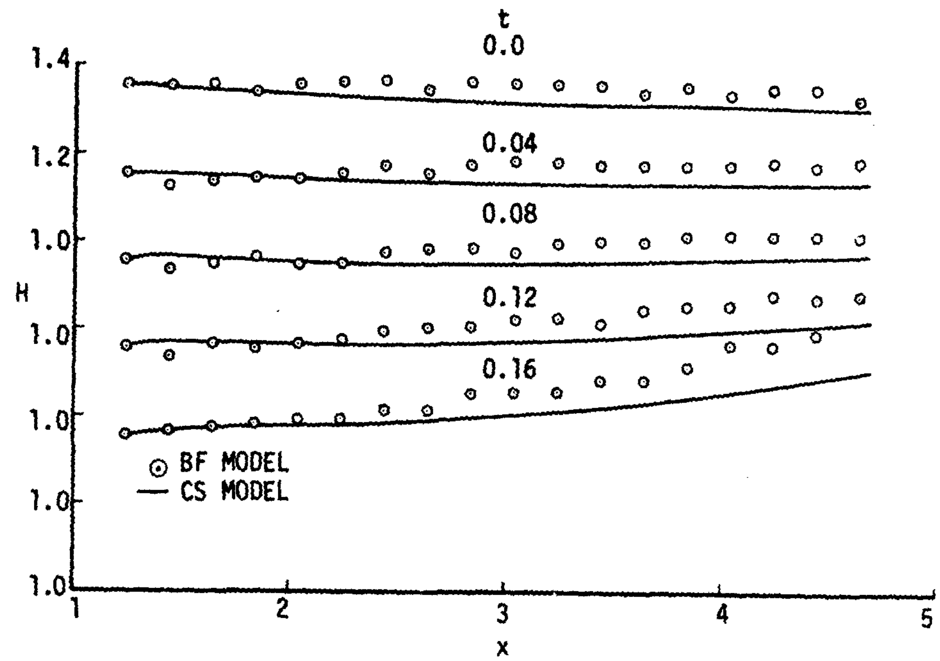

Figure 6. Computed shape factor distribution for test case 4.

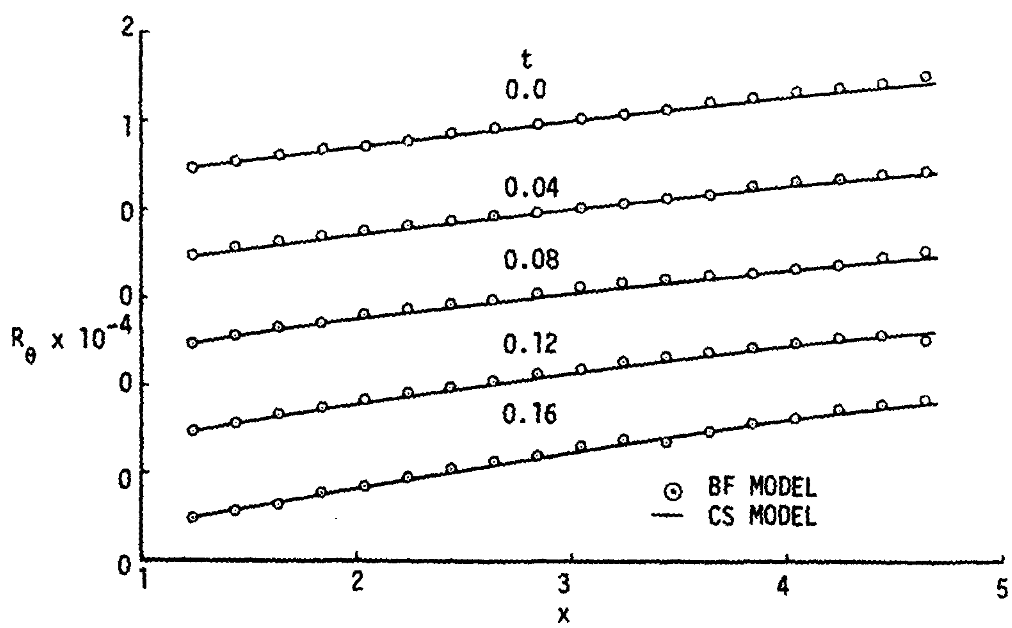

Figure 7. Computed momentum-thickness Reynolds number for test case 4.

According to the predictions of the CS model, which also has the capability of predicting unsteady boundary layers with flow reversal, the wall shear vanishes first around $t \sim 0.22$, $x = 4.69$. Since the computation of boundary layers for values of x in the range $1.24 \leq x \leq 4.69$ for $t > 0.22$ depends on the specification of a velocity profile at $x = 4.69$, we generate such a profile by assuming it is given by the extrapolation of two velocity profiles computed for $x < 4.69$. This procedure in which the extrapolated station serves as a downstream boundary condition, allows the calculations to be continued in the negative wall shear region as shown in Fig. 8.

The third case considered in our study corresponds to Case 5 in ref. [12], which in a way resembles the external velocity distribution in Eq. (34). It is given by

$$\bar{u}_e = 1 + \{A^2 + (Bt)^2\ [\xi - \xi_0]^2\}^{\frac{1}{2}} - [A^2 + (B\xi_0 t)^2]^{\frac{1}{2}} \qquad (39)$$

where $A = 0.05$, $B = 3.4\ \text{sec}^{-1}$, $\xi = (x - 1.24)/3.45$ and the range of x values are limited to $1.24 \leq x \leq 4.69$. As before, the initial velocity profiles at $x = 1.24$ for all t correspond to a steady flat-plate flow with $R_\theta = 4860$, $c_f = 2.89 \times 10^{-3}$. We again modify Eq. (39) to avoid the discontinuity in the pressure gradient. This time we multiply the right-hand side of Eq. (39) by F_1 used in Eq. (38).

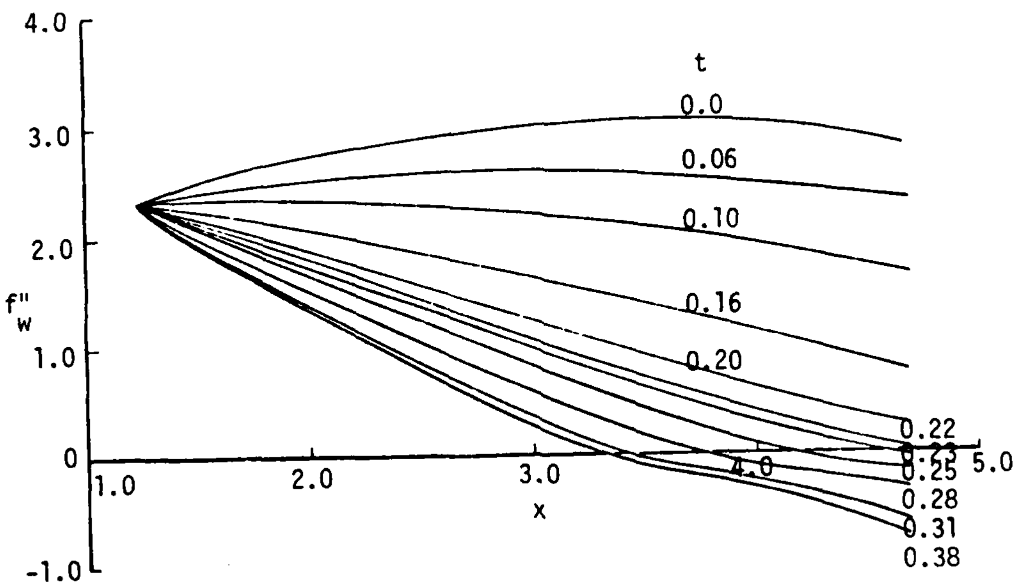

Figure 8. Variation of wall-shear parameter f''_w with distance as a function of time for test case 4.

Figures 9 and 10 show the calculated local skin-friction coefficient c_f and the momentum thickness Reynolds number R_θ for this test case. Again we present the predictions of both turbulence models. Figure 11 shows the calculated velocity profiles for several t and x-stations. As is seen from these figures, the predictions of both turbulence models are the same for all practical purposes.

Figure 12 shows the variation of wall shear parameter f''_w as a function of x and t, and Figure 13 shows the calculated velocity profiles, including the regions in which there is flow reversal across the boundary layer. These computations which are done by using the CS model provide confirmation of the general trends in test case 4, namely that as in laminar flows, the unsteady turbulent boundary layers thicken rapidly with increasing flow reversal. A new feature however is the dip in the graphs of f''_w near $x = 2.5$ which develops as t increases towards 0.40. It is possible that a singularity occurs in the solution at a later time as many authors have suggested is the case for laminar boundary layers. The most cogent argument in favor of this phenomenon has been advanced by Shen [13] but we note that the most definite sign of its occurrence appeared in his graphs of displacement thickness which showed spikey characteristics. Here the displacement thickness seems to be fairly smooth but the skin friction becomes spikey.

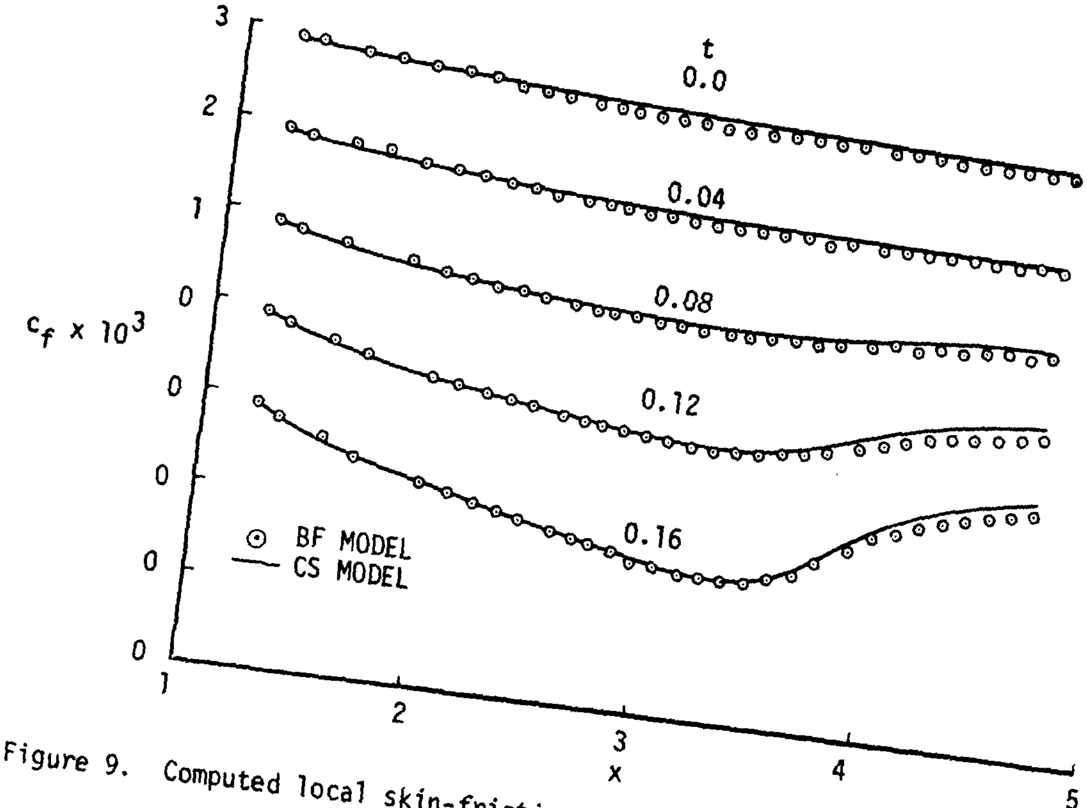

Figure 9. Computed local skin-friction distribution for test case 5.

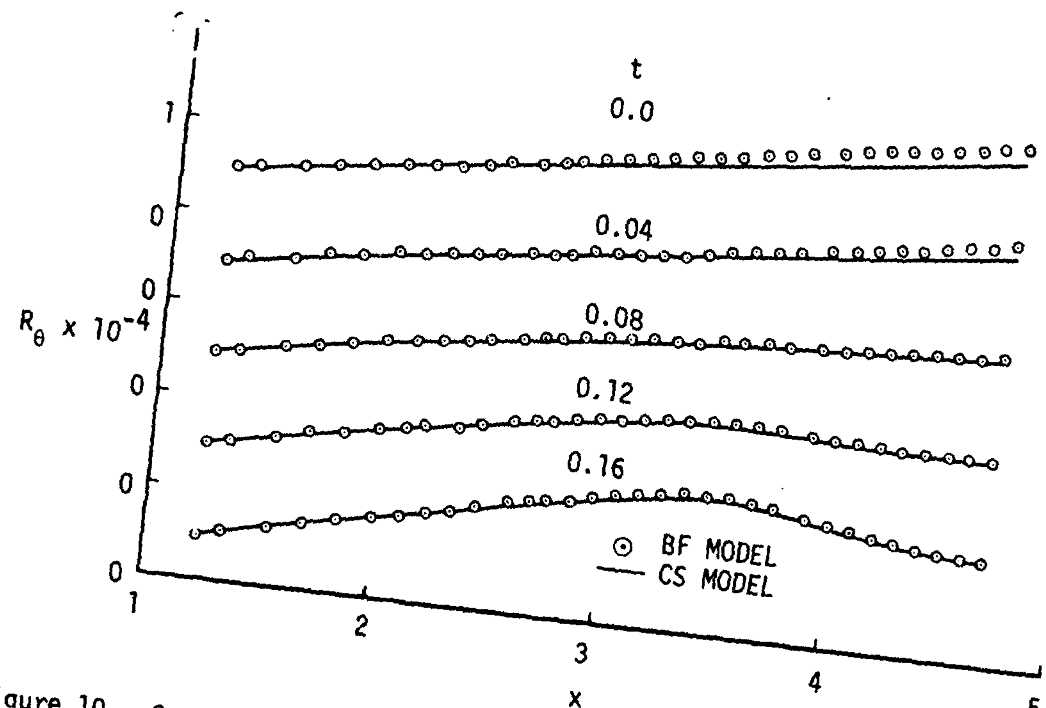

Figure 10. Computed momentum-thickness Reynolds number for test case 5.

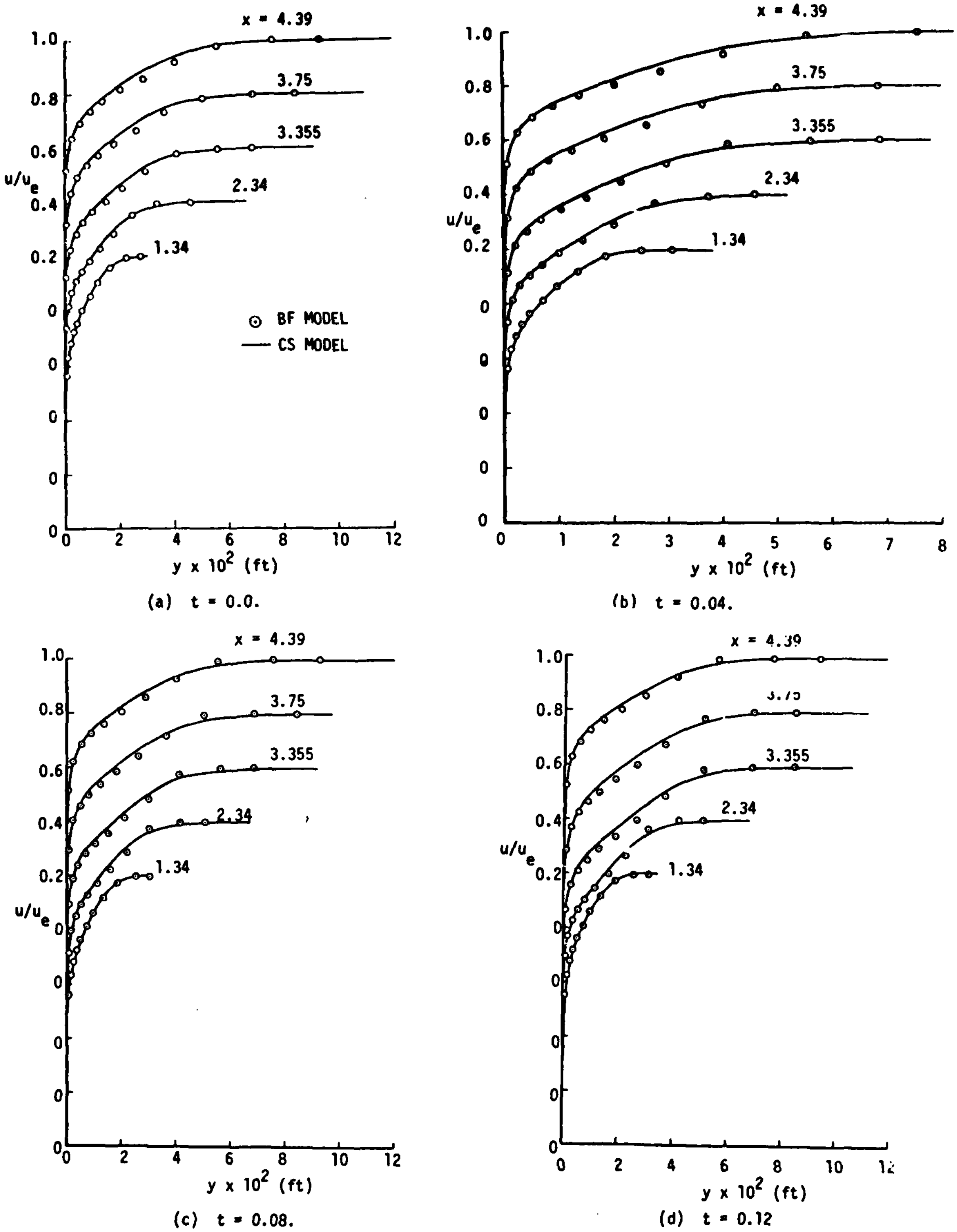

Figure 11. Comparison of calculated velocity profiles for test case 5 with no flow reversal.

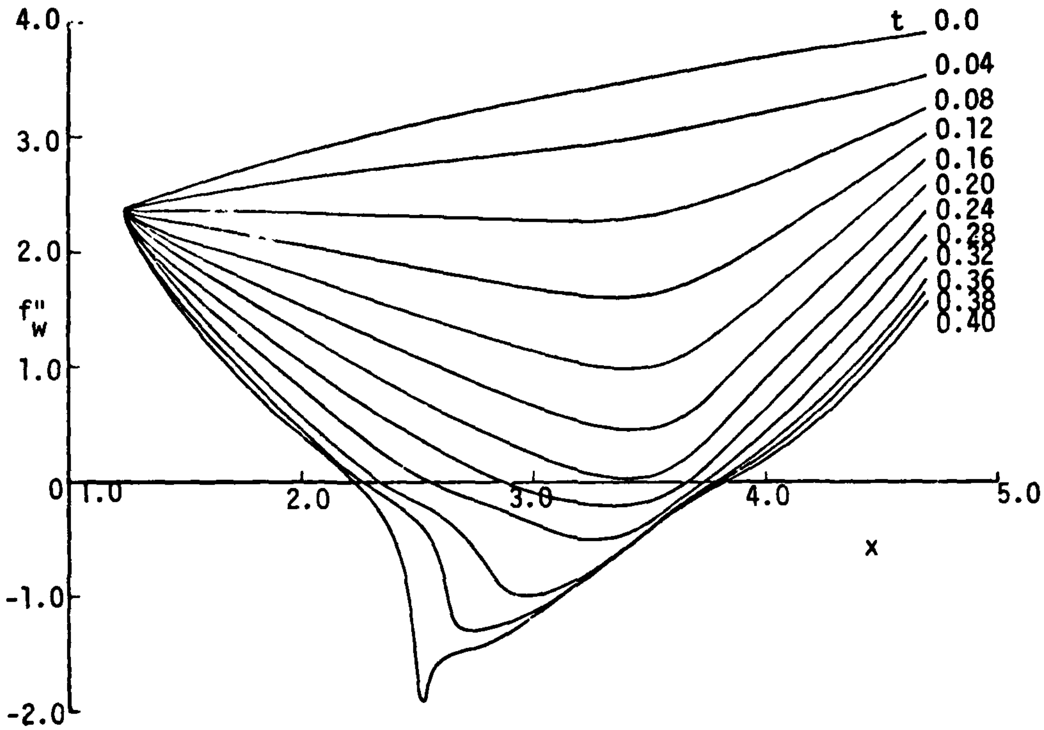

Figure 12. Variation of wall shear parameter f''_w with distance as a function of time for test case 5.

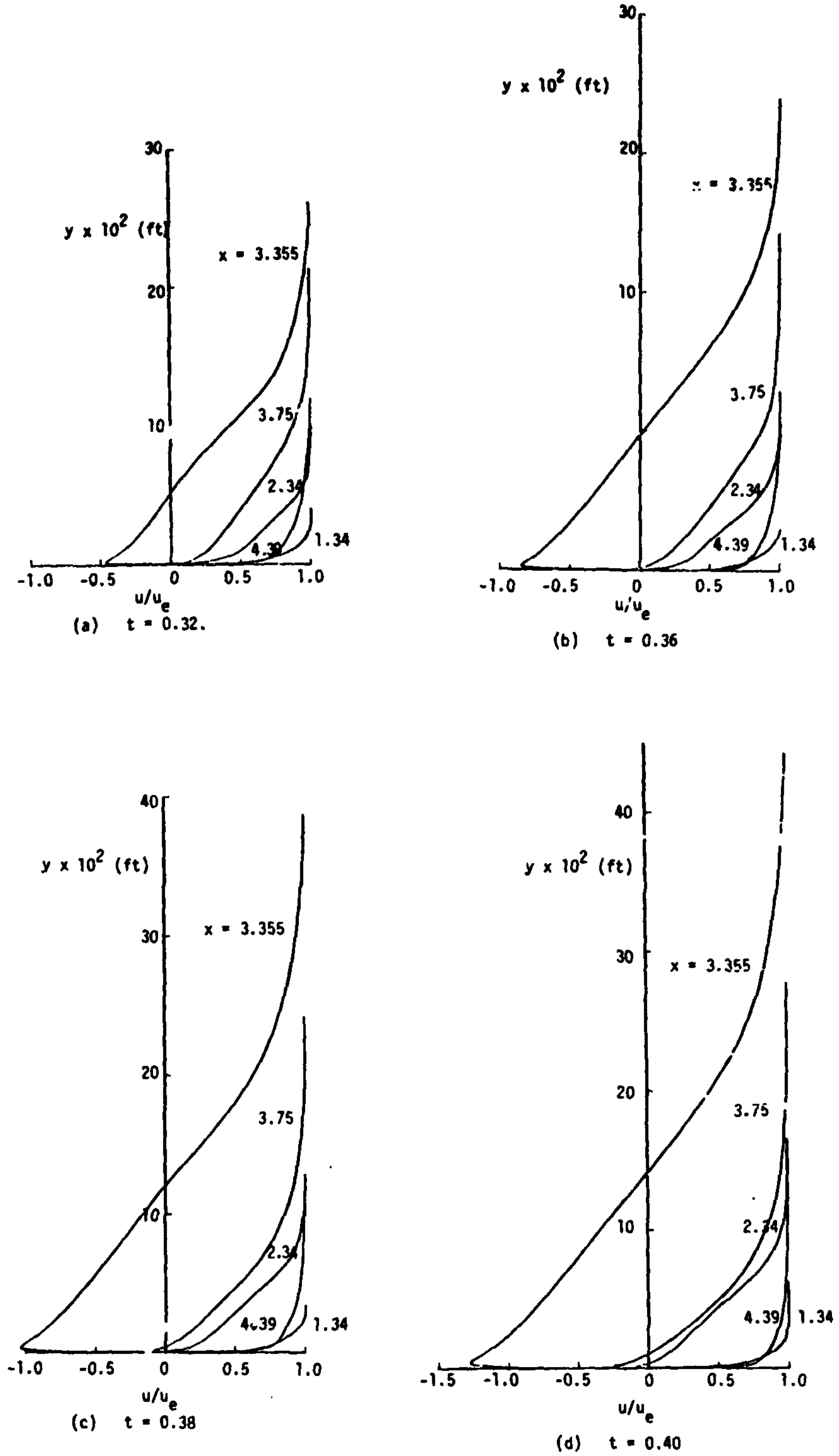

Figure 13. Calculated velocity profiles including flow reversal by the CS models for test case 5.

V. CONCLUDING REMARKS

Based on the studies conducted in this report, we observe that:

1. The numerical solution of unsteady laminar and turbulent boundary layers including the flow reversal across the layer can be obtained quite satisfactorily for a given pressure distribution. A combination of both regular and zig-zag box schemes are shown to yield accurate results for unsteady boundary layers.

2. Whether the unsteady boundary-layer equations for laminar and turbulent flows are singular for a given pressure distribution still remains to be investigated. The results for test case 5 indicate that at large times there is a puzzling "kink" in the wall shear parameter, f''_w; this may be due to a singularity or it may be due to a numerical problem. Recent studies conducted by Cebeci [14] and van Dommelen and Shen [15] for a circular cylinder started impulsively from rest indicate that at large times, $t = 1.25$ or more, there appears to be a singularity in δ^* around $\phi = 120^{\circ}$. However, these calculations do not indicate any puzzling behavior in thewall shear parameter near "singularity;" the f''_w-values are smooth and well behaved for these and larger times. On the other hand, examining the δ^*-results for test case 5, we find that while there is an abnormal behavior in f''_w at large times, the corresponding δ^*-values are smooth and well behaved, a trend which is opposite to that for a circular cylinder.

3. A comparison of the predictions of two turbulence models, namely, CS and BF models indicate that for attached flows, both models yield almost identical results. This is also true for flows which are sufficiently strong in pressure gradient to cause flow reversal across the layer.

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VII. DESCRIPTION OF THE COMPUTER PROGRAM WHICH USES THE CS MODEL

Input

Essentially the input to the computer program consists of four types of cards. Card 1 contains the title of the flow problem under consideration. Card 2 requires the following information to be specified.

NXT	Total number of t-stations to be calculated
NZT	Total number of x-stations to be calculated
NTR	x-station where transition begins. If the initial velocity profile is for turbulent flows, then NTR=1. If flow is all laminar, set NTR>NZT.
IBDY	Specifies whether the flow at x=0 starts as a flat-plate flow or as a stagnation-point flow. =1 flat-plate flow =2 stagnation-point flow
RL	Free-stream Reynolds number, $u_\infty L/\nu$.
IPRNT	Controls the print output =1 prints out only the boundary-layer parameters δ^*, θ, c_f, R_{δ^*}, R_θ, H and external velocity distribution. =2 prints out profiles as well as the boundary-layer parameters and external velocity field.

DETA(1) and VGP are the nonuniform grid parameters that control the spacing across the layer. The grid used in this report is a geometric progression with the property that the ratio of lengths of any two adjacent intervals is a constant; that is, $\Delta\eta_j = K\Delta\eta_{j-1}$. The distance to the j-th line is given by the following formula:

$$\Delta\eta_j = h_1(K^j - 1)/(K - 1) \qquad K > 1$$

There are two parameters in this equation: h_1, the length of the first step, and K, the ratio of two successive steps. The total number of points J can be calculated from the following formula:

$$J = \frac{\ln[1 + (K - 1)(\eta_e/h_1)]}{\ln K}$$

In the computer program which embodies the present solution method, h_1 and K are chosen with typical values, for moderate Reynolds numbers, of 0.01 and 1.3, respectively. In general, approximately 50 grid nodes across the boundary layer are sufficient to represent laminar and turbulent boundary-layer flows. The chosen values of h_1 and K must be such that the formula which generates the number og rid nodes according to a given or estimated η_e, i.e. Eq. ( ) does not allow J to exceed 101. Figure 14 is provided, therefore, to provide guidance in the selection of J.

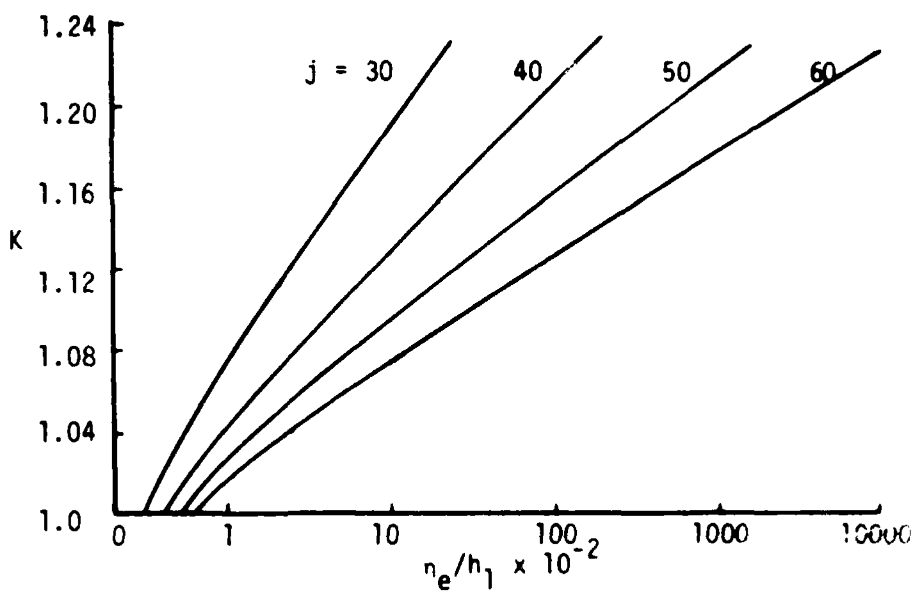

Figure 14. Variation of K with h_1 for different η_e-values.

CF and RTH are the local c_f and R_θ values which are used to start the turbulent flow calculations. The initial velocity profile is generated by using the formulas proposed by Granville (see ref. 9)

$$\frac{u}{u_\tau} = \frac{1}{\kappa} [\ln y^+ + c + \Pi(1 - \cos\pi\eta) + (\eta^2 - \eta^3)] \qquad (40)$$

From Eq. (40) and from the definitions of δ^* and θ, it can be shown that

$$\frac{\delta^*}{\delta} = \int_0^1 \frac{u_e - u}{u_\tau} \frac{u_\tau}{u_e} d\eta = \frac{u_\tau}{\kappa u_e} (\frac{11}{12} + \Pi) \qquad (41)$$

$$\frac{\theta}{\delta} = \int_0^1 \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right)d\eta = \frac{u_\tau}{\kappa u_e}\left(\frac{11}{12} + \Pi\right) - \left(\frac{u_\tau}{\kappa u_e}\right)^2 [2 + 2\Pi(1 + \frac{1}{\pi} \text{Si}(\pi)) + 1.5\Pi^2$$

$$+ \frac{1}{105} - \frac{7}{72} - 0.122925\Pi] \tag{42}$$

From Eq. (42), taking $\text{Si}(\Pi) = 1.8519$, we can also write

$$\frac{R_\theta}{R_\delta} = \frac{u_\tau}{\kappa u_e}\left(\frac{11}{12} + \Pi\right) - \left(\frac{u_\tau}{\kappa u_e}\right)^2 (1.9123016 + 3.05603 \ + 1.5\Pi^2)$$

Evaluating Eq. (40) at $\eta = 1$, we get

$$\frac{u_e}{u_\tau} = \frac{1}{\kappa}\left[\ln\left(\frac{\delta u_e}{\nu}\frac{u_\tau}{u_e}\right) + c + 2\Pi\right] \tag{43}$$

For a given value of c_f and R_θ, we can solve Eqs. (42) and (43) for δ and Π and then substitute them into Eqs. (40) and (41), thus obtaining u-profile.

Cards 3 and 4 read in the t and x stations, respectively. The present computer program specifies the external velocity distribution by a formula and computes the dimensionless pressure gradient parameters analytically as is shown in the listing which follows this section. The test case in the computer listing is for case 4 of ref. [12].

Output

Depending on the IPRNT, the computer program prints out the profiles f, f', f" and b as a function of the similarity variable η and grid parameter j together with a parameter KALC(J). Here KALC(J) = 0 when we use the standard box and = 1 when we use the zig-zag box.

ETA	η	
F	f'	
U	f'	
V	f"	
B	b $(=1 + \varepsilon_m^+)$	equals 1.0 for laminar flows

The output also includes displacement thickness δ^*, momentum thickness θ, local skin-friction coefficient c_f, Reynolds numbers based on δ^* and θ, that is, $R_{\delta *}$, R_θ and shape factor H. The definition of these parameters and their computer notation is

DELSTR $\quad \delta^* = \int_0^\infty (1 - u/u_e)dy$

THETA $\quad \theta = \int_0^\infty u/u_e(1 - u/u_e)dy$

CF $\quad c_f = 2\tau_w/\rho u_o^2$

RDELST $\quad R_\delta^* = \delta^* u_o/\nu$

RTHETA $\quad R_\theta = \theta u_o/\nu$

RZ $\quad R_z = z u_o/\nu$

H $\quad \delta^*/\theta$

In terms of transformed variables, δ^*, θ and c_f can be written as

$$\delta^* = \frac{z}{\sqrt{R_z}}[\eta_\infty - f_\infty/f'_\infty]$$

$$\theta = \frac{z}{\sqrt{R_x}}\int_0^{\eta_\infty}\frac{f'}{f'_e}(1 - \frac{f'}{f'_e})d\eta$$

$$c_f = 2\,\frac{f''_w}{\sqrt{R_z}}$$

```
      COMMON/BLCO/ NXT,NZT,NX,NZ,NP,NTR,ITMAX,IBDY,NPT,IZIG,ITURB,ETAE,
     1             VGP,A(101),ETA(101),DETA(101)
      COMMON/BLC1/ X(101),Z(81),UD(81),RZ(81),P1(81),P2(81),P3(101,81),
     1             UE(101,81)
      COMMON/BLCP/ DFLV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             B(101,81,2)
      COMMON/PRNT/ IPRNT
      COMMON/INTP/ N,IEX(81)
      COMMON/SAVE/ ITIME,NXTO,NZTO,XO(101),ZO(81),CFS(81),HS(81),RTHETA
      COMMON/BLCS/ FC(101),UC(101),VC(101),BC(101)
      DIMENSION      FF(101,3),UU(101,3),VV(101,3),BB(101,3)
      DIMENSION RTHETA(81)
C ---------------------------------------------------------------------
      READ(5,101) NXPRNT
      M      = 0
      ITMAX  = 10
      ITIME   = 0
    5 NX      = 1
      NZ      = 1
      N      = 0
      ITIME   = ITIME+1
      CALL   INPUT
      IF( ITIME .GT. 1 ) GO TO 6
      CALL  IVP
      GO TO   10
    6 DO 7 J = 1,NPT
      F(J,1,2)=FF(J,3)
      U(J,1,2)=UU(J,3)
      V(J,1,2)=VV(J,3)
      B(J,1,2)=BB(J,3)
    7 CONTINUE
   10 IF ( IPRNT .LT. 2 .AND. ITIME .GT. 2) WRITE(6,110) NX,NZ,X(NX),
     1        Z(NZ)
      IF ( ITURB .EQ. 0 ) GO TO 12
      IF ( NX .EQ. 1 .AND. NZ .EQ. 1 ) GO TO 90
   12 CONTINUE
      IF ( NZ .EQ. 1 .AND. NX .GT. 1 ) GO TO 90
      IT     = 0
      IGROW  = 0
   20 IT     = IT+1
      IF(IT .LE. ITMAX) GO TO 30
      WRITE(6, 10C ) NZ
      GO TO 90
   30 IF(NZ .GE. NTR) CALL EDDY
      IF ( NX .GT. 1 ) GO TO 50
      CALL  ICONZ
      GO TO 60
   50 CALL  COEFG(IT)
   60 CALL  SOLV3(IT)
      CALL  SMOOTH
C   CHECK FOR CONVERGENCE
      IF(NZ .GE. NTR) GO TO 70
C---LAMINAR FLOW
      IF(ABS(DFLV(1)) .GT. 0.0001)GO TO 20
      GO TO 80
C---TURBULENT FLOW
   70 IF(ABS(DFLV(1)) .LT. 2.5E-04 .OR. ABS(DFLV(1)/
```

```
     1  (V(1,NZ,2)+0.5*DELV(1))) .LT. 0.02) GO TO 80
        GO TO 20
   80   IF(NP .EQ. NPT) GO TO 90
        IF(ABS(U(NP-2,NZ,2)/U(NP,NZ,2)-1.0) .LT. 0.0015
     1  .AND. ABS(V(NP,NZ,2)) .LE. 0.001) GO TO 90
        IF(IGROW.GE.2) GOTO 90
        IGROW = IGROW+1
        WRITE(6,120)
        LL    = 2
        CALL GROWTH(LL)
        GO TO 20
   90   CALL OUTPUT
        IF ( NX .GE. NXPRNT ) IPRNT = 0
        IF ( ITIME .GT. 2 ) GO TO 10
        IF ( ITIME .EQ. 2 .AND. NZ .EQ. 3 ) GO TO 96
        IF ( ITIME .NE. 1 ) GO TO 10
        IF( M .LT. 3) GO TO 92
        DO 91 J=1,NPT
        FF(J,3)=F(J,3,2)
        UU(J,3)=U(J,3,2)
        VV(J,3)=V(J,3,2)
        BB(J,3)=B(J,3,2)
   91   CONTINUE
        GO TO 5
   92   M = M + 1
        DO 93 J=1,NPT
        FF(J,M) = F(J,M,2)
        UU(J,M) = U(J,M,2)
        VV(J,M) = V(J,M,2)
        BB(J,M) = B(J,M,2)
   93   CONTINUE
        IF ( M .LT. 3 ) GO TO 10

        NNZ   = 2
        CALL REPROF ( NPT, FF, UU, VV, BB, NNZ )

        NZ      = 3
        GO TO 10
   96   M      = 1
        DO 97 J = 1,NPT
        FF(J,M) = FC(J)
        UU(J,M) = UC(J)
        VV(J,M) = VC(J)
   97   BB(J,M) = BC(J)
        DO 98 K = 1,2
        M     = M+1
        DO 98 J = 1,NPT
        FF(J,M) = F(J,K,2)
        UU(J,M) = U(J,K,2)
        VV(J,M) = V(J,K,2)
   98   BB(J,M) = B(J,K,2)
        NZ    = 1
        CALL REPROF ( NPT, FF, UU, VV, BB, NZ )
        ITIME = ITIME+1
        IF ( IPRNT .LT. 2 .AND. ITIME .GT. 1) WRITE(6,110) NX,NZ,X(NX),
     1          Z(NZ)
        CALL OUTPUT
```

```
      GO TO 10
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
  100 FORMAT(1H0,16X,32HITERATIONS EXCEEDED ITMAX AT NZ=,I3)
 101   FORMAT(I5)
  110 FORMAT (1H0,4HNX =,I3,5X,4HNZ =,I3,5X,3HX =,F10.5,5X,3HZ =,F10.5)
  120 FORMAT(1H0,2X,'BOUNDARY LAYER HAS GROWN')
  121 FORMAT (1H0,5X,5HNX = ,I3,5X,8HX(NX) = ,F10.5,5X,5HXI = ,F10.5/
     11H0,5X,3H J ,5X,5HZ (J),5X,8HV (WALL),10X,2HCF,
     2    13X,1HH,12X,6HRTHETA,13X,2HUE,8X,6HEXTRAP / )
 122  FORMAT (1H ,5X,I3,2X,F11.6,5(2X,E13.6),5X,I1)
      END
```

```
      SUBROUTINE COEFG(IT)
      COMMON/BLCO/ NXT,NZT,NX,NZ,NP,NTP,ITMAX,IBDY,NPT,IZIG,ITURB,ETAE,
     1             VGP,A(101),ETA(101),DETA(101)
      COMMON/BLC1/ X(101),Z(81),UO(81),PZ(81),P1(81),P2(81),P3(101,81),
     1             UE(101,81)
      COMMON/BLCC/ S1(101),S2(101),S3(101),S4(101),S5(101),S6(101),
     1             R1(101),R2(101),R3(101)
      COMMON/BLCP/ DELV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             B(101,81,2)
      COMMON/ZGZG/ KALC(101)
      COMMON/SOLVE/ NZIG,NZIGS
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
      U(NP,NZ,2) = UE(NX,NZ) / UO(NZ)
      UOB = 0.5*(UO(NZ)+UO(NZ-1))
      IF ( IT .GT. 1 .AND. NZIGS .EQ. 1 ) GO TO 6
      NZIG = 0
      KALC(1)=0
      DO 4 J=2,NP
      UB   = 0.5*(U(J,NZ,2)+U(J-1,NZ,2))
      IF(UB .GE. 0.0) GO TO 4
      NZIG = 1
      KALC(1)=1
      GO TO 6
    4 CONTINUE
    6 DELX   = X(NX)-X(NX-1)
      DELZ   = Z(NZ)-Z(NZ-1)
      ZB     = 0.5*(Z(NZ)+Z(NZ-1))
      CEL    = ZB/DELZ
      BELM   = ZB/(DELX)/UOB
      CEL2   = 0.5*CEL
      CEL4   = 0.25*CEL
      P1H    = 0.5*P1(NZ)
      P2H    = 0.5*P2(NZ)
      P1B    = 0.5*(P1(NZ)+P1(NZ-1))
      P2B    = 0.5*(P2(NZ)+P2(NZ-1))
      P1BH   = 0.5*P1B
      P3B    = 0.25*(P3(NX,NZ)+P3(NX-1,NZ)+P3(NX,NZ-1)+P3(NX-1,NZ-1))
      DUSDZ   = 0.25*(U(NP,NZ,2)**2-U(NP,NZ-1,2)**2
     1          +U(NP,NZ,1)**2-U(NP,NZ-1,1)**2)/DELZ
      DUDX    = 0.5*(U(NP,NZ,2)-U(NP,NZ,1)+
     1          U(NP,NZ-1,2)-U(NP,NZ-1,1))/DELX
      P3B     =  ZB*(DUSDZ+DUDX/UOB)+P2B*(U(NP,NZ,2)**2+U(NP,NZ,1)**2
     1          +U(NP,NZ-1,2)**2+U(NP,NZ-1,1)**2)*0.25
      IF(NZ .EQ. NZT) GO TO 10
      NZP1   = NZ+1
      E1     = Z(NZ)*( (Z(NZP1)-Z(NZ))/(Z(NZP1)-Z(NZ-1))/DELZ )
      E1H    = 0.5*E1
      E2     = Z(NZ)/DELX/UO(NZ)
      E3    = Z(NZ)*DELZ/(Z(NZP1)-Z(NZ-1))/(Z(NZ)-Z(NZP1))
      P3BZ   = 0.5*( P3(NX,NZ)+P3(NX-1,NZ) )
      DUDX    = (U(NP,NZ,2)-U(NP,NZ,1))/DELX
      DUSDZ   = 0.5*((U(NP,NZ,2)**2-U(NP,NZ-1,2)**2)*E1
     1          +(U(NP,NZ,1)**2-U(NP,NZP1,1)**2)*E3)
      P3BZ    = Z(NZ)*(DUSDZ/Z(NZ)+DUDX/UO(NZ))+
     1          P2(NZ)*(U(NP,NZ,2)**2+U(NP,NZ,1)**2)*0.5
   10 CONTINUE
      DO 50  J=2,NP
```

```
      FB     = 0.5*(F(J,NZ,2)+F(J-1,NZ,2))
      FVB    = 0.5*(F(J,NZ,2)*V(J,NZ,2)+F(J-1,NZ,2)*V(J-1,NZ,2))
      UB     = 0.5*(U(J,NZ,2)+U(J-1,NZ,2))
      USB    = 0.5*(U(J,NZ,2)**2+U(J-1,NZ,2)**2)
      UB2    = 0.5*(U(J,NZ-1,2)+U(J-1,NZ-1,2))
      UB4    = 0.5*(U(J,NZ,1)+U(J-1,NZ,1))
      VB     = 0.5*(V(J,NZ,2)+V(J-1,NZ,2))
      DEPBV  = (B(J,NZ,2)*V(J,NZ,2)-B(J-1,NZ,2)*V(J-1,NZ,2))/DETA(J-1)
      FB4    = 0.5*(F(J,NZ,1)+F(J-1,NZ,1))
      USB4   = 0.5*(U(J,NZ,1)**2+U(J-1,NZ,1)**2)
      IF(NZ .EQ. NZT) GO TO 20
      IF ( NZIG .EQ. 1 ) GO TO 30
   20 FVJ2   = F(J,NZ,1)*V(J,NZ,1)+F(J,NZ-1,1)*V(J,NZ-1,1)+
     1         F(J,NZ-1,2)*V(J,NZ-1,2)
      FVJ1   = F(J-1,NZ,1)*V(J-1,NZ,1)+F(J-1,NZ-1,1)*V(J-1,NZ-1,1)+
     1         F(J-1,NZ-1,2)*V(J-1,NZ-1,2)
      FBI1   = 0.25*(F(J,NZ-1,1)+F(J-1,NZ-1,1)+F(J,NZ-1,2)+F(J-1,NZ-1,2))
      FVB234= 0.5*(FVJ2+FVJ1)
      USB2   = 0.5*(U(J,NZ-1,2)**2+U(J-1,NZ-1,2)**2)
      USB3   = 0.5*(U(J,NZ-1,1)**2+U(J-1,NZ-1,1)**2)
      USBI1  = 0.5*(USB2+USB3)
      UBK1   = 0.25*(U(J,NZ-1,1)+U(J-1,NZ-1,1)+U(J,NZ,1)+U(J-1,NZ,1))
      USJ2   = U(J,NZ,1)**2+U(J,NZ-1,1)**2+U(J,NZ-1,2)**2
      USJ1   = U(J-1,NZ,1)**2+U(J-1,NZ-1,1)**2+U(J-1,NZ-1,2)**2
      USB234= 0.5*(USJ2+USJ1)
      VJ1    = V(J-1,NZ,1)+V(J-1,NZ-1,1)+V(J-1,NZ-1,2)
      VJ2    = V(J,NZ,1)+V(J,NZ-1,1)+V(J,NZ-1,2)
      VB234  = 0.5*(VJ2+VJ1)
      BVJ1   = B(J-1,NZ,1)*V(J-1,NZ,1)+B(J-1,NZ-1,1)*V(J-1,NZ-1,1)+
     1         B(J-1,NZ-1,2)*V(J-1,NZ-1,2)
      BVJ2   = B(J,NZ,1)*V(J,NZ,1)+B(J,NZ-1,1)*V(J,NZ-1,1)+
     1         B(J,NZ-1,2)*V(J,NZ-1,2)
C
      CM1    = USB4-2.0*USBI1
      CM2    = UB2-2.0*UBK1
      CM4    = FB4-2.0*FBI1
      CM8    = (BVJ2-BVJ1)/DETA(J-1)
      CN3J   = CM1*CEL-0.5*CEL*VB234*CM4+2.0*BELM*CM2-CM8-P1B*FVB234
     1       + P2B*USB234-4.0*P3B
C     COEFFICIENTS FOR THE REGULAR BOX.
      S1(J)  = B(J,NZ,2)/DETA(J-1)+P1BH*F(J,NZ,2)+CEL4*(FB+CM4)
      S2(J)  =-B(J-1,NZ,2)/DETA(J-1)+P1BH*F(J-1,NZ,2)+CEL4*(FB+CM4)
      S3(J)  = P1BH*V(J,NZ,2)+CEL4*(VB+VB234)
      S4(J)  = P1BH*V(J-1,NZ,2)+CEL4*(VB+VB234)
      S5(J)  =-(P2B+CEL)*U(J,NZ,2)-BELM
      S6(J)  =-(P2B+CEL)*U(J-1,NZ,2)-BELM
      R2(J)  = CN3J-(DEPBV+P1B*FVB-(P2B+CEL)*USB+CEL2*(VB*FB+VB234*FB
     1              +CM4*VB)-BELM*2.0*UB)
      KALC(J)=0
      GO TO 40
C
   30 FB2    = 0.5*(F(J,NZ-1,2)+F(J-1,NZ-1,2))
      VB2    = 0.5*(V(J,NZ-1,2)+V(J-1,NZ-1,2))
      FVB4   = 0.5*(F(J,NZ,1)*V(J,NZ,1)+F(J-1,NZ,1)*V(J-1,NZ,1))
      UB6    = 0.5*(U(J,NZP1,1)+U(J-1,NZP1,1))
      FB6    = 0.5*(F(J,NZP1,1)+F(J-1,NZP1,1))
      UB46   = 0.25*(U(J,NZ,1)+U(J-1,NZ,1)+U(J,NZP1,1)+U(J-1,NZP1,1))
```

```
      VB46   = 0.25*(V(J,NZ,1)+V(J-1,NZ,1)+V(J,NZP1,1)+V(J-1,NZP1,1))
      DEBV4  = (B(J,NZ,1)*V(J,NZ,1)-B(J-1,NZ,1)*V(J-1,NZ,1))/DETA(J-1)
      C1     = E1H*(VB2*FB2-UB2**2)+E3*((UB4-UB6)*UB46-VB46*(FB4-FB6))
      C2     = DBBV4+P1(NZ)*FVB4-P2(NZ)*USB4+2.0*P3BZ
      C3     = -C2+2.0*C1-2.0*E2*UB4
C     COEFFICIENTS FOR THE ZIG-ZAG BOX.
      S1(J) = B(J,NZ,2)/DETA(J-1)+P1H*F(J,NZ,2)+E1H*(FB-FB2)
      S2(J) = -B(J-1,NZ,2)/DETA(J-1)+P1H*F(J-1,NZ,2)+E1H*(FB-FB2)
      S3(J) = P1H*V(J,NZ,2)+E1H*(VB+VB2)
      S4(J) = P1H*V(J-1,NZ,2)+E1H*(VB+VB2)
      S5(J) = -P2(NZ)*U(J,NZ,2)-E2-E1*U(J,NZ,2)
      S6(J) = -P2(NZ)*U(J-1,NZ,2)-E2-E1*U(J-1,NZ,2)
      R2(J) = C3-(DEFBV+P1(NZ)*FVB-P2(NZ)*USB-2.0*E2*UB-E1*(UB**2
     1        -VB*FB-VB2*FB+FB2*VB))
      KALC(J)=1
   40 R1(J) = F(J-1,NZ,2)-F(J,NZ,2)+DETA(J-1)*UB
      R3(J-1)=U(J-1,NZ,2)-U(J,NZ,2)+DETA(J-1)*VB
   50 CONTINUE
      IF ( IT .EQ. 1 ) NZIGS = NZIG
      R3(NP)= 0.0
      R1(1) = 0.0
      R2(1) = 0.0
      RETURN
      END
```

```
      SUBROUTINE EDDY
      COMMON/BLC0/ NXT,NZT,NX,NZ,NP,NTR,ITMAX,IBDY,NPT,IZIG,ITURB,ETAE,
     1             VGP,A(101),ETA(101),DETA(101)
      COMMON/BLC1/ X(101),Z(81),UO(81),PZ(81),P1(81),P2(81),P3(101,81),
     1             UE(101,81)
      COMMON/BLCP/ DELV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             B(101,81,2)
      COMMON/BLCD/ RL,CE,RTH
      DIMENSION EDV(101)
      DIMENSION  TB(101)
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
      GAMTR = 1.0
      IF (  ITURB  .NE.  0 ) GO TO 12
      UOI    = 0.0
      UO1    = 1.0/UE(NX,NTR-1)
      DO 10 I=NTR,NZ
      UO2    = 1.0/UE(NX,I)
      UOI    = UOI+(UO1+UO2)*0.5*(Z(I)-Z(I-1))
   10 UO1    = UO2
      GG     = 8.35E-04*UE(NX,NZ)**3 / (UE(NX,NTR-1)*Z(NTR-1))**1.34
      EXPT   = GG*RL**0.66*(Z(NZ)-Z(NTR-1))*UOI
      IF ( EXPT .LE. 10.0 ) GAMTR = 1.0 - EXP(-EXPT)
C
   12 SUM1   = 0.0
      F1     = U(1,NZ,2)/U(NP,NZ,2)*(1.0-U(1,NZ,2)/U(NP,NZ,2))
      DO 13 J=2,NP
      F2     = U(J,NZ,2)/U(NP,NZ,2)*(1.0-U(J,NZ,2)/U(NP,NZ,2))
      SUM1   = SUM1+(F1+F2)*A(J)
   13 F1     = F2
      THETA  = SQRT(Z(NZ)/(RL*UO(NZ)))*SUM1
      RT     = UE(NX,NZ)*THETA*RL
      IF ( RT .LE. 425.0 ) GO TO 14
      IF ( RT .GT. 6000.0 ) GO TO 15
      XPI    = RT/425.0-1.0
      PI     = 0.55*(1.0-EXP(-0.243*SQRT(XPI)-2.98*XPI))
      GO TO 20
   14 PI     = 0.0
      GO TO 20
   15 PI     = 0.55
C
   20 IFLG   = 0
      RZ2    = SQRT (UO(NZ)*Z(NZ)*RL)
      RZ4    = SQRT (RZ2)
      VMAX   = V(1,NZ,2)
      DO 30 J = 2,NP
      IF(ABS(V(J,NZ,2)).GT.ABS(VMAX))  VMAX= V(J,NZ,2)
   30 CONTINUE
      EDVO   = 0.0168*(1.55/(1.0+PI))*RZ2*(U(NP,NZ,2)*ETA(NP)-F(NP,NZ,2))
     1*GAMTR
      J      = 1
   80 IF(IFLG .EQ. 1) GO TO 90
      PPLUS = (P2(NZ)/RZ4)*(UE(NX,NZ)/UO(NZ))**2*(1.0/ABS(VMAX)**1.5)
      YOA    = RZ4*ETA(J)*SQRT(ABS(VMAX)*(1.0-11.8*PPLUS))/26.0
      EL     = 1.0
      IF(YOA .LT. 4.0) EL = (1.0-EXP(-YOA))**2
      EDVI   = 0.16*RZ2*ABS(V(J,NZ,2))*EL*GAMTR*ETA(J)**2
      IF(EDVI .LT. EDVO) GO TO 100
```

```
      IFLG   = 1
   90 EDV(J)= EDVO
      GO TO 110
  100 EDV(J)= EDVI
      IF(J.LE.2) GOTO 110
      IF(EDV(J).GT.EDV(J-1)) GOTO 110
      EDV(J)= EDV(J-1)+(EDV(J-1)-EDV(J-2))*VGP
      IF(EDV(J).LT.EDVO) GOTO 110
      EDV(J)= EDVO
      IFLG   = 1
  110 B(J,NZ,2)= 1.0+EDV(J)
      J        = J+1
      IF(J .LE. NP) GO TO 80
      NPM1     = NP-1
      TB(1)    = B(1,NZ,2)
      DO 150 J = 2,NP
      TB(J)    = (B(J-1,NZ,2)+B(J,NZ,2))*0.5
  150 CONTINUE
      TB(NP) = TB(NPM1)
      DO 170 J= 2,NPM1
      B(J,NZ,2)= (TB(J)+TB(J+1))*0.5
  170 CONTINUE
      B(NP,NZ,2)=B(NPM1,NZ,2)
      RETURN
      END
```

```
      SUBROUTINE GRID
      COMMON/BLCO/ NXT,NZT,NX,NZ,NP,NTR,ITMAX,IBDY,NPT,IZIG,ITURB,ETAE,
     1                VGP,A(101),ETA(101),DETA(101)
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
      IF((VGP-1.0) .LE. 0.001) GO TO 10
      NP     = ALOG((ETAE/DETA(1))*(VGP-1.0)+1.0)/ALOG(VGP) + 1.0001
      DETA(1) = ETAE*(VGP-1.0)/(VGP**(NP-1)-1.0)
      GO TO 20
   10 NP     = ETAE/DETA(1) + 1.0001
   20 IF(NP .LE. 101) GO TO 30
      WRITE(6, 50 )
      STOP
   30 ETA(1)= 0.0
      DO 40 J=2,101
      DETA(J)=VGP*DETA(J-1)
      A(J)   = 0.5*DETA(J-1)
   40 ETA(J)= ETA(J-1)+DETA(J-1)
      RETURN
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
   50 FORMAT(1H0,'  NP EXCEEDED 101 -- PROGRAM TERMINATED')
      END
```

```
      SUBROUTINE GROWTH(LL)
C
      COMMON/BLCP/ NXT,NZT,NX,NZ,NP,NTR,ITMAX,IBDY,NPT,IZIG,ITURB,ETAE,
     1             VGP,A(101),ETA(101),DETA(101)
      COMMON/BLCP/ DELV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             B(101,81,2)
C
      NPO    = NP
      NP1    = NP+1
      NPMAX  = 101
      IF ( LL .EQ. 1 ) GO TO 95
      NP     = NP + 2
      IF ( NP .GT. NPT ) NP=NPT
      NPMAX  = NP
   95 DO 100 J = NP1,NPMAX
      F(J,NZ,2)= U(NPO,NZ,2)*(ETA(J)-ETA(NPO))+F(NPO,NZ,2)
      U(J,NZ,2)= U(NPO,NZ,2)
      V(J,NZ,2) = 0.0
      B(J,NZ,2)= B(NPO,NZ,2)
  100 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE ICONZ
      COMMON/BLC0/ NXT,NZT,NX,NZ,NP,NTR,ITMAX,IBDY,NPT,IZIG,ITURB,ETAE,
     1             VGP,A(101),ETA(101),DETA(101)
      COMMON/BLC1/ X(101),Z(81),UO(81),RZ(81),P1(81),P2(81),P3(101,81),
     1             UE(101,81)
      COMMON/BLCP/ DELV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             B(101,81,2)
      COMMON/BLCC/ S1(101),S2(101),S3(101),S4(101),S5(101),S6(101),
     1             R1(101),R2(101),R3(101)
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
      BEL = 0.0
      IF(NZ .GT. 1) BEL = 0.5*(Z(NZ)+Z(NZ-1))/(Z(NZ)-Z(NZ-1))
      P1P    = P1(NZ)+BEL
      P2P    = P2(NZ)+BEL
      DO 30  J=2,NP
C   DEFINITION OF AVERAGED QUANTITIES
      FB     = 0.5*(F(J,NZ,2)+F(J-1,NZ,2))
      UB     = 0.5*(U(J,NZ,2)+U(J-1,NZ,2))
      VB     = 0.5*(V(J,NZ,2)+V(J-1,NZ,2))
      FVB    = 0.5*(F(J,NZ,2)*V(J,NZ,2)+F(J-1,NZ,2)*V(J-1,NZ,2))
      USB    = 0.5*(U(J,NZ,2)**2+U(J-1,NZ,2)**2)
      DERBV  = (B(J,NZ,2)*V(J,NZ,2)-B(J-1,NZ,2)*V(J-1,NZ,2))/DETA(J-1)
      IF(NZ .GT. 1) GO TO 10
      CFB    = 0.0
      CUB    = 0.0
      CVB    = 0.0
      CRB    = -P2(NZ)
      GO TO 20
   10 CFB    = 0.5*(F(J,NZ-1,2)+F(J-1,NZ-1,2))
      CUB    = 0.5*(U(J,NZ-1,2)+U(J-1,NZ-1,2))
      CVB    = 0.5*(V(J,NZ-1,2)+V(J-1,NZ-1,2))
      CFVB   = 0.5*(F(J,NZ-1,2)*V(J,NZ-1,2)+F(J-1,NZ-1,2)*V(J-1,NZ-1,2))
      CUSB   = 0.5*(U(J,NZ-1,2)**2+U(J-1,NZ-1,2)**2)
      CDERBV= (B(J,NZ-1,2)*V(J,NZ-1,2)-B(J-1,NZ-1,2)*V(J-1,NZ-1,2))/
     1        DETA(J-1)
      CLB    = CDERBV+P1(NZ-1)*CFVB-P2(NZ-1)*CUSB+P3(NX,NZ-1)
      CRB    = -P3(NX,NZ)+BEL*(CFVB-CUSB)-CLB
C
C   COEFFICIENTS OF THE DIFFERENCED MOMENTUM EQUATION
   20 S1(J) = B(J,NZ,2)/DETA(J-1)+(P1P*F(J,NZ,2)-BEL*CFB)*0.5
      S2(J) = -B(J-1,NZ,2)/DETA(J-1)+(P1P*F(J-1,NZ,2)-BEL*CFB)*0.5
      S3(J) = 0.5*(P1P*V(J,NZ,2)+BEL*CVB)
      S4(J) = 0.5*(P1P*V(J-1,NZ,2)+BEL*CVB)
      S5(J) = -P2P*U(J,NZ,2)
      S6(J) = -P2P*U(J-1,NZ,2)
C
C   DEFINITIONS OF RJ
      R1(J) = F(J-1,NZ,2)-F(J,NZ,2)+DETA(J-1)*UB
      R3(J-1)=U(J-1,NZ,2)-U(J,NZ,2)+DETA(J-1)*VB
      R2(J) = CRB-(DERBV+P1P*FVB-P2P*USB-BEL*(CFB*VB-CVB*FB))
   30 CONTINUE
      R1(1) = 0.0
      R2(1) = 0.0
      R3(NP)= 0.0
      RETURN
      END
```

```
      SUBROUTINE INPUT
      COMMON/BLCO/ NXT,NZT,NX,NZ,NE,NTR,ITMAX,IBDY,NPT,IZIG,ITURB,ETAE,
     1             VGP,A(101),ETA(101),DETA(101)
      COMMON/BLC1/ X(101),Z(81),UO(81),RZ(81),P1(81),P2(81),P3(101,81),
     1             UE(101,81)
      COMMON/BLCP/ DELV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             B(101,81,2)
      COMMON/BLCD/ RL,CE,RTH
      COMMON/PRNT/ IPRNT
      COMMON/SAVE/ ITIME,NXTO,NZTO,XO(101),ZO(81)
      DIMENSION TITLE(20)
C-----------------------------------------------------------------------
      PI    = 3.141593
      NPT = 101
      ITMAX = 10
      ETAE  = 8.0
      ISD   = 2
      INTR  = 2
      RL    = 1.0
      BA    = 2.4/(33.0*3.45)
      IF(ITIME .EQ. 1)GO TO 60
      NXT=NXTO
      NZT=NZTO
      DO 40 I=1,NXT
   40 X(I)=XO(I)
      DO 42 I=1,NZT
   42 Z(I)=ZO(I)
      GO TO 80
   60 ITURB = 0
      READ (5, 270) TITLE
      READ (5, 260) NXT,NZT,NTR,IBDY,RL,IPRNT,DETA(1),VGP,CE,RTH
      IF ( NTR .EQ. 1 ) ITURB = 1
      READ (5, 290) (X(I), I=1,NXT)
      READ (5, 290) (Z(I), I=1,NZT)
      NXTO=NXT
      NZTO=NZT
      DO 70 I=1,NXT
   70 XO(I)=X(I)
      DO 72 I=1,NZT
   72 ZO(I)=Z(I)
      Z1 = Z(1)
      DO 74 I=1,3
   74 Z(I) = Z1-0.01*(3-I)
      NXT=1
      NZT=3
   80 DO 100 I=1,NXT
      X(I) = X(I) * 33.0
  100 CONTINUE
      IF( ITIME .EQ. 1) GO TO 120
      WRITE(6, 330) TITLE
      WRITE(6, 340) NXT,NZT,NTR,IBDY
      WRITE(6, 342) RL, DETA(1), VGP, CE, RTH
C
  120 DO 140 I = 1,NZT
      UE(1,I)=1.0
      P3(1,I)=0.0
C     PRESSURE  GRADIENT  PARAMETER  FOR  STEADY  STATE
```

```
      P2(I) = 0.0
      P1(I) = 0.5*(1.0+P2(I))
  140 UO(I) = 1.0
      IF(ITIME .EQ. 1) RETURN
C     SAMPLE  TEST  CASE   4
      DO 150 K = 1,NXT
      DO 150 I = 1,NZT
      FUNC = 1.0
      DF   = 0.0
      FUNT = 1.0
      DT   = 0.0
      IF(Z(I) .GE. 1.34) GO TO 143
      FUNC = SIN(PI/2.0*((Z(I)-1.24)/0.1))
      DF = PI/2.0/0.1*COS(PI/2.0*((Z(I)-1.24)/0.1))
  143 IF(X(K) .GT. 1.98) GO TO 145
      FUNT = SIN(PI/2.0*(X(K)/1.98))
      DT   = PI/1.98/2.0*COS(PI/2.0*(X(K)/1.98))
  145 UE(K,I) = 1.0-BA*X(K)*(Z(I)-1.24)*FUNC*FUNT
      DUEDX =(-BA*X(K)*FUNC-BA*X(K)*(Z(I)-1.24)*DF)*FUNT
      DUEDT =(-BA*(Z(I)-1.24)*FUNC)*FUNT+DT*(-BA*(Z(I)-1.24)*X(K)*FUNC)
      P3(K,I)=Z(I)/UO(I)**2*(UE(K,I)*DUEDX+DUEDT)
  150 CONTINUE
  160 WRITE (6, 322) NXT
      WRITE (6, 326) ( X(I),I=1,NXT )
      WRITE (6, 324) NZT
      WRITE (6, 326) ( Z(I),I=1,NZT )
C
      RETURN
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
  260 FORMAT( 6I5,5F10.0 )
  270 FORMAT(20A4)
  290 FORMAT(8F10.0)
  322 FORMAT (///1H0,27HTABLE OF INPUT X FROM 1 TO , I3 / )
  324 FORMAT (1H0,27HTABLE OF INPUT Z FROM 1 TO , I3 /)
  326 FORMAT (1H , 3X, 12F10.5 )
  330 FORMAT(1H0,20A4)
  340 FORMAT( ///1H0,12H** CASE DATA/1H0,3X,6HNXT   =,I3,14X,6HNZT   =,I3,
     1          14X,6HNTR   =,I3/ 4X,6HIBDY =,I3    ,14X,6HIZIG =,I3,14X,
     2          6HITURB=,I3)
  342 FORMAT( 1H ,3X,6HRL    =,E14.6,3X,6HDETA1=,E14.6,3X,6HVGP   =,E14.6/
     1          4X,6HCF    =,E14.6,3X,6HRTHET=,E14.6 )
  350 FORMAT(1H0,3X,6HBB    =,E14.6,3X,6HBA    =,E14.6,3X,6HBL    =,E14.6,
     1          3X,6HZO    =,E14.6/)
      END
```

```
      SUBROUTINE IVP
C
      COMMON/BLCO/ NXT,NZT,NX,NZ,NP,NTR,ITMAX,IBDY,NPT,IZIG,ITURB,ETAE,
     1             VGP,A(101),ETA(101),DETA(101)
      COMMON/BLC1/ X(101),Z(81),UC(81),RZ(81),P1(81),P2(81),P3(101,81),
     1             UE(101,81)
      COMMON/BLC2/ DELV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             B(101,81,2)
      COMMON/BLC3/ RL,CF,RTHETI
      COMMON/SAVE/ ITIME,NXTC,NZTC,XC(101),ZC(81)
      DIMENSION UU(101),FF(101),GG(101),SHEAR(101)
      DATA PI,RK,C/3.14159265,0.41,2.0/
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
      IF ( ITURB .EQ. 1 ) GO TO 8
C
C     LAMINAR PROFILE
C
      CALL GRID
      NZ     = 1
      ETANPQ= 0.25*ETA(NP)
      ETAU15= 1.5/ETA(NP)
      DO 3 J=1,NP
      ETAB   = ETA(J)/ETA(NP)
      ETAB2  = ETAB**2
      F(J,NZ,2)= ETANPQ*ETAB2*(3.0-0.5*ETAB2)
      U(J,NZ,2)= 0.5*ETAB*(3.0-ETAB2)
      V(J,NZ,2)= ETAU15*(1.0-ETAB2)
      B(J,NZ,2)= 1.0
    3 CONTINUE
      RETURN
C
C     TURBULENT PROFILE
C
    8 RZ(NZ)= UE(NX,NZ)*Z(NZ)*RL
      SQRZ   = SQRT(RZ(NZ))
      CF02   = 0.5*CF
      SQCF02= SQRT(CF02)
      UTAU   = SQCF02
      SRXCF2= SQRZ*SQCF02
      SCF02K= SQCF02/RK
      YY     = 0.1
      AN     = SQCF02/RK
      AA     = AN*AN
      CC1    = -1.9123016*AA+11./12.*AN
      CC2    = AN-3.05603*AA
      CC3    = -1.5*AA
      CC4    = 1.0/AN-C-ALOG(SQCF02*RTHETI)
      AA1    = CC1+0.5*CC2*CC4+0.25*CC3*CC4**2
      AA2    = 0.5*(CC2+CC3*CC4)
      AA3    = 0.25*CC3
   10 YLOG   = ALOG(YY)
      FFC    = YY-(AA1+AA2*YLOG+AA3*YLOG**2)
      DF     = 1.0-(AA2+2.0*AA3*YLOG)/YY
      DYY    = FFC/DF
      YY     = YY-DYY
      IF(ABS(DYY) .LT. 0.00001) GO TO 20
      IT     = IT+1
```

```
      IF(IT .LT. 10) GO TO 10
   20 CONTINUE
      PIE    = 0.5*(CC4+ALOG(YY))
      RDELTA= RTHET1/YY
      ETAE   = RDELTA/SQRZ
      IF(ITIME .GT. 1)WRITE(6,150) PIE,RDELTA,ETAE,CF,RTHET1
      REGRRK= 50.0/SRXCF2
      CALL GRID
      DO 30  J=1,NP
      IF(ETA(J) .GT. REGRRK) GO TO 40
   30 CONTINUE
      NREG1 = J
      GO TO 50
   40 NREG1 = J-1
      NREG2 = J
C     CALCULATE FP,FPP PROFILES
   50 ZZ     = ETA(NREG1)/ETA(NP)
      R2     = ETA(NREG1)*SRXCF2
      R1     = 1./RK*(ALOG(R2)+C+PIE*(1.-COS(PI*ZZ))+ZZ*ZZ*(1.-ZZ))
      CY     = 0.05
      IT     = 0
   60 FFC    = R1*CY-ATAN(R2*CY)
      DF     = R1-R2/(1.+(R2*CY)**2)
      DCY    = FFC/DF/CY
      CY     = CY*(1.-DCY)
      IF(ABS(DCY).LT.0.001 .OR. IT.GT.15) GOTO 70
      IT     = IT+1
      GOTO 60
   70 UTNTM  = CY*SRXCF2
      VTM    = SQCF02*SRXCF2
      DO 80  J=1,NREG1
      U(J,NZ,2) = SQCF02*(ATAN(UTNTM*ETA(J))/CY)
   80 V(J,NZ,2) = VTM/(1.0+(UTNTM*ETA(J))**2)
      IF(NREG1 .EQ. NP) GO TO 100
      ZZ     = ETA(NREG1)/ETA(NP)
      C      = U(NREG1,NZ,2)/SCF02K-(ALOG(SRXCF2*ETA(NREG1))+PIE*
     1         (1.-COS(PI*ZZ))+ZZ*ZZ*(1.-ZZ))
      VJ     = PIE*PI/ETA(NP)
      DO 90  J=NREG2,NP
      ZZ     = ETA(J)/ETA(NP)
      COSANG= COS(PI*ZZ)
      SINANG= SQRT(1.-COSANG**2)
      U(J,NZ,2) = SCF02K*(ALOG(SRXCF2*ETA(J))+C+PIE*(1.-COSANG)+ZZ*ZZ*
     1          (1.0-ZZ))
   90 V(J,NZ,2) = SCF02K*(1./ETA(J)+VJ*SINANG+ZZ/ETA(NP)*(2.-3.*ZZ))
C     CALCULATE F PROFILE
  100 FDEN   = 2.*CY**2*SRXCF2
      F(1,NZ,2) = 0.0
      DO 110 J=2,NREG1
  110 F(J,NZ,2)=SQCF02*(ETA(J)/CY*ATAN(UTNTM*ETA(J))-1.0/FDEN*
     1         ALOG(1.0+(UTNTM*ETA(J))**2))
      IF(NREG1 .EQ. NP) GO TO 130
      ETA1   = ETA(NREG1)
      FJ     = PIE*ETA(NP)/PI
      ZZ     = ETA1/ETA(NP)
      FC = F(NREG1,NZ,2) -SCF02K*(ETA1*(ALOG(SRXCF2*ETA1)-1.0+C+PIE)-FJ*
     1          SIN(PI*ZZ)+ETA1*ZZ*ZZ*(1./3.-ZZ/4.))
```

```
      DO  120 J=NREG2,NP
      ETAJ    = ETA(J)
      ZZ      = ETAJ/ETA(NP)
  120 F(J,NZ,2)=FC+SCFO2K*(ETAJ*(ALOG(SPXCF2*ETAJ)-1.0+C+PIF)-FJ*SIN(PI
     1          *ZZ)+ETAJ*ZZ*ZZ*(1./3.-ZZ/4.))
  130 DO  140 J=1,NP
  140 U(J,NZ,2) = U(J,NZ,2)/U(NP,NZ,2)
      V(NP,NZ,2) = 0.00010
      CALL  EDDY
      RETURN
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
  150 FORMAT(1H0,4HPIE=,E14.6,3X,7HRDELTA=,E14.6,3X,5HETAE=,E14.6,3X,
     1          3HCF=,E14.6,3X,8HRTHETA1=,E14.6)
      END
```

```
      SUBROUTINE OUTPUT
      COMMON/BLCO/ NXT,NZT,NX,NZ,NP,NTR,ITMAX,IBDY,NPT,IZIG,ITUFB,ETAE,
     1             VGP,A(101),ETA(101),DETA(101)
      COMMON/BLC1/ X(101),Z(81),UC(81),RZ(81),P1(81),P2(81),P3(101,81),
     1             UE(101,81)
      COMMON/BLCB/ DELV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             B(101,81,2)
      COMMON/ZGZG/ KALC(101)
      COMMON/BLCD/ RL,CF,RTH
      COMMON/PRNT/ IPRNT
      COMMON/INTP/ N,IFX(81)
      COMMON/SAVE/ ITIME,NXTO,NZTO,XO(101),ZO(81),CFS(81),HS(81),RTHETA
      DIMENSION RTHETA(81),RTHT(81),NPK(81)
      DIMENSION FF(101,2),UU(101,2),VV(101,2),BB(101,2)
      DIMENSION FN(101),UN(101),VN(101),BN(101)
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
      NZ2    = NZT-2
      IEXTRP= 0
      IFX(NZ) = 0
      IF (ITIME .LT. 3) GO TO 5
      IF ( IPRNT .NE. 0 ) GO TO 5
      WRITE(6, 220 )
      NPM1 = NP-1
      J     = 1
      WRITE(6, 230 ) J,ETA(J),F(J,NZ,2),U(J,NZ,2),V(J,NZ,2),B(J,NZ,2),
     1                  KALC(J)
      NPM3 = NP-3
      WRITE(6, 230 )(J,ETA(J),F(J,NZ,2),U(J,NZ,2),V(J,NZ,2),B(J,NZ,2),
     1                  KALC(J),J=NPM3,NP)
    5 CONTINUE
      NPK(NZ)=NP
      IF ( NZ .EQ. 1 .AND. NTR .NE. 1 ) GO TO 90
      C1      = SQRT( Z(NZ) / (RL*UC(NZ)) )
      DELSTR= C1*(ETA(NP)-F(NP,NZ,2)/U(NP,NZ,2))
      CF      =2.0*V(1,NZ,2)/(SQRT(RL*UC(NZ)*Z(NZ))*(UE(NX,NZ)/UC(NZ))**2)
      CFS(NZ) = CF
      RDELST= UE(NX,NZ)*DELSTR*RL
      SUM1   = 0.0
      F1      = U(1,NZ,2)/U(NP,NZ,2)*(1.0-U(1,NZ,2)/U(NP,NZ,2))
      DO 10  J=2,NP
      F2      = U(J,NZ,2)/U(NP,NZ,2)*(1.0-U(J,NZ,2)/U(NP,NZ,2))
      SUM1   = SUM1+(F1+F2)*A(J)
   10 F1      = F2
      THETA  = C1*SUM1
      RTHETA(NZ) = UE(NX,NZ)*THETA*RL
      RTH    = RTHETA(NZ)
      H       = DELSTR/THETA
      HS(NZ) = H
      IF ( IEXTRP .GT. 0 .AND. NZ .EQ. NZT ) GO TO 190
   90 CALL GROWTH(1)
      IF (ITIME .LT. 3) GO TO 100
      IF ( IPRNT .GT. 1 ) GO TO 100
      WRITE ( 6, 242 ) DELSTR,THETA,CF,RDELST,RTHETA(NZ),H,
     1                   UE(NX,NZ)
  100 IF ( NZ .EQ. NZT ) GO TO 190
      IF(ITIME .LT. 3) GO TO 140
      IF( NZ .LT. NZ2) GO TO 140
```

```
      N      = N+1
      DO 120 J = 1,NPT
      FF(J,N) = F(J,NZ,2)/UE(NX,NZ)
      UU(J,N) = U(J,NZ,2)/UE(NX,NZ)
      VV(J,N) = V(J,NZ,2)/UE(NX,NZ)
  120 BB(J,N) = B(J,NZ,2)
      IF ( NZ .LT. (NZT-1)) GO TO 140
      DO 124 J = 1,NPT
      IF (U(J,NZ,2) .LT. 0. ) GO TO 130
  124 CONTINUE
      GO TO 140
  130 DZ1 = Z(NZT-1)-Z(NZT-2)
      DZ2 = Z(NZT)-Z(NZT-2)
      DO 134 J = 1,NPT
      DF = FF(J,2)-FF(J,1)
      DU = UU(J,2)-UU(J,1)
      DV = VV(J,2)-VV(J,1)
      DB = BB(J,2)-BB(J,1)
      FN(J) = FF(J,1)+DZ2*DF/DZ1
      UN(J) = UU(J,1)+DZ2*DU/DZ1
      VN(J) = VV(J,1)+DZ2*DV/DZ1
      BN(J) = BB(J,1)+DZ2*DB/DZ1
  134 CONTINUE
      IEXTRP = 1
      IEX(NZ+1)=1
  140 NZ     = NZ+1
      IF(IEXTRP .LE. 0) GO TO 148
      DO 142 J = 1,NPT
      KALC(J) = 2
      F(J,NZ,2) = FN(J)*UE(NX,NZ)
      U(J,NZ,2) = UN(J)*UE(NX,NZ)
      V(J,NZ,2) = VN(J)*UE(NX,NZ)
  142 B(J,NZ,2) = BN(J)
      IF (IPRNT  .NE. 0) GO TO 5
      WRITE(6, 220 )
      NPM1 = NP-1
      WRITE(6, 230 )(J,ETA(J),F(J,NZ,2),U(J,NZ,2),V(J,NZ,2),B(J,NZ,2),
     1                   KALC(J),J=1,NPM1,3)
      J = NP
      WRITE(6, 230 ) J,ETA(J),F(J,NZ,2),U(J,NZ,2),V(J,NZ,2),B(J,NZ,2),
     1                   KALC(J)
      GO TO 5
  148 IF (NX .GT. 1) GO TO 160
C   INITIAL GUESS FOR NEXT STATION
      DO 150 J=1,NPT
      F(J,NZ,2)= F(J,NZ-1,2)
      U(J,NZ,2)= U(J,NZ-1,2)
      V(J,NZ,2)= V(J,NZ-1,2)
  150 B(J,NZ,2)= B(J,NZ-1,2)
      IF ( NX .EQ. 1 ) RETURN
C
C
C     DETERMINE NP FOR ZIGZAG SCHEME
      IF (NZ .EQ. NZT) GO TO 170
      IF (NX .EQ. 1) GO TO 170
      IF (NP .LT. NPK(NZ+1)) NP=NPK(NZ+1)
      RETURN
```

```
 160 NP      = NPK(NZ)
     IF (NZ .EQ. 1) RETURN
     IF (NP .LT. NPK(NZ-1)) NP=NPK(NZ-1)
     IF (IZIG .EQ. 0) GO TO 170
     IF (NP .LT. NPK(NZ+1)) NP=NPK(NZ+1)
 170 UR      = UE(NX,NZ)/UE(NX,NZ-1)
     DO 180 J = 1,NPT
     F(J,NZ,2)= F(J,NZ-1,2)*UR
     U(J,NZ,2)= U(J,NZ-1,2)*UR
     V(J,NZ,2)= V(J,NZ-1,2)*UR
     B(J,NZ,2)= B(J,NZ-1,2)
 180 CONTINUE
     RETURN
190  IF(NX .EQ. NXT .AND. ITIME .EQ. 1) RETURN
     IF ( IFENT .NE. 2 ) GO TO 194
 1   XI   =   X(NX)/33.0
     WRITE ( 6, 250 ) NX, X(NX),XI
     DO 193 K  = 1, NZ
     WRITE ( 6, 260 ) K, Z(K), V(1,K,2),CFS(K),HS(K),RTHETA(K),
    1                     UE(NX,K),IFX(K)
 193 CONTINUE
 194 IF(NX .EQ. NXT .AND. ITIME .EQ. 3) STOP
     NX      = NX+1
     N       = 0
     NZ      = 1
C    SHIFT.
     DO 210 K=1,NZT
     DO 200 J=1,NPT
     F(J,K,1)= F(J,K,2)
     U(J,K,1)= U(J,K,2)
     V(J,K,1)= V(J,K,2)
 200 B(J,K,1)= B(J,K,2)
 210 CONTINUE
     GO TO 160
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
 220 FORMAT( 1H0, 2X,1HJ,4X,3HETA,10X,1HF,13X,1HU,13X,1HV,13X,1HB,8X,
    1         4HKALC)
 230 FORMAT( 1H ,I3,F10.5,4E14.6,I6)
 242 FORMAT( 1H0, 7HDELSTR=,E14.6,3X,7HTHETA  =,E14.6,3X,7HCF      =,E14.6/
    1         1H ,7HRDELST=,E14.6,3X,7HRTHETA=,E14.6                      /
    2         1H ,7HH      =,E14.6,3X,7HUE     =,E14.6 / )
 250 FORMAT ( 1H0, 5X, 5HNX = ,I3,5X,8HX(NX) = ,F10.5,5X,5HXI = ,F10.5/
    1 1H0,5X,3H J ,5X,5HZ (J),5X,8HV (WALL),10X,2HCF,
    2    13X,1HH,12X,6HRTHETA,13X,2HUE,8X,6HEXTRAP / )
260  FORMAT (1H ,5X,I3,2X,F11.6,5(2X,E13.6),5X,I1)
     END
```

```
      SUBROUTINE   SMOOTH
      COMMON/BLC0/ NXT,NZT,NX,NZ,NP,NTR,ITMAX,IBDY,NPT,IZIG,ITURB,ETAE,
     1             VGP,A(101),ETA(101),DETA(101)
      COMMON/BLC1/ X(101),Z(81),UD(81),PZ(81),P1(81),P2(81),P3(101,81),
     1             UE(101,81)
      COMMON/BLC2/ DELV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             B(101,81,2)
      DIMENSION FS(101),US(101),VS(101),BS(101)
C-----------------------------------------------------------------------
      NPM1   = NP-1
      NPM2   = NP-2
      JMAX   = 1
      VMAX   = V(1,NZ,2)
      DO 10 J = 2,NP
      IF ( V(J,NZ,2) .LT. VMAX ) GO TO 10
      VMAX   = V(J,NZ,2)
      JMAX   = J
   10 CONTINUE
      DUJ1   = U(JMAX+1,NZ,2)-U(JMAX,NZ,2)
      DVJ1   = V(JMAX+1,NZ,2)-V(JMAX,NZ,2)
      JS     = JMAX+2
      DO 20 J=JS,NP
      JJ     = J
      DUJ2   = U(J,NZ,2)-U(J-1,NZ,2)
      DVJ2   = V(J,NZ,2)-V(J-1,NZ,2)
      UJPROD = UJ2*UJ1
      VJPROD = VJ2*VJ1
      IF ( UJPROD .LT. 0.0 .OR. VJPROD .LT. 0.0 ) GO TO 30
      DUJ1   = DUJ2
      DVJ1   = DVJ2
   20 CONTINUE
   30 IF ( JJ .EQ. NP ) RETURN
      DO 40 J = JJ,NP
      FS(J)  = 0.5*(F(J-1,NZ,2)+F(J,NZ,2))
      US(J)  = 0.5*(U(J-1,NZ,2)+U(J,NZ,2))
      VS(J)  = 0.5*(V(J-1,NZ,2)+V(J,NZ,2))
      BS(J)  = 0.5*(B(J-1,NZ,2)+B(J,NZ,2))
   40 CONTINUE
      DO 50 J= JJ,NPM1
      F(J,NZ,2)=0.5*(FS(J)+FS(J+1))
      U(J,NZ,2)=0.5*(US(J)+US(J+1))
      V(J,NZ,2)=0.5*(VS(J)+VS(J+1))
      B(J,NZ,2)=0.5*(BS(J)+BS(J+1))
   50 CONTINUE
      VNP    = -V(NP-1,NZ,2)+(U(NP,NZ,2)-U(NPM1,NZ,2))/A(NP)
      IF(ABS(VNP) .LT. ABS(V(NP,NZ,2))) V(NP,NZ,2) = VNP
      RETURN
      END
```

```
      SUBROUTINE FEPROF ( NPT, FF, UU, VV, BB, N )
C
      COMMON/BLC1/ X(101),Z(81),UD(81),RZ(81),P1(81),P2(81),P3(101,81),
     1             UE(101,81)
      COMMON/BLC2/ DELV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             B(101,81,2)
      COMMON/PRNT/ IPRNT
      COMMON/BLCS/ FC(101),UC(101),VC(101),BC(101)
      DIMENSION    FF(101,3),UU(101,3),VV(101,3),BB(101,3)
      DIMENSION    FA(101,2),UA(101,2),VA(101,2),BA(101,2)
      DIMENSION    FN(101),UN(101),VN(101),BN(101),DX(2)
C
      M     = N+1
      DO 100 I = 1,2
      DO 100 J = 1,NPT
      FA(J,I)= 0.5*(FF(J,I)+FF(J,I+1))
      UA(J,I)= 0.5*(UU(J,I)+UU(J,I+1))
      VA(J,I)= 0.5*(VV(J,I)+VV(J,I+1))
      BA(J,I)= 0.5*(BB(J,I)+BB(J,I+1))
  100 CONTINUE
C
C
      DO 140 J = 1,NPT
      F(J,N,2) = 0.5*(FA(J,1)+FA(J,2))
      U(J,N,2) = 0.5*(UA(J,1)+UA(J,2))
      V(J,N,2) = 0.5*(VA(J,1)+VA(J,2))
      B(J,N,2) = 0.5*(BA(J,1)+BA(J,2))
C
      F(J,M,2) = F(J,N,2)
      U(J,M,2) = U(J,N,2)
      V(J,M,2) = V(J,N,2)
      B(J,M,2) = B(J,N,2)
      FC(J) = F(J,N,2)
      UC(J) = U(J,N,2)
      VC(J) = V(J,N,2)
      BC(J) = B(J,N,2)
  140 CONTINUE
C
      RETURN
      END
```

```
      SUBROUTINE SOLV3(IT)
      COMMON/BLC0/ NXT,NZT,NX,NZ,NP,NTR,ITMAX,IBDY,NPT,IZIG,ITUPB,ETAE,
     1             VGP,A(101),ETA(101),DETA(101)
      COMMON/BLC1/ X(101),Z(81),UC(81),RZ(81),P1(81),P2(81),P3(101,81),
     1             UE(101,81)
      COMMON/BLCC/ S1(101),S2(101),S3(101),S4(101),S5(101),S6(101),
     1             R1(101),R2(101),R3(101)
      COMMON/BLCP/ DELV(101),F(101,81,2),U(101,81,2),V(101,81,2),
     1             P(101,81,2)
      COMMON/PRNT/ IPRNT
      DIMENSION A11(101),A12(101),A13(101),A21(101),A22(101),A23(101),
     1          G11(101),G12(101),G13(101),G21(101),G22(101),G23(101),
     2          W1(101),W2(101),W3(101),DELF(101),DELU(101)
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
      RELAX  = 1.0
      IF ( IT .GT. 4 ) RELAX = 0.50
      W1(1) = R1(1)
      W2(1) = R2(1)
      W3(1) = R3(1)
      A11(1)= 1.0
      A12(1)= 0.0
      A13(1)= 0.0
      A21(1)= 0.0
      A22(1)= 1.0
      A23(1)= 0.0
      G11(2)=-1.0
      G12(2)=-0.5*DETA(1)
      G13(2)= 0.0
      G21(2)= S4(2)
      G23(2)=-2.0*S2(2)/DETA(1)
      G22(2)= G23(2)+S6(2)
      DO 20  J=2,NP
      IF(J .EQ. 2) GO TO 10
      DEN    = (A13(J-1)*A21(J-1)-A23(J-1)*A11(J-1)-A(J)*
     1         (A12(J-1)*A21(J-1)-A22(J-1)*A11(J-1)))
      G11(J)= (A23(J-1)+A(J)*(A(J)*A21(J-1)-A22(J-1)))/DEN
      G12(J)=-(1.0+G11(J)*A11(J-1))/A21(J-1)
      G13(J)= (G11(J)*A13(J-1)+G12(J)*A23(J-1))/A(J)
      G21(J)= (S2(J)*A21(J-1)-S4(J)*A23(J-1)+A(J)*(S4(J)*
     1         A22(J-1)-S6(J)*A21(J-1)))/DEN
      G22(J)= (S4(J)-G21(J)*A11(J-1))/A21(J-1)
      G23(J)= (G21(J)*A12(J-1)+G22(J)*A22(J-1)-S6(J))
   10 A11(J)= 1.0
      A12(J)=-A(J)-G13(J)
      A13(J)= A(J)*G13(J)
      A21(J)= S3(J)
      A22(J)= S5(J)-G23(J)
      A23(J)= S1(J)+A(J)*G23(J)
      W1(J) = R1(J)-G11(J)*W1(J-1)-G12(J)*W2(J-1)-G13(J)*W3(J-1)
      W2(J) = R2(J)-G21(J)*W1(J-1)-G22(J)*W2(J-1)-G23(J)*W3(J-1)
      W3(J) = R3(J)
   20 CONTINUE
      DELU(NP)  = W3(NP)
      E1        = W1(NP)-A12(NP)*DELU(NP)
      E2        = W2(NP)-A22(NP)*DELU(NP)
      DELV(NP)  = (E2*A11(NP)-E1*A21(NP))/(A23(NP)*A11(NP)-A13(NP)*
     1             A21(NP))
```

```
      DELF(NP) = (F1-A13(NP)*DELV(NP))/A11(NP)
      J        = NP
   30 J        = J-1
      F3       = W3(J)-DELU(J+1)+A(J+1)*DELV(J+1)
      DELV(J)  = (A11(J)*(W2(J)+F3*A22(J))-A21(J)*W1(J)-F3*A21(J)*A12(J)
     1          )/(A21(J)*A12(J)*A(J+1)-A21(J)*A13(J)-A(J+1)*
     2          A22(J)*A11(J)+A23(J)*A11(J))
      DELU(J)  =-A(J+1)*DELV(J)-F3
      DELF(J)  = (W1(J)-A12(J)*DELU(J)-A13(J)*DELV(J))/A11(J)
      IF(J .GT. 1) GO TO 30
      IF ( IPRNT .LT. 2 ) WRITE(6, 50) V(1,NZ,2), DELV(1)
      DO 40  J=1,NP
      F(J,NZ,2)= F(J,NZ,2)+DELF(J)*RELAX
      U(J,NZ,2)= U(J,NZ,2)+DELU(J)*RELAX
   40 V(J,NZ,2)= V(J,NZ,2)+DELV(J)*RELAX
      U(1,NZ,2)= 0.0
      RETURN
C - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
   50 FORMAT(1H ,5X,8HV(WALL)=,F14.6,5X,6HDELVW=,F14.6)
      END
```

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16. Abstract

Recently a new procedure, which solves the governing boundary-layer equations with Keller's box method, has been developed for calculating unsteady laminar flows with flow reversal. In this report, we extend this method to turbulent boundary layers with flow reversal. Using the algebraic eddy viscosity formulation of Cebeci and Smith, we consider several test cases to investigate the proposition that unsteady turbulent boundary layers also remain free of singularities. We also perform turbulent flow calculations by using the turbulence model of Bradshaw, Ferriss, and Atwell; we solve the governing equations for both models by using the same numerical scheme and compare the predictions with each other.

The study reveals that, as in laminar flows, the unsteady turbulent boundary layers are free from singularities, but there is a clear indication of rapid thickening of the boundary layer with increasing flow reversal. The study also reveals that the predictions of both turbulence models are the same for all practical purposes.

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