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SOME GENERALIZATIONS OF KANTOROVICH INEQUALITY*

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C. R. Rao on the occasion of his sixtieth birthday.

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SOME GENERALIZATIONS OF KANTOROVICH INEQUALITY

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Summary: Kantorovich gave an upper bound to $(x'Vx)(x'V^{-1}x)$ where x is an n -vector of unit length and V is an $n \times n$ positive definite matrix. Bloomfield, Watson and Knott found the bound to $|X'VXX'V^{-1}X|$, and Khatri and Rao to the trace and determinant of $X'VYY'V^{-1}X$ where X and Y are $n \times k$ matrices such that $X'X = Y'Y = I$. In the present paper we establish bounds for traces and determinants of $X'VYY'V^{-1}X$ and $X'BY Y'CX$ when X and Y are matrices of different orders. A review of previous results on generalizations of Kantorovich inequality and a number of new results of independent interest are also given.

Key Words: Kantorovich inequality, Least squares estimation.

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1. INTRODUCTION

Let V be an $n \times n$ positive definite matrix with eigen values $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$, and define $\omega_i = (\lambda_i + \lambda_{n+i-1})^2 / 4 \lambda_i \lambda_{n+i-1}$, $i = 1, \dots, \alpha(\leq n/2)$. Kantorovich (1948) established the inequality

$$1 \leq (x' V x) (x' V^{-1} x) / (x' x)^2 \leq \omega_1 \quad (1.1)$$

for all non-null vectors x . A natural generalization of (1.1) is to compare the matrices $A = (X'X)^{-1} X' V X (X'X)^{-1}$ and $B = (X' V^{-1} X)^{-1}$ when X varies over $n \times k$ matrices of rank k . Let $\theta_1, \dots, \theta_k$ be the roots of $|A - \theta B| = 0$, i.e., the eigen values of A with respect to B . It is easy to establish that $\theta_i \geq 1$ for all i . Bloomfield and Watson (1975) and Knott (1975) showed that

$$|AB^{-1}| = \theta_1 \dots \theta_k \leq \prod_{i=1}^{\min(k, n-k)} \omega_i \quad (1.2)$$

while Khatri and Rao (1980) established that

$$\begin{aligned} \text{tr}(AB^{-1}) = \theta_1 + \dots + \theta_k &\leq \omega_1 + \dots + \omega_k \quad \text{if } n \geq 2k \\ &\leq \omega_1 + \dots + \omega_{n-k} + (2k-n), \quad \text{if } n < 2k \end{aligned} \quad (1.3)$$

where $\text{tr} C$ represents the trace of matrix C . It may be noted that $\sum \theta_i = \text{tr } P_X V P_X V^{-1}$ where P_X is the projection operator on the column space of X . Bloomfield and Watson (1975) gave another inequality

$$\text{tr } P_X V (I - P_X) V \leq \frac{1}{4} \sum_{i=1}^k (\lambda_i - \lambda_{n-i+1})^2 \quad (1.4)$$

when $n \geq 2k$. The inequalities (1.2) - (1.4) are useful in comparing the efficiencies of simple least squares estimators with the minimum variance unbiased estimators of parameters in the Gauss-Markoff model (see Khatri and Rao, 1980).

Khatri (1978, 1980) considered the matrix $(I - BA^{-1})$ which arises in a different context and proved the following results. Let $\underline{i} = (i_1, i_2, \dots, i_n)$ be a permutation of $(1, 2, \dots, n)$ and P denote the class of all permutations of $(1, 2, \dots, n)$. Further, let

$$\xi_{\underline{i}(\alpha)} = (\lambda_{i_\alpha} - \lambda_{i_{n-\alpha+1}})^2 / (\lambda_{i_\alpha} + \lambda_{i_{n-\alpha+1}})^2 \quad (1.5)$$

for $\alpha = 1, 2, \dots, k$ with $n \geq 2k$. Then

$$|I - BA^{-1}| \leq \sup_{\underline{i} \in P} \prod_{\alpha=1}^{\min(k, n-k)} \xi_{\underline{i}(\alpha)} = \prod_{j=1}^{\min(k, n-k)} \frac{(\lambda_j - \lambda_{n-j+1})^2}{(\lambda_j + \lambda_{n-j+1})^2}, \quad (1.6)$$

$$\text{tr } (I - BA^{-1}) \leq \sup_{\underline{i} \in P} \sum_{\alpha=1}^k \xi_{\underline{i}(\alpha)} \quad \text{if } n \geq 2k, \quad (1.7)$$

$$\text{tr } (I - BA^{-1})^{-1} \geq \inf_{\underline{i} \in P} \sum_{\alpha=1}^k \xi_{\underline{i}(\alpha)}^{-1} \quad \text{if } n \geq 2k. \quad (1.8)$$

Khatri (1978) further showed that if

$$C = X' V Y (Y' V Y)^{-1} Y' V X (X' V X)^{-1} \quad (1.9)$$

where Y is an $n \times s$, ($s \leq n-k$), matrix such that $X'Y = 0$,
then

$$|C| \leq |Q|, \text{tr } C \leq \text{tr } Q, \text{ and } \text{tr } C^{-1} \geq \text{tr } Q^{-1} \quad (1.10)$$

where $Q = I - BA^{-1}$. Eaton (1976) established the results (1.6) and (1.10) when $k=s=1$.

Strang (1960) generalized the Kantorovich inequality (1.1) in the form

$$[(x'Ax)(y'A^{-1}x)/(x'x)(y'y)] \leq \omega_1 \quad (1.11)$$

for all non-null vectors x and y where A is an $n \times n$ nonsingular matrix with singular values $\delta \geq \delta_2 \geq \dots \geq \delta_n > 0$ and

$$\omega_1 = (\delta_1 + \delta_{n-1+1})^2 / 4 \delta_1 \delta_{n-1+1}. \quad (1.12)$$

Greub and Rheinboldt (1959) proved that

$$\frac{(x'G^2x)(x'H^2x)}{(x'GHx)^2} \leq \frac{(\lambda_1\mu_1 + \lambda_n\mu_n)^2}{4\lambda_1\lambda_n\mu_1\mu_n} \quad (1.13)$$

for all non-null vectors x , when G and H are positive definite commuting matrices with eigen values $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n > 0$ respectively. The result (1.13) can be proved using (1.11).

A natural generalization of the expression in (1.11) is of the form

$$g(X,Y) = |X' A P_Y A^{-1} X| / |X' X| \quad (1.14)$$

or

$$f(X,Y) = \text{tr} (P_X A P_Y A^{-1}) \quad (1.15)$$

where X and Y are $(n \times k)$ and $(n \times s)$ matrices of ranks k and s respectively with $s \geq k$, and A is an $n \times n$ non-singular matrix with singular values $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n > 0$. When $s = k$, Khatri and Rao (1980) established the inequalities

$$g(X,Y) \leq \prod_{i=1}^{\min(k, n-k)} \omega_i \quad (1.16)$$

$$\begin{aligned} f(X,Y) &\leq \sum_{i=1}^k \omega_i \quad \text{if } n \geq 2k \\ &\leq \sum_{i=1}^{n-k} \omega_i + 2k - n \quad \text{if } n < 2k \end{aligned} \quad (1.17)$$

where ω_i are as defined in (1.12). In the next section we establish the bounds for (1.14) and (1.15) when s is not necessarily equal to k .

We also consider determinants and traces of matrices of the type

$$(X' G^2 Y) (Y' G H Y)^{-1} (Y' H^2 X) (X' G H X)^{-1} \quad (1.18)$$

which are natural extensions of the expression in (1.13) and establish bounds under very general conditions on X , Y , G and H .

2. THE MAIN THEOREMS

In all the theorems stated in this section, X and Y stand for $n \times k$ and $n \times s$ matrices of ranks k and s respectively with $s \geq k$ and P_Z stands for the projection operator on the column space of matrix Z .

Theorem 1. Let $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n > 0$ be the singular values of an $n \times n$ nonsingular matrix A , and

$$\omega_i = \frac{(\delta_i + \delta_{n-i+1})^2}{4\delta_i \delta_{n-i+1}}, \quad i = 1, \dots, m(\leq n/2). \quad (2.1)$$

Then

$$\frac{|X' A P_Y A^{-1} X|}{|X' X|} \leq \prod_{i=1}^{\min(k, n-s)} \omega_i \quad (2.2)$$

and

$$\begin{aligned} \text{tr}(P_X A P_Y A^{-1}) &\leq \sum_{i=1}^k \omega_i \quad \text{if } n \geq s+k \\ &\leq \sum_{i=1}^{n-s} \omega_i + (s+k-n) \quad \text{if } n < s+k \end{aligned} \quad (2.3)$$

Note 1. If X_1 and Y_1 are $n \times (n-k)$ and $n \times (n-s)$ matrices which are orthogonal complements of X and Y respectively, then $(n-k) \geq (n-s)$ and

$$\frac{|X' A P_Y A^{-1} X|}{|X' X|} = \frac{|Y_1' A^{-1} P_{X_1} A Y_1|}{|Y_1' Y_1|} \quad (2.4)$$

and

$$\text{tr} (P_X A P_Y A^{-1}) = (s+k-n) + \text{tr} (P_{Y_1} A^{-1} P_{X_1} A). \quad (2.5)$$

The results (2.4) and (2.5) show that we need only consider the case $s \geq k$ and $n \geq s+k$ in proving Theorem 1. If $n < s+k$, then $n \geq (n-s) + (n-k)$ in which case we consider the expressions on the right-hand sides of (2.4) and (2.5) and apply the same proof.

Note 2. If $A = P D_\delta Q'$ is the singular value decomposition of A , then we can write the left-hand side expressions of (2.2) and (2.3) as

$$|X'_* D_\delta Y_* Y'_* D_\delta^{-1} X_*| \text{ and } \text{tr} (X'_* D_\delta Y Y'_* D_\delta^{-1} X_*) \quad (2.6)$$

choosing $X_* = P'X(X'X)^{-\frac{1}{2}}$ and $Y_* = Q'Y(Y'Y)^{-\frac{1}{2}}$ so that $X'_* X_* = I_k$ and $Y'_* Y_* = I_s$, and $D_\delta = \text{Diag}(\delta_1, \dots, \delta_n)$ (i.e., a diagonal matrix with $\delta_1, \dots, \delta_n$ as diagonal elements). In view of (2.6), we need only prove Theorem 1 with the restrictions $X'X = I_k$, $Y'Y = I_s$ and A is a diagonal matrix.

Theorem 2. Let X and Y be $n \times k$ and $m \times s$ matrices of ranks k and s respectively with $k \leq s$, and B and C be $n \times m$ and $m \times n$ matrices such that $C = B^+$ (the Moore-Penrose inverse of B , see Rao, 1973, p. 26). Further, let $\delta_1 \geq \delta_2 \geq \dots \geq \delta_t > 0$ be the nonzero singular values of B and $t \geq s$. Then

$$\frac{|X' B P_Y C X|}{|X' X|} \leq \prod_{i=1}^{\min(k, t-s)} \omega_i \quad (2.7)$$

and

$$\begin{aligned} \text{tr}(P_X B P_Y C) &\leq \sum \omega_i \quad \text{if } t \geq s+k \\ &\leq s+k-t + \sum_{i=1}^{t-s} \omega_i \quad \text{if } t < s+k \end{aligned} \quad (2.8)$$

where $\omega_i = (\delta_i + \delta_{t-i+1})^2 / 4 \delta_i \delta_{t-i+1}$.

Theorem 3. Let V and W be $n \times n$ and $m \times m$ non-negative definite matrices, and X and Y be $n \times k$ and $m \times s$ matrices such that $X' V X$ and $Y' W Y$ are positive definite. Further, let B and C be $n \times m$ and $m \times n$ matrices such that

$$(a) \quad t = R(B) = R(C) \leq s$$

$$(b) \quad \rho(W) = \rho\left(\frac{B}{W}\right) = \rho(C:V), \quad \rho(V) = \rho(B:V) = \rho\left(\frac{C}{V}\right)$$

where $\rho(A)$ stands for the rank of matrix A .

$$(c) \quad B W^+ C \quad \text{and} \quad C V^+ B \quad \text{are symmetric of rank } t \\ \text{and} \quad B W^+ C V^+ B = B.$$

If $\delta_1^2 \geq \delta_2^2 \geq \dots \geq \delta_t^2 > 0$ are the nonzero eigen values of $B W^+ B' V^+$ and $\omega_i = (\delta_i + \delta_{t-i+1})^2 / 4 \delta_i \delta_{t-i+1}$, then

$$\frac{|X' B Y (Y' W Y)^{-1} Y' C X|}{|X' V X|} \leq \prod_{i=1}^{\min(k, t-s)} \omega_i \quad (2.9)$$

and

$$\begin{aligned} \text{tr}[(X' V X)^{-1} X' B Y (Y' W Y)^{-1} Y' C X] &\leq \sum_{i=1}^k \omega_i, \quad \text{if } t \geq s+k \\ &\leq (s+k-t) + \sum_{i=1}^{t-s} \omega_i, \quad \text{if } t < s+k. \end{aligned} \quad (2.10)$$

Theorem 4. Let S and R be $n \times m$ and $m \times n$ matrices such that $t = \rho(S) = \rho(R) = \rho(SR)$ with $t \geq s \geq k$, and SR and RS are symmetric, nonnegative definite and idempotent matrices. Further let $\delta_1 \geq \delta_2 \geq \dots \geq \delta_t > 0$ be the nonzero singular values of S and $\omega_i = (\delta_i + \delta_{t-i+1})^2 / 4\delta_i \delta_{t-i+1}$. Then

$$\frac{|X' S Y (Y' R S Y)^{-1} Y' R X|}{|X' S R X|} \leq \prod_{i=1}^{\min(k, t-s)} \omega_i \quad (2.11)$$

and

$$\begin{aligned} \text{tr}[(X' S R X)^{-1} X' S Y (Y' R S Y)^{-1} Y' R X] \\ \leq \sum_{i=1}^k \omega_i \quad \text{if } t \geq s + k \\ \leq (s + k - t) + \sum_{i=1}^{t-s} \omega_i \quad \text{if } t < s + k. \end{aligned} \quad (2.12)$$

Proof. Theorem 4 follows from Theorem 3 by choosing $V = SR$, $W = RS$, $B = S$ and $C = R$.

Theorem 5. Let G and H be symmetric and commuting matrices such that GH is nonnegative definite and $\rho(G) = \rho(H) = \rho(GH) = t \geq s \geq k$. Further let $\delta_1 \geq \delta_2 \geq \dots \geq \delta_t > 0$ be the nonzero eigen values of $H^{-1}G$ where H^{-1} is any g -inverse of H , and $\omega_i = (\delta_i + \delta_{t-i+1})^2 / 4\delta_i \delta_{t-i+1}$. Then

$$\frac{|X' G^2 Y (Y' G H Y)^{-1} Y' H^2 X|}{|X' G H X|} \leq \prod_{i=1}^{\min(k, t-s)} \omega_i \quad (2.13)$$

and

$$\begin{aligned} & \text{tr}[(X'GHX)^{-1}X'G^2Y(Y'GHY)^{-1}Y'H^2X] \\ & \leq \sum_{i=1}^k \omega_i \quad \text{if } t \geq k+s \\ & \leq (k+s-t) + \sum_{i=1}^{t-s} \omega_i \quad \text{if } t < k+s. \quad (2.14) \end{aligned}$$

Proof. Theorem 5 follows from Theorem 3 by choosing $V = W = GH$, $B = G^2$ and $C = H^2$.

Theorem 6. Let V and W be positive definite, and B and C be two matrices such that $\rho(B) = \rho(C) = t (\geq s)$, $BW^{-1}C$ and $CV^{-1}B$ are symmetric and $BW^{-1}CV^{-1}B = B$. Further let $\delta_1 \geq \delta_2 \geq \dots \geq \delta_t > 0$ be the nonzero singular values of $V_1^{-1}B(W_1')^{-1}$ where $W = W_1W_1'$ and $V = V_1V_1'$. Then the inequalities (2.9) and (2.10) hold.

3. PROOFS OF MAIN THEOREMS

Proof of Theorem 1. The proof depends on a number of lemmas which are also of independent interest. As observed in Notes 1 and 2 following the statement of Theorem 1, we can take X and Y such that $X'X = I_k$, $Y'Y = I_s$, $n \geq s+k$ and A as $D_\delta = \text{Diag}(\delta_1, \dots, \delta_n)$ with all δ_i positive.

Lemma 1. Let V be an n.n.d. matrix of order n with non-zero eigen values $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0$ and X be an $n \times k$ matrix of rank k ($\leq s$). Then

$$\sup_X \frac{|X'VX|}{|X'X|} = \prod_{i=1}^k \lambda_i \quad \text{and} \quad \sup_X \text{tr}(P_X V) = \sum_{i=1}^k \lambda_i \quad (3.1)$$

Proof. The result (3.1) is an immediate consequence of the Poincaré Separation Theorem (see Rao, 1979, p. 364) which states that

$$\mu_i \leq \lambda_i, \quad i = 1, \dots, k (\leq s) \quad (3.2)$$

where $\mu_i, i = 1, 2, \dots$ are the roots of the determinantal equation

$$|X'VX - \mu X'X| = 0 \quad (3.3)$$

and the equality in (3.2) is attained for a suitably chosen X .

Lemma 2. Let X and Y be $n \times k$ and $n \times s$ matrices such that $X'X = I_k$, $Y'Y = I_s$ and $s \geq k$. Let D_δ be a positive definite diagonal matrix and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0$ be the eigen values of $(Y'D_\delta Y)(Y'D_\delta^{-1} Y)$. Then

$$|X'D_\delta Y Y'D_\delta^{-1} X| \leq \left(\prod_{i=1}^k \lambda_i \right)^{\frac{1}{2}} |X'D_\delta X X'D_\delta^{-1} X|^{\frac{1}{2}} \quad (3.4)$$

and

$$\text{tr}(P_X D_\delta P_Y D_\delta^{-1}) \leq \left(\sum_{i=1}^k \lambda_i \right)^{\frac{1}{2}} (\text{tr} P_X D_\delta P_X D_\delta^{-1})^{\frac{1}{2}} \quad (3.5)$$

Proof of (3.4). Consider

$$|X'D_{\delta} Y Y' D_{\delta}^{-1} X|^2 = |(DX)' B (DX)|$$

where $D^2 = D_{\delta}$ and

$$B = D Y Y' D_{\delta}^{-1} X X' D_{\delta}^{-1} Y Y' D$$

which is an n.n.d. matrix of rank k . If $\alpha_1, \dots, \alpha_k$ are the nonzero eigen values of B , then

$$\left(\prod_{i=1}^k \alpha_i \right) = |X'D_{\delta}^{-1} Y Y' D_{\delta}^{-1} X|. \quad (3.6)$$

Hence using the result (3.1) of Lemma 1 with X as DX and V as B we have

$$|X'D_{\delta} Y Y' D_{\delta}^{-1} X|^2 \leq |X'D_{\delta} X| \cdot |X'D_{\delta}^{-1} Y Y' D_{\delta} Y Y' D_{\delta}^{-1} X|. \quad (3.7)$$

Now,

$$|X'D_{\delta}^{-1} Y Y' D_{\delta} Y Y' D_{\delta}^{-1} X| = |(D^{-1}X)' C (D^{-1}X)| \quad (3.8)$$

where $C = D^{-1} Y Y' D_{\delta} Y Y' D^{-1}$ is an n.n.d. matrix of rank s and its eigen values are $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0$. Then, again applying the result (3.1) to the expression on the right hand side of (3.8), we have from (3.7)

$$|X'D_{\delta} Y Y' D_{\delta}^{-1} X|^2 \leq |X'D_{\delta} X| \cdot |X'D^{-1} X| \left(\prod_{i=1}^k \lambda_i \right)$$

which proves (3.4).

Proof of (3.5). Consider the singular value decompositions

$$D^{-1} X X' D = T_1 D_1 T_2' \quad \text{and} \quad D Y Y' D^{-1} = T_3 D_2 T_4'$$

where T_1, T_2, T_3 and T_4 are orthogonal matrices and D_1 and D_2 are diagonal matrices such that $D_1 = \text{Diag}(\beta_1, \dots, \beta_k, 0, \dots)$ and $D_2 = \text{Diag}(\gamma_1, \dots, \gamma_s, 0, \dots)$, where $\beta_1 \geq \dots \geq \beta_k > 0$ and $\gamma_1 \geq \dots \geq \gamma_s > 0$. Then using a theorem of von Neumann (see equation (2.11) in Rao, 1979),

$$\text{tr}(P_X D_\delta P_Y D_\delta^{-1}) = \text{tr}(D_1 T_2' T_3 D_2 T_4' T_1) \leq \text{tr} D_1 D_2 \quad (3.9)$$

and the equality holds iff

$$T_2' T_3 = \begin{pmatrix} I_k & 0 \\ 0 & \Delta_1 \end{pmatrix} \quad \text{and} \quad T_4' T_1 = \begin{pmatrix} I_k & 0 \\ 0 & \Delta_2 \end{pmatrix}$$

where Δ_1 and Δ_2 are arbitrary orthogonal matrices.

Now,

$$\text{tr} D_1 D_2 = \sum_{i=1}^k \gamma_i \beta_i \leq \left(\sum_{i=1}^k \gamma_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^k \beta_i^2 \right)^{\frac{1}{2}}, \quad (3.10)$$

$$\sum_{i=1}^k \beta_i^2 = \text{tr}(P_X D_\delta P_X D_\delta^{-1})$$

and $\lambda_i = \gamma_i^2, i = 1, \dots, s$ are the nonzero eigen values of $P_Y D_\delta P_Y D_\delta^{-1}$. Hence using (3.10) in (3.9) we get

$$\text{tr}(P_X D_\delta P_Y D_\delta^{-1}) \leq (\text{tr} P_X D_\delta P_X D_\delta^{-1})^{\frac{1}{2}} \left(\sum_{i=1}^k \lambda_i \right)^{\frac{1}{2}}$$

which proves (3.5). Thus Lemma 2 is established.

Lemma 3. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0$ be the eigen values of $Y' D_\delta Y Y' D_\delta^{-1} Y$ where Y is an $n \times s$ matrix such that $Y' Y = I_s$ and $\text{diag } D_\delta = (\delta_1, \dots, \delta_n)$ with all δ_i positive, and $\omega_i = (\delta_i + \delta_{n-i+1})^2 / 4\delta_i \delta_{n-i+1}$ for $i = 1, \dots, k (\leq s)$. Then

$$\prod_{i=1}^k \lambda_i \leq \prod_{i=1}^k \omega_i \quad \text{and} \quad \sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \omega_i. \quad (3.11)$$

Proof. Let

$$\begin{aligned} \phi(B) &= |B' Y' D_\delta Y B| / |B' (Y' D_\delta^{-1} Y)^{-1} B| \\ \psi(B) &= \text{tr}[B' Y' D_\delta Y B (B' (Y' D_\delta^{-1} Y)^{-1} B)^{-1}] \end{aligned} \quad (3.12)$$

where B is an $s \times k$ matrix with $\rho(B) = k \leq s$. Then applying (3.1)

$$\sup_B \phi(B) = \prod_{i=1}^k \lambda_i \quad \text{and} \quad \sup_B \psi(B) \leq \sum_{i=1}^k \lambda_i. \quad (3.13)$$

Next we observe that

$$B' (Y' D_\delta^{-1} Y)^{-1} B - B' B (B' Y' D_\delta^{-1} Y B)^{-1} B' B \quad (3.14)$$

is non-negative definite. (see example 33 on p. 77 of Rao, 1973). Then, substituting the second expression in (3.14) for the first in (3.12), we get

$$\phi(B) \leq |(YB)' D_{\delta}(YB)| \quad |(YB)' D_{\delta}^{-1}(YB)| / |(YB)'(YB)|^2$$

$$\Psi(B) \leq \text{tr}\{(YB)' D_{\delta}(YB) | (YB)' YB |^{-1} (YB)' D_{\delta}^{-1}(YB) | (YB)'(YB) |^{-1}\}. \quad (3.15)$$

Now, writing $YB = L$ which is of rank k and applying the inequalities (1.2) and (1.3) to the right hand sides of (3.15) we have

$$\phi(B) \leq \prod_{i=1}^k \omega_i \quad \text{and} \quad \Psi(B) \leq \sum_{i=1}^k \omega_i$$

which in conjunction with (3.13) proves (3.11), since ω_i are independent of B .

Combining the results (3.4) and (3.11) we get

$$\begin{aligned} |X' D_{\delta} Y Y' D_{\delta}^{-1} X| &\leq \left(\prod_{i=1}^k \omega_i \right)^{\frac{1}{2}} |X' D_{\delta} X X' D_{\delta}^{-1} X|^{\frac{1}{2}} \\ \text{tr}(X' D_{\delta} Y Y' D_{\delta}^{-1} X) &\leq \left(\sum_{i=1}^k \omega_i \right)^{\frac{1}{2}} (\text{tr} X' D_{\delta} X X' D_{\delta}^{-1} X)^{\frac{1}{2}}. \end{aligned} \quad (3.16)$$

A further application of (1.2) and (1.3) to the right hand side expressions of (3.16) proves Theorem 1.

Proof of Theorem 2. Consider the singular value decomposition, $B = \Delta_1' D_{\delta} \Delta_2'$ where $\Delta_i' \Delta_i = I_t$ for $i=1,2$ and $D_{\delta} = \text{Diag}(\delta_1, \dots, \delta_t)$ with all δ_i positive. Then $C = B^+ = \Delta_2 D_{\delta}^{-1} \Delta_1'$. Let $X_0 = \Delta_1' X$ and $Y_0 = \Delta_2' Y$ so that $X'X - X_0' X_0$ and $Y'Y - Y_0' Y_0$ are n.n.d. matrices and so also

$$P_{Y_0} = Y_0 (Y'Y)^{-1} Y_0' \quad \text{and} \quad P_{X_0} = X_0 (X'X)^{-1} X_0'.$$

Then it is easily seen that

$$\frac{|X' B P_Y C X|^2}{|X' X|^2} \leq \frac{|X'_O D_\delta P_{Y_O} D_\delta^{-1} X_O|^2}{|X'_O X_O|^2} \quad (3.17)$$

and

$$\text{tr}(P_X B P_Y C) \leq \text{tr}(P_{X_O} D_\delta P_{Y_O} D_\delta^{-1}). \quad (3.18)$$

Now using Theorem 1 on the right hand side expressions of (3.17) and (3.18), we get the results (2.7) and (2.8) of Theorem 2.

Proof of Theorem 3. Let V and W be of ranks n_1 and m_1 respectively and write $V = V_1 V_1'$ and $W = W_1 W_1'$ where V_1 and W_1 are $n \times n_1$ and $m \times m_1$ matrices of ranks n_1 and m_1 respectively. Let

$$B_1 = (V_1' V_1)^{-1} V_1' B W_1 (W_1' W_1)^{-1} \text{ and } C_1 = (W_1' W_1)^{-1} W_1' C V_1 (V_1' V_1)^{-1}.$$

Then, under the given conditions, it is easy to verify that $C_1 = B_1^+$ and taking $X_O = V_1' X$ and $Y_O = W_1' Y$,

$$X' B Y (Y' W Y)^{-1} Y' C X = X'_O B_1 Y_O (Y'_O Y_O)^{-1} Y'_O C_1 X'_O \quad (3.19)$$

and $X' V X = X'_O X_O$. Now applying Theorem 2 to the right-hand side expression of (3.19), we get the results of Theorem 3.

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