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CHARACTERIZATIONS OF GENERALIZED MARKOV-POLYA AND  
GENERALIZED POLYA-EGGENBERGER DISTRIBUTIONS.

K. G. Janardan\*  
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and  
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CHARACTERIZATIONS OF GENERALIZED MARKOV-POLYA AND  
GENERALIZED POLYA-EGGENBERGER DISTRIBUTIONS

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ABSTRACT

A discrete model is considered where the original observation is subjected to partial destruction according to the generalized Markov-Polya damage model. A characterization of the generalized Polya-Eggenberger distribution is given in the context of the Rao-Rubin condition. Several other characterization theorems are also proved concerning these probability distributions

Key Words & Phrases: Generalized Markov-Polya distribution; generalized Polya-Eggenberger distribution; damaged model; conditional distribution characterizations.

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# 1. INTRODUCTION

Urn models are readily adapted to the development of probability distributions used in the analysis of complex problems in real life situations. In many natural phenomena involving individuals or living organisms, the probability of success seems to increase or decrease with the number of successes or failures. Thus, with the aid of urn models, research workers have developed a number of discrete probability distributions (see Johnson and Kotz, 1977; chp. 4) when the probability of success of an event is a linear function of the number of successes. Among the principal researchers who have used urn models for developing discrete probability models for contagious events, Markov (1917) and Polya and Eggenberger (1923) are pioneers in the field.

Following Hald's (1960) approach, Janardan (1973), and Janardan and Schaeffer (1977) have recently considered a new three urn model with predetermined strategy and have derived the generalized Markov-Polya distribution (GMPD) as a model (1.1) for voting in small groups where contagion is present within each group and the group leader devises some new strategies for bringing success to his candidate:

$$P(X=x) = \binom{n}{x} \frac{a \cdot b(a+b+nt) \cdot (a+xt)^{(x,c)} \cdot (b+(n-x)t)^{(n-x,c)}}{(a+b)(a+xt)(b+(n-x)t)(a+nt)^{(n,c)}} \quad (1.1)$$

where  $a > 0$ ,  $b > 0$ ,  $0 \leq t \leq 1$ ,  $c \neq 0$ , and  $x = 0, 1, 2, \dots, n$ . If  $t = 0$  this reduces to Markov-Polya distribution (see equation (1.1) of Johnson-Kotz, 1977, p. 177). When  $c = 1$  and  $t = 1$  this is the Polya-Eggenberger distribution.

The GMPD (1.1) has several interesting properties and a large number of applications (see Janardan and Schaeffer, 1977 and Janardan, 1978). Under certain limiting conditions, the probability distribution (1.1) gives the limiting form:

$$P(X=x) = \frac{h(h+xt)^{(x,c)}}{(h+nt)^{(n,c)}} \cdot \rho^x (1-\rho)^{(n-x)/c} \quad (1.2)$$

where  $x=0,1,2,\dots$ ,  $h>0$ ,  $0<\beta<1$ ,  $0\leq\alpha\leq 1$  and  $c\neq 0$ .

This distribution was named "The generalized Polya-Eggenberger" distribution (GPED) by Janardan (1973) since (1.2) reduces to the Polya-Eggenberger distribution when  $\alpha=0$  (see Patil and Joshi, 1968 p. 20).

Situations often arise where the original observation produced by nature undergoes a destructive process and what is recorded is only the damaged portion of the actual (original) observation. This problem was first brought to light by Rao (1963) when he considered the resultant models after the observations, produced by some probability model, were ruined by other probability models. Subsequently, Rao and Rubin (1964) proved that if the observation generated by nature (denoted by r.v.  $X$ ) is reduced to  $Y$  by a binomial destructive process and if  $Y$  satisfies the condition:

$$P(Y=k) = P(Y=k/\text{no damage}) = P(Y=k/\text{partial damage}), \quad (1.3)$$

then the original r.v.  $X$  must have the Poisson distribution. In the literature, this result is known as Rao-Rubin characterization of the Poisson distribution and the condition (1.3) is called RR-condition.

In this paper, we consider the generalized Markov-Polya distribution as a damage model subject to RR condition and characterize the generalized Polya-Eggenberger distribution (GPED) as a model of contagion for the production of observations in nature. In addition to this result which is given in the next section, we prove several other theorems which characterize the generalized Markov-Polya and generalized Polya-Eggenberger distributions.

## 2. NOTATION AND IDENTITIES

To begin with, we shall discuss the notation and two identities required in this paper. The notation  $\mu^{(x,c)}$  used in the definitions of probability distributions (1.1) and (1.2), and in sequel stands for the ascending factorial:

$$\begin{aligned}
m(x, c) &= m(m+c)(m+2c)\dots(m+(x-1)c), \\
m(x, 0) &= m^x, \quad m(0, c) = 1, \\
m(x, -1) &= m^{(x)} = m(m-1)\dots(m-x+1), \\
m(x, +1) &= m^{[x]} = m(m+1)\dots(m+x-1).
\end{aligned} \tag{2.1}$$

From Janardan (1973), we record the following two identities:

$$\sum_{k=0}^n J_k(A, t, c) J_{n-k}(B, t, c) = J_n(A+B, t, c) \tag{2.2}$$

$$\text{where } J_m(a, t, c) = a(a+ct)^{(m, c)} / (a+ct)m! \tag{2.3}$$

$$\text{and } J_0(a, b, c) = 1. \tag{2.4}$$

$$\sum_{k=0}^{\infty} J_k(A, t, c) V^k = W^{-A/c}, \quad \text{with } V = (1-W)W^t/c.$$

These identities can be derived by using Lagrange's expansion.

With this notation, the probability distributions (1.1) and (1.2) can be written respectively as

$$P(X=x) = J_x(a, t, c) J_{n-x}(b, t, c) / J_n(a+b, t, c), \tag{2.5}$$

$$P(X=x) = f_x = J_x(h, t, c) (\beta/c)^x (1-\beta)^{(h+xt)/c} \tag{2.6}$$

### 3. CHARACTERIZATIONS BASED ON CONDITIONAL DISTRIBUTIONS

We now prove the following theorem:

**THEOREM 1:** If  $X$  and  $Y$  are two independent random variables having the generalized Polya-Eggenberger distributions with parameters  $(a, t, c, \beta)$  and  $(b, t, c, \beta)$  respectively, then the conditional distribution of  $X$  given  $X+Y=n$  is a generalized Markov-Polya distribution as given in (1.1).

**PROOF:** By definition of conditional probability,

$$P(X=x/X+Y=n) = P(X=x, Y=n-x) / P(X+Y=n)$$

$$= \frac{J_x(a, t, c) (\beta/c)^x (1-\beta)^{(a+xt)/c} \prod_{n-x} J_{n-x}(b, t, c) (\beta/c)^{n-x} (1-\beta)^{(b+(n-x)t)/c}}{\sum_{x=0}^n J_x(a, t, c) (\beta/c)^x (1-\beta)^{(a+xt)/c} \prod_{n-x} J_{n-x}(b, t, c) (\beta/c)^{n-x} (1-\beta)^{(b+(n-x)t)/c}}$$

$$= \frac{J_x(a, t, c) J_{n-x}(b, t, c)}{\sum_{x=0}^n J_x(a, t, c) J_{n-x}(b, t, c)}$$

By identity (2.2), the denominator equals  $J_n(a+b, t, c)$ , which proves the theorem. The following theorem gives the converse of the above.

**THEOREM 2:** Let  $X$  and  $Y$  be two independent discrete r.v.'s such that the conditional distribution of  $X=x$  given  $X+Y=n$  is the generalized Markov-Polya distribution given by (1.1) or (2.5) then each  $X$  and  $Y$  has a generalized Polya-Eggenberger distribution as in (1.2).

**PROOF:** By hypothesis of the theorem,  $P(X=x/X+Y=n)$  is given by

$$J_x(a, t, c) J_{n-x}(b, t, c) / J_n(a+b, t, c) \text{ which is of the form}$$

$$\alpha(x)\beta(n-x)/\gamma(n) \text{ with } \alpha(x) = J_x(a, t, c), \beta(n-x) = J_{n-x}(b, t, c) \text{ and } \gamma(n) = J_n(a+b, t, c).$$

Applying theorem 1 of Janardan (1975), we get

$f(x) = p\alpha(x)e^{rx}$  and  $g(y) = q\beta(y)e^{ry}$ , where  $p, q$  and  $r$  are some positive constants. Setting  $e^r = (1-\beta)^{1/c}$ , the functions  $f(\cdot)$  and  $g(\cdot)$  can be written as

$$f(x) = pJ_x(a, t, c) (1-\beta)^{tx/c}, \text{ and}$$

$$g(y) = qJ_y(b, t, c) (1-\beta)^{ty/c}.$$

Since  $1 = \sum_{x=0}^{\infty} f(x) = \sum_{y=0}^{\infty} g(y)$ , applying the identity (2.4), we get

$$p = (1-\beta)^{-a/c} \text{ and } q = (1-\beta)^{-b/c} \text{ completing the proof of the theorem.}$$

**THEOREM 3 :** If a non-negative integer random variable  $N$  is sub-divided into two components  $N_A$  and  $N_B$  such that the conditional distribution  $P(N_A = x, N_B = n-x | N=n)$  is the generalized Markov-Polya distribution (2.5) then the r.v's  $N_A$  and  $N_B$  are independent if, and only if,  $N$  has a generalized Polya-Eggenberger distribution.

**PROOF:** The joint probability of  $N_A$  and  $N_B$  becomes

$$P[N_A=x, N_B=n-x] = \frac{J_x(a, t, c) J_{n-x}(b, t, c)}{J_n(a+b, t, c)} \cdot P(N=n) \quad (3.1)$$

If  $N$  has a generalized Polya-Eggenberger distribution, then with  $h = a+b$ , its probability function is

$$P(N=n) = J_n(a+b, t, c) (\beta/c)^n (1-\beta)^{(a+b+nt)/c} \quad (3.2)$$

$$a>0, b>0, 0 \leq t \leq 1, 0 < \beta < 1, c \neq 0.$$

Inserting the value of  $P(N=n)$ , we can easily write (3.1) as

$$\begin{aligned} P[N_A=x, N_B=n-x] &= \\ J_x(a, t, c) (\beta/c)^x (1-\beta)^{(a+xt)/c} J_{n-x}(b, t, c) (\beta/c)^{n-x} (1-\beta)^{(b+(n-x)t)/c} \\ &= P(N_A = x) P(N_B = n-x). \end{aligned}$$

Conversely, if  $N_A$  and  $N_B$  are independent r.v's such that the conditional distribution of  $N_A$  and  $N_B$  given  $N_A + N_B = n$  is the GMPD, then  $N_A$  and  $N_B$  have the generalized Polya-Eggenberger distributions. This follows from theorem 2.

**THEOREM 4:** If  $X$  and  $Y$  are two independent r.v's defined on non-negative integers such that  $P(X=x) = f(x) > 0, \sum_{x=0}^{\infty} f(x) = 1$  and

$$P(Y=y) = g(y) > 0,$$

$$\sum_{y=0}^{\infty} g(y) = 1 \text{ and further if for } \lim_{n \rightarrow \infty} \frac{h_n}{n} = A,$$

$$P(X=k/X+Y=n) =$$

$$\binom{n}{k} \frac{a_n b_n (\lambda + nt) (a_n + kt)^{(k,c)} (b_n + (n-k)t)^{(n-k,c)}}{(a_n + kt) (b_n + (n-k)t) \lambda (\lambda + nt)^{(n,c)}} \quad (3.3)$$

for  $k = 0, 1, 2, \dots, n$ .

$$= 0 \text{ for } k > n$$

Then (i)  $a_n$  is independent of  $n$  and equals a constant 'a' for all values of  $n$ , and

(ii)  $X$  and  $Y$  must have generalized Polya-Eggenberger distributions with parameters  $(a, t, c, \lambda)$  and  $(b, t, c, \lambda)$  respectively.

PROOF: Since  $X$  and  $Y$  are independent r.v.'s we have

$$P(X=k/Y=n) = f(k)g(n-k) / \sum_{k=0}^n f(k)g(n-k) \quad (3.4)$$

Using (3.3) and (3.4) we can derive the functional relation (3.5) for all values of  $0 \leq k \leq n$ , and  $n \geq 1$ :

$$\frac{f(k)g(n-k)}{f(k-1)g(n-k+1)} = \frac{n-k+1}{k} \frac{(a_n + kt)^{(k,c)} (b_n + (n-1)t)^{(n-k,c)} (a_n + (k-1)t) (b_n + (n-k+1)t)}{(a_n + (k-1)t)^{(k-1,c)} (b_n + (n-k+1)t)^{(n-k+1,c)} (a_n + kt) (b_n + (n-k)t)} \quad (3.5)$$

Replacing  $k$  by  $k+1$  and  $n$  by  $n+1$  in (3.5) and dividing the resulting expression by (3.5) we get  $f(k+1)f(k-1)/f^2(k)$  on the left side and a very complex untidy expression on the right side. Since the left side is independent of  $n$ , the right side must also be independent of  $n$ . Thus,  $a_{n+1} = a_n = a$  and  $b_{n+1} = b_n = b$  for all  $n$ , and hence ignoring the subscripts on  $a$ 's and  $b$ 's we will have

$$\text{have } \frac{f(k+1)f(k-1)}{f^2(k)} = \frac{k}{k+1} \frac{(a+(k-1)t)^{(k-1,c)} (a+(k+1)t)^{(k+1,c)} (a+kt)^2}{(a+(k-1)t) (a+(k+1)t) [(a+kt)^{(k,c)}]}$$

(3.6)

which by continued substitution for  $k=1, 2, \dots, (n-1)$ , and multip-

lication together yields

$$\frac{f(n)f(0)}{f(n-1)f(1)} = \frac{1}{n!} \frac{(a+nt)^{(n,c)} (a+(n-1)t)}{a(a+nt)(a+(n-1)t)^{(n-1,c)}} \quad (3.7)$$

Setting  $B = f(1)/f(0)a$ , the recurrence relation (3.7) becomes

$$f(n) = \frac{B}{n!} \frac{(a+nt)^{(n,c)} (a+(n-1)t)}{(a+nt)(a+(n-1)t)^{(n-1,c)}} f(n-1), \quad (3.8)$$

which is true for all integral  $n$ . Thus,

$$f(n) = B^n a(a+nt)^{(n,c)} f(0) / (a+nt)n! \quad (3.9)$$

Since  $\sum_{n=0}^{\infty} f(n) = 1$ , the series (3.9) must converge to unity. Let

the unknown positive quantity  $\beta$  be equal to  $(\beta/c)(1-\beta)^{t/c}$ ,  $0 < \beta < 1, t > 0$ , and  $c \neq 0$ . Thus,

$$1 = \sum_{n=0}^{\infty} \frac{a(a+nt)^{(n,c)}}{(a+nt)n!} \left(\frac{\beta}{c}\right)^n (1-\beta)^{nt/c} f(0).$$

By applying identity (2.4), we can easily see that  $f(0) =$

$$(1-\beta)^{-a/c} \text{ and } f(x) = \sum_x (a,t,c) \left(\frac{\beta}{c}\right)^x (1-\beta)^{(a+tx)/c},$$

which proves that the r.v.  $X$  is distributed as the GPED with parameters  $(a,t,c,\beta)$ . By putting  $k=1$  in (3.5), one can easily see that

$$\frac{g(n)}{g(n-1)} = \frac{B}{n!} \frac{(b+nt)^{(n,c)} (b+(n-1)t)}{(b+nt)(b+(n-1)t)^{(n-1,c)}}.$$

Hence  $g(n) = B^n b(b+nt)^{(n,c)} g(0) / (b+nt)n!$

and the fact  $\sum_{n=0}^{\infty} g(n) = 1$  will similarly give  $g(0) = (1-\beta)^{-b/c}$  for

$B = (\beta/c)(1-\beta)^{t/c}$ . Thus, the r.v.  $Y$  must be the GPED with parameters  $(b,t,c,\beta)$ .

**REMARK:** It was shown in theorem 3, that if  $X$  and  $Y$  are independent generalized Polya-Eggenberger random variables, then the conditional distribution of  $X$  given  $X+Y$  is generalized Markov-

Polya distribution. The above theorem, which was motivated by theorem 1 of Chatterji (1963), shows that a weak form of this property characterizes the Polya.

#### 4. CHARACTERIZATION THEOREMS BASED ON THE PR CONDITION

Let  $(X,Y)$  be a random vector of non-negative integer-valued components such that

$$P(X=n, Y=k) = f_n S(k/n) \quad (4.1)$$

where  $\{f_n: n=0,1,2,\dots\}$  and  $\{S(k/n): k=0,1,2,\dots,n\}$  for each  $n \geq 0$  are discrete probability distributions. That is, the marginal distribution of  $X$  is  $\{f_n\}$  and for each  $n \geq 0$  with  $f_n > 0$ , the conditional distribution of  $Y$  given  $X=n$  is  $\{S(k/n): k=0,1,2,\dots,n\}$ . Further,

$$P\{Y=k/\text{no damage}\} = \frac{f_k S(k/k)}{\sum_{j=0}^{\infty} f_j S(j/j)} \quad (4.2)$$

$$P\{Y=k/\text{damaged}\} = \sum_{n=k+1}^{\infty} f_n S(k/n) / \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} f_n S(k/n). \quad (4.3)$$

**THEOREM 5:** If a r.v.  $X$  defined on non-negative integers is distributed in nature as a GPFD (1.2) with parameters  $(a,b,t,c,1)$  and if it is damaged and reduced to  $k$  by the GPFD (1.1) and further, if  $Y$  is the resulting random variable, then (i) RP Condition (1.3) is satisfied, and (ii)  $Y$  has a GPFD with parameters  $(a,t,c,\beta)$ .

**PROOF:**  $P(Y=k) = \sum_{n=k}^{\infty} f_n S(k/n)$  because  $P(Y=0) = 1$ .

$$\sum_{n=k}^{\infty} [J_n(a+b,t,c)(\beta/c)^n (1-\beta)^{(a+k+nt)/c}] J_k(a,t,c) J_{n-k}(b,t,c) J_n(a+b,t,c]$$

$$= J_k(a,t,c)(\beta/c)^k (1-\beta)^{(a+kt)/c} \left[ \sum_{n=k}^{\infty} J_{n-k}(b,t,c)(\beta/c)^{n-k} (1-\beta)^{(a+(n-k)t)/c} \right]$$

$$= J_k(a,t,c)(\beta/c)^k (1-\beta)^{(a+kt)/c} \text{ since the sum of the terms in the square brackets is one.}$$

From equation (4.2),

$$P(Y=k/\text{no damage}) = J_k(a, t, c) (c/c)^k (1-c)^{(a+t)/c} = P(Y=k).$$

From equation (4.3),

$$\begin{aligned} P(Y=k/\text{damaged}) &= \frac{J_k(a, t, c) (c/c)^k (1-c)^{(a+t)/c}}{\sum_{k=0}^{\infty} J_k(a, t, c) (c/c)^k (1-c)^{(a+t)/c}} \\ &= P(Y=k). \end{aligned}$$

THEOREM 6: Let  $X$  be a non-negative integer-valued r.v. and let the probability that an observation  $n$  of  $X$  is reduced to  $k$  during a destructive process be given by the GMFD:

$$S(k/n) = J_k(a, t, c) J_{n-k}(b, t, c) / J_n(1, t, c),$$

for  $0 < a < 1$ ,  $a+b=1$ ,  $0 \leq t < 1$ ,  $c > 0$ , and  $k=0, 1, 2, \dots, n$ . If the resulting r.v.  $Y$  is such that it satisfies the RR-Condition (1.3), then  $X$  has a GPED with parameters  $(1, t, c, f)$ .

PROOF: The RR-Condition is equivalent to

$$\sum_{n=k}^{\infty} f_n S(k/n) = f_k S(k/k) / \sum_{j=0}^{\infty} f_j S(j/j),$$

where  $f_k = P(X=k)$ . This gives

$$\sum_{n=k}^{\infty} f_n \frac{J_k(a, t, c) J_{n-k}(b, t, c)}{J_n(1, t, c)} = \frac{f_k J_k(a, t, c) / J_k(1, t, c)}{\sum_{j=0}^{\infty} f_j J_j(a, t, c) / J_j(1, t, c)} \quad (4.4)$$

setting  $n=k+s$  and cancelling  $J_k(a, t, c)$  on both sides we get

$$\sum_{s=0}^{\infty} f_{k+s} \frac{J_s(b, t, c)}{J_{k+s}(1, t, c)} = \frac{f_k / J_k(1, t, c)}{\sum_{j=0}^{\infty} f_j J_j(a, t, c) / J_j(1, t, c)} \quad (4.5)$$

$$\text{Define } f_k = F(k)J_k(1, t, c)V^k, \quad (4.6)$$

for all integral values of  $k$ , where  $V$  is some arbitrary quantity to be set later. Substituting (4.6) in (4.5) we get

$$\sum_{s=0}^{\infty} F(k+s)J_s(b, t, c)V^{k+s} = F(k)V^k \sum_{j=0}^{\infty} F(j)J_j(a, t, c)V^j \quad (4.7)$$

$$\text{Let } G(az, t) = \sum_{k=0}^{\infty} F(k)J_k(az, t, c)V^k,$$

where  $-\infty < z < \infty$  so that  $G(0, t) = F(0)$  and  $G(1, t) = 1$ .

Multiplying both sides of (4.7) by  $J_k(az, t, c)$  and summing over  $k$  from 0 to  $\infty$  (4.8) becomes

$$\sum_{n=0}^{\infty} F(n)V^n \left\{ \sum_{k=0}^n J_k(az, t, c)J_{n-k}(b, t, c) \right\} = G(az, t)/G(a, t). \quad (4.9)$$

By identity (2.2) the inner sum on the left side of (4.9) is equivalent to  $J_n(az+b, t, c)$ , and hence (4.9) gives the bivariate functional equation:

$$G(az+b, t)G(a, t) = G(az, t). \quad (4.10)$$

Clearly  $G(a+b, t) = G(1, t) = 1$ . Setting  $x = a(a-1)$  and  $b = 1-a$  (4.10) gives

$$G(x+1, t)G(a, t) = G(x+a, t) \quad (4.11)$$

Setting  $\phi(x) = G(x+1, t)$ , and  $a-1=y$  (4.11) reduces to the Cauchy functional equation,  $\phi(x)\phi(y) = \phi(x+y)$ . whose non-trivial solution is given by  $\phi(x) = e^{\lambda x}$ . Thus,  $G(x, t) = e^{\lambda(x-1)}$  which is the probability generating function of the Poisson distribution. Now replacing  $x$  by  $az$ , assigning a value  $(1-e^{-\lambda c})e^{-\lambda t/c}$  to  $V$ , using the definition of  $G(az, t)$ , we get

$$e^{\lambda az} = \sum_{k=0}^{\infty} F(k)e^{\lambda J_k(az, t, c)} V^k \quad (4.12)$$

To determine the value of  $F(k)$ , consider the identity (2.4) with  $A = az$  and  $w = e^{-\lambda c}$  which gives

$$e^{\lambda az} = \sum_{k=0}^{\infty} J_k(az, t, c) v^k. \quad (4.13)$$

Subtracting (4.13) from (4.12),

$$0 = \sum_{k=0}^{\infty} [F(k)e^{\lambda} - 1] J_k(az, t, c) v^k. \quad (4.14)$$

Since (4.14) is true for all values of  $\lambda$ , it is obvious that  $F(k) = e^{-\lambda}$  for all  $k$ . Hence, by definition (4.6), we get

$$\begin{aligned} f(k) &= F(k) J_k(1, t, c) v^k \\ &= e^{-\lambda} J_k(1, t, c) [e^{-\lambda t} (1 - e^{-\lambda c}) / c]^k \\ &= J_k(1, t, c) (\delta / c)^k (1 - \delta)^{(1+kt)/c} \\ &\quad \text{with } \delta = 1 - e^{-\lambda c}. \end{aligned}$$

That is,  $X$  has a GPED with parameter  $(1, t, c, \delta)$ .

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