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NEARLY EFFICIENT ESTIMATORS BASED ON
ORDER STATISTICS

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ABSTRACT

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1. INTRODUCTION

In 1946, to estimate the location parameter θ of normal distribution $N(\theta, 1)$, Mosteller considered estimators of the form

$$\hat{\theta} = \sum_{i=1}^k c_i X_{(n_i)}, \quad (1.1)$$

where $X_{(1)} \leq \dots \leq X_{(n)}$ are the order statistics of a random sample

$X_1, \dots, X_n$ from $N(\theta, 1)$, $n_i = [\alpha_i] + 1$ with $0 < \alpha_1 < \dots < \alpha_k < 1$, c_i 's are

constants and $[\alpha_i]$ is the largest integer less than or equal to α_i .

Assuming equal weights $c_1 = \dots = c_k = 1/k$, for each $k = 1, 2, \dots$, he

found the optimal choice for $(\alpha_1, \dots, \alpha_k)$ for which the asymptotic vari-

ance of $\hat{\theta}$ is minimized. In this note, we generalize his work into two

directions. First, rather than normal distribution, an arbitrary dis-

tribution of a location parameter θ with pdf $f(x-\theta)$ is considered, where

f is assumed to be known. Second, instead of giving equal weights for

$(c_1, \dots, c_k)$, the optimum weights are determined along with the optimum

spacings for $(n_1, \dots, n_k)$ or $(\alpha_1, \dots, \alpha_k)$. Although our argument is appli-

cable to general $k = 1, 2, \dots$, because of analytical complication and

because of satisfactory achievement of relative asymptotic efficiency,

we discuss the case of $k = 3$ in details and briefly treat the case of

$k = 5$. In section 3, the result is applied to the Cauchy distribution

$$f(x-\theta) = 1/\pi(1+(x-\theta)^2)^{-1}, \quad -\infty < x < \infty \quad (1.2)$$

and the logistic distribution with pdf

$$f(x-\theta) = e^{-(x-\theta)} / [1 + e^{-(x-\theta)}]^2, \quad -\infty < x < \infty \quad (1.3)$$

as well as normal distribution. As is well known, a computational dif-

ficulty is involved in deriving the MLE's (maximum likelihood estimators)

of 0 for the distributions (1.2) and (1.3). On the other hand, it is also well known (e.g. [1]) that for any $f(x-\theta)$ satisfying certain regularity conditions, with $k = n$ and suitable $c_i = c_i(n, f)$ (known as score function), $\hat{\theta}$ in (1.1) becomes a BAN (best asymptotically normal) estimator. Compared to the BAN estimators, our estimators have a great computational merit and yet are nearly efficient. In fact, the estimators in this note use only 3 or 5 order statistics out of n order statistics however large n may be and attain at least 90% asymptotic efficiency, relative to the BAN estimators. More specifically, in the Cauchy case the RAE (relative asymptotic efficiency) of the median alone is 81%, but the RAE's of our optimum estimators with $k = 3$ and $k = 5$ are 87% and 90% respectively. In the logistic case, while the RAE of the median alone is only 75%, the RAE of our optimum estimator with $k = 3$ is surprisingly 93.8%. It is noted that the optimal weights c_i 's in these estimators are constants independently of n . Further, our estimators are briefly compared to similar estimators considered in Gastwirth (1966).

We remark that, associated with robust estimators, estimators of the form (1.1) with $k = n$ have been treated in great many papers (e.g. [5]).

2. Main result. Let $\mathcal{F}$ be the class of continuously differentiable and symmetric pdf's on $\mathbb{R}^1$ with respect to the Lebesgue measure, and suppose $X_1, \dots, X_n$ are i.i.d. with pdf $f(x-\theta)$ where $f \in \mathcal{F}$ is known and let $X_{(1)} \leq \dots \leq X_{(n)}$ be the corresponding order statistics. Further, for any k , let $0 < \alpha_1 < \dots < \alpha_k < 1$ be real numbers and let x_{α_1} be the α_1 -quantile of the population when $\theta = 0$, i.e.,

$$\alpha_i = \int_{-\infty}^{\xi_i} f(x) dx = F(\xi_i) \quad (2.1)$$

We assume that $f(\xi_i) > 0$ ($i=1, \dots, k$). Define the sample quantiles by $X_{(n_i)}$'s where $n_i = [n\alpha_i] + 1$ ($i=1, \dots, k$). To estimate the location parameter θ , we consider the class of linear estimators of $X_{(n_i)}$'s of the form,

$$\hat{\theta} = a_1 X_{(n_1)} + \dots + a_k X_{(n_k)}, \text{ where } (a_1, \dots, a_k) \in R^k. \quad (2.2)$$

As is well known (4) p. 17), as $n \rightarrow \infty$, the joint distribution of $(X_{(n_1)}, \dots, X_{(n_k)})$ converges to a k -variate normal distribution with mean $(\theta + \xi_1, \dots, \theta + \xi_k)$ and covariance matrix $\frac{1}{n}((\sigma_{ij}))$, where

$$\sigma_{ij} = \alpha_i(1-\alpha_j)/f(\xi_i)f(\xi_j) \quad (i \leq j).$$

Therefore, for any θ in (2.2), the asymptotic distribution of $\hat{\theta}$ is normal with mean $\mu = \sum_{i=1}^k a_i(\theta + \xi_i)$ and variance σ^2/n , where

$$\begin{aligned} \sigma^2 = & \sum_{i=1}^k a_i^2 \alpha_i(1-\alpha_i)/f^2(\xi_i) + \sum_{i < j} a_i a_j \alpha_i(1-\alpha_j)/f(\xi_i)f(\xi_j) \\ & + \sum_{i > j} a_i a_j \alpha_j(1-\alpha_i)/f(\xi_j)f(\xi_i) \end{aligned} \quad (2.4)$$

For $\hat{\theta}$ to be consistent for θ , the conditions

$$\sum_{i=1}^k a_i = 1 \quad \text{and} \quad \sum_{i=1}^k a_i \xi_i = 0 \quad (2.5)$$

are necessary and sufficient. Let C_k be the class of estimators of the form (2.2) satisfying (2.5). Then finding a minimum asymptotic variance estimator in C_k is equivalent to minimizing (2.4) with respect to $(a_1, \dots, a_k)$ and $(\alpha_1, \dots, \alpha_k)$ subject to (2.5). Due to the symmetry of f , for k

odd, we include the median $X_{(\lceil \frac{n}{2} \rceil + 1)}$ and choose the other quantiles

symmetrically about the median, i.e., $\alpha_1 = 1 - \alpha_k$, $\alpha_2 = 1 - \alpha_{k-1}$, ..., or

equivalently $\xi_1 = -\xi_k$, $\xi_2 = -\xi_{k-1}$, ..., and the weights $a_1 = a_k$, $a_2 =$

a_{k-1} , Then, clearly the second condition of (2.5) holds and

$f(\xi_1) = f(\xi_k)$, $f(\xi_2) = f(\xi_{k-1})$,

Case $k = 3$. In this case, $\hat{\theta}$ in (2.2) is of the form

$$\hat{\theta}(a, b, \alpha_1) = aX_{(1)} + bX_{(2)} + aX_{(3)}$$

where $X_{(1)}$ and $X_{(3)}$ are the sample α_1 and $(1 - \alpha_1)$ quantiles respectively and $X_{(2)}$ is the median. From the first condition of (2.5), $2a + b = 1$.

Writing $f(\xi_1) = f_1$ and $f(\xi_2) = f(0) = f_0$, the asymptotic variance of $\hat{\theta}$

in (2.4) is computed as

$$\sigma^2(a, \alpha_1) = a^2 \left[\frac{2\alpha_1}{f_1^2} - \frac{4\alpha_1}{f_1 f_0} + \frac{1}{f_0^2} \right] + a \left[\frac{2\alpha_1}{f_1 f_0} - \frac{1}{f_0^2} \right] + \frac{1}{4f_0^2}. \quad (2.7)$$

Minimizing this with respect to a yields

$$a_* = a_*(\alpha_1) = \left[\frac{1}{f_0^2} - \frac{2\alpha_1}{f_1 f_0} \right] / 2 \left[\frac{2\alpha_1}{f_1^2} - \frac{4\alpha_1}{f_1 f_0} + \frac{1}{f_0^2} \right] \quad \text{and} \quad (2.8)$$

$$\sigma^2(a_*, \alpha_1) = \alpha_1(1 - 2\alpha_1) / 2[2\alpha_1 f_0^2 - 4\alpha_1 f_1 f_0 + f_1^2]. \quad (2.9)$$

To further minimize (2.9) with respect to α_1 ($0 < \alpha_1 < 1/2$), let

$$\eta(\alpha_1) = f_1/f_0 = f(F^{-1}(\alpha_1))/f(0), \quad (2.10)$$

where F^{-1} is the inverse of F in (2.7). Then minimizing (2.9) is equivalent to maximizing

$$1/f_0^2 \sigma^2(a_*, \alpha_1) = (\eta - 2\alpha_1)^2 / \alpha_1(1-2\alpha_1) + 2, \quad (2.10)$$

which results in the equation

$$\eta'(\alpha_1) - [(1-4\alpha_1)/2\alpha_1(1-2\alpha_1)] \eta(\alpha_1) = 1/(1-2\alpha_1) \quad (2.11)$$

In any specific problem, we first solve (2.11) for α_1 ($0 < \alpha_1 < 1/2$),

and then compute a_* via (2.8). Finally from (2.9) the minimum asymptotic variance $\sigma_*^2 \equiv \sigma^2(a_*(\alpha_{1*}), \alpha_{1*})$ is obtained, where α_{1*} is the solution of (2.11).

As a remark, in order that σ_*^2 be less than the asymptotic variance of the median $X_{(2)}$, i.e., $1/4f_0^2$, it is easy to see that

$$(f_{1*} - 2f_0\alpha_{1*})^2 > 0 \quad (2.12)$$

is necessary and sufficient. Hence, unless

$$f(F^{-1}(\alpha_{1*})) = 2f_0\alpha_{1*}, \quad (2.13)$$

$\hat{\theta}_* = \hat{\theta}(a_*, b_*, \alpha_{1*})$ is asymptotically better than the median $X_{(2)}$.

Since (2.13) is implied by

$$f(x) = 2f(0)F(x) \quad \text{for all } x, \quad (2.14)$$

it follows in particular that in the case of the double exponential density $f(x) = \exp(-|x|)/2$, $\hat{\theta}_*$ cannot improve over $X_{(2)}$. Further, we remark

that one of the optimal weights (a^*, b^*) may take negative value.

Case $k=5$. In this case, $\hat{\theta}$ in (2.2) is of the form

$$\hat{\theta} = \hat{\theta}(a, b, c, \alpha_1, \alpha_2) = aX_{(1)} + bX_{(2)} + cX_{(3)} + dX_{(4)} + eX_{(5)}, \quad (2.15)$$

where $X_{(1)}$, $X_{(2)}$, $X_{(4)}$ and $X_{(5)}$ are respectively the α_1 , α_2 , $(1-\alpha_2)$,

and $(1-\alpha_1)$ quantiles, and $X_{(3)}$ is the median ($0 < \alpha_1 < \alpha_2 < 1/2$). The

first condition of (2.5) yields $2a + 2b + c = 1$. Let $f_1 = f(\xi_1)$,

$f_2(\xi_2) = f_2$ and $f(0) = f_0$. Then σ^2 in (2.4) reduces to

$$\sigma^2(a, b, c, \alpha_1, \alpha_2) = a^2\Lambda + b^2C + abB + a^2D + b^2E + 1/4f_0^2$$

where

$$\left. \begin{aligned} \Lambda &= \frac{2\alpha_1}{f_1^2} - \frac{4\alpha_1}{f_1 f_0} + \frac{1}{f_0^2}, & B &= \frac{4\alpha_1}{f_1 f_2} - \frac{4\alpha_1}{f_1 f_0} - \frac{4\alpha_2}{f_2 f_0} + \frac{2}{f_0^2} \\ C &= \frac{2\alpha_2}{f_2^2} - \frac{4\alpha_2}{f_2 f_0} + \frac{1}{f_0^2}, & D &= \frac{2\alpha_1}{f_1 f_0} - \frac{1}{f_0^2} \\ E &= \frac{2\alpha_2}{f_2 f_0} - \frac{1}{f_0^2} \end{aligned} \right\} (2.17)$$

Minimizing (2.16) with respect to (a, b) yields

$$\left. \begin{aligned} a_* &= (BE - 2CD)/(4AC - B^2) \\ b_* &= (2AE - BD)/(B^2 - 4AC) \end{aligned} \right\} (2.18)$$

and

$$\sigma^2(a_*, b_*, c_*, \alpha_1, \alpha_2) = \frac{CD^2 + BE^2 - 2BDE}{B^2 - 4AC} + \frac{1}{4f_0^2}. \quad (2.19)$$

For ready reference, we record below

$$\begin{aligned} B^2 - 4AC &= \frac{(4\alpha_1 - 1)^2}{f_1^2} \left(\frac{1}{f_2} - \frac{1}{f_0} \right)^2 - \left(\frac{4\alpha_2 - 1}{f_2 f_0} \right)^2 + \frac{16\alpha_1}{f_1 f_2 f_0} \left(\frac{1}{f_0} - \frac{1}{f_1} \right) \\ &+ \frac{8\alpha_1}{f_1 f_2^2} - \frac{1}{f_2^2 f_0^2} + \frac{16\alpha_1 \alpha_2}{f_1 f_2} \left(\frac{2}{f_1 f_0} - \frac{1}{f_1 f_2} - \frac{2}{f_0^2} \right) - \frac{1}{f_1^2} \left(\frac{1}{f_2} - \frac{1}{f_1} \right)^2 \end{aligned}$$

and

$$CV^2 + EBN - 3AV^2 = \left(\frac{3\alpha_1}{f_1 f_0} - \frac{1}{f_0} \right)^2 \left[\frac{\alpha_1}{f_0^2} - \frac{4\alpha_0}{f_2 f_0} + \frac{1}{f_0^2} \right] \quad (2.21)$$

$$+ \left(\frac{3\alpha_2}{f_2 f_0} - \frac{1}{f_0} \right)^2 \left[\frac{3\alpha_1}{f_1 f_0} - \frac{1}{f_0^2} - \frac{6\alpha_1}{f_1^2} \right] - \frac{4\alpha_1}{f_1} \left(\frac{1}{f_0} - \frac{1}{f_2} \right) \left(\frac{3\alpha_2}{f_2 f_0} - \frac{1}{f_0} \right) \left(\frac{3\alpha_1}{f_1 f_0} - \frac{1}{f_0} \right).$$

Exact algebraic minimization of $\sigma^2(a_*, b_*, c_*, \alpha_1, \alpha_2)$ with respect

to α_1 and α_2 seems difficult although in specific problems, the solution

can be obtained numerically. It is remarked that even an approximate

solution of (α_1, α_2) coupled with the optimal weights (a^*, b^*) in (2.13)

may be useful in getting $\hat{\theta}$ better than the median and closer to Chernoff,

Gastwirth and Johns' PAF estimator (see [14]).

3. Comparison of estimators. The table 1 below summarizes features of our optimum estimators for $k = 3$ for normal, Cauchy and logistic distributions.

TABLE 1 ($k = 3$)

	a_{1*}	a_*	$b^*=1-2a^*$	σ_4^2	$1/I$	PAF
Normal	0.1632	0.3054	0.3892	1.1100	1	90%
	(1/3)	(0.3)	(0.4)	(1.2830)		(78%)
Cauchy	0.0791	-0.0207	1.0414	2.3023	2	86.9%
	(1/3)	(0.3)	(0.4)	(2.5000)		(80%)
Logistic	0.2500	0.3	0.4	3.2	3	93.8%
	(1/3)	(0.3)	(0.4)	(3.2367)		(91%)

Here the numbers in the parentheses are those obtained from the viewpoint of robustness by Gastwirth (1966), and $1/I$ is the asymptotic minimum variance of the PAF estimator where I denotes the Fisher information. In the normal case, while the PAF of Mosteller's estimator with $k = 3$, $\alpha_1 = 0.1826$ and equal weights is 88%, the PAF of our estimator is 90%. Compared to Gastwirth's robust estimator, which is the

same form as our estimators, naturally our estimators perform better since in our case, knowing the form of f , the optimal weights with optimal spacings are obtained. This might suggest that, when n is large, it will be better to try to know the form of f rather than to apply "robust" procedures. However, for example, testing the normality in an efficient way is not an easy task. Since for $k = 3$, the RAE of Cauchy case is still less than 90%, we computed the case of $k = 5$ numerically. The resulting optimum values are as follows

$$a_* = -0.0729, b_* = 0.0435, c_* = 0.0312$$

$$\alpha_{1*} = 0.16, \alpha_{2*} = 0.19 \text{ and } \sigma_*^2 = 2.22 .$$

In this case, the RAE of $\hat{\theta}_*$ is 90.1%. It is noted that when $k = 3$ and $k = 5$, some of the optimal weights are negative, which is consistent with the result by Chernoff, Gastwirth and John (1967).

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