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Some Tests of Randomness With Applications

by

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A brief survey of tests of randomness is made. Commonly used tests of randomness for samples, for generated random numbers, for time series and for discrete random events are discussed. Applications are made to coal mining disasters in England, and present lottery games.

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1. Introduction

Statistical techniques as a rule require that some sort of randomization has been used in obtaining data. It is generally assumed that the design of experiment or design of a survey resulting in the data did use some available method of randomization. Most of the statistical procedures have been developed on the assumption that the available data arose from certain uses of the principle of randomness. However, in statistical practice, very little attention is paid to confirming this basic assumption of randomness in data analysis. There may be possibilities of grave errors in assuming the randomness of a given set of data while it may have occurred from a known or an unknown type of bias.

In simulation studies, extensive use is made of random numbers generated from computers. Many theoretical studies in modern science depend heavily on the generation of random numbers or pseudo-random numbers. In the present state lotteries and daily number games, random numbers are produced in millions. Highly sophisticated routines have been developed to test generation of such numbers. The users, as a rule, do not test the randomness of numbers as generated by well known computer packages though important uses of such random numbers are made in real problems.

There are several situations in problems of sciences and industry where data are generated in series. Examples of time series data abound in problems in economics, geology, medicine, climatology, energy, etc. and judicious application of statistical time series requires answers to questions of randomness.

Randomness implies different meanings in different contexts. Usually a random sample from a population means that we have observed a set of identically and independently distributed random variables from a population with a specified probability model. Sir Ronald A. Fisher made a major contribution to scientific experimentation by introducing the concept of randomization at the stage of design. Fisher developed tests for the sample resulting from randomization process and recommended alternative strategies.

When randomization leads to a bad looking experiment or sample, Fisher said that the experimenter should, with discretion and judgement, put the sample aside and draw another, Savage (1976).

Importance of testing for randomness in statistical practice is not recognized in spite of warnings of statisticians. In a recent paper, Federer (1978) noted that "many statisticians and teachers of statistics, assume, but do not verify, that they have a random sample from a prescribed population." The literature on randomness is extensive and scattered over many disciplines. The list of references includes many such studies. In this paper a brief account is given of commonly used tests of randomness for samples, for generated random numbers, for time series and for discrete random events. An example from coal mining disasters in England, recently discussed by Jarrett (1979) is studied in detail and the data is subjected to analysis. Random numbers obtained from lottery games in Ohio are used to illustrate some other tests.

2. Hypothesis of Randomness

A common formulation of the hypothesis of randomness is in terms of identically and independently distributed observations from an unknown distribution function. Given that $X_1, X_2, \ldots, X_n$ is a sample from a population having the cumulative distribution function $F(x)$, the hypothesis of randomness is stated in the following manner.

H_0: $X_1, X_2, \ldots, X_n$ are independently and identically distributed with probability distribution function $F(x)$ versus the alternative hypothesis that they are not. For convenience, a further assumption is made about the continuity of $F(x)$ so that its density function $f(x)$ exists. The test statistic is highly dependent on the form of the alternative. Suppose the alternative is

K: $X_1, X_2, \ldots, X_n$ are distributed independently with distributions $F_1(x), F_2(x), \ldots, F_N(x)$ respectively.

The most powerful test of H_0 in the Neyman-Pearson set up, is a test based on the sample ranks such that it provides a critical region of given size α. Except in simple cases, the distribution theory of the test statistic cannot be obtained explicitly and the test is of limited use. Theorem 6.A of Hájěk (1969) which is essentially a reformulation of Neyman-Pearson lemma, provides the main result.

There are, however, many useful nonparametric tests which test the hypothesis of randomness, against alternatives which are in terms of two samples. Suppose now that the alternative is given by

H_2: $X_1, X_2, \ldots, X_m$ is independently and identically distributed as $F_1(x)$ and $X_{m+1}, \ldots, X_N$ is distributed independently and identically as $F_2(x)$.

A special case of the alternative hypothesis is obtained if F_1 and F_2 differ in location or scale. Several of these situations are discussed by Hájěk resulting in many classical nonparametric tests.

3. Tests for Random Numbers

There is a large variety of tests available for testing whether the given set of digits are independently uniformly distributed. This is what is ordinarily meant by the word "random" in the generation of random numbers. Knuth (1968) has provided a long list for such tests.

Usually a generator is not regarded as good random number generator unless it passes at least half a dozen tests of randomness. The reason is that "randomness" has various "attributes". The alternative hypotheses to that of testing the null hypothesis that X_i's are independently and identically uniformly distributed, are too many. We shall formulate these alternative hypotheses in the following and give the tests which are commonly used in practice. Many of the following tests are found in Knuth (1968).

(i) Chi-squared test

H_0: The digits are distributed independently and identically with equal probability of being 0, 1, ..., 9.

H_1: They are not.

The usual statistic counts the observed numbers of digits 0, 1, 2, ..., 9 in the sample and compares these frequencies with the expected frequency for the sample. For a large sample of the digits, this statistic has a χ^2-distribution. Power studies of this test are available in the literature.

(ii) Equidistribution test. This can be performed by using Kolmogorov-Smirnov statistic to test uniformity of the real valued sequence so generated. The discrete form of Kolmogorov-Smirnov is applied if the numbers are rounded off to integers.

(iii) Serial test. It uses Chi-squared statistic to compare adjacent pairs or triples of numbers. Good (1957) has discussed the distribution of Chi-squared statistic if both equidistribution test and serial test are used on the same data.

(iv) Gap test. Two numbers, α, β are chosen so that the length of runs of the numbers between α, β is used as a statistic. If $p = \beta-\alpha$, the test of goodness-of-fit is used to compare the observed values with the expected values p, $p(1-p)$, $p(1-p)^2$, ..., $p(1-p)^t$, when the number of possible runs is $t+1$.

(v) Poker list: Using any set of five successive integers, the frequency of various combinations, as found in the game of poker, are used to test goodness-of-fit. Under the hypothesis of randomness these probabilities are given below.

Bust (a b c d e) = .3024

Pair (a a b c d) = .5040

2 pairs (a a b b c) = .1080

3 of a kind (a a a b c) = 0.0720

Full house (a a a b b) = .0090

4 of a kind (a a a a b) = 0.0045

5 of a kind (a a a a a) = .0001

(vi) Coupon collector's test. The length of sequences are observed so as to get integers from 0, 1, 2, ..., d-1.

If p_r is the probability that r digits are needed, $r = 0, 1, 2, \ldots, d-1$, we have

$$p_r = \frac{1}{d^{r-1}} \sum_{\nu=0}^{d-1} (-1)^{\nu} \binom{d-1}{\nu} (d-1-\nu)^{r-1} ,$$

$$r = d, d+1, \ldots$$

(vii) Permutation test. The sequence is divided into n groups of t elements. The number of times each ordering appears is counted. A Chi-squared test is utilized, since the probability of a given ordering is $1/(t!)$.

(viii) Runs test The statistic used is the number of runs up or down. By a run is meant the length of increasing (or decreasing) sequences of integers. The distribution of runs of various length is well known and is given in recent text books.

(ix) Maximum of t. For the given sequence of random numbers $U_1, U_2, \ldots, U_n$, consider subsequences of length t. Let

$$V_j = \max (U_j, U_{j+1}, \ldots, U_{j+t-1})$$

for $j = 1, 2, \ldots, n$. Using $V_0, V_1, \ldots, V_{n-1}$ as the observations from $F(x) = x^t$, $0 \leq x < 1$, we have the usual Kolmogorov-Smirnov statistic for testing randomness

(ix) Serial correlation test. The test uses the statistic which computes the serial correlation for the generated numbers. If the hypothesis of randomness is to be rejected, the sample serial correlation should be large.

4. Randomness of Series of Events

In a given series of events such as coal mining accidents, computer failures, occurrence of prizes in a lottery or arrival of a cancer patient at a clinic, interest may center around the randomness of these events. Cox and Lewis (1966) have given several applications where problems of randomness of series of events are discussed. We consider first the case of binary events.

In a recent paper, Larsen et. al (1973) have studied the hypothesis of randomness of binary events with the alternative of unimodal clustering. Assume that a sequence of n Bernoulli trials with m successes has the order of ith success given by y_i with

$$1 \leq y_1 < y_2 < \ldots < y_m \leq n,$$

y_i are integers between 1 and n. Let m = 2r if m is even and 2r + 1 if m is odd. Then

$$K_1 = \sum_{i=1}^{m} |y_i - y_{r+1}|$$

is used to study unimodal clustering. The hypothesis of randomness is rejected when K_1 is small. When the data consists of s such sequences, the statistic K_s can be formed by summing K_1 for each sequence.

It is shown by Larsen et al that,

$$E(K_1) = \frac{(n+1)[\frac{m}{2}][\frac{m+1}{2}]}{m+1}$$

$$V(K_1) = \frac{r(n+1)(n-m)\{(m+1)^2 - 2r^2 - \delta(m)\}}{12\{2[\frac{m+1}{2}] + 1\}^2}$$

where [x] = largest integer not exceeding x and

$$\delta(m) = r-2 \text{ when } m \text{ is odd}$$

$$= 2r-1, \quad m \text{ is even}$$

Approximate tests are constructed using asymptotic theory for K_s.

Other alternatives to randomness are considered by several authors. For example, O'Brien (1976) studied the alternative of multiple clustering to randomness in the case of binary data. It is assumed that in the number of N trials observed, m are successes and, $N-m = n$ failures such that $m \geq n$.

Let y_i be the number of successes prior to ith failure but subsequent to (i-1)th failure. Let $\bar{y}$ be the average length, $\bar{y} = \sum_{i=1}^{n+1} \frac{y_i}{n+1}$. Let

$$s^2 = \frac{\sum_{i=1}^{n+1} (y_i - \bar{y})^2}{n}$$

The distribution of $\frac{ns^2}{m^2}$ has been tabulated by Dixon (1940) for $m, n \leq 10$. Dixon also showed that for $m, n > 10$, distribution of cs^2 is approximately chi-squared with $c = \frac{n^3}{2m(m+n)}$, $\nu = \frac{n}{2}$. The hypothesis of randomness is rejected when s^2 is large or small.

A test of randomness using the coefficient of variation (CV) where $X_1, X_2, \ldots, X_n$ are independently and identically exponentially distributed was studied by Moran (1951). He showed that $k(CV)^2$ has χ^2_ν where

$$k \simeq n/2$$

$$\nu \simeq n/2$$

approximately. Asymptotically the distribution of $(CV)^2$ is normal. O'Brien (1976) has given comparison of the actual Chi-squared statistic and the observed value based on simulation. This approximation does not seem very good except for upper percentile points.

5. Tests of Randomness in Spacial Situations.

There are several situations in which random phenomena occur in space. For example, one may be interested in the distribution of points on a line or in plane. Related problems occur in tests using geometrical methods such as tests of certain hypotheses utilizing graphical techniques. In the case of multivariate data, several procedures have been found to be useful for preliminary study of data.

Some of the early tests of randomness for points on a line can be transformed to the test of hypothesis of independent and identically distributed uniform random variables. Pearson (1963) has given comparisons of four tests based on Kolmogorov-Smirnov and von Mises statistics and their standardized forms.

In two-dimensions, several models have been discussed by authors to derive tests for the randomness of points in a plane.

Brown and Rothery (1978) have discussed the hypothesis of randomness of points in a plane formulated as the points forming a two-dimensional Poisson point process. They have proposed two statistics which are sensitive to the alternative hypothesis of local regularity. One is the squared coefficient of variation of squared nearest neighbor distances and the other is the ratio of the geometric mean to the arithmetic mean of the squared distances. Distribution of these statistics are given with the help of computer simulations and numerical approximations.

Let n points be distributed in a given region. Suppose $v_1, v_2, \ldots, v_n$ are the squared distances to the nearest neighbor of a randomly selected individual.

Let

$$D = \frac{\Sigma v_i}{n}$$

$$S = \frac{\Sigma(v_i - D)^2}{(n-1)D^2}$$

$$G = \frac{(v_1 v_2 \cdots v_n)^{1/n}}{D}$$

Tests based on S and G reject the hypothesis of randomness if S is small or G is large.

In the case of points in an infinite plane, -2 log G is a special case of Bartlett's statistic and has been extensively studied, see for example, Glaser (1976). However, for finite planes for various shapes and sizes simulations have to be used. Brown and Rothery (1978) have obtained the values for estimated probabilities in the upper tails of the distribution of G and lower tails of the distribution of S based on 1500 realization for the circle, 2000 realizations for the square and 1200 realizations for various rectangle sizes as given in Tables I and II, for sample sizes of 25 and 36. Recently a survey of tests of randomness for spatial joint patterns has been given by Ripley (1979). The asymptotic distribution theory and power of tests based on the nearest-neighbor distances and estimates of the variance function are investigated in this study.

6. Tests of Randomness for Time Series Data

When the data in an experiment arises in a sequence, the natural question arises about dependence of observations. Ths usual alternatives to randomness in time series are those of trend and periodicity. Kendall and Stuart (1968) give several tests for randomness in time series. Commonly

Table I

Estimated probabilities of G exceeding upper

5% and 2.5% point

Shape	n = 25		n = 36	
	α = 0.05	α = 0.025	α = 0.05	α = 0.025
Circle	0.066	0.038	0.085	0.049
Square	0.075	0.042	0.067	0.045
1x2 rectangle	0.063	0.034	0.063	0.039
1x4 rectangle	0.073	0.046	0.060	0.036
1x8 rectangle	0.055	0.038	0.041	0.023
1x16 rectangle	0.024	0.015	0.038	0.021
1x32 rectangle	0.009	0.007	0.012	0.007

Table II

Estimated probabilities of S falling below
5% and 2.5% points

Shape	n = 25		n = 36	
	α = .05	α = .025	α = .05	α = 0.025
Circle	0.048	0.028	0.048	0.032
Square	0.046	0.031	0.042	0.029
1x2 rectangle	0.041	0.025	0.042	0.027
1x4 rectangle	0.046	0.033	0.036	0.023
1x8 rectangle	0.036	0.025	0.032	0.013
1x16 rectangle	0.022	0.013	0.028	0.018
1x32 rectangle	0.005	0.003	0.007	0.003

used tests are the turning points test, run test, rank correlation test and difference-sign test. For example, the turning points test is performed as follows. Let $u_1, u_2, \ldots, u_n$ be the time series.

Let p be the number of turning points of this series. It can be shown that mean and variance of p are given by

$$E(p) = \frac{2(n-2)}{3}$$

and

$$\sigma_p^2 = \frac{16n-29}{90}$$

under the assumption of randomness. Using normal approximation, the test is usually performed. The rank correlation test is performed as follows.

Let P be the number of pairs where $u_j > u_i$, $j > i$ for $i, j = 1, 2, \ldots, n$. Then $E(P) = \frac{n(n-1)}{4}$. If Q_j is the number of pairs such that $u_j < u_i$, $j > i$, then Kendall's τ's

$$\tau = 1 - \frac{4Q}{n(n-1)} .$$

It is well known that

$$E(\tau) = 0$$

and

$$V(\tau) = \frac{2(2n + 5)}{9n(n-1)} ,$$

and has approximately normal distribution which is then used to test the hypothesis of randomness. The details of these tests are available in Kendall and Stuart (1976).

7. Randomness of Treatment Allocation in Experiments

To verify the claim that the treatments were assigned at random in an experiment, often tests using randomization test or permutation test are utilized. To perform the permutation test, the fact that the conditional distribution of any arrangement of the ordered observations given the values of the ordered statistics is uniform, can be utilized. In such a case, an appropriate test statistic is chosen and the value of this statistic is calculated for each arrangement. The hypothesis tested is that there is no difference among the treatments assigned at random to experimental units. The hypothesis is rejected for $\alpha n!$ arrangements which give the most extreme test statistic. Here α is the level of the test and n is the number of observations. Since the critical region depends on the sample values of the ordered statistics, unlike the usual case, the critical region can only be obtained after the sample has been observed. Since such computations are laborious these tests are not usually carried out. Details of these tests are given by Hajek (1969).

8. Miscellaneous Tests

(i) Test of Randomness of Several Rankings

When several judges rank the same items, one is interested in the test of randomness of these ranking. Usually this can be tested by Friedman Statistic. Let there be I items and R_i be the sum of m ranks assigned to item i, i = 1, 2, ..., I.

Friedman's statistic is given by

$$\chi_r^2 = \frac{12}{mI(I+1)} \sum R_i^2 - 3m(I+1)$$

For large m and I, Friedman's statistics is distributed as Chi-squared with I-1 degrees of freedom asymptotically.

(ii) Tests for multivariate normality

Andrews (1972) has proposed tests for assessing multivariate normality for p-dimensional data. Suppose the data are transformed so as to have zero means and identity covariance matrix. Then using the probability transform we have points in a hypercube. The nearest distance statistic is given by

$$d(\underset{\sim}{X}_i, \underset{\sim}{X}_j) = \max\{\min[|x_{ki} - x_{kj}| , |x_{ki} - x_{kj}| - 1]\}$$

Volume of the set enclosed by a distance d from the point $\underset{\sim}{X}_j$ is

$$V(d) = (2d)^p .$$

Since $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ are uniformly distributed, V(d) has exponential distribution, with a parameter, say λ. Then the conditional probability of $V(d) \leq V(d_i)$ given that $d_i \leq d_0$ is

$$p(d_i) = \frac{1-e^{-\lambda V(d_i)}}{1-e^{-\lambda V(d_0)}}$$

Let

$$w_i = \Phi^{-1}(p(d_i)) .$$

If w_i are calculated from disjointed parts of the unit hypercube, they should not show any dependence on the center, $\underset{\sim}{x}_i$ of the part of the cube. This dependence may be tested by using quadratic regression of w_i on x_i. The regression sum of squares has a Chi-squared distribution with (p+1)(p+2)/2 degrees of freedom. For further details, the reader is referred to Gnanadesikan (1977, p. 169) wherein other relevant tests are also given.

Graphical tests, as generalizations of univariate plots on probability paper to assess univariate normality, are based on radii and angles.

Let

$$\underset{\sim}{Z}_i = \underset{\sim}{S}^{-1/2}(\underset{\sim}{Y}_i - \underset{\sim}{Y})$$

$$\gamma_i^2 = Z_i' \, Z_i$$

$$\sim \chi_k^2 \text{ approximately}$$

Plot γ_i^2, i = 1, 2, ..., n ordered in magnitude against the quantile of a χ_p^2 distribution corresponding to a cumulative probability of

$$\frac{i-\frac{1}{2}}{n}$$

Let θ_i = angle which $\underset{\sim}{Z}_i$ makes with the abscissa-axis in bivariate normal case. Plot of θ_i against $\frac{i-1/2}{n}$ If there is bivariate normality in data, both of these plots should be linear. In the case of p-dimensions, there would be p-1 angles. For one of the angles, a probability plot is still appropriate since it is still uniformly distributed on $(0, 2\pi)$. For the remaining (p-2) angles, the distributions are proportional to $(\sin \theta_j)^{p-1-j}$, $0 \leq \theta_j \leq \pi$, $j = 1, 2, \ldots, p-2$. For these angles, the appropriate plots are obtained by plotting n ordered values against n quantiles of this distribution.

(iii) Tests of randomness using stochastic processes

Liebetrau (1979) has studied some statistics utilized in tests of randomness based on the variance-time curve of the Poisson process. Consider the following notation.

$\{T_j\}$ = real weakly stationary point process,

$N(A)$ = the number of $\{T_j\}$ in A for Borel set A,

The mean and variance of N(A) are given by,

$$M(A) = E(N(A)),$$

$$V(A) = E[N(A) - M(A)]^2.$$

When A is an interval $(x, x + t)$, the mean and variance of the process are denoted by

$$M(A) = M\{(x, x+t)\} = M(t),$$

$$V(A) = V(t).$$

Suppose $T_1, T_2, \ldots, T_n$ have been observed in $(0, T)$. $V(t)$ is estimated by

$$\hat{V}(t) = \frac{1}{T} \int_0^T [(N(x, x+t) - \frac{nt}{T}]^2 dx$$

If $\{T_j\}$ is a Poisson process with rate μ and $V(u) = \frac{nu}{T}$, then Liebetrau (1976) showed that

$$\eta_T^{\gamma}(t) = T^{\beta}[\hat{V}(tT^{\gamma}) - V(tT^{\gamma})],$$

$$\beta = \frac{1}{2}(1-3\gamma), \quad 0 < \gamma < 1$$

converges weakly as $T \to \infty$ to Gausian Process η^{γ} with covariance $K(t,u) = \frac{2}{3}\mu^2(3t^2u-t^3)$, $0 \leq t \leq u < 1$. Similarly $\hat{\eta}_T^C(t) = T^{1/2}[V(tc) - V(tc)]$ converges weakly as $T \to \infty$ to $\hat{\eta}^C$ which is Gaussian with covariance

$$K^C(t, u) = C^3K(t,u) .$$

Tests of randomness are based on

$$\zeta_1^C = \int_0^1 \hat{\eta}^C(t)dt$$

$$\zeta_2^C = \int_0^1 [\hat{\eta}^C(t))^2 dt$$

Upper percentage points are given for ζ_2^C by Liebetrau (1979, p. 38).

9. Applications

Tests of randomness are routinely applied in many areas of science and engineering. A few recent examples from lottery games are discussed and examples are given from evolutionary paleontology and mining disasters.

8.1 Evolutionary Paleontology

Several tests of randomness have been applied in studying the pattern of evolution in the fossil record which is basic to the study of evolutionary paleontology. The model of progressive specialization through the phanerozoic is studied using taxonomic and morphological evidence. In a recent paper, Flessa and Levinton (1975) have used tests of goodness of fit of the Poisson distribution and also the run test for studying randomness. Using data obtained from patterns of origination within taxa and patterns of dominance diversity, they have distinguished random and non-random components.

8.2 Mining Disasters

Jarrett (1979) has given a revised table of time intervals in days between explosions in mines in England during 1851 - 1962. These data were originally given by Maguire, Pearson and Wynn (1952). Table 1 of Jarrett (1979) is reproduced here as Table III. One of the earliest problems studied for the data, has been the test of hypothesis of the randomness of the occurrence of coal mining disasters.

Assuming that the mining disasters occur at random, the time interval between them has a Gamma distribution. Since the data given are in terms of these time intervals, the test of randomness is carried through the test of goodness of fit of the Gamma distribution.

Table III

Time intervals in days between explosions in mines, from March 15, 1851 to March 22, 1962 (to be read down columns)

157	65	53	93	127	176	22	1205	1643	312
123	186	17	24	218	55	61	644	54	536
2	23	538	91	2	93	78	467	326	145
124	92	187	143	0	59	99	871	1312	75
12	197	34	16	378	315	326	48	348	364
4	431	101	27	36	59	275	123	745	37
10	16	41	144	15	61	54	456	217	19
216	154	139	45	31	1	217	498	120	156
80	95	42	6	215	13	113	49	275	47
12	25	1	208	11	189	32	131	20	129
33	19	250	29	137	345	388	182	66	1630
66	78	80	112	4	20	151	255	292	29
232	202	3	43	15	81	361	194	4	217
826	36	324	193	72	286	312	224	368	7
40	110	56	134	96	114	354	566	307	18
12	276	31	420	124	108	307	462	336	1358
29	16	96	95	50	188	275	228	19	2366
190	88	70	125	120	233	78	806	329	952
97	225	41	34	203	28	17	517	330	632

Using the density of the Gamma as

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\,\beta^{\alpha}} e^{-x/\beta} x^{\alpha-1}, & x > 0 \\ 0, & \text{elsewhere,} \end{cases}$$

the maximum likelihood estimates of α and β are obtained by solving the following equations

$$\hat{\alpha}\,\hat{\beta} = \bar{X}$$

$$\log \hat{\alpha} - \frac{\Gamma'(\hat{\alpha})}{\Gamma(\hat{\alpha})} = \log \bar{X} - \overline{(\log X)}$$

Omitting 0 as an observation, we use $n = 189$ observations with $\bar{X} = 213.53$, $\overline{\log X} = 4.556$, we use tables of Pearson and Hartley (1966) to obtain

$$\hat{\alpha} = 0.7384$$

and

$$\hat{\beta} = 290.53$$

Using Kolmogorov-Smirnov statistic, we accept the hypothesis at .05 level of significance.

9.3 Occurrence of Prizes in a State Lottery Game

In legalized lottery games, presently being played in many states in American and various countries throughout the world, the games are designed in such a way that the prizes occur at random with preassigned probabilities. One of the main problems in these games is to keep control of the integrity and honesty of the game. When several persons receive prizes in the same locality or several prizes occur within a short interval of time during the progress of the game, the process of randomness is likely to be challenged. Most often, tests of randomness are performed to check whether the prize structure in the game is random. Consider the situation that the probability of getting a prize is 0.01. Then the hypothesis of randomness means that in the independent trials resulting in losers and winners, the winners occur with probability 0.01. The test statistic in this case can be obtained from goodness of fit test for the geometric distribution with probability p = 0.01. The data given in the following were obtained from a game designed for a state lottery. The frequency distribution of the number of winning tickets in a given sequence of 1500 tickets generated, is given in Table IV. The number of losers between successive winners is called waiting time. Table V provides the list of 180 waiting times for certain large tier prizes in another state lottery game. The test for randomness in this case is the usual Chi-squared test for goodness of fit. The Chi-squared statistic is 4.98 in this case and the comparison is made with the tabulated Chi-squared value of 9.24 with 5 degrees of freedom at 10% level of significance. Hence we say that the prizes occur at random.

It can be seen from the following straightforward argument that the geometric distribution is the discrete analog of the exponential distribution. Consider the

Table IV

Frequency of occurrence of prizes

Number of winning tickets	Observed frequency	Expected frequency
1	51	54
2	40	43
3	27	35
4-5	51	50
6-7	37	32
8-11	35	34
12 or more	32	25
	273	273

Table V

Waiting times for high tier denomination prizes in a state lottery game (given in multiples of 300 and arranged in increasing order in every column.

33	5	1	6	13	20	4	5	5	8
35	15	7	24	32	21	7	6	15	18
47	18	9	28	34	43	17	25	17	22
49	29	26	29	43	43	24	29	27	26
60	29	40	42	47	62	37	38	57	47
63	38	42	45	65	68	42	40	68	52
72	55	50	55	66	75	60	46	70	59
77	58	55	68	71	75	79	60	89	84
86	77	72	70	74	77	80	63	106	88
91	92	93	77	80	80	88	70	108	121
106	112	94	81	94	89	96	79	113	137
120	120	105	87	98	115	102	94	113	140
126	122	119	109	130	116	156	118	117	141
132	172	153	119	134	155	189	129	147	143
154	195	164	190	163	162	206	134	148	162
160	195	166	232	163	219	210	151	212	198
235	253	282	266	191	227	219	259	219	216
274	335	442	392	422	273	307	574	289	258

$\bar{x} = 106.67 \quad n = 180$

exponential distribution with parameter λ. The discretized probability between [X] - 1 and [X] is

$$\int_{[x]-1}^{[x]} \lambda e^{-\lambda t} dt = e^{-\lambda[x]-\lambda}[1-e^{-\lambda}] .$$

Let $p = 1-e^{-\lambda}$, then we have for $[x] = y$ a positive integer, $= p(1-p)^{y-1}$ which is the geometric distribution.

We test the hypothesis that the waiting times are exponentially distributed since the continuous analog of the geometric distribution is the exponential distribution. The frequency distribution is given in Table VI for the waiting times in Table V. The Chi-squared value for the table under the exponential model is 6.68 and we again accept the hypothesis of randomness at 10 percent level of significance as the tabulated Chi-squared value is 7.78 for 4 degrees of freedom.

9.4 <u>Randomness of Digits in a Daily Lottery Numbers Game</u>

The tests of randomness for numbers generated for lottery games as well as for awarding prizes are made in the same manner. Consider the following sequence of three-digit random numbers in a state lottery "Number Game" for 100 drawings as given in Table VII. To obtain the frequency test for the random digits we form the Table VIII giving the distribution of 300 digits in 100 numbers. Testing for uniformity provides the confirmation of the hypothesis of randomness at 10 percent level of significance.

Table VI

Frequency distribution of waiting times

Interval	Observed frequency	Expected frequency
0 - 60	64	77
61 - 120	59	45
121 - 180	27	29
181 - 240	16	12
241 - 300	8	10
301 or more	6	7
	180	180

Table VII

Drawings in a number game in a state lottery

308	967	521	492	407
646	514	559	458	145
554	991	751	259	730
804	657	432	972	407
098	109	986	261	748
130	743	551	167	682
037	691	717	002	688
709	146	544	706	909
089	503	163	758	710
613	340	081	114	036

876	758	972	580	738
519	123	568	854	760
810	351	742	392	810
892	983	988	415	460
392	623	533	743	454
726	190	714	750	407
516	951	024	253	107
953	080	035	988	798
969	547	158	472	216

Table VIII

Frequency distribution of digits

digit	frequency	expected
0	34	30
1	32	30
2	21	30
3	25	30
4	29	30
5	33	30
6	22	30
7	34	30
8	30	30
9	30	30
	300	300

$$\chi^2 = 7.20$$

$$\chi^2_{0.90} = 14.68 \qquad \text{with 9 degrees of freedom}$$

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References

Andrews, D. F. (1972). Plots of high-dimensional data. Biometrics 28, 125-36.

Bartholomew, D. J. (1956a). A sequential test for randomness of intervals. J. R. Statist. Soc. Ser. B 18, 95-103.

Bartholomew, D. J. (1956b). Tests of randomness in a series of events when the alternative is a trend. J. R. Statist. Soc. B 18, 234-239.

Barton, D. and Mallow, C. (1965). Some aspects of a random sequence. Ann. Math. Statist., 236-260.

Barton, D. E., David, F. N., and Fix, E. (1963). Random points in a circle and the analysis of chromosome patterns. Biometrika, 50, 23-29.

Bellhouse, D. R. (1979). Towards a definition of randomness, Technical Report, University of Western Ontario, London, Ontario.

Berkson, J. (1966). Examination of randomness of α-particular emissions. Research Papers in Statistics, (F. N. David, Editor), J. Wiley and Sons, New York, 37-54.

Blum, J. R., J. Kiefer, and M. Rosenblatt (1961). Distribution free tests of independence based on the sample distribution function. Ann. Math. Statist., 32, 485-498.

Bohrer, Robert and Imrey, Petter B. (1977). Statistics, stationarity and random number generation. Simuletter 9, 64-71.

Bohrer, Robert and Putnam, Daniel (1976). Statistical tests of random number generators. Simuletter 8, 85-91.

Brown, D. (1975). A test of randomness of nest spacing. Wildfowl 26, 102-3.

Brown, D. and Rothery, P. (1978). Randomness and local regularity of points in a plane. Biometrika, 65, 115-22.

Butcher, J. C. (1961). Random sampling from normal distribution. Comp. J. 3, 251-253.

Cox, D. R. and Lewis, P. A. W. (1966). The Statistical Analyses o. Series of Events. London: Methuen.

David, F. N. (1947). A power function for tests of randomness in a sequence of alternatives. Biometrika, 34, 335-339.

Dodd, E. L. (1942). Certain tests of randomness applied to data grouped into small sets. Econometrica 10, 249-257.

Durbin, J. and Watson, G. S. (1950, 1951). Testing for serial correlation in least squares regression I and II. Biometrika 37, 409-428; 38, 159-178.

Ederer, F., Myers, M. H. and Mantel, N. (1964). A statistical problem in space and time: Do Leukemia cases come in clusters. Biometrics, 20, 626-638.

Federer, Walter T. (1978). Some remarks on statistical education. The American Statistician 32, 117-121.

Flessa, Karl. W. and Levinton, Jeffery S. (1975). The density dependence of phanerozoic diversity patterns: Tests of randomness. J. Geology 83, 239-248.

Glaser, R. E. (1976). The ratio of the geometric mean to the arithmetic mean for a random sample from a gamma distribution. J. Am. Statist. Assoc. 71, 480-7.

Gnanadesikan, R. (1977). Methods for Statistical Data Analysis of Multivariate Observations. John Wiley & Sons: New York.

Good, I. J. (1957). On the serial test for random sequence. Ann. Math. Statist., 28, 262-264.

Good, I. J. (1953). The serial test for sampling numbers and other tests of randomness. Proc. Camb. Philis. Soc. 49, 276-284.

Goodman, L. (1958). Simplified runs tests and likelihood ratio tests for Markov chains. Biometrika, 45, 181-197.

Granger, G. W. J. (1963). A quick test for serial correlation suitable for use with non-stationary time series. J. Am. Statist. Assoc. 58, 728-736.

Greenberg, B. G. (1951). Why randomize? Biometrics, 7, 309-322.

Gupta, G. D. and Govindarajulu, Z. (1980). Nonparametric tests of randomness against autocorrelated normal alternatives. Biometrika, 67, 375-380.

Hájek, J. (1969). A Course in Nonparametric Statistics. Holden-Day: San Francisco

Hammer, Preston (1976). Randomness is nonsense. Simuletter 8, 11.

Hoeffding, W. (1948). A nonparametric test of independence. Ann. Math. Statist. 19, 546-57.

Holgate, P. (1965). Some new tests of randomness. J. Ecol., 53, 261-6.

Hsu, S. and Mason, J. D. (1972). The nearest neighbor statistics for testing randomness of point distributions in a bounded two-dimensional space. Proceedings of Meetings of IGU Commission on Qualitative Geography. (M. H. Yates, Ed.). Montreal: McGill-Queen's University Press, 32-54.

Hušková, Marie (1975). Multivariate rank statistics for testing randomness concerning some marginal distributions. J. Mult. An., 5, 487-496.

Jarrett, R. G. (1979). A note on the intervals between coal-mining disasters. Biometrika, 66, 191-193.

Kempthorne, O. (1977). Why randomize. J. Statist. Pl. Inf., 1, 1-26.

Kendall, Maurice G. and Stuart, Alan (1976). The Advanced Theory of Statistics, Vol. 3. Hafner Publishing Company: New York.

Kennedy, Jr., William J., and Gentle, James E. (1980). Statistical Computing. Marcel Dekker: New York, pp. 169-175.

Knoke, James D. (1975). Testing randomness against autocorrelated alternatives: The parametric case. Biometrika, 62, 571-6.

Knuth, D. E. (1968). The Art of Computer Programming, Vol. II, Addison-Wesley Publishing Co.

Koppelaar, Hank and Kanst, Leo (1976). Randomness is a belief. Simuletter, 8, 12.

Kuper, N. H. (1960). Tests concerning random points on a circle. Proc. Koninkl. Nederl. Akad. Van Welter-schappen, Series A, 63, 38-47.

Lars n, R. J., Holmes, C. L, and Heath, C. W. (1973). A statistical test for measuring unimodal clustering: A description of the test and its application to cases of acute Leukemia in Metropolitan Atlanta, Georgia. Biometrics, 29, 301-9.

Lehmann, E. L. (1975). Nonparametrics: Statistical Methods Based on Ranks. Holden-Day, Inc.

Lehmann, E. L. (1966). Some concepts of dependence. Ann. Math. Statist., 37, 1137-1153.

Liebetrau, A. M. (1979). Some tests of randomness based upon the variance-time curve of the Poisson process. J. Roy. Stat. Soc. B, 41, 32-39.

Liebetrau, A. M. (1977). Tests for randomness in two dimensions. Comm. Statist. Theor. Meth., A6 (14) 1367-1383.

Liebetrau, A. M. (1976). On the weak convergence of a class of estimators of the variance-time curve of a weakly stationary point process. J. Appl. Prob., 14, 114-126.

Mann, H. B. (1945). Nonparametric tests against trend. Biometrika, 13, 245-59.

Martin-Lof, P. (1969). Algorithms and randomness. Review of the International Statistical Institute. 37, 265-272.

Moran, P. A. P. (1951). The random division of an interval II. J. R. Statist. Soc. B 13, 147-50.

Nance, Richard E. (1976). A rejoinder to randomness is nonsense. Simuletter 8, 13-15.

O'Brien, Peter C. (1976). A test for randomness. Biometrics, 32, 391-401.

Olmstead, P. S. (1946). Distribution of sample arrangements for runs up and down. Ann. Math. Statist. 17, 24-33.

Pearson, E. S. (1963). Comparison of tests of randomness on a line. Biometrika 50, 315-326.

Pearson, E. S. and Hartley, H.). (1966). Biometrika Tables for Statisticians, Volumes I and II, Charles Griffin & Co., London.

Rao, C. R. (1961). Generation of random permutations of given number of elements using random sampling numbers, Sankhyā, A23, 305-307.

Raoul, A., and Sathe, P. T. (1975). A run test for sample nonrandomness. J. Quality Tech. 7, 196-199.

Ripley, B. D. (1979). Tests of randomness for spacial point patterns. J. Roy. Statist. Soc. 41, 368-374.

Savage, L. J. (1976). On rereading R. A. Fisher. Ann. Statist. 4, 441-500.

Stephens, M. A. (1964). The testing of unit vectors for randomness. J. Am. Statist. Assoc. 59, 160-167.

Stuart, A. (1956). The efficiencies of tests of randomness against normal regression. J. Am. Statist. Assoc. 51, 285-7.

Stuart, A. (1954). The asymptotic relative efficiencies of distribution-free tests of randomness against normal alternatives. J. Am. Statist. Assoc. 49, 147-57.

Wald. A. and Wolfowitz, J. (1943). An exact test for randomness in the nonparametric case based on serial correlation. Ann. Math. Statist., 14, 378-388.

Watson, G. S. (1956). A test for randomness of directions. Monthly Notices, Royal Astronomical Society, Geophysical Supplement 7, 160-1.