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DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION

WEAPONS SYSTEMS RESEARCH LABORATORY

DEFENCE RESEARCH CENTRE SALISBURY
SOUTH AUSTRALIA

TECHNICAL MEMORANDUM

WSRL-0169-TM

AN EXTENSION OF A PREVIOUS RESULT ON THE ANALYSIS OF

A METHOD OF MEASURING THE SIGNAL TO NOISE RATIO OF

A SINUSOID IN NOISE

A.P. CLARKE

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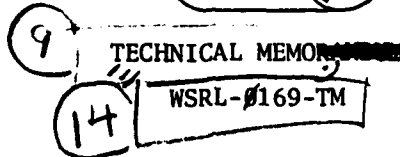Memoranda usually tentative in nature and reflect the personal views of the author.

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S U M M A R Y

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1. INTRODUCTION

The statistical properties of estimators of signal-to-noise ratio of acoustic tones are needed in ongoing studies of acoustic signal processing (Task No.DST 79/069 - Signal Processing for Underwater Detection) in areas as widely separated as measurement of acoustic transmission loss and signal detection theory.   In a previous paper by the author(ref. 1) an expression was derived for the probability density function of an estimate of the signal-to-noise ratio of a sine wave in Gaussian noise.   The estimate was obtained from the power spectrum of a sampled data sequence.   The analytic results in reference 1 have been used to obtain functional forms for the coefficient of variation of signal-to-noise ratio estimates derived from data obtained from several sea-going experiments.   The agreement between analysis and experiment was found to be excellent.

Reference 1 did not address the situation where the spectrum used for estimation was obtained as the average of a number of spectra derived from consecutive sequences of data samples.   This present paper gives a solution to this problem by using a characteristic function approach with the aid of some integrals presented in Appendix I.   The problem is to find the distribution of z

$$z = \frac{1}{r}\left\{ \frac{\dfrac{1}{N}\displaystyle\sum_{}^{N} x_i}{\dfrac{1}{p}\displaystyle\sum_{j=1}^{p}\dfrac{1}{N}\displaystyle\sum_{i=1}^{N} y_{ij}} - 1\right\} \tag{1}$$

where x_i is the power in the signal bin for the i-th spectral estimate and y_{ij} is the power in the j-th noise bin for the i-th spectral estimate.

The noise bin must not include the signal bin or contain extraneous signal residues.

N is the number of spectra from which an average spectrum is obtained.
p is the number of noise bins in the average spectrum used to estimate noise power
$\frac{1}{r}$ is the bin width in Hertz.

Equation (1) can be rewritten as

$$z = \left\{ \frac{\dfrac{p}{r}\displaystyle\sum_{}^{N} x_i}{\displaystyle\sum_{j=1}^{p}\displaystyle\sum_{i=1}^{N} y_{ij}}\right\} - \frac{1}{r} \tag{2}$$

Before proceeding with the development of the solution, some results of a standard nature are summarised from reference 2 in order to aid in reading this paper, viz:

(1) The characteristic function of a random variable x with probability density function f(x) is given by

$$\phi(t) = \int_{-\infty}^{\infty} dx \, \exp(jtx) \, f(x) \tag{3}$$

(2) The characteristic function of the sum S of N random variables whose individual characteristic functions are $\phi(t)$ is

$$\phi_S(t) = \{\phi(t)\}^N \tag{4}$$

(3) If $\phi(t)$ is absolutely integrable over the range $(-\infty, \infty)$ then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, \exp(-jtx) \, \phi(t) \tag{5}$$

2. THE CHARACTERISTIC FUNCTION OF SIGNAL PLUS NOISE POWER

For a single spectral estimate the probability density function of the signal + noise power is given by equation (7) of reference 1. Note that the nomenclature is the same, ie:

$$f(x) = A^2 \, \exp\{-A^2(x + K^2)\} \, I_0(2A^2 \, K\sqrt{x}), \; x \geqslant 0 \tag{6}$$

$$= 0, \; x < 0$$

where $I_0(.)$ is the modified Bessel function of zero order. The characteristic function of this density function is then, after a simple rearrangement, given by

$$\phi(t) = A^2 \, \exp(-A^2 \, K^2) \int_0^{\infty} dx \, \exp\{-(A^2 - jt) \, x\} \, I_0(2A^2 K \sqrt{x}) \tag{7}$$

By making the elementary substitutions

$$\alpha = A^2 \, \exp(-A^2 \, K^2)$$

$$\beta = A^2 - jt$$

$$\gamma = 2j \, A^2 \, K \tag{8}$$

$$\mu = \sqrt{x}$$

and using the relation $I_0(\mu) = J_0(j\mu)$ the characteristic function can be written
as

$$\phi(t) = 2a \int_0^\infty d\mu\, \mu\, \exp(-\beta\,\mu^2)\, J_0(\gamma\mu) \tag{9}$$

Applying the formula I.1 from the Appendix I gives

$$\phi(t) = \frac{a}{\beta}\, \exp\left(-\frac{\gamma^2}{4\beta}\right) \tag{10}$$

ie

$$\phi(t) = A^2\left[\frac{A^2 + jt}{A^4 + t^2}\right]\, \exp(-A^2\, K^2)\, \exp\left\{-A^4\, K^2\left(\frac{A^2 + jt}{A^4 + t^2}\right)\right\} \tag{11}$$

3.   THE PROBABILITY DENSITY FUNCTION OF SIGNAL PLUS NOISE POWER

The density function derived in this section is for the summation in the numerator
of equation (2).

For the sum of N terms each distributed as in equation (6)

$$\phi_{sum} = \{\phi(t)\}^N = \left(\frac{a}{\beta}\right)^N\, \exp\left\{-\frac{N\gamma^2}{4\beta}\right\} \tag{12}$$

.... from equation (10)

Hence the probability density function for the sum is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty dt\left(\frac{a}{\beta}\right)^N\, \exp(-jt\, x)\, \exp\left(\frac{-N\gamma^2}{4\beta}\right) \tag{13}$$

.... from equation (5)

As β is a function of t, viz $\beta = A^2 - jt$, the integral can be rewritten in terms
of β to give, after some rearrangement:

$$f(x) = \frac{ja^N}{2\pi}\, \exp(-A^2\, x) \int_{A^2 + j\infty}^{A^2 - j\infty} d\beta\, \frac{1}{\beta^N}\, \exp(x\beta - \frac{N\gamma^2}{4\beta}) \tag{14}$$

Using the formula I.2 in the Appendix I gives, after some algebra:

$$f(x) = \frac{A^2}{j^{N-1}}\left(\frac{x}{NK^2}\right)^{(N-1)/2}\, \exp -A^2\,(x + NK^2)\, J_{N-1}(2jA^2\, K\sqrt{Nx})$$

The Bessel function of the first kind of order n is related to the modified Bessel function of order n by

$$I_n(x) = (-j)^n J_n(jx) \quad .$$

Hence

$$f(x) = A^2 \left(\frac{x}{NK^2}\right)^{(N-1)/2} \exp\{-A^2(x + NK^2)\} I_{N-1}(2A^2 K \sqrt{Nx}) \qquad (15)$$

4. THE PROBABILITY DENSITY FUNCTION OF THE NOISE POWER

From reference 1, equation (8), each term y_{ij} in the denominator of equation (2) is distributed as $\chi^2_{2, 1/A\sqrt{2}}$.

Hence, using the well-known addition theorem(ref.2) for χ^2 variates, the double summation term in equation (2) is distributed as $\chi^2_{2pN, 1/A\sqrt{2}}$ ie:

if

$$\mu = \sum_{j=1}^{p} \sum_{i=1}^{N} y_{ij}$$

then

$$f_U(y) = \frac{A^{2pN} . 2^{pN} . y^{pN-1} . \exp(-A^2 y)}{2^{pN} . \Gamma(pN)}$$

$$= \frac{A^{2pN} . y^{pN-1} . \exp(-A^2 y)}{\Gamma(pN)}$$

$$(16)$$

5. THE PROBABILITY DENSITY FUNCTION FOR THE SIGNAL-TO-NOISE RATIO ESTIMATOR

The probability density function for the ratio of the summation terms in equation (2) can now be obtained by applying the classical relation given in Parzen(ref.2) for the density of the ratio of two independent positive random variables, viz:

$$f_{X/Y}(y) = \int_0^{\infty} f_X(yx) f_Y(x) \, dx \qquad (17)$$

ie

$$f_{X/Y}(y) = \int\limits_{0}^{\infty} dx\ x\ A^2 \left\{\frac{yx}{NK^2}\right\}^{(N-1)/2} \exp(-A^2(yx+NK))$$

$$\times\ \frac{I_{N-1}(2A^2K\sqrt{Nxy})\ A^{2pN}\ x^{pN-1}\ \exp(-A^2x)}{\Gamma(pN)}$$

(18)

Gathering terms independent of x gives

$$\frac{A^{2pN+2}}{\Gamma(pN)}\ (\frac{y}{NK^2})^{(N-1)/2}\ \exp(-NA^2K^2)$$

(19)

The integral rearranges to

$$\int\limits_{0}^{\infty} dx\ x^{pN+(N-1)/2}\ \exp(-A^2x(y+1))\ I_{N-1}(2A^2K\sqrt{Nxy})$$

(20)

As stated previously a simple relation holds between the modified Bessel functions and the Bessel functions of the first kind, ie:

$$I_k(x) = j^{-k}\ J_k(jx)$$

(21)

Hence the integral becomes

$$\int\limits_{0}^{\infty} dx\ x^{pN+(N-1)/2}\ \exp(-A^2x(y+1))\ j^{-N+1}\ J_{N-1}(2A^2Kj\sqrt{Nyx})$$

(22)

With the substitution $\mu^2 = x$ the integral becomes

$$2j^{-N+1} \int\limits_{0}^{\infty} d\mu\ \mu^{2pN+N}\ \exp(-A^2(y+1)\mu^2)\ J_{N-1}(2A^2Kj\sqrt{Ny}\ \mu)$$

$$= 2j^{-N+1} \int\limits_{0}^{\infty} d\mu\ \mu^{a-1}\ \exp(-\beta^2\mu^2)\ J_{n-1}(\gamma\mu)$$

(23)

where the substitutions

$$\alpha \quad = \quad 2pN+N+1$$

$$\beta \quad = \quad A(y+1)^{\frac{1}{2}}$$

$$\gamma \quad = \quad 2A^2 Kj\sqrt{Ny}$$

are distinct from the use previously made in manipulating equation (7).

A simple application of the formula given by equation I.1 in the Appendix I enables the integral (22) to be evaluated to be evaluated to give

$$\frac{2j^{-N+1} \Gamma(\frac{a+n-1}{2}) \gamma^{N-1} M(\frac{a+N-1}{2},N, -\frac{\gamma^2}{4\beta^2})}{\Gamma(N) 2^N \beta^{a+N-1}} \tag{24}$$

Substitution for α, β, γ gives, after some rearrangement:

$$\frac{\Gamma(pN+N) K^{N-1} N^{(N-1)/2} y^{(N-1)/2} M(pN+N,N, NA^2K^2 (\frac{y}{y+1}))}{\Gamma(N) A^{2pN+2} (y+1)^{pN+N}} \tag{25}$$

Forming the product with (19) gives, after some algebra:

$$f_{X/Y}(y) \quad = \quad \frac{\exp(-NA^2K^2) y^{N-1}}{\beta(pN,N) (y+1)^{pN+N}} M(pN+N,N,NA^2K^2 (\frac{y}{y+1})) \tag{26}$$

where $\beta(...)$ is the bivariate β-function.

The distribution of z as defined by equation (2) then follows from

$$f_Z(z) \quad = \quad \frac{r}{p} f_Y\left(\frac{rz+1}{p}\right) \tag{27}$$

where $z = \frac{p}{r} y - \frac{1}{r}$ from equation (2).

The above is simply derived from the classic relation(ref.2)

$$f_{aX+B}(y) \quad = \quad \frac{1}{a} f_X (\frac{y-b}{a}) \tag{28}$$

ie

$$f_Z(z) \quad = \quad \frac{r p^{pN} e^{-NA^2K^2} (rz+1)^{N-1} M(pN+N,N,NA (\frac{rz+1}{rz+p+1}))}{\beta(pN,N) (rz+p+1)^{pN+N}} \tag{29}$$

A simple consideration of equation (2) indicates that z will only take values in

$$-\frac{1}{r} \leqslant z < \infty$$

6. THE MEAN AND VARIANCE OF THE SIGNAL-TO-NOISE RATIO ESTIMATOR

The mean and variance of z follow readily from the moments of y, where y is the ratio of the summation terms appearing in equation (2). The moments of y can be derived from the density equation (26), ie:

$$E[y^n] = \frac{\exp(-NA^2 K^2)}{\beta(pN,N)} \int_0^\infty dy \frac{y^{n+(N-1)}}{(y+1)^{pN+N}} M(pN+N,N,NA^2 K^2 (\frac{y}{y+1})) \tag{30}$$

A series expansion for $M(.,.,.)$ written in the form

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{r=0}^{\infty} \frac{\Gamma(a+r)}{\Gamma(b+r)} \frac{z^r}{r!} \tag{31}$$

can be substituted in the above equation to give, after reversing the order of integration and summation and using the Γ-function form of the β-function:

$$E[y^n] = \frac{\exp(-NA^2 K^2)}{\Gamma(pN)} \sum_{r=0}^{\infty} \frac{(pN+N+r)(NA^2 K^2)^r}{\Gamma(N+r) r!} \int_0^\infty dy \frac{y^{n+N-1+r}}{(y+1)^{pN+N+r}} \tag{32}$$

As shown in the derivation I.3 in the appendix the integral is simply the β-function

$$\beta((n+N+r),(pN-n))$$

This β-function can be written in terms of appropriate Γ-functions and after a rearrangement of terms the n-th moment of y can be written as

$$E[y^n] = \exp(-NA^2 K^2) \frac{\beta(pN-n,N+n)}{\beta(pN,N)} M(N+n,N,NA^2 K^2) \tag{33}$$

Hence

$$E[y] = \frac{N \exp(-NA^2 K^2)}{pN-1} M(N+1,N,NA^2 K^2) \tag{34}$$

A simple application of the formulae 13.4.1 and 13.6.12 in Abromawitz and Stegun (ref.3) gives

$$E[y] = \frac{N(1+A^2 K^2)}{pN-1} \tag{35}$$

A similar application of the same formulae enables the second moment of y to be written

$$E[y^2] = \frac{N}{(pN-1)(pN-2)} \{(NA^2 K^2 +N+2)(1+A^2 K^2)-1\} \tag{36}$$

From equation (2)

$$E[z] = \frac{p}{r} E[y] - \frac{1}{r} \qquad (37)$$

and

$$var(z) = \frac{p^2}{r^2} \left\{ E[y^2] - E^2[y] \right\} \qquad (38)$$

Hence precise expressions can be written for the mean and variance of z, ie:

$$E[z] = \frac{pN(1+A^2 K^2)}{r(pN-1)} - \frac{1}{r} \qquad (39)$$

and

$$var(z) = \frac{p^2}{r^2} \left\{ \frac{N^2 A^4 K^4 + 2N(pN+N-1)A^2 K^2 + (pN^2 +N^2 -N)}{(pN-1)^2 (pN-2)} \right\} \qquad (40)$$

And, the coefficient of variation defined as

$$C = \frac{\sigma}{E[z]} \qquad (41)$$

where σ is the standard deviation can be written down immediately.

Substituting N=1 in equations (39) and (40) gives

$$E[z] = \frac{p(1+A^2 K^2)}{r(p-1)} - \frac{1}{r} \qquad (42)$$

$$var(z) = \frac{p^2}{r^2} \left\{ \frac{A^4 K^4 + 2pA^2 K^2 + p}{p-2} \right\}^{\frac{1}{2}} \qquad (43)$$

and after some algebra

$$C = \frac{p}{pA^2 K^2 + 1} \left\{ \frac{A^4 K^4 + 2pA^2 K^2 + p}{p-2} \right\}^{\frac{1}{2}} \qquad (44)$$

These are precisely the respective equations developed in reference 1 and hence equations (42), (43) and (44) supply a partial validation of the analysis contained in this memorandum.

7. AN ALTERNATIVE DERIVATION

On realising (note the acknowledgement at the end of this paper) that the quotient in equation (1) is the ratio of a non-central chi-squared variate (with 2N degrees of freedom) to a chi-squared variate with 2pN degrees of freedom ie the quotient has the distribution of a non-central F distribution, it is possible to derive the formulas equations (29), (38) and (40) as particular cases of a more general formula. The non-central F with ν_1 degrees of freedom in the numerator and ν_2 degrees of freedom in the denominator has a probability density function

$$f_Y(y) = \frac{\exp(-\lambda/2)}{\Gamma(\nu_2/2)} \left\{\frac{\nu_1}{\nu_2}\right\}^{\nu_1/2} z^{\nu_1/2 - 1}$$

$$\times \left\{1 + \frac{\nu_1}{\nu_2} z\right\}^{-(\nu_1 + \nu_2)/2} \times \sum_{j=0}^{\infty} \frac{\Gamma((\nu_1 + \nu_2)/2 + j)}{\Gamma(\nu_1/2 + j)\; j!}$$

$$\times \left\{\frac{z\lambda\nu_1}{2\nu_2}\right\}^{j} \left\{\frac{1 + \nu_1\, z}{\nu_2}\right\}^{-j} \tag{45}$$

where λ is the non-centrality parameter. This equation can obviously be written as

$$f_Y(y) = \frac{\exp(-\lambda/2)}{\Gamma(\nu_2/2)} \left\{\frac{\nu_1}{\nu_2}\right\}^{\nu_1/2} z^{\nu_1/2 - 1}$$

$$\times \left\{1 + \frac{\nu_1}{\nu_2} z\right\}^{-(\nu_1 + \nu_2)/2} \frac{\Gamma((\nu_1 + \nu_2)/2)}{\Gamma(\nu_1/2)} \times$$

$$M\left(\frac{\nu_1 + \nu_2}{2}, \frac{\nu_1}{2}, \frac{z\lambda\nu_1}{2\nu_2 + 2\nu_1 z}\right) \tag{46}$$

On making the substitutions

$$\nu_1 = 2N, \quad \nu_2 = 2pN, \quad \lambda = 2NA^2 K^2$$

and carrying out some elementary algebra

$$f_Y(y) = \frac{\exp(-N A^2 K^2)\; y^{N-1}\; p^{pN}\; M(pN + N, N, \frac{yNA^2 K^2}{p + y})}{\beta(pN,N)\; (y + p)^{pN + N}} \tag{47}$$

In this alternative derivation $z = \frac{1}{r} y - \frac{1}{r}$

Hence

$$f_Z(z) = r f_Y(rz + 1)$$

as a result of applying equation (2)

ie

$$f_Z(z) = \frac{rp^{pN} \exp(-NA^2K^2) (rz + 1)^{N-1} M(pN+N, N, NA^2K^2 \left\{\frac{rz+1}{rz+p+1}\right\})}{\beta(pN,N) (rz + p + 1)p^{N+N}}$$

which agrees with equation (29).

The nth order moment of the non-central F-distribution to be derived from equation (45) involves evaluating an integral

$$\int_0^\infty \frac{dy \, y^n \, y^{\nu_2/2^{-1}}}{(1 + \frac{\nu_1}{\nu_2} y)^{(\nu_1 + \nu_2)/2}} \quad \frac{y^j}{(1 + \frac{\nu_1}{\nu_2} y)^j} \tag{48}$$

The integral can be written as

$$\int_0^\infty dz \, \frac{z^{C_1}}{(1 + az)^{C_2}} \tag{49}$$

where

$$C_1 = n + j + \frac{\nu_1}{2} - 1, \quad C_2 = \frac{\nu_1 + \nu_2}{2} + j, \quad a = \frac{\nu_1}{\nu_2}$$

On making the substitution $az = \mu$ the integral becomes

$$\frac{1}{a^{C_1 + 1}} \int_0^\infty d\mu \, \frac{\mu^{C_1}}{(1 + \mu)^{C_2}}$$

$$= \frac{1}{a^{C_1 + 1}} \beta(C_2 - C_1 - 1, C_1 + 1) \tag{50}$$

On substituting this integral in the appropriate expression for $E[y^n]$ and carrying out some algebra

$$E[y^n] = \exp(-\lambda/2) \left(\frac{\nu_2}{\nu_1}\right)^n M\left(\frac{\nu_1}{2} + n, \frac{\nu_1}{2}, \frac{\lambda}{2}\right) \left\{\frac{\Gamma(\frac{\nu_2}{2} - n) \Gamma(\frac{\nu_1}{2} + n)}{\Gamma(\nu_2) \Gamma(\nu_1)}\right\}$$

Using the substitutions ν_1 = 2N, ν_2 = 2pN, λ = $2NA^2K^2$ for values of n of 1 and 2 in this expression then enables E[z] and E[z^2] to be derived from

$$z = \frac{1}{r} y - \frac{1}{r}$$

Although not carried out in detail the results are easily shown to be identical to equations (39) and (40).

8. SOME PRACTICAL RESULTS

In order to give some idea of how to use the formulae, two steps were taken. First, a program (program A in Appendix II) was developed to examine the variation in the form of the density function as the number of spectra used to obtain an average spectrum is increased. Results are shown in figure 1 for a moderately high input signal-to-noise ratio of 5 dB, using 10 noise bins and a bin width of 1 Hz, ie p=10, r=1. It is surprising that even at this signal level the spread in a number of estimates can be high. This spread should be compared with the results in figure 2 for a much lower input signal-to-noise ratio of -5 dB. The comparison indicates a necessity for a further detailed examination.

The second step taken was to develop a program (program B in the appendix) to examine the variation with signal-to-noise ratio of the mean, standard deviation, and coefficient of variation of the estimate for a specified number of bins used to estimate noise power. Typical results for the mean and coefficient of variation are shown in figures 3 and 4 respectively, using p=10 and r=1 as in figure 1. In figure 3 the bias (due to the assymetry of the distribution) is clear at even high signal levels. At low signal levels it is obvious that it is necessary to average a large number of spectra to minimise the bias in the mean. In figure 4 the behaviour in the coefficient of variation indicates the large number of spectra to be analysed to keep the frequency of occurrence of negative estimates of signal-to-noise ratio to a minimum. This need is of course going to conflict with the non-stationarity of the statistics of any physical medium in which an experiment is taking place such as when measuring acoustic signal transmission properties in the ocean.

9. CONCLUSIONS

This paper presents a rigorous mathematical analysis and a brief look at some experimental implications of a definition and associated measurement technique of the signal-to-noise ratio of a sine wave in white noise. The computer programs developed as a result of this analysis can be used to carry out a detailed examination of any proposed experimental scenario. During the development of the analysis it has appeared that the probability distribution function for both signal plus noise and signal-to-noise ratio are both emenable to an analysis that extends techniques reported in this paper and it is proposed to publish results on this shortly. It is anticipated that this further work will throw some interesting light on some of the problems of detection theory.

10. ACKNOWLEDGEMENT

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APPENDIX I

MATHEMATICAL DETAILS

I.1 A general integral formula first due to Hankel(ref.4)

$$\int_0^\infty dt \; t^{\mu-1} \exp(-p^2 t^2) \, J_N(at) \;=\; \frac{\Gamma(\frac{\mu+\nu}{2}) \, a^\nu}{\Gamma(\nu+1) \, 2^{\nu+1} \, p^{\nu+\mu}} \, M(\frac{\mu+\nu}{2}, \nu+1, \frac{-a^2}{4p^2})$$

$$|\arg p | < \frac{\pi}{4} \;, \; Re(\mu+\nu) > 0$$

If the substitutions $a = \gamma$, $\nu = 0$, $\mu = 2$, $p = \sqrt{\beta}$ are made then

$$\int_0^\infty dt \; t \exp(-\beta t^2) \, J_0(\gamma t) \;=\; \frac{1}{2\beta} \, M(1,1, -\frac{\gamma^2}{4})$$

$$=\; \frac{1}{2\beta} \, \exp(\frac{-\gamma^2}{4})$$

from applying formula 13.6.12(ref.3).

I.2 An integral formula(ref.4), generally attributed to Sonine, for the ordinary Bessel function is

$$J_\nu(z) \;=\; \frac{(\frac{1}{2}z)^\nu}{2\pi j} \int_{C+j\infty}^{C+j\infty} dt \frac{1}{(t)^{\nu+1}} \, \exp(t - \frac{z^2}{4t})$$

where the path of integration is the straight line $R\ell(t) = C > 0$

If the substitutions $\nu+1 = N$, $t = x\beta$, $z^2 = N\gamma^2 x$, $C = A^2$ are made, then

$$J_{\nu-1}(\gamma\sqrt{Nx}) \;=\; \frac{\gamma^{N-1} \, N^{(N-1)/2}}{2^N \, \pi j \, x^{(N-1)/2}} \int_{A^2-j\infty}^{A^2+j\infty} d\beta \, \frac{1}{\beta^N} \exp(x\beta - \frac{N\gamma^2}{4\beta})$$

I.3 β-functions can be written in the integral form

$$\beta(p,q) \;=\; \int_0^\infty \frac{y^{q-1}}{(y+1)^{p+q}} \, dy$$

Make the parameter substitutions

$$q = n + N + r$$

$$p = pN - n$$

Then

$$p + q = pN + N + r$$

Hence

$$\int_0^\infty dy \, \frac{y^{n+N-1+r}}{(y+1)^{pN+N+r}} = \beta(n+N+r, pN-n)$$

as $\beta(p,q) = \beta(q,p)$

APPENDIX II

COMPUTER PROGRAM

A
```
00020 C
00030 C$$$ EVALUATE SNR PDF FOR : $$$$$$$$$$$$$$$$$
00040 C        1/R=1.
00050 C        N SPECTRA (XX.)
00060 C        P NOISE BINS (XX.)
00070 C        S SIGNAL TO NOISE RATIO(DB)
00080 C$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
00090      FUNCTION CHGF(A,C,Z)
00100      S=1.
00110      Y=1.
00120      N=1
00130   10 RM=N-1
00140      Y=Y$(Z/N)$(A+RM)/(C+RM)
00150      SL=S
00160      S=S+Y
00170      N=N+1
00180      IF(N.LE.100.AND.S.NE.SL) GO TO 10
00190      CHGF=S
00200      RETURN
00210      END
00220 C$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
00230      REAL N,P,S,Z(320),LGDEN(320),DEN(320)
00240      Z(1)=-1.
00250      R=1.
00260      WRITE(6,100)
00270      READ(5,$) N,P,S
00280      S=10.$$(S/10.)
00290      DO 10 I=1,320
00300      D=CHGF((P$N+N),(N),(N$S$(R$Z(I)+1.)/(R$Z(I)+P+1.)))
00310      IF(D.LT.0.) GO TO 20
00320      D=ALOG(D)
00330      GO TO 25
00340   20 D=-ALOG(ABS(D))
00350   25 CONTINUE
00360      LGDEN(I)=ALOG(R)+P$N$ALOG(P)+(N-1.)$
00370     1    ALOG(R$Z(I)+1.)+ALGAMA(P$N+N)-ALGAMA(P$N)
00380     1    -ALGAMA(N)-(P$N+N)$ALOG(R$Z(I)+P+1.)
00390     1    +D-N$S
00400      DEN(I)=EXP(LGDEN(I))
00410   10 Z(I+1)=Z(I)+0.1
00420      S=10.$ALOG10(S)
00430      WRITE(6,200) S
00440      WRITE(6,210)
00450      DO 50 I=1,320
00460   50 WRITE(6,230) Z(I),DEN(I)
00470  100 FORMAT(' ENTER N,P,S (REAL)...')
00480  200 FORMAT(' S= ',F5.2)
00490  210 FORMAT('    Z         DEN')
00500  230 FORMAT(2(2X,F8.2,2X,F8.2))
00510      STOP
00520      END
```

B
```
00020 C
00030 C$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
00040 C    COMPUTE COEFF. OF VARIATION, MEAN, SD OF
00050 C    SNR ESTIMATION OF TONE IN WHITE NOISE
00060 C    FROM AVERAGE SPECTRA.
00070 C    N  SPECTRA
00080 C    P  NOISE BINS
00090 C$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
00100      REAL SS(12),EZ(12),SZ(12),C(12),N
00110      R=1.
00120      X=-40.
00130      WRITE(6,100)
00140  100 FORMAT(' ENTER N,P (REAL)...')
00150      READ(5,$) N,P
00160      DO 10 I=1,12
00170      X=X+5.
00180      S=10.$$(X/10.)
00190      EZ(I)=N$P$(1.+S)/(R$(P$N-1))-1./R
00200      SZ(I)=(P/R)$SQRT((((N$$2)$(S$$2)
00210     1    +2.$N$(P$N+N-1.)$S+(P$(N$$2)
00220     1    +(N$$2)-N))/(((P$N-1.)$$2)$(P$N-2.)))
00230      C(I)=SZ(I)/EZ(I)
00240   10 SS(I)=X
00250      WRITE(6,200)
00260  200 FORMAT('    SS        EZ        SZ        C ')
00270      DO 20 I=1,12
00280   20 WRITE(6,300) SS(I),EZ(I),SZ(I),C(I)
00290  300 FORMAT(2X,F5.2,3(2X,F8.3))
00300      STOP
00310      END
```
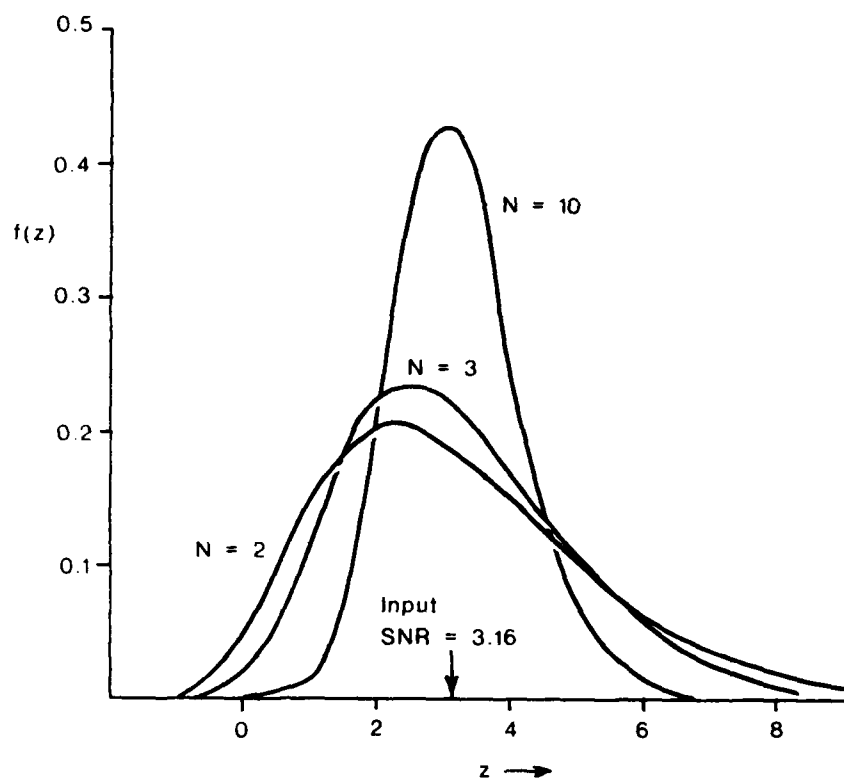
Figure 1. Variation in the probability density function for the estimate of
signal-to-noise ratio from an average of N spectra using 10 noise
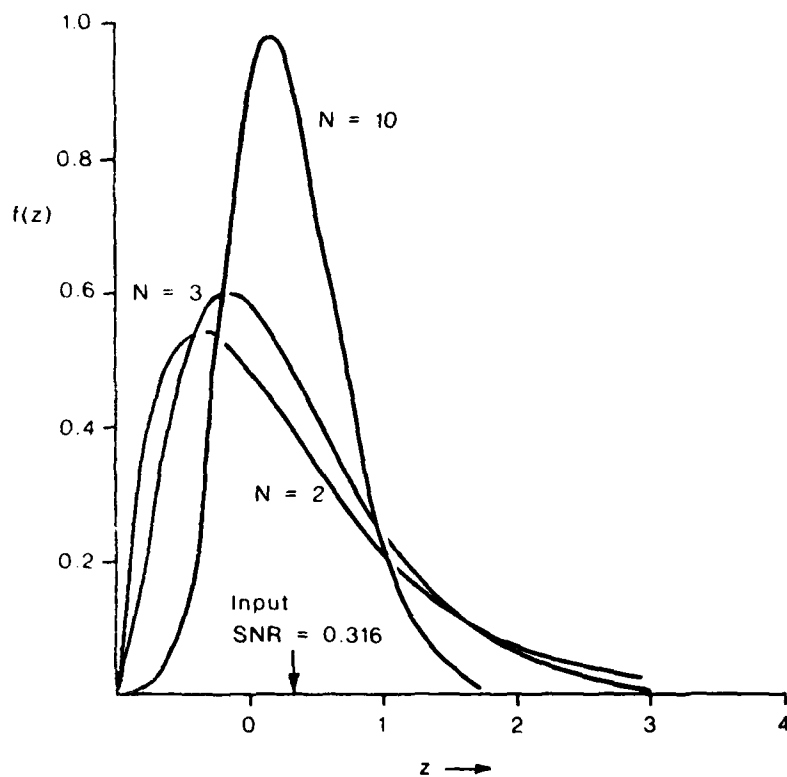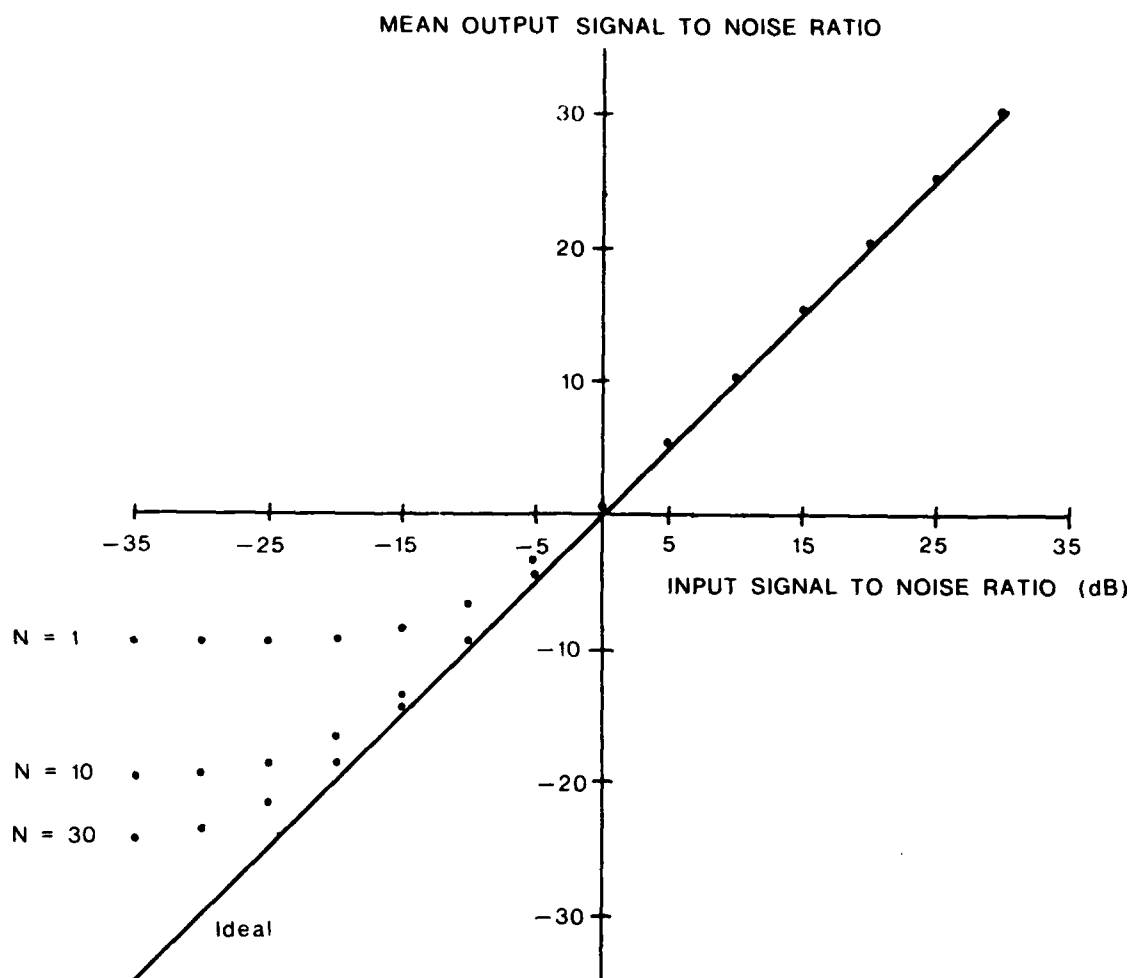bins at an input signal-to-noise ratio of 5 dB

Figure 2. Variation in the probability density function for the estimate of
signal-to-noise ratio from an average of N spectra using 10 noise
bins at an input signal-to-noise ratio of -5 dB

Figure 3.   Deviation of the mean when using N spectra to estimate signal-to-
noise ratio for a 1 Hz bin width and using 10 noise bins for
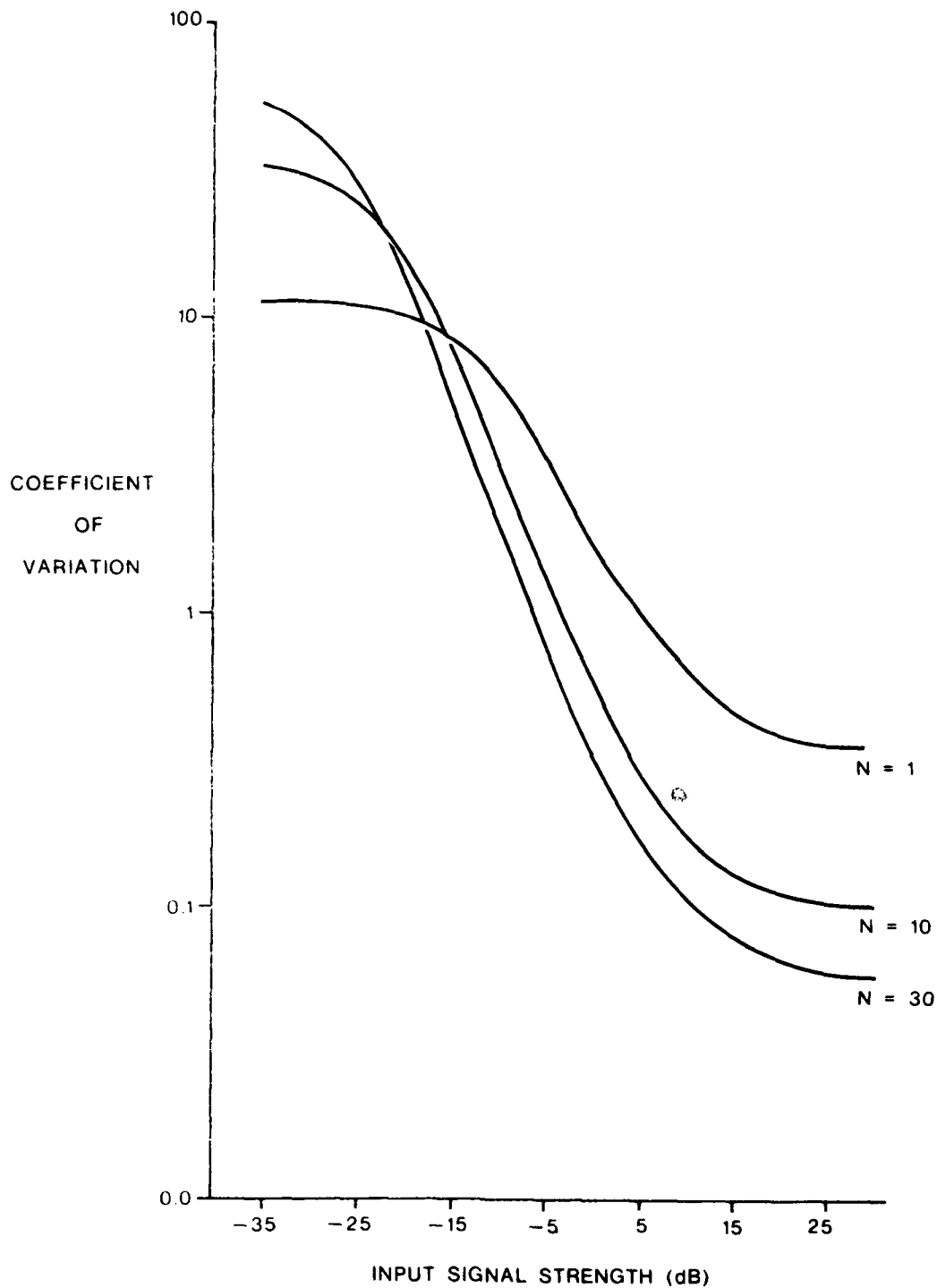estimating noise power

Figure 4. A demonstration of the effect of using N spectra to estimate signal-to-noise ratio for a 1 Hz bin width and using 10 noise bins for estimating noise power

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