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(10) Ralph W. Carr
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Mathematics Research Center
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RESOLVENT FORMULAS FOR A VOLTERRA EQUATION IN HILBERT SPACE

Ralph W. Carr[1] and Kenneth B. Hannsgen[2]

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ABSTRACT

Let $\mathbf{y}(t,\mathbf{x},\mathbf{f})$ denote the solution of the Cauchy problem

$$\mathbf{y}'(t) + \int_0^t [d + a(t-s)]\mathbf{L}\,\mathbf{y}(s)ds = \mathbf{f}(t),\ t \geq 0,\ \mathbf{y}(0) = \mathbf{x}\ ,$$

where $d \geq 0$ and $\mathbf{L}$ is a self-adjoint densely defined linear operator on a Hilbert space H with $\mathbf{L} \geq \lambda_1\mathbf{I}$. Let $\mathbf{U}(t)\mathbf{x} = \mathbf{y}(t,\mathbf{x},\mathbf{0})$, $\mathbf{V} = \mathbf{U}'$. By analyzing a related scalar equation with parameter, we find sufficient conditions on the kernel a in order that $\int_0^\infty \|\mathbf{V}(t)\mathbf{L}^{-\gamma}\|dt < \infty$ $(\gamma > 0)$. These results and certain resolvent formulas can be used to study the asymptotic behavior of the solution $\mathbf{y}(t,\mathbf{x},\mathbf{f})$ as t ~~$\to \infty$.~~ approaches infinity. An application to a semilinear integro-partial differential equation is presented.

AMS (MOS) Subject Classifications: 45J05, 45M05, 45M10

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Work Unit Number 1 - Applied Analysis

[1] Department of Mathematics and Computer Science, St. Cloud State University, St. Cloud, Minnesota 56301.

[2] Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061.

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SIGNIFICANCE AND EXPLANATION

The resolvent formula for a nonhomogeneous Volterra integrodifferential equation enables one to study the behavior of solutions of the equation for large values of the time variable in terms of general properties of the forcing terms in the equation. This technique depends on having "good" a priori estimates obtained for the resolvent kernel.

When the solution takes its values in a Hilbert space, the resolvent kernel is a function whose values are operators on that space. It is important to know whether the norm of the resolvent kernel (or of its derivative) is integrable on $(0,\infty)$. For a class of equations which includes linear models for the dynamics of viscoelastic materials, we develop sufficient conditions for the derivative of the resolvent kernel to be integrable. Results of this study →

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

RESOLVENT FORMULAS FOR A VOLTERRA EQUATION IN HILBERT SPACE

Ralph W. Carr[1] and Kenneth B. Hannsgen[2]

1. <u>Introduction</u>. We continue our study, begun in [2], of the nonhomogeneous linear equation

$$(1.1)\qquad \underset{\sim}{y}'(t) + \int_0^t [d + a(t-s)]\underset{\sim}{L}\,\underset{\sim}{y}(s)ds = \underset{\sim}{f}(t) \quad (t \geq 0)$$

$$\underset{\sim}{y}(0) = \underset{\sim}{y}_0 \in \mathcal{H} \quad , \quad ' = d/dt \quad ,$$

where $\underset{\sim}{L}$ is a positive self-adjoint linear operator defined on a dense subspace $\mathcal{D}$ of the Hilbert space $\mathcal{H}$. The kernel $d + a(t)$ satisfies

$$(1.2)\qquad a \in L^1_{\ell oc}(\mathbb{R}^+, \overline{\mathbb{R}}^+) \quad (\mathbb{R}^+ = (0,\infty), \ \overline{\mathbb{R}}^+ = [0,\infty));\ a \ \underline{\text{is nonincreasing and convex}}$$
$$\underline{\text{with}} \ a(\infty) = 0 < a(0+) \leq \infty, \ \underline{\text{and}} \ d \geq 0 \quad ,$$

and $\underset{\sim}{f}$ belongs to $B^1_{\ell oc}(\overline{\mathbb{R}}^+, \mathcal{H})$, the class of locally Bochner integrable functions from $\overline{R}^+$ to $\mathcal{H}$.

Let $u(t,\lambda)$ denote the solution of the real equation

$$(1.3)\qquad u'(t) + \lambda \int_0^t [d + a(t-s)]u(s)ds = 0, \ u(0) = 1 \quad ;$$

define $v = \partial u/\partial t$,

$$\underset{\sim}{U}(t) = \int_{\mathbb{R}} u(t,\lambda)d\underset{\sim}{E}_\lambda \quad , \quad \underset{\sim}{V}(t) = \int_{\mathbb{R}} v(t,\lambda)d\underset{\sim}{E}_\lambda \quad ,$$

[1] Department of Mathematics and Computer Science, St. Cloud State University, St. Cloud, Minnesota 56301.

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where $\{E_\lambda\}$ is the spectral family corresponding to L. In [2] we established the resolvent formula

$$(1.4) \qquad y(t) = U(t)y_0 + \int_0^t U(t-s)f(s)ds$$

for the solution of (1.1), and we gave sufficient conditions for

$$(1.5) \qquad \int_0^\infty \|U(t)\|dt < \infty \quad .$$

In particular, (1.5) holds if $-a'$ is convex. (See Theorem A after Theorem 2.4 below; here and below we use the norm symbol for a space to indicate the operator norm for linear operators from that space to itself.)

We are principally concerned here with V, the formal derivative of U. V can be used with (1.4) to express $y'(t)$, and it appears in the alternate resolvent formula

$$(1.6) \qquad y(t) = F(t) + \int_0^t V(t-s)F(s)ds$$

for the integrated version of (1.1), that is

$$(1.7) \qquad y(t) + \int_0^t [(t-s)d + A(t-s)]L\, y(s)ds = F(t) \quad ,$$

where $A(t) = \int_0^t a(s)ds$, $F(t) = \int_0^t f(s)ds$.

Estimate (1.5), with V in place of U, is always false (see Corollary 2.1 below). Our main results, Theorems 2.3 and 2.4, contain the following:

THEOREM 1.1. Let (1.2) hold, and assume that $-a'$ is convex. Then

$$(1.8) \qquad t\|V(t)L^{-1/2}\| \text{ is bounded on } \mathbb{R}^+, \text{ and } \int_0^\infty \|V(t)L^{-1/2}\|dt < \infty \quad .$$

The conditions of Theorem A for (1.5) do imply

$$\int_0^\infty \|V(t)L^{-1}\|dt < \infty \quad . \tag{1.9}$$

Estimates (1.8) and (1.9) can be used with (1.4) and (1.6) to study the asymptotic behavior of $y(t)$ under various assumptions on the forcing term.

A variant of (1.1) is

$$z'(t) + \int_0^t [d + a(t-s)]\ [L\ z(s) + g(s)]ds = f(t) \quad (t \geq 0) \tag{1.10}$$

$$z(0) = z_0 \quad ,$$

with $g : \mathbb{R}^+ \to H$. Proceeding formally from (1.4) and the formal identity

$$V(t) = -\int_0^t [d + a(t-s)]L\ U(s)ds \quad ,$$

we obtain

$$z(t) = U(t)z_0 + \int_0^t U(t-s)f(s)ds + \int_0^t V(t-s)L^{-1}g(s)ds \quad . \tag{1.11}$$

In Section 3 we state a theorem justifying (1.11), and we use it to study the semilinear equation

$$y'(t) + \int_0^t [d + a(t-s)]\ [L\ y(s) + N\ y(s)]ds = f(t) \tag{1.12}$$

$$y(0) = y_0 \quad .$$

Here N is a nonlinear operator with

$$N(0) = 0 \quad , \tag{1.13}$$

$$\sup_{\|x_1\|_{\mathcal{D}_1}\ \|x_2\|_{\mathcal{D}} \leq \Delta} \|N\ x_1 - N\ x_2\|_{\mathcal{D}_1} \leq \epsilon(\Delta)\|x_1 - x_2\|_{\mathcal{D}} \quad ,$$

where $\epsilon : (0,\alpha) \to \mathbb{R}^+$ and $\epsilon \to 0$ as $\Delta \to 0$,

$$\|\underset{\sim}{x}\|_{\mathcal{D}}^2 = \|\underset{\sim}{x}\|^2 + \|\underset{\sim}{L}\,\underset{\sim}{x}\|^2, \quad \|\underset{\sim}{x}\|_{\mathcal{D}_1}^2 = \|\underset{\sim}{x}\|^2 + \|\underset{\sim}{L}^{1/2}\underset{\sim}{x}\|^2 \quad .$$

We also give an example of an integro-partial differential equation of the form (1.12), to which our result applies.

The spectrum of $\underset{\sim}{L}$ is contained in a closed subinterval of $\mathbb{R}^+$; without loss of generality we take this interval to be $[1,\infty)$. Then for $0 \leq \gamma < \infty$,

$$\|\underset{\sim}{V}(t)\underset{\sim}{L}^{-\gamma}\| \leq \sup_{1\leq\lambda<\infty} |v(t,\lambda)|\lambda^{-\gamma} \equiv v_\gamma(t) \quad , \tag{1.14}$$

$$\|\underset{\sim}{V}(t)\underset{\sim}{L}^{-\gamma}\|_{\mathcal{D}} \leq v_\gamma(t) \quad .$$

We shall develop estimates for v_γ from (1.3) and deduce estimates such as (1.9) from (1.14).

In Section 2, we state our results from v_γ; they are proved in Sections 4 through 8. In particular, Section 8 contains a correction for the proof of [2, Lemma 5.2]. We discuss the operator $\underset{\sim}{V}$ and equations (1.11) and (1.12) in Section 3; proofs follow in Section 9.

2. <u>Statement of results for</u> v_γ. Throughout this paper, it is assumed that $d + a(t)$ satisfies (1.2). We define

$$A(t) = \int_0^t a(s)ds, \quad A_1(t) = \int_0^t s\, a(s)ds \quad ,$$

$$\hat{a}(\tau) = \int_0^\infty a(t)e^{-i\tau t}dt \equiv \varphi(\tau) - i\tau\theta(\tau) \quad (\tau > 0)$$

(with φ and θ real; note that $\hat{a}$ is continuous),

$$D(\tau) = D(\tau,\infty) = \hat{a}(\tau) - id\tau^{-1}, \; D(\tau,\lambda) = D(\tau) + i\tau\lambda^{-1} \quad .$$

Formally, the Fourier transform of $v(t,\lambda)$ (defined to be zero for $t < 0$) is given by

$$\hat{v}(\tau,\lambda) = \frac{-D(\tau)}{D(\tau,\lambda)} \quad , \tag{2.1}$$

so $v(\cdot,\lambda) \notin L^1(\mathbb{R}^+)$ if $D(\tau,\lambda) = 0$ for some τ. By [4], $\varphi(\tau) \geq 0$; moreover, $\varphi(\tau) > 0$ $(\tau > 0)$ unless $a(t)$ is piecewise linear with changes of slope only at integral multiples of a fixed number t_0 (taken as large as possible) and τ is an integral multiple of $2\pi/t_0$. In all other cases, $D(\tau,\lambda) \neq 0$ $(\tau > 0)$; then the hypotheses of [15, Theorem 2] hold, and $v(\cdot,\lambda) \in L^1(\mathbb{R}^+)$ and (2.1) holds. Throughout this paper, we restrict ourselves to this case by assuming

$$\varphi(\tau) > 0 \quad (\tau > 0) \quad . \tag{2.2}$$

Estimates for v_γ depend crucially on the size of $\hat{v}(\tau,\lambda)$ when $\operatorname{Im} D(\tau,\lambda) = \tau[\lambda^{-1}-\theta(\tau) - d\tau^{-2}]$ is zero. Choose and fix $t_1 > 0$ with $a(t_1) > 0$, and let $\rho = 6/t_1$. We showed in [2] that $\theta \downarrow 0$ $(\tau \uparrow \infty)$ and that the equation

$$\lambda^{-1} - \theta(\omega) - d\omega^{-2} = 0 \tag{2.3}$$

defines a continuous, strictly increasing function $\omega(\lambda)$ on the interval $[\lambda_0,\infty)$, where

$$\lambda_0 = \max\{1, \; [\theta(\rho) + d\rho^{-2}]^{-1}\} \quad .$$

We extend ω to $[1,\infty)$ if necessary by defining $\omega(\lambda) = \rho$ $(1 \le \lambda \le \lambda_0)$. We showed in [2, Eqs. (4.3), (4.24), (4.27)] that

$$(2.4) \qquad \frac{1}{5} A_1(\tau^{-1}) \le \theta(\tau) \le 12A_1(\tau^{-1}) \qquad (\tau > 0) \quad ,$$

$$(2.5) \qquad 10\omega^2 \ge a(t_1)\lambda \qquad (\lambda \ge 1) \quad ,$$

$$(2.6) \qquad \frac{1}{5} A_1(\omega^{-1}) \le \lambda^{-1} \le C_1A_1(\omega^{-1}) \qquad (\lambda \ge 1) \quad ,$$

where $C_1 = \lambda_0[12 + (2d/a(t_1))] \ge 12$. (We shall often suppress λ as in (2.5) and (2.6).)

For $\lambda \ge \lambda_0$ we then have

$$\lambda^{-\gamma} \int_0^\infty |v(t,\lambda)|dt \ge \left(\frac{\theta(\omega)}{60}\right)^\gamma |\hat{v}(\omega,\lambda)| \ge \left(\frac{\theta(\omega)}{60}\right)^\gamma \frac{\omega\theta(\omega)}{\varphi(\omega)} \quad .$$

This proves our first result.

THEOREM 2.1. <u>Let</u> (1.2) <u>and</u> (2.2) <u>hold, and let</u> $\gamma > 0$. <u>If</u> $v_\gamma \in L^1(\mathbb{R}^+)$, <u>then</u>

$$(2.7) \qquad \sup_{\rho \le \tau < \infty} \frac{\tau[\theta(\tau)]^{1+\gamma}}{\varphi(\tau)} < \infty \quad .$$

Suppose, in particular, that $a(0+) < \infty$. From (2.6) we see that

$$\frac{1}{10} a(t_1)\lambda \le \omega^2 \le \frac{1}{2} C_1 a(0+)\lambda \quad .$$

In this case, for $\gamma = \frac{1}{2}$, (2.7) is equivalent to

$$(2.8) \qquad \sup_{0 < \tau < \infty} \frac{1 + \tau^2}{\varphi(\tau)} < \infty \quad ;$$

that is, a is strongly positive.

To find upper bounds for v_γ, we first define $\sigma = \sigma(\lambda)$ to be the unique solution of

$$(2.9) \qquad \sigma^{-1}A(\sigma^{-1}) = \lambda^{-1} \quad .$$

Then $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing, since $\alpha(t) \equiv t\,A(t)$ is strictly increasing. Using (2.6), we see that for $\lambda \geq 1$,

$$\alpha(\frac{C_1}{\omega}) \geq \frac{C_1}{\omega} A(\frac{1}{\omega}) \geq C_1 A_1(\frac{1}{\omega}) \geq \frac{1}{\lambda} = \alpha(\frac{1}{\sigma}) \quad .$$

Therefore, since (2.5) holds,

$$\omega \leq C_1\sigma \quad \text{and} \quad \lambda \leq C_2\sigma^2 \qquad (\lambda \geq 1) \quad , \tag{2.10}$$

with $C_2 = 10C_1^2/a(t_1)$. σ can grow faster than ω; for example, if $a(t) = t^{-1}(-\log t)^{-3/2}$ for small t, one shows from (2.6) and (2.9) that

$$K_1\omega \log \omega \leq K_2\lambda(\log \lambda)^{-1/2} \leq \sigma \leq K_3\lambda(\log \lambda)^{-1/2} \leq K_4\omega \log \omega \quad ,$$

where the K_j are positive constants. Note, however, that

$$\lim_{\lambda\to\infty} \frac{\sigma}{\lambda} = \lim_{\lambda\to\infty} A(\frac{1}{\sigma}) = 0 \quad . \tag{2.11}$$

The next result relates σ to v.

THEOREM 2.2. _If_ (1.2) _holds, then_

$$\frac{\sigma}{8(8 + dC_2)} \leq \sup_{t\geq 0} |v(t,\lambda)| \leq (8 + dC_2)\sigma \qquad (\lambda \geq 1) \quad . \tag{2.12}$$

The proof of Theorem 2.2 contains the following:

COROLLARY 2.1. _Let_ (1.2) _hold. There exist_ $\varepsilon, K > 0$ _such that_ $v_0(t) \geq K/t$ $(0 < t < \varepsilon)$; _in particular,_ $\int_0^1 v_0(t)dt = \infty$.

By (2.6) and (2.10), (2.12) shows that $\lambda^{-1/2}v(t,\lambda)$ is not bounded if $a(0+) = \infty$. If $a(0+) < \infty$, (2.9) shows that $\sigma^2 \leq a(0+)\lambda$, so $\lambda^{-1/2}v(t,\lambda)$ is bounded. The latter conclusion strengthens [6, Lemma 5.2]; it thus improves Theorems 1 and 2 of that paper by showing that one may omit the term $\log(\lambda/\Lambda)$ from the definition of u^1 without changing the conclusions. Our main results, Theorems 2.3 and 2.4, generalize this part of [6] to cases where $a(0+) = \infty$. As in [2], we shall need the technical hypothesis

(2.13) $a(t) = b(t) + c(t)$, *where* b *and* c *each satisfy* (1.2), *except that either* $b(0+) = 0$ *or* $c(0+) = 0$ *is permitted. Moreover,* $\int_1^\infty t^{-1} b(t)dt < \infty$ *and* $-c'$ *is convex.*

THEOREM 2.3. *Suppose* (1.2) *and* (2.2) *hold, and let* $0 \le \gamma < \infty$. (i) *If*

$$\sup_{\frac{1}{2}\rho \le \tau < \infty} \frac{\tau[\theta(\tau)]^{1+\gamma}}{\varphi(\tau)} < \infty \quad , \tag{2.14}$$

then $\sup_{t>0} t\, v_\gamma(t) < \infty$. (ii) *If* (2.13) *holds and either*

$$\sup_{\frac{1}{2}\rho \le \tau < \infty} \frac{\tau[\theta(\tau)]^{1+\gamma-\varepsilon}}{\varphi(\tau)} < \infty \quad \textit{for some} \quad \varepsilon,\ 0 < \varepsilon < \gamma \quad , \tag{2.15}$$

or $\gamma \ge 1$ *and*

$$\sup_{\frac{1}{2}\rho \le \tau < \infty} \frac{\tau[\theta(\tau)]^{2+\gamma}}{\varphi^2(\tau)} < \infty \quad , \tag{2.16}$$

then

$$\int_0^\infty v_\gamma(t)dt < \infty \quad . \tag{2.17}$$

When $\gamma = \frac{1}{2}$, the following criterion is sometimes weaker than (2.15).

THEOREM 2.4. *If* (1.2), (2.2), *and* (2.13) *hold, and if*

$$\sup_{\frac{1}{2}\rho \le \tau < \infty} \frac{\tau^2\theta^2(\tau)}{\varphi(\tau)} < \infty \quad , \tag{2.18}$$

then $\int_0^\infty v_{1/2}(t)dt < \infty$.

For purposes of comparison, we restate our conditions for (1.5) from [2].

THEOREM A. *Suppose* (1.2), (2.2), *and* (2.13) *hold. Then*

$$\int_0^\infty \sup_{1 \le \lambda < \infty} |u(t,\lambda)| dt < \infty \tag{2.19}$$

if and only if

$$\sup_{\frac{1}{2}\rho \le \tau < \infty} \frac{\theta(\tau)}{\varphi(\tau)} < \infty \quad . \tag{2.20}$$

The hypotheses in these results satisfy the following implications:

$$(2.18) \Rightarrow (2.14)\ (\gamma = \tfrac{1}{2}) \Rightarrow (2.20) \Rightarrow (2.16)\ (\gamma \ge 1) \tag{2.21}$$

(see (2.4)). If $a(0+) < \infty$, (2.4) gives us

$$\frac{1}{10} a(t_1)\tau^{-2} \le \theta(\tau) \le 6a(0+)\tau^{-2} \quad ,$$

so (2.18), (2.14) $(\gamma = \frac{1}{2})$, and (2.20) all are equivalent to strong positivity. Thus while (2.15) obviously implies (2.14), the kernel $a(t) = e^{-t}$ provides an example where (2.18) holds but (2.15) $(\gamma = \frac{1}{2})$ is false.

If $0 < \beta < 1$, the example $a(t) = t^{-\beta}$ satisfies (2.14) $(\gamma = 0)$ and hence satisfies (2.15) for all positive γ.

By considering a certain family of piecewise linear kernels, we can demonstrate other differences among our hypotheses. We defer the proof to Section 7.

THEOREM 2.5. *There are kernels* a_1, $a_{2,\gamma}(\gamma = \frac{1}{2}, 1, \frac{3}{2}, \ldots)$, a_3, *and* a_4 *satisfying* (1.2), (2.2), *and* (2.13) *and such that*

(i) a_1 *satisfies* (2.15) $(\gamma = \frac{1}{2})$ *but not* (2.18).

(ii) *For each fixed* γ, $a_{2,\gamma}$ *satisfies* (2.14), *but neither* (2.15) *nor* (2.18) *nor* (2.16) *when* $\gamma \geq 1$ *holds.*

(iii) a_3 *satisfies* (2.20) *but not* (2.14) $(\gamma = \frac{1}{2})$.

(iv) a_4 *satisfies* (2.16) $(\gamma = 1)$ *but not* (2.20).

By (2.21), Theorem 2.3, and (1.14), the sufficient condition (2.20) of Theorem A implies (1.9), as asserted in Section 1. The following corollary shows that Theorems 2.3 and 2.4 contain Theorem 1.1.

COROLLARY 2.2. *If* (1.2), (2.2), *and* (2.13) *hold, and if*

$$\limsup_{t \to 0+} \frac{\int_0^t b(s)ds}{\int_0^t c(s)ds} < \infty \quad , \tag{2.22}$$

then (2.18) *holds, so* [*by* (2.21) *and Theorems* A *and* 2.3] $\sup_{t \geq 0} tv_{1/2}(t) < \infty$, *and* (2.17) $(\gamma = \frac{1}{2})$ *and* (2.19) *are valid.*

3. Statement of results for equations in H. A solution of (1.1) (or (1.10) or (1.12)) is a continuously differentiable function $\underset{\sim}{y} : \overline{\mathbb{R}}^+ \to H$ such that $\underset{\sim}{L}\,\underset{\sim}{y} : \overline{\mathbb{R}}^+ \to H$ is defined and continuous (in brief, $\underset{\sim}{y} \in C(\overline{\mathbb{R}}^+, \mathcal{D})$) and (1.1) (or (1.10) or (1.12)) holds. Unless otherwise specified, integrals $\int_a^b$ of H-valued functions are Bochner integrals in $B^1((a,b), H)$; Hille and Phillips [7, pp. 59-89] give the theory of this integral. We recall from [2, Theorem 2.1(i)] that if (1.2) holds, then $\underset{\sim}{U}(t)$ is strongly continuous on H and $\|\underset{\sim}{U}(t)\| \le 1$ $(t \in \overline{\mathbb{R}}^+)$.

Our first result concerns $\underset{\sim}{V}(t)$ as an operator from $\mathcal{D}_1$ to H. The results of Section 2 can also be used to study $\underset{\sim}{V}(t)\underset{\sim}{L}^{-\gamma}$ $(\gamma \ne \frac{1}{2})$.

THEOREM 3.1. (i) Suppose (1.2) and (2.2) hold and

$$\sup_{\frac{1}{2}\rho \le \tau < \infty} \frac{\tau[\theta(\tau)]^{3/2}}{\varphi(\tau)} < \infty \quad . \tag{3.1}$$

Then for $t > 0$, $\underset{\sim}{V}(t)\underset{\sim}{L}^{-1/2}$ is a bounded operator on H, strongly continuous on $\mathbb{R}^+$. Moreover,

$$\underset{\sim}{V}(t)\underset{\sim}{y} = \frac{d}{dt}\underset{\sim}{U}(t)\underset{\sim}{y} \qquad (t > 0,\ \underset{\sim}{y} \in \mathcal{D}_1). \tag{3.2}$$

(ii) If $a(0+) < \infty$, we may omit (2.2) and (3.1) in (i); moreover, $\underset{\sim}{V}(t)\underset{\sim}{L}^{-1/2}$ is strongly continuous and uniformly bounded on $\overline{\mathbb{R}}^+$.

Next we state a representation theorem for solutions of (1.10).

THEOREM 3.2. (i) Let the hypotheses of Theorem 2.3(ii) $(\gamma = \frac{1}{2})$ or of Theorem 2.4 hold. Let $\underset{\sim}{z}_0 \in \mathcal{D}$, let $\underset{\sim}{f} \in C(\overline{\mathbb{R}}^+, H)$ with $\underset{\sim}{f}(t) \in \mathcal{D}$ $(t \ge 0)$ and $\underset{\sim}{L}\,\underset{\sim}{f} \in B^1_{\ell oc}(\overline{\mathbb{R}}^+, H)$. Assume that $\underset{\sim}{g} \in B^\infty_{\ell oc}(\overline{\mathbb{R}}^+, \mathcal{D}_1)$. Then the function $\underset{\sim}{z}(t)$ given by (1.11) is the unique solution of (1.10).

(ii) Let (1.2) hold with $a(0+) < \infty$. Let $\underset{\sim}{z}_0$ and $\underset{\sim}{f}$ satisfy the hypotheses of (i), and let $\underset{\sim}{g} \in B^1_{\ell oc}(\overline{\mathbb{R}}^+, \mathcal{D}_1)$. Then the conclusion of (i) is valid

__Remark__. In (i) above, we need $\|V(\cdot)L^{-1/2}\| \in L^1_{\ell oc}(\overline{\mathbb{R}}^+)$, and by (1.14), the conclusions of Theorems 2.3(ii) ($\gamma = \frac{1}{2}$) and 2.4 imply this.

Miller [13] shows how to combine the resolvent formula for Volterra equations with fixed point theorems in order to prove global existence theorems for nonlinear equations. We use this method and Theorem 3.2 to obtain a result for (1.12).

THEOREM 3.3. __Let the hypotheses of Theorem__ 2.3(ii) ($\gamma = \frac{1}{2}$) __or of Theorem__ 2.4 __hold, and let__ $y_0 \in \mathcal{D}$. __Let__ f __satisfy the hypotheses of Theorem__ 3.2(i) __with__ $f = f_1 + f_2$, $f_1 \in \mathcal{B}^1(\mathbb{R}^+, \mathcal{D})$, $f_2 \in \mathcal{B}^\infty(\mathbb{R}^+, \mathcal{D})$. __Let__

$$N : \{x \in \mathcal{D} \mid \|x\|_{\mathcal{D}} < \alpha\} \to \mathcal{D}_1$$

__satisfy conditions__ (1.13). __Then if__ $\mu \equiv \|y_0\|_{\mathcal{D}} + \|f_1\|_{\mathcal{B}^1(\mathbb{R}^+,\mathcal{D})} + \|f_2\|_{\mathcal{B}^\infty(\mathbb{R}^+,\mathcal{D})}$ __and__ $\Delta > 0$ __are sufficiently small,__ (1.12) __has one and only one solution__ y __such that__ $\|y(t)\|_{\mathcal{D}} \leq \Delta$ $(t \in \overline{\mathbb{R}}^+)$.

A simple example illustrating Theorem 3.3 is the problem

$$u_t(t,x) = \int_0^t a(t-s)[u_{xx}(s,x) + u(s,x)u_x(s,x)]ds + F(t,x) \tag{3.3}$$

$$u(t,0) = u(t,\pi) = 0 \ (t \geq 0), \ u(0,x) = u_0(x) \quad .$$

We take $\mathcal{H} = L^2(0,\pi)$, $L\,y = -y''$ on $\mathcal{D}$, the space of differentiable functions y on $[0,\pi]$ with $y(0) = y(\pi) = 0$, y' absolutely continuous, and $y'' \in \mathcal{H}$. $\mathcal{D}_1$ consists of absolutely continuous functions which vanish at 0 and π and have square integrable first derivatives.

In terms of Fourier sine series

$$y(x) = \sum_{n=1}^{\infty} c_n \sin nx \quad ,$$

$\mathcal{D}$ and $\mathcal{D}_1$ are characterized respectively by the conditions $\sum n^4 c_n^2 < \infty$ and $\sum n^2 c_n^2 < \infty$, and

$$\underset{\sim}{L}^{1/2} \underset{\sim}{y}(x) = \sum_{n=1}^{\infty} n c_n \sin nx \quad .$$

Thus $\|\underset{\sim}{L}^{1/2} \underset{\sim}{y}\| = \|\underset{\sim}{y}'\| \quad (\underset{\sim}{y} \in \mathcal{D}_1)$. Note also that if $\underset{\sim}{y} \in \mathcal{D}$,

$$|\underset{\sim}{y}'(x)|^2 \leq \left(\sum_{n=1}^{\infty} n|c_n|\right)^2 \leq \sum_{n=1}^{\infty} n^{-2} \sum_{n=1}^{\infty} n^4 c_n^2 = B^2 \|\underset{\sim}{L}\, \underset{\sim}{y}\|^2$$

$(0 \leq x \leq \pi)$, so also $|y(x)| \leq \frac{1}{2} B\pi \|\underset{\sim}{L}\, \underset{\sim}{y}\| \; (0 \leq x \leq \pi)$. Using these facts, one easily shows that $\underset{\sim}{N}\, \underset{\sim}{y} = y\underset{\sim}{y}'$ satisfies (1.13).

The nonlinearity uu_x in (3.3) could be generalized, but our theorem does not cover such nonlinearities as u_x^2 or $\underset{\sim}{N}_1 \underset{\sim}{u} = [h(u_x)]_x$; $\underset{\sim}{N}_1$ is important in viscoelasticity theory.

MacCamy [11, 12], Dafermos and Nohel [3], and Staffans [17] have established global existence results for (3.3) with $\underset{\sim}{N}$ replaced by $\underset{\sim}{N}_1$ and $a(0+) < \infty$. Londen's global existence results [10] deal with (1.1) with $\underset{\sim}{L}$ replaced by a maximal monotone (nonlinear) operator and $a(0+) < \infty$, $a'(0+) = -\infty$. Travis and Webb [18] prove a general local existence result for hyperbolic semilinear equations, including (1.12) when $a(0+) < \infty$.

4. <u>Proofs of Theorem 2.2 and Corollary 2.1</u>. We redefine a', b', c' where necessary to make them continuous from the left on $\mathbb{R}^+$. da' denotes the Lebesgue-Stieltjes measure on $\mathbb{R}^+$. We adopt the conventions

$$\int_0^y f(t)\,da'(t) \equiv \int_{(0,y)} F\,da', \quad \int_x^y f(t)\,da'(t) \equiv \int_{[x,y)} f\,da'$$

$(0 < x < y)$. For this proof we define $\delta = \sigma^{-1}$.

Recall that when (1.2) holds,

$$|u(t,\lambda)| \le 1 \qquad (t \ge 0,\ \lambda > 0) \tag{4.1}$$

(see [5], [2, p. 965]). Then (1.3), (4.1), and (2.10) imply

$$|v(t,\lambda)| \le \lambda(td + A(t)) \le \sigma + \lambda d\delta \le \sigma(1 + dC_2) \quad (0 \le t \le \delta) \quad . \tag{4.2}$$

For $\delta \le t < \infty$, we make the change of variable $s \to t-s$ in (1.3) and integrate by parts to obtain the identity

$$\begin{aligned} v(t,\lambda) &= \lambda \int_0^\delta a'(s) \int_{t-s}^t u(r,\lambda)\,dr\,ds \\ &\quad + \lambda \int_\delta^t a'(s) \int_{t-s}^t u(r,\lambda)\,dr\,ds - \lambda(d + a(t)) \int_0^t u(s,\lambda)\,ds \\ &\equiv v_1(t,\lambda) + v_2(t,\lambda) + v_3(t,\lambda) \quad . \end{aligned}$$

Clearly,

$$|v_1(t,\lambda)| \le -\lambda \int_0^\delta s\,a'(s)\,ds \le \lambda A(\delta) = \sigma \quad . \tag{4.3}$$

Since a' is monotone, we can use Fubini's Theorem to see (with λ suppressed) that

$$\int_\delta^t \int_\delta^s a'(r)[u(s) - u(s-r)]drds$$

$$= \int_\delta^t a'(r) \int_r^t [u(s) - u(s-r)]dsdr$$

$$= \int_\delta^t a'(s)[\int_s^t - \int_0^{t-s}]u(r)drds$$

$$= \lambda^{-1}v_2(t,\lambda) - \int_\delta^t a'(s) \int_0^s u(r)drds \, (t \geq \delta) \quad .$$

Therefore $v_2(t,\lambda)$ is locally absolutely continuous in t, and

$$\frac{1}{\lambda}\frac{\partial v_2}{\partial t} = a'(t) \int_0^t u(s,\lambda)ds + \int_\delta^t a'(s)[u(t,\lambda) - u(t-s,\lambda)]ds$$

a.e. $(t \geq \delta)$. Integration by parts then yields

$$\frac{1}{\lambda}\frac{\partial v_2}{\partial t} = u(t,\lambda)[a(t) - a(\delta)] + a'(\delta) \int_{t-\delta}^t u(r,\lambda)dr$$

$$+ \int_\delta^t [\int_{t-s}^t u(r,\lambda)dr]da'(s) \quad \text{a.e.} \quad (t \geq \delta) \quad ,$$

so

$$\text{(4.4)} \qquad \frac{1}{\lambda}\left|\frac{\partial v_2}{\partial t}\right| \leq 2a(\delta) - 2\delta a'(\delta) - 2a(t) + ta'(t) \quad \text{a.e.}$$

Since

$$\frac{1}{\lambda}\frac{\partial v_3}{\partial t} = - a'(t) \int_0^t u(s,\lambda)ds - (d + a(t))u(t,\lambda) \quad \text{a.e.} \quad ,$$

$$\frac{1}{\lambda}\left|\frac{\partial v_3}{\partial t}\right| \leq -ta'(t) + d + a(t) \quad \text{a.e.} \quad .$$

Adding this to (4.4) yields

$$\text{(4.5)} \qquad \frac{1}{\lambda}\left|\frac{\partial(v_2 + v_3)}{\partial t}\right| \leq 2a(\delta) - 2\delta a'(\delta) + d \quad \text{a.e.} \quad .$$

Suppose there exists $t^* > \delta$ such that

$$(4.6) \qquad |v(t^*,\lambda)| > (8 + dC_2)\sigma \quad .$$

Let $I = [t^*-\delta,\ t^*+\delta]$, and observe that if $s \in I$,

$$(4.7) \quad |v(s,\lambda)| \geq |v(t^*,\lambda)| - 2 \sup_{r\in I}|v_1(r,\lambda)| - \delta \operatorname{ess\,sup}_{r\in I}\left|\frac{\partial(v_2 + v_3)}{\partial t}(r,\lambda)\right|$$

$$> (8 + dC_2)\sigma - 2\sigma - 2\lambda(\delta a(\delta) - \delta^2 a'(\delta)) - \lambda d\delta \quad ;$$

here (4.3), (4.4) and the absolute continuity of $v_2 + v_3$ have been used. Integration by parts shows that

$$0 \leq \int_0^\delta t^2 da'(t) = 2A(\delta) - 2\delta a(\delta) + \delta^2 a'(\delta) \quad .$$

Combining this with (4.7), we obtain

$$|v(s,\lambda)| > (6 + dC_2)\sigma - 4\lambda A(\delta) - \lambda d\delta \qquad (s \in I) \quad .$$

But $\delta^{-1} = \sigma = \lambda A(\delta)$, and since (2.10) holds,

$$|v(s,\lambda)| > 2\delta^{-1} \qquad (s \in I) \quad .$$

Thus by (4.1) and the Mean Value Theorem, (4.6) has led us to the contradiction

$$2 \geq |u(t^*,\lambda) - u(t^*-\delta,\lambda)| > \delta\cdot 2\delta^{-1} = 2 \quad .$$

Since (4.2) holds, the second inequality in (2.12) is established. It follows that

$$u(t,\lambda) \geq 1 - (8 + dC_2)\sigma t \qquad (t \geq 0) \quad ,$$

so $u(t,\lambda) \geq \frac{1}{2}$ for $0 \leq t \leq [2\sigma(8 + dC_2)]^{-1} \equiv 2T$. Then by (1.2) and (1.3),

$$(4.8) \quad |v(t,\lambda)| \geq \frac{1}{2}\lambda A(T) \geq \frac{\lambda A(\sigma^{-1})}{8(8 + dC_2)} = \frac{\sigma}{8(8 + dC_2)} \qquad (T \leq t \leq 2T) \quad .$$

This proves Theorem 2.2.

If $a(0+) < \infty$, the second inequality in (2.12) is essentially contained in Levin [8]. The idea of writing $v = v_1 + v_2 + v_3$ in case $a(0+) = \infty$ was introduced by Londen [9, Lemma 2].

For Corollary 2.1, let $T = T(\lambda)$ as in (4.8). If $t > 0$ is sufficiently small, we can find $\lambda = \lambda_t$ such that $T(\lambda) \leq t \leq 2T(\lambda)$. Then by (4.8),

$$v_0(t) \geq v(t,\lambda) \geq \frac{\sigma}{8(8 + dC_2)} \geq \frac{1}{16t(8 + dC_2)^2} ,$$

as asserted.

5. Proof of Theorem 2.3. Throughout this paper, the symbol M denotes a finite positive constant, independent of $\lambda (1 \le \lambda < \infty)$; the numerical value of M can change each time M appears. We assume (1.2) and (2.2).

(2.11) and (2.12) immediately yield

$$v_\gamma(t) \le M \qquad (\gamma \ge 1,\ t \ge 0) \quad . \tag{5.1}$$

Choose $\omega^* = \omega^*(\lambda)$ so that

$$\tfrac{1}{2}\omega \le \tau \le 2\omega \quad \text{and} \quad \varphi(\omega^*) = \min_{\frac{1}{2}\omega \le \tau \le 2\omega} \varphi(\tau) \quad :$$

for instance, ω^* could be the smallest such number.

We shall establish the following estimates.

$$|v(t,\lambda)| \le M\left(1 + \frac{\omega^*\theta(\omega^*)}{\varphi(\omega^*)}\right)t^{-1} \qquad (t > 0) \quad . \tag{5.2}$$

If (2.13) holds,

$$|v(t,\lambda)| \le M\left[\left(1 + \frac{\omega^*\theta(\omega^*)}{\varphi(\omega^*)}\right)Q(t) + \left(1 + \frac{\omega^{*2}\theta^2(\omega^*)}{\varphi^2(\omega^*)}\right)t^{-2}\right] \qquad (t \ge 1) \quad , \tag{5.3}$$

where $Q \in L^1(1,\infty)$.

Before proving (5.2) and (5.3), we show that they imply the conclusions of Theorem 2.3. Note that

$$\int_t^{2t} sa(s)ds \le a(t) \int_t^{2t} s\, ds = 3a(t) \int_0^t s\, ds \le 3A_1(t) \quad .$$

Therefore,

$$A_1(2t) \le 4A_1(t) \qquad (t > 0) \quad . \tag{5.4}$$

Using (5.4), we can combine (2.4) and (2.6) to see that

$$(5.5)\qquad \frac{1}{M} \le \lambda\theta(\tau) \le M \qquad (\tfrac{1}{2}\omega \le \tau \le 2\omega) \;.$$

Then if (2.14) holds, (5.2) gives us the conclusion of Theorem 2.3(i).

If $\gamma \ge 1$ and (2.13) and (2.16) hold, we use the algebraic inequality

$$(5.6)\qquad 2\theta/\varphi \le 1 + (\theta/\varphi)^2$$

to deduce from (5.3) that

$$|v(t,\lambda)| \le M[Q(t)\;(1+\omega^*) + \frac{\omega^*\theta^2(\omega^*)}{\varphi^2(\omega^*)}\,t^{-2}] \qquad (t \ge 1) \;.$$

Then by (5.5), (2.10), (2.11), and (2.16),

$$\lambda^{-\gamma}|v(t,\lambda)| \le M(Q(t) + t^{-2}) \qquad (t \ge 1) \;.$$

Since (5.1) holds, (2.17) is valid.

Now assume (2.13) and (2.15). If $0 < \gamma < 1$, we deduce from (2.15), (5.2), and (5.5) that

$$(5.7)\qquad |v(t,\lambda)| \le M\, t^{-1}\lambda^{\gamma-\epsilon} \qquad (t > 0) \;.$$

If $p = (1-\gamma)/(1-\gamma+\epsilon)$, then $0 < p < 1$ and $p(\gamma-\epsilon) + (1-p) = \gamma$, so (2.11), (2.12), and (5.7) tell us that

$$(5.8)\qquad |v(t,\lambda)| = |v(t,\lambda)|^{p+(1-p)} \le M\lambda^{\gamma}t^{-p} \qquad (t > 0)$$

if $\gamma < 1$. We conclude from (5.1) and (5.8) that

$$(5.9)\qquad \int_0^1 v_\gamma(t)dt < \infty \text{ if (2.15) holds} \;.$$

Choose $\delta < \epsilon/(\gamma-\epsilon)$, $0 < \delta < 1$. If $1 \le t^\delta \le \lambda^\epsilon$, (2.15), (5.2), and (5.5) imply that

$$(5.10) \qquad |v(t,\lambda)| \le M\left(1 + \frac{\omega^* \theta^{1+\gamma-\epsilon}(\omega^*)}{\varphi(\omega^*)}\right)\lambda^{\gamma-\epsilon}\left(\frac{\lambda^{\epsilon}}{t^{1+\delta}}\right)$$

$$\le M\lambda^{\gamma} t^{-1-\delta} \quad .$$

If $\lambda^{\epsilon} < t^{\delta}$, then $\lambda^{-\epsilon+\epsilon/\delta} < t^{1-\delta}$, so (2.15), (5.3), and (5.5) yield

$$|v(t,\lambda)| \le M\left[\lambda^{\gamma-\epsilon}Q(t) + \left(\frac{\omega^* \theta^{1+\gamma-\epsilon}(\omega^*)}{\varphi(\omega^*)}\right)^2 \frac{\lambda^{2\gamma-2\epsilon} t^{1-\delta}}{t^2 \lambda^{-\epsilon+\epsilon/\delta}}\right] \quad .$$

Since $\gamma - \epsilon < \epsilon/\delta$, another application of (2.15) shows that

$$|v(t,\lambda)| \le M\lambda^{\gamma}[Q(t) + t^{-1-\delta}] \qquad (\lambda^{\epsilon} < t^{\delta}) \quad .$$

This inequality, taken together with (5.9) and (5.10), gives us (2.17).

We have shown that Theorem 2.3 is a consequence of (5.2) and (5.3), which we prove next.

When (1.2) and (2.2) hold, one has the inversion formula

$$(5.11) \qquad \pi v(t,\lambda) = \mathrm{Re}\left\{\frac{1}{t\lambda}\int_0^{\infty} e^{i\tau t}\left(\frac{\tau D'(\tau) - D(\tau)}{D^2(\tau,\lambda)}\right)d\tau\right\} \qquad (t > 0) \quad ,$$

where the integral is absolutely convergent at both $\tau = 0$ and $\tau = \infty$. This was established in [1].

The next lemmas will enable us to estimate D and D'.

LEMMA 5.1. _If_ (1.2) _holds, then_

$$(5.12) \qquad \varphi(\tau) \ge \frac{1}{2}[A(\tau^{-1}) - 3\tau A_1(\tau^{-1})] \qquad (\tau > 0) \quad .$$

Proof. Two integrations by parts yield

$$\varphi(\tau) = \tau^{-2} \int_0^\infty (1 - \cos \tau t)\, da'(t)$$

$$\geq \frac{1}{4} \int_0^{1/\tau} t^2 da'(t)$$

$$\geq \frac{1}{4} \int_0^{1/\tau} (t^2 - \tau t^3) da'(t)$$

$$= \frac{1}{2}[A(\tau^{-1}) - 3\tau A_1(\tau^{-1})] + \frac{1}{4\tau} a(\tau^{-1}) \quad .$$

Here we have used $1 - \cos x \geq \frac{1}{4} x^2$ $(0 \leq x \leq 1)$ and the fact that da' is a positive measure. Since $a \geq 0$, the lemma is proved.

LEMMA 5.2. <u>If</u> (1.2) <u>holds, then</u>

$$\varphi^2(\tau) + \left(\frac{\tau-\omega}{\lambda}\right)^2 \leq M|D(\tau,\lambda)|^2 \qquad (\tau \geq \tfrac{1}{2}\rho) \quad , \tag{5.13}$$

$$A(\tau^{-1}) \leq M|D(\tau,\lambda)| \qquad (\tau \in [\tfrac{1}{2}\rho, \tfrac{1}{2}\omega] \cup [2\omega,\infty)) \quad . \tag{5.14}$$

<u>Proof</u>. [2, Lemma 5.2] states that (1.2) implies

$$|\tau-\omega| \leq M\lambda|D(\tau,\lambda)| \qquad (\tau \geq \tfrac{1}{2}\omega) \quad , \tag{5.15}$$

$$\tau A_1(\tau^{-1}) \leq M|D(\tau,\lambda)| \qquad (\tfrac{1}{2}\rho \leq \tau \leq \tfrac{1}{2}\omega) \quad . \tag{5.16}$$

In Section 8 below, we give a corrected proof of this lemma.

For $\tau \geq \frac{1}{2}\omega$, (5.13) is a trivial consequence of (5.15). (2.6) and (5.15) show that

$$\frac{1}{10}\tau A_1(\tau^{-1}) \leq \frac{1}{10}\tau A_1(\omega^{-1}) \leq \frac{1}{2}\tau/\lambda \leq (\tau-\omega)/\lambda \tag{5.17}$$

$$\leq M|D(\tau,\lambda)| \qquad (\tau \geq 2\omega) \quad .$$

Thus if $\tau \in [\frac{1}{2}\rho, \frac{1}{2}\omega] \cup [2\omega,\infty)$, (5.12), (5.16), and (5.17) imply that

$$A(\tau^{-1}) \le 2\varphi(\tau) + 3\tau A_1(\tau^{-1}) \le M|D(\tau,\lambda)|$$

as asserted in (5.14).

If $\frac{1}{2}\rho \le \tau \le \frac{1}{2}\omega$, by (2.6),

$$\frac{\omega-\tau}{\lambda} \le \frac{\omega}{\lambda} \le C_1\omega A_1(\frac{1}{\omega}) \le C_1 A(\frac{1}{\tau}) \quad .$$

By (5.14), this implies (5.13) for such τ, and our proof is complete.

Recall from [2, Lemma 4.1] that when (1.2) holds we have

$$(5.18) \quad 2^{-3/2}A(\tau^{-1}) \le |\hat{a}(\tau)| \le 4A(\tau^{-1}), \ |\hat{a}'(\tau)| \le 40A_1(\tau^{-1}) \quad (\tau > 0) \quad .$$

We now deduce (5.2) from (5.11). If $d > 0$, (5.18) shows that

$$|\tau D'(\tau) - D(\tau)| \le M\tau^{-1} \quad (0 < \tau \le \rho) \quad ,$$

while (2.2) gives

$$(5.19) \quad |D(\tau,\lambda)| \ge \max\{\varphi(\tau), (d-\tau^2)/\tau\} \ge 1/M\tau \quad (0 < \tau \le \rho) \quad .$$

Thus

$$(5.20) \quad \int_0^{\rho/2} \frac{|\tau D'(\tau) - D(\tau)|}{\lambda|D(\tau,\lambda)|^2}\, d\tau \le M \quad .$$

On the other hand, if $d = 0$, (5.18) implies that $|\tau D'(\tau) - D(\tau)| \le MA(\tau^{-1})$ and

$$(5.21) \quad |D(\tau,\lambda)| \ge \max\{2^{-3/2}A(\tau^{-1}) - \tau, \varphi(\tau)\} \quad ,$$

so (5.20) is again valid.

By (2.10), (5.18), and (5.14),

$$(5.22)\qquad \left[\int_{\rho/2}^{\frac{\omega}{2}} + \int_{2\omega}^{2C_1\sigma}\right] \frac{|\tau D'(\tau) - D(\tau)|}{\lambda |D(\tau,\lambda)|^2}\, d\tau \le M \int_{\rho/2}^{2C_1\sigma} \frac{A(\tau^{-1})d\tau}{\lambda A^2(\tau^{-1})}$$

$$\le \frac{M\sigma}{\lambda A(\sigma^{-1})} = M \quad .$$

Next we use (5.18) and (5.13) to obtain

$$(5.23)\qquad \int_{2C_1\sigma}^{\infty} \frac{|\tau D'(\tau) - D(\tau)|}{\lambda |D(\tau,\lambda)|^2}\, d\tau \le M\lambda \int_{2C_1\sigma}^{\infty} \frac{A(\tau^{-1})}{\tau^2}\, d\tau$$

$$\le M\lambda A(\sigma^{-1})\sigma^{-1} \le M \quad .$$

Before estimating the final piece in (5.11), note that (5.18) implies

$$(5.24)\qquad MA(\tau^{-1}) \le \varphi(\omega^*) + \omega^*\theta(\omega^*) \qquad (\tfrac{1}{2}\,\omega \le \tau \le 2\omega) \quad .$$

Now (5.13), (5.18), and (5.24) give us

$$(5.25)\qquad \int_{\frac{\omega}{2}}^{2\omega} \frac{|\tau D'(\tau) - D(\tau)|}{\lambda |D(\tau,\lambda)|^2}\, d\tau \le M\lambda A(2\omega^{-1}) \int_{\frac{\omega}{2}}^{2\omega} \frac{d\tau}{[\lambda\varphi(\omega^*)]^2 + |\tau-\omega|^2}$$

$$\le \frac{MA(2\omega^{-1})}{\varphi(\omega^*)} \le M\left(1 + \frac{\omega^*\theta(\omega^*)}{\varphi(\omega^*)}\right) \quad .$$

Thus from (5.22), (5.23), and (5.25) we obtain (5.2).

Next we turn to (5.3). Assume (2.13) in addition to (1.2) and (2.2), and write (5.11) as

$$(5.26)\quad v(t,\lambda) = \mathrm{Re}\{\lambda^{-1}v_1(t) + i\lambda^{-2}v_2(t) + \lambda^{-3}v_3(t) - v_4(t,\lambda) - v_5(t,\lambda)\}$$

where (these v_j are unrelated to those of Section 4)

$$tv_1(t) = \int_0^\rho e^{i\tau t} \frac{\tau D'(\tau)}{D^2(\tau)} d\tau$$

$$tv_2(t) = \int_0^\rho \frac{\tau e^{i\tau t}}{D^2(\tau)} [1 - \frac{2\tau D'(\tau)}{D(\tau)}] d\tau$$

$$tv_3(t) = \int_0^\rho e^{i\tau t} \frac{2\tau^2}{D^3(\tau)} d\tau$$

$$t\lambda v_4(t,\lambda) = \int_0^\rho e^{i\tau t} \left\{ \left(\frac{\tau^3 D_\tau(\tau,\lambda)}{\lambda^2 D^3(\tau) D(\tau,\lambda)} \right) \left(\frac{2}{D(\tau)} + \frac{1}{D(\tau,\lambda)} \right) + \frac{1}{D(\tau,\lambda)} \right\} d\tau$$

$$t\lambda v_5(t,\lambda) = \int_\rho^\infty e^{i\tau t} \left(\frac{\tau D'(\tau) - D(\tau)}{D^2(\tau,\lambda)} \right) d\tau \quad .$$

We shall show that

$$(5.27) \quad |v_4(t,\lambda)| + |v_5(t,\lambda)| \le M\{[1 + \frac{\omega^* \theta(\omega^*)}{\varphi(\omega^*)}] q(t) + \frac{\omega^* \theta^2(\omega^*)}{\varphi^2(\omega^*) t^2}\} \qquad (t \ge 1) \quad ,$$

where

$$q(t) = t^{-2} + t^{-2} \int_0^t b(s)ds + t^{-1} b(t) - b'(t) \qquad (t \ge 1) \quad .$$

We know from [2, p. 972] that $q \in L^1(1,\infty)$. Moreover, from [15, Theorem 2] and the fact that

$$\operatorname{Re} \int_0^\infty e^{-st} a(t) dt > 0 \qquad (\operatorname{Re} s \ge 0, \ s \ne 0)$$

under our hypotheses [4], it follows that $v(\cdot,\lambda) \in L^1(\mathbb{R}^+)$. Then by (5.26) and (5.27), each of v_1, v_2, v_3 belongs to $L^1(1,\infty)$. (5.3) now follows from (5.26) and (5.27) with $Q = |v_1| + |v_2| + |v_3| + q$. We have reduced (5.3) to (5.27).

Let $J(u) = iu(1 - e^{iu}) - 2(1 - iu - e^{iu})$ and recall from [2, (4.9)] that

$$b'(\tau) = \tau^{-3} \int_0^\infty J(-\tau s)\,db'(s) \qquad (\tau > 0) \quad .$$

For $t, \tau > 0$ define

$$\beta^0(t,\tau) = \tau^{-3} \int_0^t J(-\tau s)\,db'(s)$$

$$\beta^\infty(t,\tau) = \tau^{-3} \int_t^\infty J(-\tau s)\,db'(s)$$

$$(5.28) \qquad \Delta(t,\tau) = \beta^0(t,\tau) + \hat{c}'(\tau) + id\tau^{-2} = D'(\tau) - \beta^\infty(t,\tau) \quad .$$

In [2, Lemma 5.1] we proved by direct estimates that $\hat{c} \in C^2(\mathbb{R}^+)$, $\partial\beta^0/\partial\tau \in C(\mathbb{R}^+ \times \mathbb{R}^+)$ and

$$(5.29) \qquad |\hat{c}''(\tau)| \le 6000 \int_0^{1/\tau} s^2 c(s)\,ds \qquad (\tau > 0) \quad ,$$

$$(5.30) \qquad |\beta^\infty(t,\tau)| \le 40\tau^{-2}(b(t) - tb'(t)) \qquad (t,\tau > 0) \quad ,$$

$$(5.31) \qquad \left|\frac{\partial\beta^0}{\partial\tau}(t,\tau)\right| \le 500\tau^{-2} \int_0^t b(s)\,ds \qquad (t,\tau > 0) \quad ,$$

$$(5.32) \qquad |\beta^0(t,\tau)| \le 40 \int_0^{1/\tau} sb(s)\,ds \qquad (t,\tau > 0) \quad ,$$

$$(5.33) \qquad |\hat{c}'(\tau)| \le 40 \int_0^{1/\tau} sc(s)\,ds \qquad (\tau > 0) \quad .$$

Write $v_4 = v_{41} + v_{42}$, where

$$(5.34) \quad t\lambda v_{41}(t,\lambda) = \int_0^\rho e^{i\tau t}\left[\left(\frac{\tau^3[\Delta(t,\tau) + i\lambda^{-1}]}{\lambda^2 D^3(\tau) D(\tau,\lambda)}\right)\left(\frac{2}{D(\tau)} + \frac{1}{D(\tau,\lambda)}\right) + \frac{1}{D(\tau,\lambda)}\right]d\tau \quad ,$$

$$(5.35) \qquad t\lambda v_{42}(t,\lambda) = \int_0^\rho e^{i\tau t}\left[\frac{\tau^3\beta^\infty(t,\tau)}{\lambda^2 D^3(\tau) D(\tau,\lambda)}\left(\frac{2}{D(\tau)} + \frac{1}{D(\tau,\lambda)}\right)\right]d\tau \quad .$$

Likewise let $v_5 = v_{51} + v_{52}$, where

$$(5.36) \qquad t\lambda v_{51}(t,\lambda) = \int_\rho^\infty e^{i\tau t} \frac{\tau\Delta(t,\tau) - D(\tau)}{D^2(\tau,\lambda)} d\tau \quad ,$$

$$(5.37) \qquad t\lambda v_{52}(t,\lambda) = \int_\rho^\infty e^{i\tau t} \frac{\tau\beta^\infty(t,\tau)}{D^2(\tau,\lambda)} d\tau \quad .$$

Now integrate by parts in (5.34) and (5.36) to obtain

$$(5.38) \quad i\lambda t^2 v_{41}(t,\lambda) = e^{i\rho t}\left\{\frac{\rho^3[\Delta(t,\rho) + i\lambda^{-1}]}{\lambda^2 D^3(\rho)D(\rho,\lambda)}\left(\frac{2}{D(\rho)} + \frac{1}{D(\rho,\lambda)}\right) + \frac{1}{D(\rho,\lambda)}\right\}$$

$$- \frac{1}{D(0+)} - \frac{1}{\lambda^2}\int_0^\rho e^{i\tau t}\left\{\frac{3\tau^2[\Delta(t,\tau) + i\lambda^{-1}] + \tau^2\Delta_\tau(t.\tau)}{D^3(\tau)D(\tau,\lambda)}\left[\frac{2}{D(\tau)} + \frac{1}{D(\tau,\lambda)}\right]\right.$$

$$- \frac{\tau^3[\Delta(t,\tau) + i\lambda^{-1}][3D'(\tau)D(\tau,\lambda) + D(\tau)D_\tau(\tau,\lambda)]}{D^4(\tau)D^2(\tau,\lambda)}\left[\frac{2}{D(\tau)} + \frac{1}{D(\tau,\lambda)}\right]$$

$$\left. - \frac{\tau^3[\Delta(t,\tau) + i\lambda^{-1}]}{D^3(\tau)D(\tau,\lambda)}\left[\frac{2D'(\tau)}{D^2(\tau)} + \frac{D_\tau(\tau,\lambda)}{D^2(\tau,\lambda)}\right] - \frac{\lambda^2 D_\tau(\tau,\lambda)}{D^2(\tau,\lambda)}\right\} d\tau \quad ,$$

$$(5.39) \quad - i\lambda t^2 v_{51}(t,\lambda) = e^{i\rho t}\left(\frac{\rho\Delta(t,\rho) - D(\rho)}{D^2(\rho,\lambda)}\right)$$

$$+ \int_\rho^\infty e^{i\tau t}\left[\frac{\Delta(t,\tau) + \tau\Delta_\tau(t,\tau) - D'(\tau)}{D^2(\tau,\lambda)} - 2\left(\frac{D_\tau(\tau,\lambda)(\tau\Delta(t,\tau) - D(\tau))}{D^3(\tau,\lambda)}\right)\right] d\tau \quad .$$

Here (5.18), (5.32), and (5.33) have been used to simplify the boundary terms. In (5.38), $1/D(0+)$ is zero unless $d = 0$ and $a \in L^1(\mathbb{R}^+)$. Our estimates (5.18), (5.19), (5.21) and (5.40) below show that the integrals in (5.38) converge absolutely.

By (5.18),

$$\left|\frac{D_\tau(\tau,\lambda)}{D^2(\tau,\lambda)}\right| \le M \frac{A_1(\tau^{-1}) + d\tau^{-2} + \lambda^{-1}}{|D(\tau,\lambda)|^2} \quad .$$

If $d > 0$, (5.19) shows that

$$\int_0^\rho \left|\frac{D_\tau(\tau,\lambda)}{D^2(\tau,\lambda)}\right| d\tau \le M \quad . \tag{5.40}$$

If $d = 0$, we recall from [15, (1.21)] that

$$\int_0^1 \frac{A_1(\tau^{-1})}{A^2(\tau^{-1})}\, d\tau < \infty \quad .$$

Thus by (5.21), (5.40) holds in this case as well. It is now a straightforward matter to use (5.18), (5.19), (5.21), (5.40), and (5.29) through (5.33) to estimate the terms in (5.35) and (5.38) and deduce

$$|v_4(t,\lambda)| \le Mq(t) \qquad (t \ge 1) \quad . \tag{5.41}$$

We turn now to v_5. The following estimates, direct consequences of (2.6), (5.18), and (5.29) through (5.33), will be used without explicit mention for estimates of the numerators.

$$|\Delta(t,\tau)| + |\tau\Delta_\tau(t,\tau)| + |D'(\tau)| \le M[A_1(\tau^{-1}) + t^2q(t)\tau^{-1}] \qquad (t \ge 1,\ \tau \ge \tfrac{1}{2}\rho) \quad ,$$

$$|D_\tau(\tau,\lambda)| \le MA_1(\tau^{-1}) \le M\tau^{-1}A(\tau^{-1}) \qquad (\tau \ge \tfrac{1}{2}\rho) \quad ,$$

$$\tau^2|\beta^\infty(t,\tau)| \le Mtq(t) \qquad (t \ge 1,\ \tau > 0) \quad ,$$

$$\tau|\Delta(t,\tau)| + |D(\tau)| \le MA(\tau^{-1}) \qquad (t \ge 1,\ \tau \ge \tfrac{1}{2}\rho) \tag{5.42}$$

$$|\Delta(t,\tau)| + |\tau\Delta_\tau(t,\tau)| + |D'(\tau)| \le M(\lambda^{-1} + \tau^{-1}t^2q(t)) \qquad (t \ge 1,\ \tau \ge \tfrac{1}{2}\omega)$$

$$|D_\tau(\tau,\lambda)| \le M\lambda^{-1} \qquad (\tau \ge \tfrac{1}{2}\omega) \quad .$$

We recall as well that

$$A(\tau^{-1}) \ge A(1/2C_1\sigma) \ge A(\sigma^{-1})/2C_1 = \sigma/2C_1\lambda \qquad (\tau \le 2C_1\sigma) \tag{5.43}$$

and

$$\tau A(\tau^{-1}) \geq a(\tau^{-1}) \geq a(t_1) \qquad (\tau \geq \tfrac{\rho}{2}) \quad .$$

We use Lemma 5.2 and its simple consequence

$$\tau\lambda^{-1} \leq M|D(\tau,\lambda)| \qquad (2\omega \leq \tau < \infty) \tag{5.44}$$

to get

$$\left[\int_{\rho/2}^{\omega/2} + \int_{2\omega}^{2C_1\sigma}\right] \frac{|\Delta(t,\tau) + \tau\Delta_\tau(t,\tau) - D'(\tau)|}{\lambda|D(\tau,\lambda)|^2} d\tau \tag{5.45}$$

$$\leq \frac{M}{\lambda} \int_{\rho/2}^{2C_1\sigma} \frac{A_1(\tau^{-1}) + \tau^{-1}t^2 q(t)}{A^2(\tau^{-1})} d\tau$$

$$\leq \frac{M}{\lambda} \int_{\rho/2}^{2C_1\sigma} \frac{1}{\tau A(\tau^{-1})} + \frac{t^2 q(t)}{\tau A^2(\tau^{-1})} d\tau$$

$$\leq \frac{M\sigma}{\lambda a(t_1)} \left[1 + \frac{t^2 q(t)}{A(\sigma^{-1})}\right] \leq Mt^2 q(t)$$

and (here (2.6) is used as well)

$$\int_{2C_1\sigma}^{\infty} \frac{|\Delta(t,\tau) + \tau\Delta_\tau(t,\tau) - D'(\tau)|}{\lambda|D(\tau,\lambda)|^2} d\tau \tag{5.46}$$

$$\leq \frac{M}{\lambda} \int_{2C_1\sigma}^{\infty} \frac{\lambda^{-1} + \tau^{-1}t^2 q(t)}{(\tau/\lambda)^2} d\tau \leq Mt^2 q(t) \quad .$$

Similarly,

$$\left[\int_{\rho/2}^{\omega/2} + \int_{2\omega}^{2C_1\sigma} + \int_{2C_1\sigma}^{\infty}\right] \frac{|D_\tau(\tau,\lambda)||\tau\Delta(t,\tau) - D(\tau)|}{\lambda|D(\tau,\lambda)|^3} d\tau \tag{5.47}$$

$$\leq M(1 + \lambda A(\sigma^{-1}) \int_{2C_1\sigma}^{\infty} \frac{d\tau}{\tau^3}) \leq M \quad .$$

On $[\frac{1}{2}\omega, 2\omega]$ we use (5.13) to estimate the denominator. This yields

$$(5.48)\qquad \int_{\omega/2}^{2\omega} \frac{|\Delta(t,\tau) + \tau\Delta_\tau(t,\tau) - D'(\tau)|}{\lambda|D(\tau,\lambda)|^2}\, d\tau$$

$$\le M\lambda \int_{\omega/2}^{2\omega} \frac{\lambda^{-1} + \tau^{-1}t^2 q(t)}{[\lambda\varphi(\omega^*)]^2 + |\tau-\omega|^2}\, d\tau$$

$$\le M(1 + t^2 q(t)\lambda\omega^{-1}) \int_0^\infty \frac{ds}{[\lambda\varphi(\omega^*)]^2 + s^2}$$

$$\le \frac{Mt^2 q(t)\lambda\omega^{-1}}{\lambda\varphi(\omega^*)} \le Mt^2 q(t) \frac{\omega^*\theta(\omega^*)}{\varphi(\omega^*)} .$$

The last inequality above used (2.5) and (5.5). Similarly, using (5.24), (5.25), and (5.6), we obtain

$$(5.49)\qquad \int_{\omega/2}^{2\omega} \frac{|D_\tau(\tau,\lambda)|\,|\tau\Delta(t,\tau) - D(\tau)|}{\lambda|D(\tau,\lambda)|^3}\, d\tau \le M \int_{\omega/2}^{2\omega} \frac{\lambda A(2\omega^{-1})\, d\tau}{[(\lambda\varphi(\omega^*))^2 + |\tau-\omega|^2]^{3/2}}$$

$$\le \frac{MA(2\omega^{-1})}{\lambda\varphi^2(\omega^*)} \le M \frac{[\varphi(\omega^*) + \omega^*\theta(\omega^*)]\theta(\omega^*)}{\varphi^2(\omega^*)}$$

$$= M\left(\frac{\theta(\omega^*)}{\varphi(\omega^*)} + \frac{\omega^*\theta^2(\omega^*)}{\varphi^2(\omega^*)}\right) \le M\left(1 + \frac{\omega^*\theta^2(\omega^*)}{\varphi^2(\omega^*)}\right) .$$

Thus the representation (5.39), along with the estimates (5.45) through (5.49), gives us

$$(5.50)\qquad |v_{51}(t,\lambda)| \le M\left\{q(t)\left[1 + \frac{\omega^*\theta(\omega^*)}{\varphi(\omega^*)}\right] + t^{-2}\left[1 + \frac{\omega^*\theta^2(\omega^*)}{\varphi^2(\omega^*)}\right]\right\} .$$

As in (5.45) through (5.48), we derive

$$(5.51) \qquad \left[\int_{\rho/2}^{\omega/2} + \int_{2\omega}^{2C_1\sigma} + \int_{2C_1\sigma}^{\infty}\right] \frac{\tau|\beta^{\infty}(t,\tau)|}{\lambda|D(\tau,\lambda)|^2}\, d\tau \le Mtq(t) \quad .$$

Again we use (5.13) on $[\frac{1}{2}\omega, 2\omega]$. This gives us

$$(5.52) \qquad \int_{\omega/2}^{2\omega} \frac{\tau|\beta^{\infty}(t,\tau)|}{\lambda|D(\tau,\lambda)|^2}\, d\tau \le \frac{Mtq(t)}{\omega^{*}\varphi(\omega^{*})} \le Mtq(t)\left[\frac{\omega^{*}\theta(\omega^{*})}{\varphi(\omega^{*})}\right] \quad ,$$

where the last inequality invokes (2.5) and (5.5).

Then (5.37), (5.51), and (5.52) imply

$$(5.53) \qquad |v_{52}(t,\lambda)| \le Mq(t)\left[1 + \frac{\omega^{*}\theta(\omega^{*})}{\varphi(\omega^{*})}\right] \quad .$$

But $v_5 = v_{51} + v_{52}$, so (5.41), (5.50), and (5.53) give us (5.27). This, in turn, gives us (5.3).

This completes the proof of Theorem 2.3.

6. Proofs of Theorem 2.4 and Corollary 2.2. To prove Theorem 2.4 we need (6.1), (6.2), and (6.3) below, which are consequences of (2.18), (5.18), and (5.24).

$$(6.1)\quad \frac{A^2(\tau^{-1})}{\varphi(\omega^*)} \le M \frac{|\hat{a}(1/\omega^*)|^2}{\varphi(\omega^*)} \le M(\varphi(\omega^*) + \frac{(\omega^*\theta(\omega^*))^2}{\varphi(\omega^*)}) \le M \qquad (\tfrac{1}{2}\omega \le \tau \le 2\omega) \quad .$$

Thus,

$$(6.2)\qquad 1 + \frac{\omega^*\theta(\omega^*)}{\varphi(\omega^*)} \le 2\,\frac{|\hat{a}(1/\omega^*)|}{\varphi(\omega^*)} \le M\,\frac{A(2/\omega)}{\varphi(\omega^*)} \le \frac{M}{A(2/\omega)}$$

$$\le \frac{M}{A(1/C_1\sigma)} \le \frac{C_1M}{A(\sigma^{-1})} \le M\lambda\sigma^{-1} \quad .$$

Furthermore, by (2.21), (2.18) implies (2.20), so

$$(6.3)\qquad 1 + \frac{\omega^*\theta^2(\omega^*)}{\varphi^2(\omega^*)} \le (1 + \frac{\omega^*\theta(\omega^*)}{\varphi(\omega^*)})(1 + \frac{\theta(\omega^*)}{\varphi(\omega^*)}) \le M\lambda\sigma^{-1} \quad .$$

Comparing (6.2) and (6.3) with (5.3) shows that

$$(6.4)\qquad |v(t,\lambda)| \le MQ(t)\lambda\sigma^{-1} \le MQ(t)\lambda^{\frac{1}{2}}, \ (\lambda \ge 1,\ t \ge 1) \quad .$$

Using (5.2) and (6.2) it follows that

$$(6.5)\qquad |v(t,\lambda)| \le M\lambda\sigma^{-1}t^{-1} \quad . \qquad (\lambda \ge 1,\ t > 0) \quad .$$

Combining (2.12) and (6.5) yields

$$(6.6)\qquad |v(t,\lambda)| = |v(t,\lambda)|^{\frac{1}{2}+\frac{1}{2}} \le M\sigma^{\frac{1}{2}}(\lambda\sigma^{-1}t^{-1})^{\frac{1}{2}} = M\lambda^{\frac{1}{2}}t^{-\frac{1}{2}}$$

$$(\lambda \ge 1,\ 0 < t \le 1) \quad .$$

Theorem 2.4 is an easy consequence of (6.4) and (6.6).

Proof of Corollary 2.2. If $a(0+) < \infty$, then [14, Corollaries 2.1 and 2.2] imply that c is strongly positive. Then a is strongly positive. As noted in Section 2, strong positivity implies (2.18) (which in turn implies (2.14)) in this case, so our assertion follows from Theorems 2.3(i) and 2.4.

If $a(0+) = \infty$, we follow the proof of [2, Cor. 2.1(ii)] for this case. There we invoked [16, Thm. 2(iii)] to obtain

$$\varphi(\tau) \geq \frac{\alpha}{8\beta^2} A^2(\tau^{-1}) \qquad (\tau \geq \max\{\rho, x_0^{-1}\}) \tag{6.7}$$

(α, β, x_0 are positive constants whose values are irrelevant here) at an intermediate stage of the proof. Since $A(\tau^{-1}) \geq \tau A_1(\tau^{-1})$ and (2.4) holds, we deduce from (6.7) that

$$\varphi(\tau) \geq \frac{\alpha}{8\beta^2} \left[\frac{\tau\theta(\tau)}{12}\right]^2 \qquad (\tau \geq \max\{\rho, x_0^{-1}\}) \quad .$$

But φ and θ are continuous, so (2.18) holds, and our conclusions follow as before. This completes the proof.

7. Proof of Theorem 2.5. Each example has the form

$$a(t) = \sum_{k=0}^{\infty} c_k b_k(t) \tag{7.1}$$

where

$$b_k(t) = (1 - 2^{\beta^k} t)\chi_k(t) \quad ;$$

χ_k is the characteristic function of the interval $[0, 2^{-\beta^k}]$ and β is an integer greater than or equal to 2.

Each c_k will be positive and we shall have $2A(\infty) = \sum_{k=0}^{\infty} c_k 2^{-\beta^k} < \infty$; then (1.2) and (2.2) hold. (2.13) is clear because $a(\frac{1}{2}) = 0$.

For any kernel of the form (7.1),

$$\varphi(\tau) = \sum_{k=0}^{\infty} c_k 2^{\beta^k} \frac{(1 - \cos 2^{-\beta^k}\tau)}{\tau^2} \quad . \tag{7.2}$$

Note that

$$\frac{1}{4} u^2 \le 1 - \cos u \le \frac{1}{2} u^2 \qquad (0 \le u \le 1) \quad . \tag{7.3}$$

Therefore,

$$\varphi(\tau) \ge \frac{c_{m+1}}{4} 2^{-\beta^{m+1}} \ge \frac{c_{m+1}}{4\tau^\beta} \qquad (2^{\beta^m} \le \tau \le 2^{\beta^{m+1}}) \quad . \tag{7.4}$$

On the other hand, if we let $\tau_n = 2^{\beta^n}(2\pi)$, (7.2) and (7.3) show that

$$\varphi(\tau_n) \le \frac{1}{2} \sum_{k=n+1}^{\infty} c_k 2^{-\beta^k} \quad . \tag{7.5}$$

From (2.4) we get

$$(7.6)\quad \frac{1}{12}\,\theta(\tau) \le \sum_{k=0}^{m} c_k \int_0^{1/\tau} t(1 - 2^{\beta^k} t)\,dt + \sum_{k=m+1}^{\infty} c_k \int_0^{2^{-\beta^k}} t(1 - 2^{\beta^k})\,dt$$

$$\le \frac{1}{2\tau^2} \sum_{k=0}^{n} c_k + \frac{1}{6} \sum_{k=m+1}^{\infty} c_k 2^{-2\beta^k} \qquad (2^{\beta^m} \le \tau \le 2^{\beta^{m+1}}) \quad ,$$

$$(7.7)\qquad 5\theta(\tau_n) \ge \sum_{k=0}^{n} c_k \int_0^{1/\tau_n} t(1 - \tau_n t)\,dt \ge \frac{1}{6\tau_n^2} \sum_{k=0}^{n} c_k \quad .$$

Now we need only choose β and c_k appropriately.

For a_1, let $\beta = 2$, $c_k = k2^{2^{k-2}}$, $0 < \epsilon < 1/14$. By (7.4), if

$2^{2^m} \le \tau \le 2^{2^{m+1}}$,

$$\varphi(\tau) \ge \frac{m+1}{4}\, 2^{-\frac{3}{2}(2)^m} \ge \log_2\log_2 \tau/4\tau^{3/2} \quad .$$

Since (7.6) holds and

$$(7.8)\qquad (x+y)^r \le (2x)^r + (2y)^r \qquad (x,y,r > 0) \quad ,$$

$$\frac{1}{150}\,[\theta(\tau)]^{\frac{3}{2}-\epsilon} \le \Big(\frac{1}{2}\,\tau^{-2} \sum_{k=0}^{m} c_k\Big)^{\frac{3}{2}-\epsilon} + \Big(\frac{1}{6} \sum_{k=m+1}^{\infty} k2^{-7(2)^{k-2}}\Big)^{\frac{3}{2}-\epsilon} \quad (2^{2^m} \le \tau \le 2^{2^{m+1}}) .$$

The first sum on the right is dominated by c_m; the second sum is dominated by its first term. Thus

$$\frac{1}{150}\,[\theta(\tau)]^{\frac{3}{2}-\epsilon} \le (\tau^{-2} c_m)^{\frac{3}{2}-\epsilon} + \big((m+1)\,2^{-7(2)^{m-1}}\big)^{\frac{3}{2}-\epsilon}$$

$$\le (1 + \log_2\log_2\tau)^{3/2}\big(\tau^{-\frac{21}{8}+\frac{7\epsilon}{4}} + 2^{-5(2)^m}\big) \qquad (2^{2^m} \le \tau \le 2^{2^{m+1}}) \quad .$$

Since θ and φ are continuous, we deduce that (2.15) holds with $\gamma = \frac{1}{2}$. by (7.5) and (7.7),

$$\varphi(\tau_n) \le (n+1)2^{-3(2)^{n-1}} ,$$

$$900\ \tau_n^2\theta^2(\tau_n) \ge \frac{n^2}{4\pi^2} 2^{-3(2)^{n-1}} ,$$

so $\tau_n^2\theta^2(\tau_n)/\varphi(\tau_n) \to \infty$ $(n\to\infty)$, and (2.18) does not hold.

For $a_{2,\gamma}$, let $\beta = 1 + 2\gamma$, $c_k = 2^{(1+\gamma)^k}$. Note that

$$\frac{c_{k+1}2^{-2\beta^{k+1}}}{c_k 2^{-2\beta^k}} < 2^{-3\gamma} < 1, \quad \frac{c_{k+1}}{c_k} > 2^{\gamma} > 1 .$$

Therefore (7.6) implies

$$\text{(7.9)} \qquad \theta(\tau) \le K(\gamma)\left[\frac{c_m}{\tau^2} + c_{m+1}2^{-2\beta^{(m+1)}}\right] \quad (2^{\beta^m} \le \tau \le 2^{\beta^{m+1}})$$

for some number $K(\gamma) < \infty$. Using (7.4), (7.8), and (7.9), we get

$$\frac{\tau\theta^{1+\gamma}(\tau)}{\varphi(\tau)} \le 2^{1+\gamma}K(\gamma)\left[\frac{c_{m+1}\tau^{-2(\gamma+1)\ +\ 1}}{c_{m+1}\tau^{-(1+2\gamma)}} + \frac{c_{m+1}^{1+\gamma}2^{(-1-2\gamma)\beta^{m+1}}}{c_{m+1}2^{-\beta^{(m+1)}}}\right]$$

$$\le 2^{1+\gamma}K(\gamma)[1 + 2^{\gamma[(1+\gamma)^{m+1}-2(1+2\gamma)^{m+1}]}]$$

$$\le 2^{2+\gamma}K(\gamma) \qquad (2^{\beta^m} \le \tau \le 2^{\beta^{m+1}}) .$$

Thus (2.14) holds. On the other hand, (7.5) and (7.7) yield

$$\text{(7.10)} \qquad \varphi(\tau_n) \le \frac{1}{2}\left(1 + \frac{1}{1-2^{-\gamma}}\right)c_{n+1}\left(\frac{2\pi}{\tau_n}\right)^{\beta},\ 30\ \theta(\tau_n) \ge c_n\tau_n^{-2} .$$

From (7.10), our final conclusions about $a_{2,\gamma}$ follow easily.

For a_3 we take $\beta = 2$, $c_k = 2^k$, and for a_4 we take $\beta = 2$, $c_k = 1$. The estimates are similar to those given above, so we omit them. Example a_4 appeared in [2].

8. <u>A lemma</u>. In this section we prove

LEMMA 8.1. *If* (1.2) *holds, then*

$$(8.1) \qquad |D(\tau,\lambda)| \geq M\tau A_1(\tau^{-1}) \qquad (\tfrac{1}{2}\rho \leq \tau \leq \tfrac{1}{2}\omega) \quad ,$$

$$(8.2) \qquad |D(\tau,\lambda)| \geq M \frac{|\tau-\omega|}{\lambda} \qquad (\tau \geq \tfrac{1}{2}\omega) \quad .$$

This is the same as [2, Lemma 5.2], but our proof in [2] contains an error.

<u>Proof</u>. When $\frac{1}{2}\rho \leq \tau < \rho = \omega$ or $\tau \geq \omega$, the proof in [2] is valid, so we exclude those cases here. [2, (5.11)] is not correct when $\tau < \omega$. When we integrate the inequality

$$-\theta'(\tau) \geq \frac{\tau}{5}\int_0^{1/\tau} r^3 a(r)dr \geq \frac{1}{80\tau^3} a(\tau^{-1})$$

[2, (4.4)] from τ to ω, we obtain

$$(8.3) \qquad |\mathrm{Im}\ D(\tau,\lambda)| \geq \frac{\tau(\omega-\tau)(\omega+\tau)}{20}\int_0^{1/\omega} r^3 a(r)dr$$

$$+ \frac{\tau}{160}\int_{1/\omega}^{1/\tau} ra(r)dr \qquad (\rho < \omega,\ \tfrac{1}{2}\rho \leq \tau < \omega) \quad .$$

Since

$$\int_{1/\omega}^{2/\omega} r^3 a(r)dr \leq 15\int_0^{1/\omega} r^3 a(r)dr \quad ,$$

we have

$$(8.4) \qquad 16\int_0^{1/\omega} r^3 a(r)dr \geq \int_0^{2/\omega} r^3 a(r)dr \geq \int_0^{1/\tau} r^3 a(r)dr \qquad (\tau \geq \tfrac{1}{2}\omega) \quad .$$

By (8.3) and (8.4),

$$(8.5) \qquad |\mathrm{Im}\ D(\tau,\lambda)| \geq \frac{\tau(\tau-\omega)(\tau+\omega)}{320}\int_0^{1/\tau} r^3 a(r)dr + \frac{\tau}{160}\int_{1/\omega}^{1/\tau} ra(r)dr$$

for $\rho < \omega$, $\frac{1}{2}\omega \leq \tau < \omega$. Except for a constant, this is the same as [2, (5.11)] for these τ, ω, so the remainder of the proof of (8.2) as given in [2] is valid. We need only establish (8.1).

Note that

$$(8.6) \qquad |D(\tau,\lambda)| \geq \varphi(\tau) \geq M \geq M\tau A_1(\tau^{-1}) \qquad (\tfrac{1}{2}\rho \leq \tau \leq \rho) \quad .$$

If $\frac{1}{2}\omega \leq \rho$, then (8.6) implies (8.1). Otherwise we consider two cases.

Case 1. If $\rho \leq \tau \leq \frac{1}{2}\omega$ and $A(\omega^{-1}) \geq 6\omega A_1(\omega^{-1})$, then, as in the proof of Lemma 5.1,

$$(8.7) \qquad |\mathrm{Re}\ D(\tau,\lambda)| = \varphi(\tau) \geq \frac{1}{4}\int_0^{1/\tau} t^2 da'(t)$$

$$\geq \frac{1}{4}\int_0^{1/\omega} t^2 da'(t)$$

$$\geq \frac{1}{2}\,[A(\omega^{-1}) - 3\omega A_1(\omega^{-1})] + \frac{1}{4\omega}\,a(\omega^{-1})$$

$$\geq \frac{1}{4}A(\omega^{-1}) \geq \frac{1}{4}\omega A_1(\omega^{-1}) \geq \frac{1}{4}\tau A_1(\omega^{-1}) \quad .$$

Thus (8.3) and (8.7) imply

$$(8.8) \qquad \sqrt{2}\,|D(\tau,\lambda)| \geq \frac{\tau}{160}\int_{1/\omega}^{1/\tau} ra(r)dr + \frac{\tau}{4}\int_0^{1/\omega} ra(r)dr$$

$$\geq \frac{\tau}{160}\,A_1(\tau^{-1}) \quad \text{in Case 1} \ .$$

Case 2. If $\rho \leq \tau \leq \frac{1}{2}\omega$ and $A(\omega^{-1}) < 6\omega A_1(\omega^{-1})$, then let $g(t) = (6\omega t - 1)a(t)$. In Case 2, then, $\int_0^{1/\omega} g(t)dt > 0$.

It is easy to see that $(6\omega t)^n g(t) \geq g(t)$ $(t > 0,\ n = 1,2)$, so we conclude that

$$(8.9) \qquad (6\omega)^n \int_0^{1/\omega} t^n g(t)dt > 0 \qquad (n = 1,2) \quad .$$

From (8.9) it follows that

$$(8.10) \qquad 36\omega^2 \int_0^{1/\omega} t^3 a(t)dt > \int_0^{1/\omega} ta(t)dt \quad .$$

Now (8.3) implies

$$(8.11) \qquad |\mathrm{Im}\ D(\tau,\lambda)| \geq \frac{\tau\omega^2}{40} \int_0^{1/\omega} r^3 a(r)dr + \frac{\tau}{160} \int_{1/\omega}^{1/\tau} ra(r)dr \ \left(\rho \leq \tau \leq \frac{1}{2}\,\omega\right) \quad .$$

(8.10) and (8.11) combine to yield

$$(8.12) \qquad |\mathrm{Im}\ D(\tau,\lambda)| \geq \frac{\tau}{1440} \int_0^{1/\omega} ra(r)dr + \frac{\tau}{160} \int_{1/\omega}^{1/\tau} ra(r)dr$$

$$\geq \frac{\tau}{1440} A_1(\tau^{-1}) \quad \text{in Case 2.}$$

Finally, (8.6), (8.8), and (8.12) establish (8.1) in all cases. This completes the proof of Lemma 8.1.

9. <u>Proofs of Theorems 3.1, 3.2, and 3.3.</u> For Theorem 3.1(i), first observe that Theorem 2.3(i) implies

$$(9.1) \qquad |v(t,\lambda)| \leq Mt^{-1}\lambda^{1/2} \qquad (t > 0) \quad .$$

By (1.14), $\underset{\sim}{V}(t)\underset{\sim}{L}^{-1/2}$ is bounded, for each $t > 0$. Moreover, if $t, s > 0$ and $\underset{\sim}{y} \in \mathcal{H}$,

$$\|[\underset{\sim}{V}(t) - \underset{\sim}{V}(s)]\underset{\sim}{L}^{-1/2}\underset{\sim}{y}\|^2 = \int_1^\infty [v(t,\lambda) - v(s,\lambda)]^2\lambda^{-1}d(\underset{\sim}{E}_\lambda\underset{\sim}{y}, \underset{\sim}{y}) \quad .$$

Since $v(t,\lambda)$ is continuous in t and (9.1) holds, Lebesgue's Dominated Convergence Theorem shows that $\underset{\sim}{V}(t)\underset{\sim}{L}^{-1/2}\underset{\sim}{y} \to \underset{\sim}{V}(s)\underset{\sim}{L}^{-1/2}\underset{\sim}{y}$ $(t \to s)$. $v(t,\lambda)$ is differentiable in t, so the Mean Value Theorem implies

$$(9.2) \qquad \|h^{-1}[\underset{\sim}{U}(t+h) - \underset{\sim}{U}(t) - h\underset{\sim}{V}(t)]\underset{\sim}{y}\|^2$$

$$= \int_1^\infty \left|\frac{v(t+\eta,\lambda) - v(t,\lambda)}{\lambda^{1/2}}\right|^2 \lambda d(\underset{\sim}{E}_\lambda\underset{\sim}{y}, \underset{\sim}{y})$$

$(\underset{\sim}{y} \in \mathcal{D}_1)$, where $\eta = \eta(t,\lambda,h)$ is between 0 and h. For $\underset{\sim}{y} \in \mathcal{D}_1$, $\lambda d(\underset{\sim}{E}_\lambda\underset{\sim}{y}, \underset{\sim}{y})$ is a finite measure, so by (9.1) and dominated convergence, the integral in (9.2) tends to zero as $h \to 0$. Therefore $\underset{\sim}{U}(t)\underset{\sim}{y}$ is differentiable $(t > 0)$ and (3.1) holds. This proves Theorem 3.1(i).

Under the hypotheses of Theorem 3.1(ii),

$$(9.3) \qquad \sup_{t \geq 0} |v(t,\lambda)| \leq M\sigma \leq Ma(0+)\lambda^{1/2}$$

(see Theorem 2.2 and the remarks following it). Using (9.3) in place of (9.1), we can argue as above and prove the results on the closed interval $\overline{\mathbb{R}}^+$. This completes the proof of Theorem 3.1.

<u>Proof of Theorem 3.2.</u> To simplify formulas, we take $d = 0$, since this does not change the argument. For (i), the uniqueness assertion and the special case $\underset{\sim}{g} \equiv \underset{\sim}{0}$ are just Theorem 2.1(ii) of [2]. Therefore it suffices to establish (1.11) when $\underset{\sim}{f} \equiv \underset{\sim}{0}$ and $\underset{\sim}{z}_0 = 0$.

Let n be a positive integer, and let $\underset{\sim}{g}_n = \underset{\sim}{E}_n \underset{\sim}{g}$,

$$\underset{\sim}{h}_n(t) = \int_0^t a(t-s)\underset{\sim}{g}_n(s)ds \quad .$$

Then $\underset{\sim}{g}_n \in \mathcal{B}_{\ell oc}(\overline{\mathbb{R}}^+, \mathcal{H})$. Since $\|\underset{\sim}{L}\underset{\sim}{E}_n\| \leq n$ and $\underset{\sim}{L}\underset{\sim}{g}_n = \underset{\sim}{E}_n\underset{\sim}{L}\underset{\sim}{g}$ is measurable, $\underset{\sim}{g}_n$ belongs to $\mathcal{B}^\infty_{\ell oc}(\overline{\mathbb{R}}^+, \mathcal{D})$. Therefore $\underset{\sim}{h}_n : \overline{\mathbb{R}}^+ \to \mathcal{D}$ is continuous. By [2, Theorem 2.1], the unique solution of

$$\underset{\sim}{z}'(t) + \int_0^t a(t-s)[\underset{\sim}{L}\underset{\sim}{z}(s) + \underset{\sim}{g}_n(s)]ds = \underset{\sim}{0}, \; \underset{\sim}{z}(0) = \underset{\sim}{0} \quad ,$$

is

$$\underset{\sim}{z}_n(t) = -\int_0^t \underset{\sim}{U}(t-s)\underset{\sim}{h}_n(s)ds \quad .$$

Then $\underset{\sim}{z}_n \in C(\overline{\mathbb{R}}^+, \mathcal{D})$ and

$$\underset{\sim}{z}_n(t) = \int_0^t \int_0^s a(s-r)[\underset{\sim}{L}\underset{\sim}{z}_n(r) + \underset{\sim}{g}_n(r)]drds \quad (t \geq 0) \quad . \tag{9.4}$$

But for $\underset{\sim}{y}_0 \in \mathcal{D}$, $\underset{\sim}{y}(t) = \underset{\sim}{U}(t)\underset{\sim}{y}_0$ is the solution of

$$\underset{\sim}{y}'(t) + \int_0^t a(t-s)\underset{\sim}{L}\underset{\sim}{y}(s)ds = 0, \quad \underset{\sim}{y}(0) = \underset{\sim}{y}_0$$

[2, Theorem 2.1(i)]. Since $\underset{\sim}{L}$ is closed and (3.1) holds, this means (see Theorem 3.1)

$$L^{-1}V(t)y_0 = V(t)L^{-1}y_0 = -\int_0^t a(t-s)U(s)y_0 ds$$

$$= -\int_0^t U(t-s)a(s)y_0 ds \qquad (y_0 \in \mathcal{D}) \quad .$$

Therefore

$$\int_0^t V(t-s)L^{-1}g_n(s)ds$$

$$= -\int_0^t [\int_0^{t-s} a(t-s-r)U(r)g_n(s)dr]ds$$

$$= -\int_0^t U(r) \int_0^{t-r} a(t-r-s)g_n(s)ds\, dr$$

$$= z_n(t) \quad .$$

Since $z_0 = f = 0$, (1.11) reduces to

$$z(t) = \int_0^t V(t-s)L^{-1}g(s)ds$$

but $V(\cdot)L^{-1/2}$ is strongly continuous on $\mathbb{R}^+$ and $\|V(\cdot)L^{-1/2}\| \in L^1(\mathbb{R}^+)$, while $g \in \mathcal{B}^{\infty}_{\ell oc}(\overline{\mathbb{R}}^+,\mathcal{D}_1)$. Therefore

$$V(t-s)L^{-1}g(s) \quad \text{and} \quad LV(t-s)L^{-1}g(s) = V(t-s)L^{-1/2}\cdot L^{1/2}g(s)$$

are strongly measurable in s (a modified version of [2, Lemma 3.1] shows this), and standard estimates show that $z \in C(\overline{\mathbb{R}}^+, \mathcal{D})$. Then by (1.14),

$$\|L[z(t) - z_n(t)]\| \leq \int_0^t \|V(t-s)L^{-1/2}\| \; \|L^{1/2}[g(s) - g_n(s)]\|ds$$

$$\leq \int_0^t v_{1/2}(s) \; \|(I-E_n)L^{1/2}g(t-s)\|ds \quad .$$

But $\underset{\sim}{E}_n \to \underset{\sim}{I}$ strongly $(n \to \infty)$, and the integrand here is dominated by the L^1 function

$$w(s) = v_{1/2}(s) \operatorname*{ess\,sup}_{0\le r\le t} \|\underset{\sim}{L}^{1/2}\underset{\sim}{g}(r)\| \quad ,$$

so

$$\|\underset{\sim}{L}[\underset{\sim}{z}(t) - \underset{\sim}{z}_n(t)]\| \le \int_0^T w(s)ds \qquad (0 \le t \le T < \infty) \quad ,$$

$$\underset{\sim}{L}\,\underset{\sim}{z}_n(t) \to \underset{\sim}{L}\,\underset{\sim}{z}(t) \quad \text{in } H \qquad (n \to \infty,\ t \ge 0) \quad .$$

Similarly, $\underset{\sim}{z}_n(t) \to \underset{\sim}{z}(t)$ $(n \to \infty)$ and $\underset{\sim}{z} - \underset{\sim}{z}_n$ is bounded on finite intervals. Therefore we can let $n \to \infty$ in (9.4), using dominated convergence, and deduce that

$$\underset{\sim}{z}(t) = \int_0^t \int_0^s a(s-r)[\underset{\sim}{L}\,\underset{\sim}{z}(r) + \underset{\sim}{g}(r)]dr\,ds \quad .$$

Therefore $\underset{\sim}{z}(t)$ is a solution of (1.10) with $\underset{\sim}{z}_0 = \underset{\sim}{f} = \underset{\sim}{0}$, as asserted. For (ii), the hypotheses imply $v_{1/2}(t) \le M$ (see Theorem 2.2), so the proof of (i) can be repeated with minor changes. This proves Theorem 3.2.

<ins>Proof of Theorem 3.3</ins>. By (4.1), Theorem A of Section 2, and the fact that (2.14) $(\gamma = \frac{1}{2})$ implies (2.21), our hypotheses yield

$$\text{(9.5)} \qquad \|\underset{\sim}{U}(t)\|_{\mathcal{D}} \le 1 \ (t \ge 0), \ \int_0^\infty \|\underset{\sim}{U}(t)\|_{\mathcal{D}} dt \equiv \nu < \infty \quad .$$

Let $\underset{\sim}{T} : \underset{\sim}{g} \to \underset{\sim}{z}$ be the operator defined formally by the right-hand side of (1.11) with $\underset{\sim}{y}_0$ in place of $\underset{\sim}{z}_0$, but interpret the integrals as Bochner integrals in $\mathcal{B}^1((0,t),\mathcal{D})$. If $\underset{\sim}{g} \in \mathcal{B}^\infty(\mathbb{R}^+,\mathcal{D}_1)$, Theorem 3.2(i) shows that $\underset{\sim}{T}\underset{\sim}{g} \in C(\overline{\mathbb{R}}^+,\mathcal{D})$. Moreover, by (9.5) and (1.14),

$$\|\underset{\sim}{T}\underset{\sim}{g}(t)\|_{\mathcal{D}} \le \|\underset{\sim}{y}_0\|_{\mathcal{D}} + \|\underset{\sim}{f}_1\|_{\mathcal{B}^1(\mathbb{R}^+,\mathcal{D})} + \nu\|\underset{\sim}{f}_2\|_{\mathcal{B}^\infty(\mathbb{R}^+,\mathcal{D})}$$

$$+ \|\underset{\sim}{g}\|_{\mathcal{B}^\infty(\mathbb{R}^+,\mathcal{D}_1)} \int_0^\infty v_{1/2}(t)dt \qquad (t \in \overline{\mathbb{R}}^+) \quad .$$

With $K = 1 + \nu + \|v_{1/2}\|_{L^1}$,

$$\|\underset{\sim}{T}\underset{\sim}{g}(t)\|_{\mathcal{D}} \le K(\mu + \|\underset{\sim}{g}\|_{\mathcal{B}^\infty(\mathbb{R}^+,\mathcal{D}_1)}) \qquad (t \in \overline{\mathbb{R}}^+) \quad .$$

Referring to our hypotheses, we choose Δ, $0 < \Delta < \alpha$, so small that $K\,\epsilon(\Delta) < \frac{1}{2}$, and choose $\mu \le \Delta/2K$. The $\underset{\sim}{T}\underset{\sim}{N}$ maps the ball

$$S_\Delta = \{\underset{\sim}{y} \mid \|y(t)\|_{\mathcal{D}} \le \Delta,\ t \in \overline{\mathbb{R}}^+\}$$

in the Banach space $C(\overline{\mathbb{R}}^+,\mathcal{D})$ into itself. $\underset{\sim}{y} \in S_\Delta$ is a fixed point of $\underset{\sim}{T}\underset{\sim}{N}$ if and only if $\underset{\sim}{y}$ is a solution of (1.12) in S_Δ.

We complete the proof by showing that $\underset{\sim}{T}\underset{\sim}{N}$ is a contraction on S_Δ. For $\underset{\sim}{z}_1, \underset{\sim}{z}_2 \in S_\Delta$,

$$\|\underset{\sim}{T}\underset{\sim}{N}\underset{\sim}{z}_1(t) - \underset{\sim}{T}\underset{\sim}{N}\underset{\sim}{z}_2(t)\|_{\mathcal{D}} =$$

$$\|\int_0^t \underset{\sim}{V}(t-s)\underset{\sim}{L}^{-1}[\underset{\sim}{N}\underset{\sim}{z}_1(s) - \underset{\sim}{N}\underset{\sim}{z}_2(s)]ds\|_{\mathcal{D}}$$

$$\le K\|\underset{\sim}{N}\underset{\sim}{z}_1 - \underset{\sim}{N}\underset{\sim}{z}_2\|_{\mathcal{B}^\infty(\mathbb{R}^+,\mathcal{D}_1)}$$

$$\le K\,\epsilon(\Delta)\ \|z_1 - z_2\|_{\mathcal{B}^\infty(\mathbb{R}^+,\mathcal{D})}$$

$$< \frac{1}{2}\|\underset{\sim}{z}_1 - \underset{\sim}{z}_2\|_{C(\overline{\mathbb{R}}^+,\mathcal{D})} \quad .$$

This proves Theorem 3.3

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Let $y(t,x,f)$ denote the solution of the Cauchy Problem

$$y'(t) + \int_0^t [d + a(t-s)] L\, y(s)\, ds = f(t), \quad t \geq 0, \quad y(0) = x \quad ,$$

where $d \geq 0$ and L is a self-adjoint densely defined linear operator on a Hilbert space H with $L \geq \lambda_1 I$. Let $U(t)x = y(t,x,0)$, $V = U'$. By analyzing a related scalar equation with parameter, we find sufficient conditions on the

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ABSTRACT (continued)

kernel a in order that $\int_0^\infty \|V(t)L^{-\gamma}\|dt < \infty$ $(\gamma > 0)$. These results and certain resolvent formulas can be used to study the asymptotic behavior of the solution $y(t,x,f)$ as $t \to \infty$. An application to a semilinear integro-partial differential equation is presented.

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