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ENTROPY-BASED RANDOM NUMBER EVALUATION*

by

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ABSTRACT

Recent work has shown how to test a simple hypothesis of uniformity on the interval (0,1) by using estimates of entropy. In this paper we use Monte Carlo methods to extend previous tables of critical points and power for such entropy tests to the large sample sizes likely to be desirable when evaluating the output of one or more random number generators. A comparison with asymptotic critical points and power is made. The results are used to evaluate a number of commonly used random number generators. At least one random number generator judged "good" by the spectral test (multiplier 5**15) is found unsuitable for use.

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1. INTRODUCTION

An estimator of entropy was proposed by Vasicek (1976) and studied by Dudewicz and van der Meulen (1979), who showed how this estimator could be used to test whether a random sample comes from the uniform distribution on (0,1) and gave Monte Carlo and asymptotic evaluations of the percentage points of the distribution of the estimator. In this paper we utilize this test for uniformity to evaluate random number generators. A comparison of asymptotic with Monte Carlo for percent points and power is also given.

The differential entropy (or entropy) of a random variable X with density function f is defined as

$$H(X) = -\int_{-\infty}^{\infty} f(x) \log f(x)dx \qquad (1.1)$$

and has the properties that if $X \epsilon [0,1]$ w.p. 1 then

$$H(X) \leq 0, \qquad (1.2)$$

and among all densities f concentrated on [0,1] the uniform f_0 maximizes H(f) to

$$H(f_0) = 0. \qquad (1.3)$$

The following two definitions were given by Dudewicz and van der Meulen (1979), and use the above.

Definition (1.4). Two densities f_1 and f_2 are said to be entropy-distinguishable if $H(f_1) \neq H(f_2)$.

Definition (1.5). A density f^* is said to be entropy-unique (or e-unique) in a class $\mathcal{C}$ of densities if $f^* \in \mathcal{C}$ and $\nexists$ $f \in \mathcal{C}$, $f \neq f^*$, such that $H(f) = H(f^*)$.

By property (1.3), we know that in the class $\mathcal{C}$ of densities on (0,1), f_0 is e-unique. This fact will be central below.

Now let $X_1, \ldots, X_n$ be a random sample from an absolutely continuous distribution F with density f and let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics. The estimator of H(f) (Vasicek (1976)) is

$$H_{m,n} = n^{-1} \sum_{i=1}^{n} \log\{\frac{n}{2m} [X_{(i+m)} - X_{(i-m)}]\} \qquad (1.6)$$

where $1 \leq m < n/2$, $X_{(j)} = X_{(1)}$ for $j < 1$, and $X_{(j)} = X_{(n)}$ for $j > n$.

Dudewicz and van der Meulen (1979) proposed a test for uniformity based on the e-uniqueness of f_0. We are concerned with this test and so describe it here.

Let $X_1, \ldots, X_n$ be a random sample from an absolutely continuous distribution F with density f concentrated on [0,1]. Let f_0 denote the U(0,1) density. The level α test rejects $H_0 : f = f_0$ in favor of $H_A : f \neq f_0$ if and only if

$$H_{m,n} \leq H_\alpha^*(m,n) \tag{1.7}$$

where $H_\alpha^*(m,n)$ is the 100α percentile point of the distribution of $H_{m,n}$ under f_0.

By the e-uniqueness of f_0 among all densities concentrated on [0,1] and by the consistency of the estimator (Vasicek (1976)), it follows that the above test is consistent against all alternatives f on [0,1]. Further, in Dudewicz and van der Meulen (1979) it is shown that if f is concentrated on [0,1] then, w.p. 1, $H_{m,n} \leq 0$.

In Dudewicz and van der Meulen (1979), the above test is studied in detail using both analytical and Monte Carlo techniques. The latter are necessary since the form of the distribution of $H_{m,n}$ appears to be analytically intractable. From Dudewicz and van der Meulen (1979) we know

<u>Theorem 1.8</u>. Under H_0, if $m = o(n^{1/3-\delta})$, $\delta > 0$, the quantities

$$(6mn)^{1/2}[H_{m,n} - \log(\tfrac{n}{2m}) + \log(n + 1) + \gamma - R(1, 2m-1)] \tag{1.9}$$

and

$$(6mn)^{1/2}[H_{m,n} + \log(2m) + \gamma - R(1, 2m-1)] \tag{1.10}$$

are asymptotically N(0,1) as $n \to \infty$.

Here γ is Euler's constant, $\gamma = 0.5772\ldots$ and, for $j \leq k$, $R(j, k) = k^{-1} + (k-1)^{-1} + \ldots + j^{-1}$. Using (1.9) we then have that

$$\hat{H}_\alpha^*(m,n) = \Phi^{-1}(\alpha)(6mn)^{-1/2} - \log(2m) - \gamma + R(1, 2m-1)$$

$$- \log\left(\frac{n+1}{n}\right) \qquad (1.11)$$

is an asymptotic approximation to the 100α percentile point of the distribution of $H_{m,n}$ where $\Phi(\cdot)$ is the standard normal distribution function.

For asymptotic power, Dudewicz and van der Meulen (1979) show

<u>Theorem 1.12</u>. For any bounded positive step function alternative density f_1 on $(0,1)$,

$$n^{1/2}\left(\frac{n}{n+2-2m}\right)^{1/2}[H_{m,n} + \log(2m) + \gamma - R(1, 2m-1) - H(f_1)]$$

is asymptotically normal with mean zero and variance $\tau(2m) + \mathrm{Var}_{f_1} \log f_1$, where

$$\tau(k) = \begin{cases} \pi^2/6 - 1 & \text{if } k = 1 \\ (2k^2-2k+1)\left\{\frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \ldots + \frac{1}{(k-1)^2}\right)\right\}-2k+1 & \text{if } k \geq 2 . \end{cases}$$

Hence an asymptotic approximation to the power of a level α test under alternative f_1 is

$$\Phi\left(\frac{n^{1/2}\left(\frac{n}{n+2-2m}\right)^{1/2}[H_\alpha{}^*(m,n) + \log(2m) + \gamma - R(1,2m-1) - H(f_1)]}{\sqrt{\tau(2m) + \mathrm{Var}_{f_1}\log f_1}}\right). \tag{1.13}$$

2. ONE-SAMPLE EVALUATION BASED ON ASYMPTOTIC PERCENTILE RANKS

We now use test (1.7) to evaluate whether each of nine of the uniform random number generators available in Dudewicz and Ralley (1981) have entropies consistent with uniformity. At the same time, the form of the test (1.7) suggests that if, for two random number generators $\mathcal{G}_1$ and $\mathcal{G}_2$,

$$H_{m,n}(\mathcal{G}_1) < H_{m,n}(\mathcal{G}_2),$$

then $\mathcal{G}_2$ is "better than" $\mathcal{G}_1$. This provides a basis for comparative evaluation of the random number generators. Thus we first obtain $H_{m,n}(\mathcal{G}_i)$ for each of nine of the random number generators $\mathcal{G}_i$ in Dudewicz and Ralley (1981): RNCG (using IX = 1, L1 = 452807053, C = 0.0, PIP = 2.**31), SRAND, KERAND, UNI, RN1, RN2, RN3, RN4 and RN5.

Fixing n = 10,000 and letting m = 1(1)10(5)20(10)40, comparison between the results obtained by Monte Carlo methods and by asymptotic methods is one of our goals. This value of n automatically excluded RN5 from consideration for m < 5 since the period of RN5 is only 2048 ($=2^{11}$), hence leading to definite rejection due to gaps of size zero for small m.

For each of the 9 generators, the statistics $H_{m,10000}$ were computed from a single sample of 10,000 numbers for 14 values of m (10 for RN5) as above. The percentile ranks of these statistics, expressed in terms of their asymptotic distributions under H_0 (see equation (1.10)), were plotted in Figure 1.

Figure 1

Examination of that graph reveals that, among all the generators considered, RN2, RN4, and KERAND have entropy estimates most consistent, almost <u>too</u> consistent, with uniformity, while UNI gives good entropy estimates in the same sense. The poorest generator is RNCG, while the rest performed reasonably well. (To be rejected, the $H_{m,n}$ - value of a generator needs to have a percentile rank $\leq$ 5.00, at the top of the graph scale.) These interpretations are based on a single sample of size 10,000 and therefore should be regarded more as an illustration of use of $H_{m,n}$ for comparing random number generators than as a definitive statement regarding them. More replications are used below in a way reminiscent of the extensive testing of random number generators reported in Dudewicz and Ralley (1981).

Since the critical region for the entropy test involves the lower (left) tail of the null distribution of $H_{m,n}$, the (asymptotic) cutoff point, say for α = .05, corresponds to the 5% point of the asymptotic null distribution and is seen to be to the left of most points in Figure 1. Thus, it is clear from the plots in Figure 1 that all generators under study except RNCG pass this test of uniformity for all the values of m considered. (Of course RN5 fails for $m < 5$.)

3. CONVERGENCE OF $H_{m,n}$ TO ASYMPTOTIC NULL DISTRIBUTIONS

Dudewicz and van der Meulen (1979) reported Monte Carlo percentage points of the null distribution of $H_{m,n}$ for 10,000 replications of sizes n = 10(10)50(50)100 each using the generator UNI (using ISEED = 524287 and JSEED = 654345465). A pilot study showed that extending this to 10,000 replications of size 10,000 each would be feasible but expensive.

In the same report, the above authors also studied the normal approximation to the percentage point $H_\alpha^*(m,n)$ given by (1.11) for $\alpha = .05$, and a variety of m,n up to $n \leq 100$. It was found that the asymptotic cutoffs deviate by a large amount from the simulated cutoff points, at least for $n \leq 100$.

In this report we continue the investigations of the above authors and investigate the normal approximation to percentage points for n = 10,000. For this we first obtain estimates of the true percentage points based on a Monte Carlo study using 1,000 samples of size 10,000. The resulting Monte Carlo estimates are given in Table 1, for various levels of $\alpha(\alpha = .40, .30, .20, .10, .05, .01)$ and m = 1(1)10(5)20(10)40. We also obtained the corresponding asymptotic values of $H_\alpha^*(m,n)$, for n = 10,000 and the same values of α and m. These asymptotic values are listed in

Table 1, together with the Monte Carlo estimates, and it is seen that - for such a large sample size as n = 10,000 - the asymptotic cutoffs are - for m up to 10 - in close agreement with the simulated ones using UNI, and based on 1000 replications.

Table 1

In addition to the above we plotted the asymptotic distributions of the $H_{m,n}$ - statistics for n = 10,000 and m = 1(1)10(5)20(10)50 , 250 in Figures 2a-2p. This asymptotic distribution is, for $n \to \infty$, $m = o(n^{1/3-\delta})$, and $\delta > 0$, given in an explicit form by (cf. 1.10)

$$\mathcal{N}(-\log(2m) - \gamma + R(1, 2m-1), \frac{1}{6mn}), \qquad (3.1)$$

where $\mathcal{N}(\mu,\sigma^2)$ denotes a normal distribution with mean μ and variance σ^2. In each figure the asymptotic distribution function of $H_{m,n}$ is overlayed with the corresponding simulated percentage points of $H_{m,n}$ obtained from the Monte Carlo study described above for α = .005, .010, .025, .050, .100, .200, .300, .400. It is seen from these figures that the normal approximation to the simulated values is good for all m-values up to 4, bad beyond 10, and questionable in the range inbetween. The goodness depends on the use to which the approximation is to be put. For the use in Section 5 below it is bad in the range $m > 4$. (From the restriction on m one expects goodness at best for $m \leq n^{1/3} \approx 22$.)

FIGURES 2a-2p

The fact that the normal approximation to the distribution of $H_{m,n}$ deteriorates for $n = 10,000$ as $m > 10$, as observed from Figures 2a-2p, also has a bearing on the study reported in Section 2. There, for one sample of size $n = 10,000$ the percentile ranks of the observed values of $H_{m,n}$ with respect to the asymptotic distributions are plotted for nine random number generators as functions of m. Strictly speaking, this approximation is not justified for values of $m > n^{1/3} \approx 22$ on the basis of Theorem 1.8. The figures 2a-2p indicate that the asymptotic distribution moves more and more to the right of the percentile points obtained through simulation as m increases, $m \geq 15$, i.e. the asymptotic distribution is more and more significantly stochastically larger than the actual (simulated) distribution. From this it can be surmised that for $m \geq 15$ the actual percentile ranks of the observed $H_{m,n}$ - values in the study leading to Figure 1 are larger than the ones plotted there, which are based on the asymptotic distribution, and that the discrepancies get larger as m increases. This may explain the general upward trend in Figure 1 of the graphs as m increases beyond 15; the actual percentile ranks are larger than the plotted ones for $m \geq 15$, which, when corrected for, would yield a more horizontal behavior of the percentile rank curves for $m \geq 15$ for the random number generators under consideration.

4. CONVERGENCE OF $H_{m,n}$ TO ASYMPTOTIC ALTERNATIVE DISTRIBUTIONS

Expression (1.13) for the asymptotic power of the test against alternatives which are bounded positive step functions on [0,1] was used by Dudewicz and van der Meulen (1979) to compute an approximation to the power for nine alternative distirbutions for $n = 20$, $m = 1, 2, 4, 9$ and $\alpha = .05$. Some of these alternative distributions belong to the class $\mathcal{F}$ of bounded positive step functions, others do not. As a general conclusion, the above authors found that the asymptotic power approximation agreed poorly with the Monte Carlo power evaluations for the values of n, m, and α considered. These investigations were continued for $n = 100$ and it was found that the normal approximation is still not accurate.

In this paper we extend the evaluation of the normal approximation to power for a particular alternative (called F in Dudewicz and van der Meulen (1979)) for $n = 100$, 1000, and 10,000 and $\alpha = .05$. This alternative which, belongs to $\mathcal{F}$, is defined by the following distribution function:

$$
F: \quad \begin{aligned}
F(x) &= x & &\text{if } 0 \le x \le 1/4-\varepsilon/2, \\
F(x) &= \delta(1/4-\varepsilon/2)+x(1-\delta), & &\text{if } 1/4-\varepsilon/2 \le x \le 1/4+\varepsilon/2, \\
F(x) &= x-\varepsilon\delta, & &\text{if } 1/4+\varepsilon/2 \le x \le 3/4-\varepsilon/2, \\
F(x) &= -\delta(3/4+\varepsilon/2)+x(1+\delta), & &\text{if } 3/4-\varepsilon/2 \le x \le 3/4+\varepsilon/2, \\
F(x) &= x, & &\text{if } 3/4+\varepsilon/2 \le x \le 1,
\end{aligned}
$$

(with $0 < \varepsilon < 0.5$, $0 < \delta \le 1$; we used $\varepsilon = .02$, $\delta = 0.1$).

In order to carry out the asymptotic power evaluation for alternative F we first need the cutoff points $H_\alpha^*(m,n)$ of our test procedure (1.7) for various values of α, m, n, and, moreover, Monte Carlo estimates of the power to compare the asymptotic approximation with. Monte Carlo estimates of $H_\alpha^*(m,n)$ for n = 100 and various values of α and m are given in Dudewicz and van der Meulen (1979). Monte Carlo estimates of $H_\alpha^*(m,n)$ for n = 1000 (based on N = 10,000 replications, and for m = 1(1)10(5)20(10)50, 250) and for n = 10,000 (based on N = 1000 replications, and for m = 1(1)10(5)20(10)50, 250, 500, 2500) are given in Table 2 and Table 3 respectively for α = .005, .01, .025, .05, .10, .20, .30, .40.

Table 2
Table 3

Monte Carlo estimates for the power of the entropy-based test against alternative F for n = 100, 1000, and

10,000 (based on 10,000, 10,000, and 1000 replications respectively) are given in Table 4 for α = .05 and m = 1(1)10(5)20(10)40, together with the asymptotic power calculated in each case using formula (1.13). Both the Monte Carlo estimates ("simulated power") and the asymptotic power are obtained using the Monte Carlo estimates of $H_\alpha^*(m,n)$ ("simulated cutoffs") described above.

In calculating the asymptotic power note that - for the density f_1 of F - $H(f_1)$ = -.000002, $\mathrm{Var}_{f_1} \log f_1(X)$ = -.000003842, and γ = .5772...

It is seen from Table 4 that there is reasonable agreement for n = 10,000 (and $1 \le m \le 3$), but for n = 100 and n = 1,000 there seems to be poor agreement overall, except for m = 1 at n = 1,000.

Table 4

5. EVALUATION OF NINE RANDOM NUMBER GENERATORS BASED ON A χ^2-TEST

In Section 2 we presented a one-sample evaluation of nine random number generators based on the percentile ranks of the observed $H_{m,n}$-values within the asymptotic distribution.

As a more sensitive approach we ran a χ^2-test using $N = 100$ sample-values of $H_{m,n}$ ($n = 10{,}000$; $m = 1(1)4$) from the same nine (without RN5) random number generators to determine whether the distributions of $H_{m,n}$ fit the corresponding asymptotic distributions which, for $m \leq 4$, have been verified in Section 3 as being very close to the true distributions. The results are given in Table 5.

Table 5

For carrying out the χ^2-test for goodness of fit we used 10 equi-probable cells. The null-hypothesis entertained in each case is that the 100 sample values of $H_{m,n}$ come from the asymptotic normal distribution given by (3.1). The entries in Table 5 are the observed χ^2-values for the eight random number generators and four values of m. Those χ^2-values which are significant at the .05 level are starred.

From the observed χ^2-values we conclude that seven random number generators produce numbers which are consistent with the hypothesis of randomness as measured by the Chi-square entropy-test, whereas one random number generator (RN3) is to be rejected due to the lack of proper distribution of the $H_{m,n}$ - statistic on the basis of samples of size n = 10,000.

6. CONCLUSIONS

In this paper we looked at the performance of nine random number generators in terms of the entropy-based test statistic $H_{m,n}$. A one-sample evaluation of size n = 10,000 indicated that all (except two) random number generators easily pass the entropy-based test, but it also showed that among the various generators some are closer to the uniform distribution than others as measured by the value of $H_{m,n}$. One random number generator was excluded from further study since its small period leads to rejection at every level.

The study of the convergence of $H_{m,n}$ to the asymptotic null distribution (initiated in Dudewicz and van der Meulen (1979) for sample sizes up to n = 100, and continued here for n = 10,000) revealed that for n = 10,000 and $m \leq 10$ the normal approximation to the percentage points (and thus to the distribution of $H_{m,n}$) is good. This comparison was carried out in two ways: i) by a comparison of simulated cutoff points with those found through the asymptotic formula (1.11), and (ii) by a plot of the asymptotic normal distribution of $H_{m,n}$ with the simulated values drawn into it. The latter study confirmed the first one.

Next we investigated the convergence of $H_{m,n}$ to the asymptotic alternative distribution under a specific alternative F which, because it is so close to uniform, is hard to detect. We observed that the simulated power of the entropy-based test is small for n = 10,000 but for some m's definitely larger than the significance level. We noticed that the asymptotic approximation to power for n = 10,000 as given by formula (1.13) has validity only for $m \leq 3$. In the course of these investigations we obtained Monte Carlo estimates of the percentage points for n = 1,000 and n = 10,000 which supplement the tables already provided in Dudewicz and van der Meulen (1979).

Finally we evaluated the eight remaining random number generators on basis of a χ^2-test. The purpose of this test was to investigate whether the true distribution of $H_{m,n}$ fitted the asymptotic null distributions closely, which we know from the results in Section 3 they should for m = 1, 2, 3, 4 if the generators are truely random. On basis of this study the random number generator RN3 was rejected.

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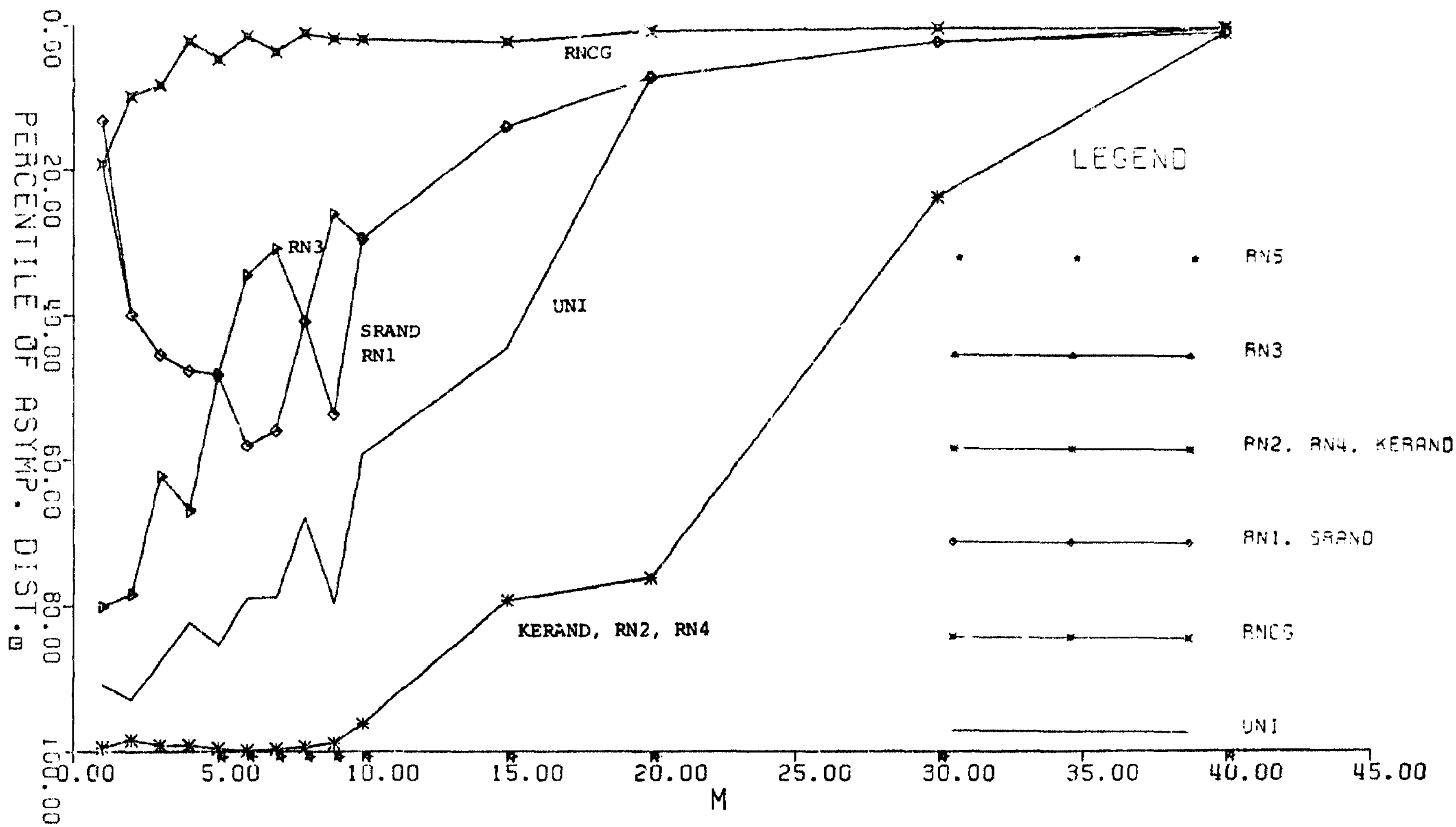

<u>Figure 1</u>. Comparison of 9 random number generators using percentile ranks of the asymptotic distribution of $H_{m,n}$ for n=10,000 and m=1(1)10(5)20(10)40.

<u>Figures 2a-2p</u>. The asymptotic distribution function of $H_{m,n}$ for n=10,000 and m=1(1)10(5)20(10)50,250; * represents a Monte Carlo value at α=.005,.01,.025,.05,.10,.20,.30,.40.

<u>Figure 2a</u>.

N= 10000

M= 1

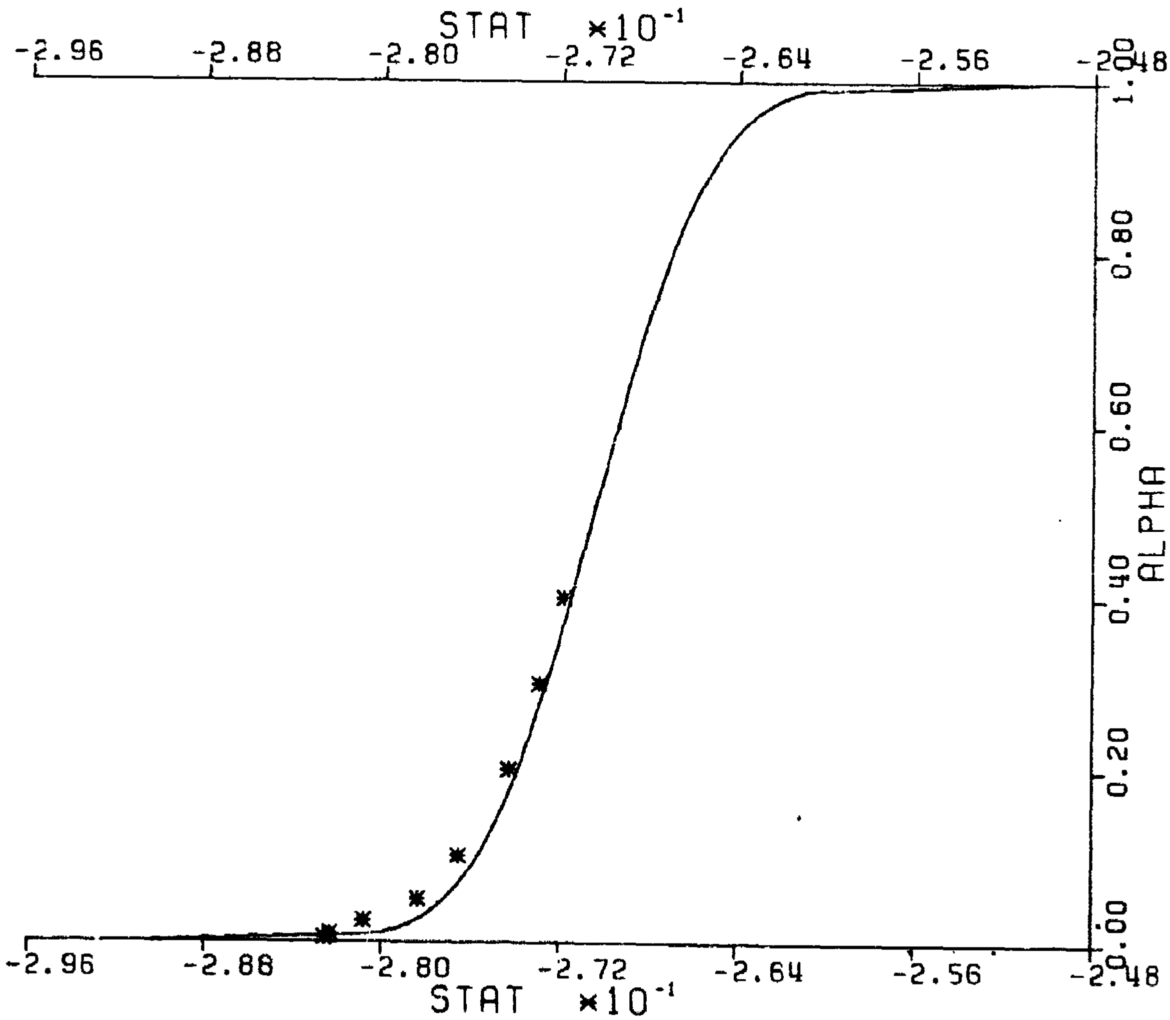

Figure 2b.

N= 10000

M= 2

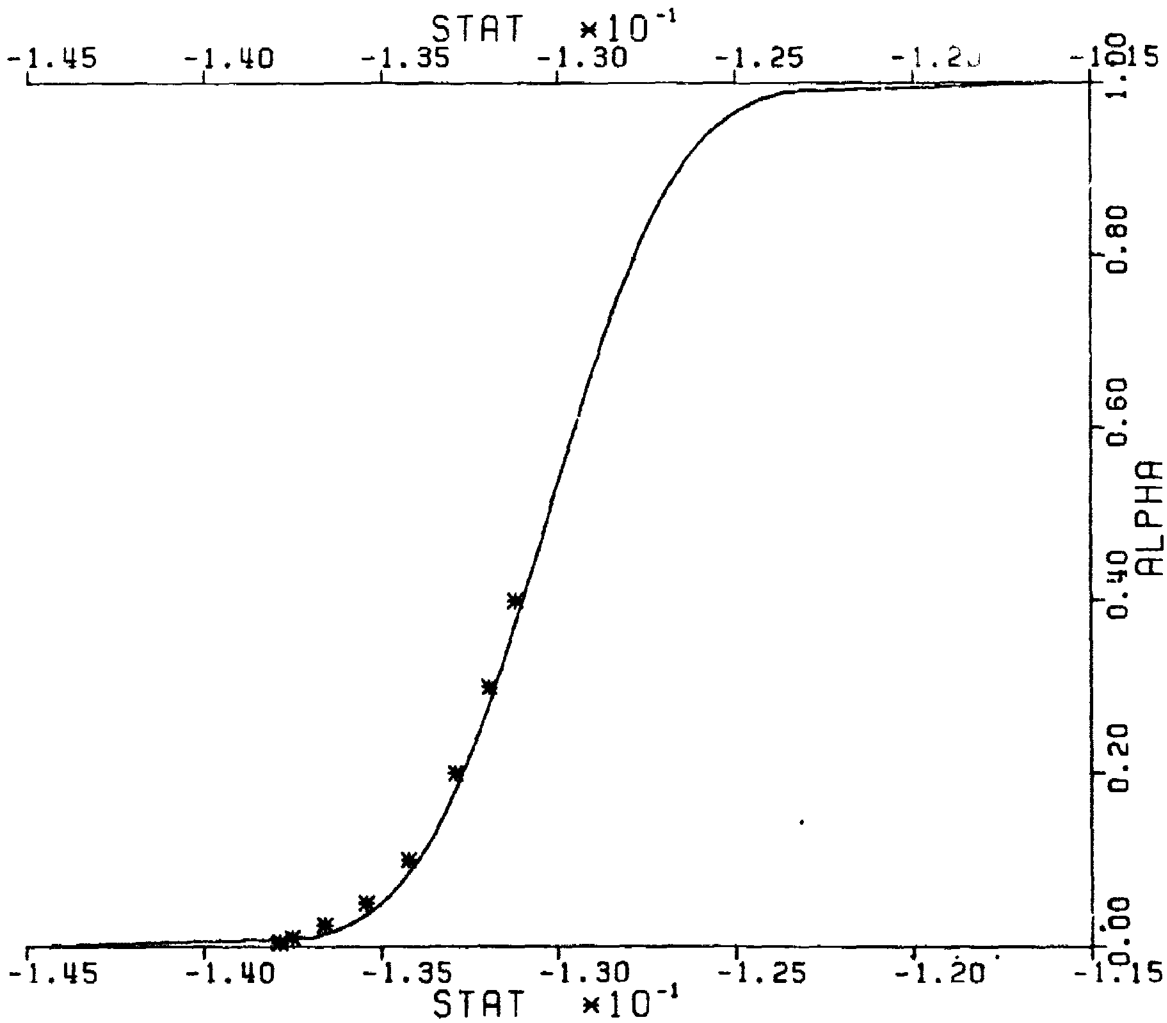

Figure 2c.

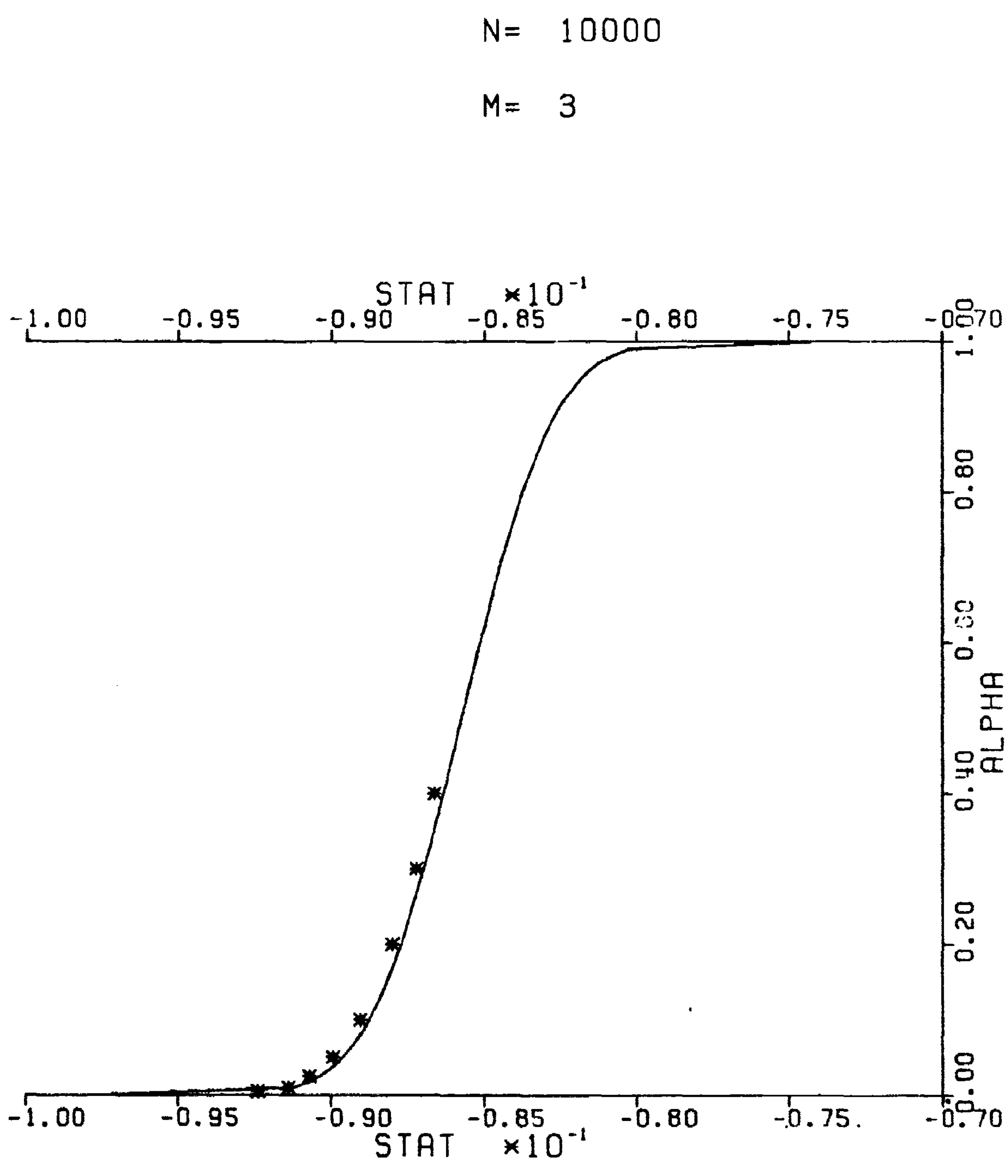
N= 10000
M= 3
STAT ×10⁻¹
-1.00
-0.95
-0.90
-0.85
-0.80
-0.75
-0.70
0.00
0.20
0.40
0.60
0.80
1.00
ALPHA
-1.00
-0.95
-0.90
-0.85
-0.80
-0.75
-0.70
STAT ×10⁻¹

Figure 2d.

N= 10000

M= 4

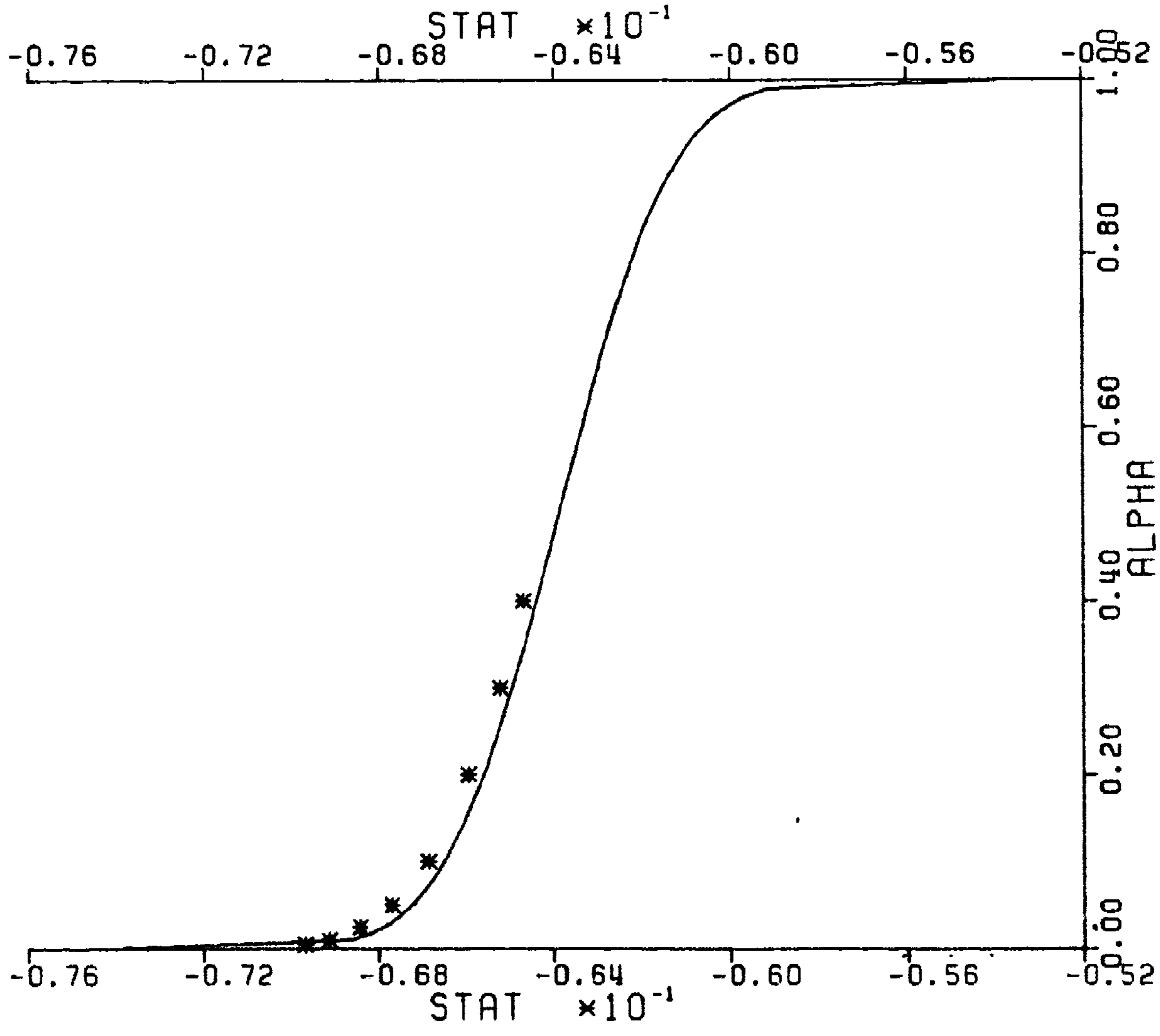

Figure 2e.

N= 10000

M= 5

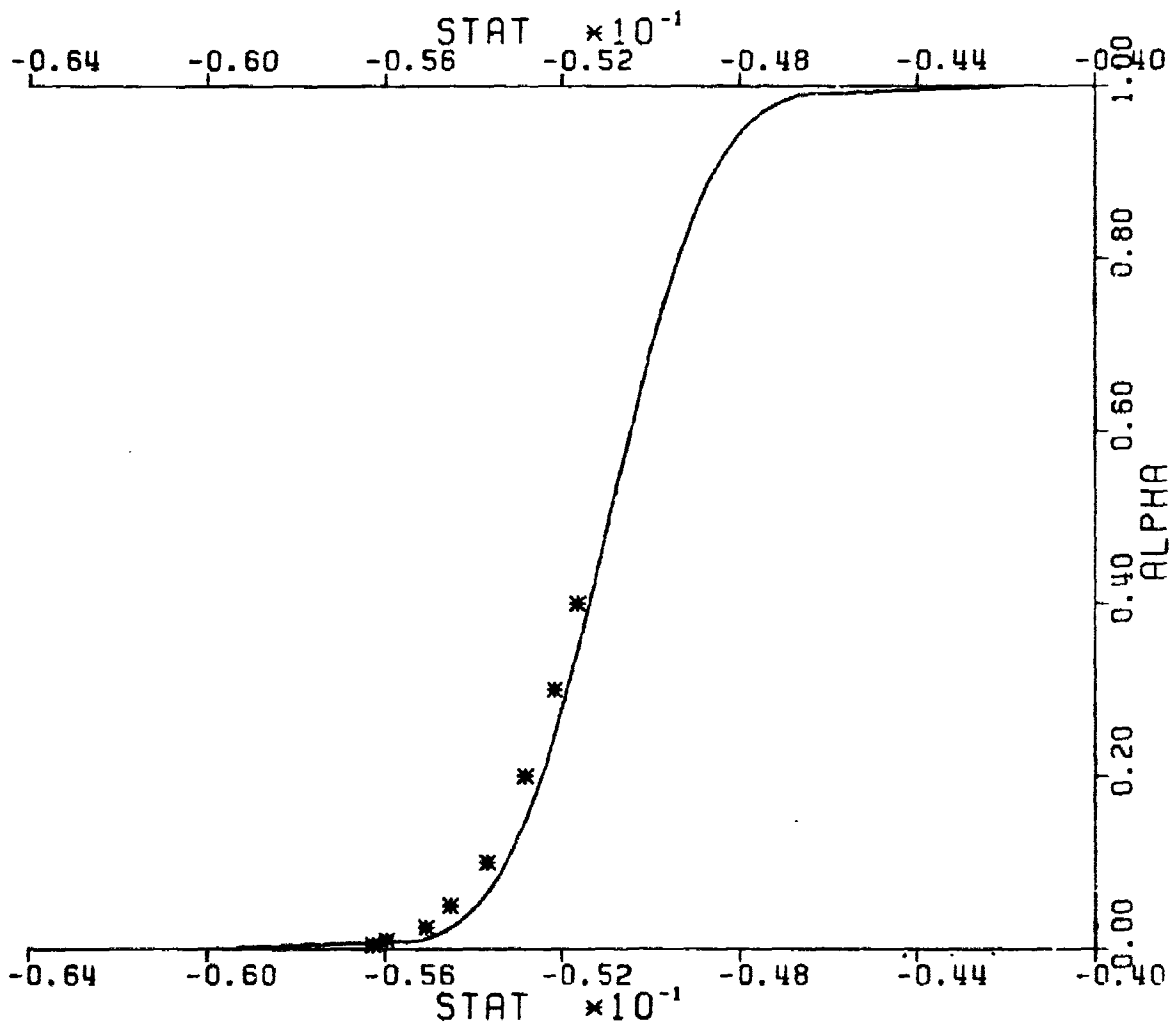

Figure 2f.

N= 10000

M= 6

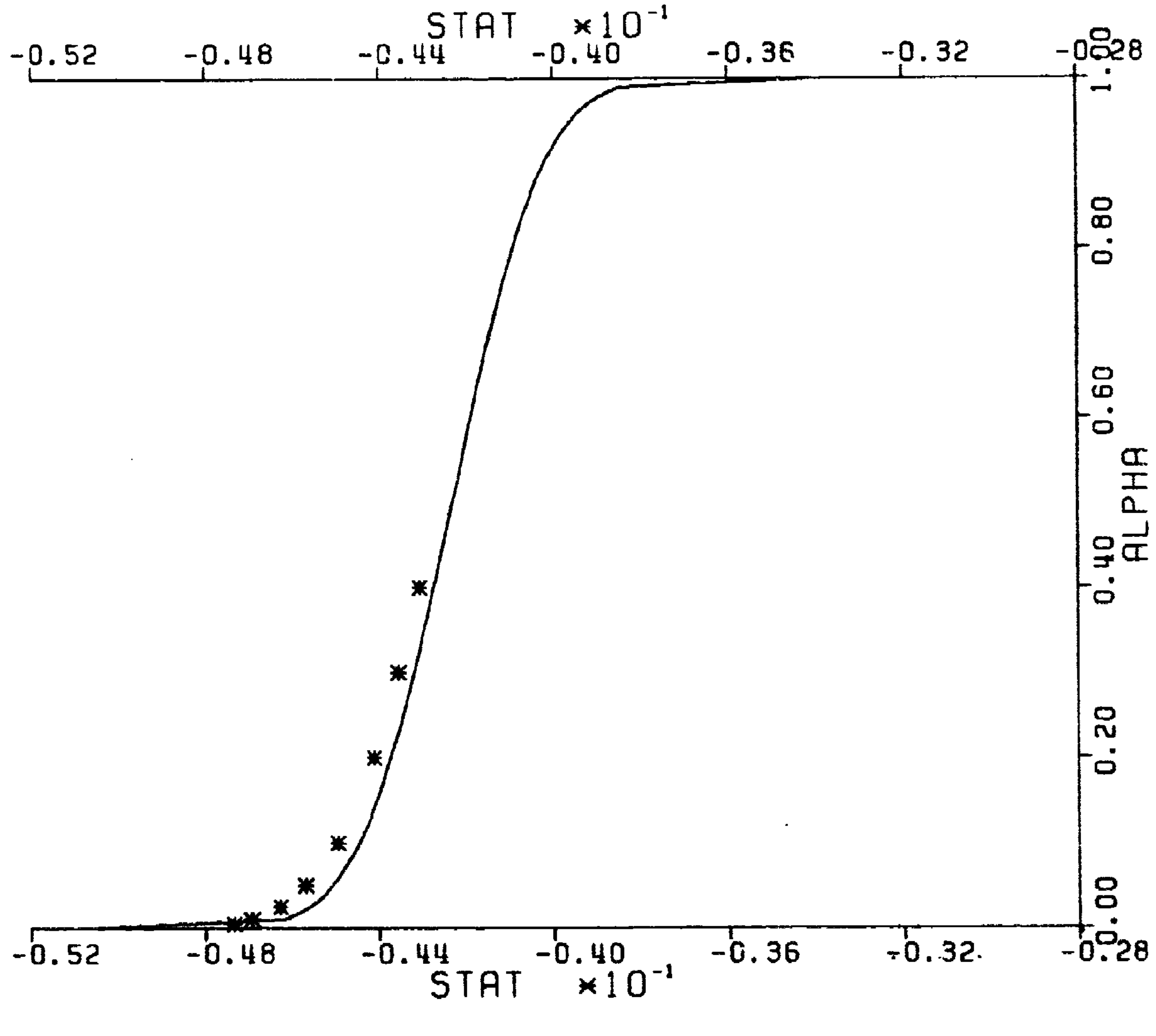

Figure 2g.

N= 10000

M= 7

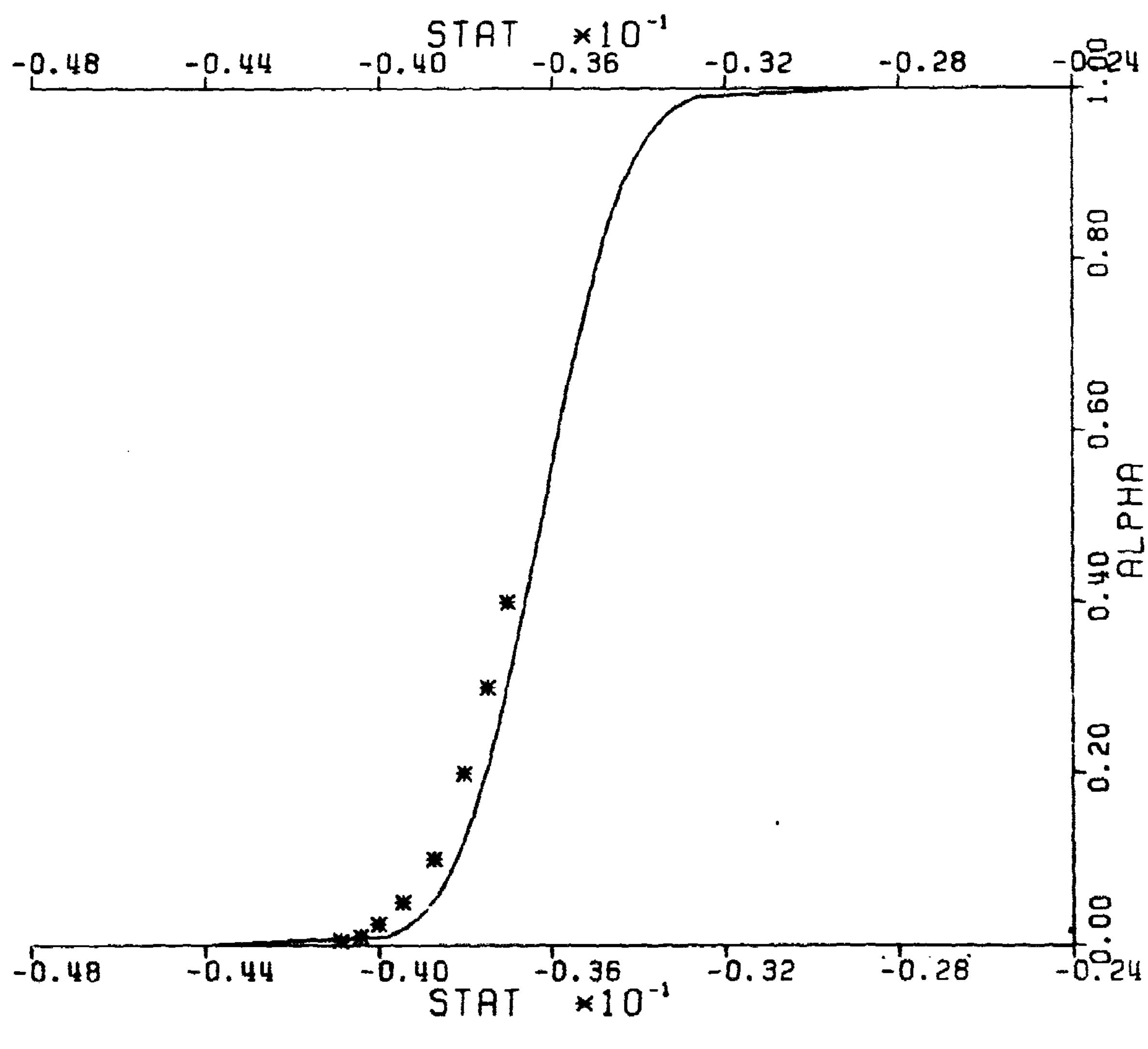

Figure 2h.

N= 10000

M= 8

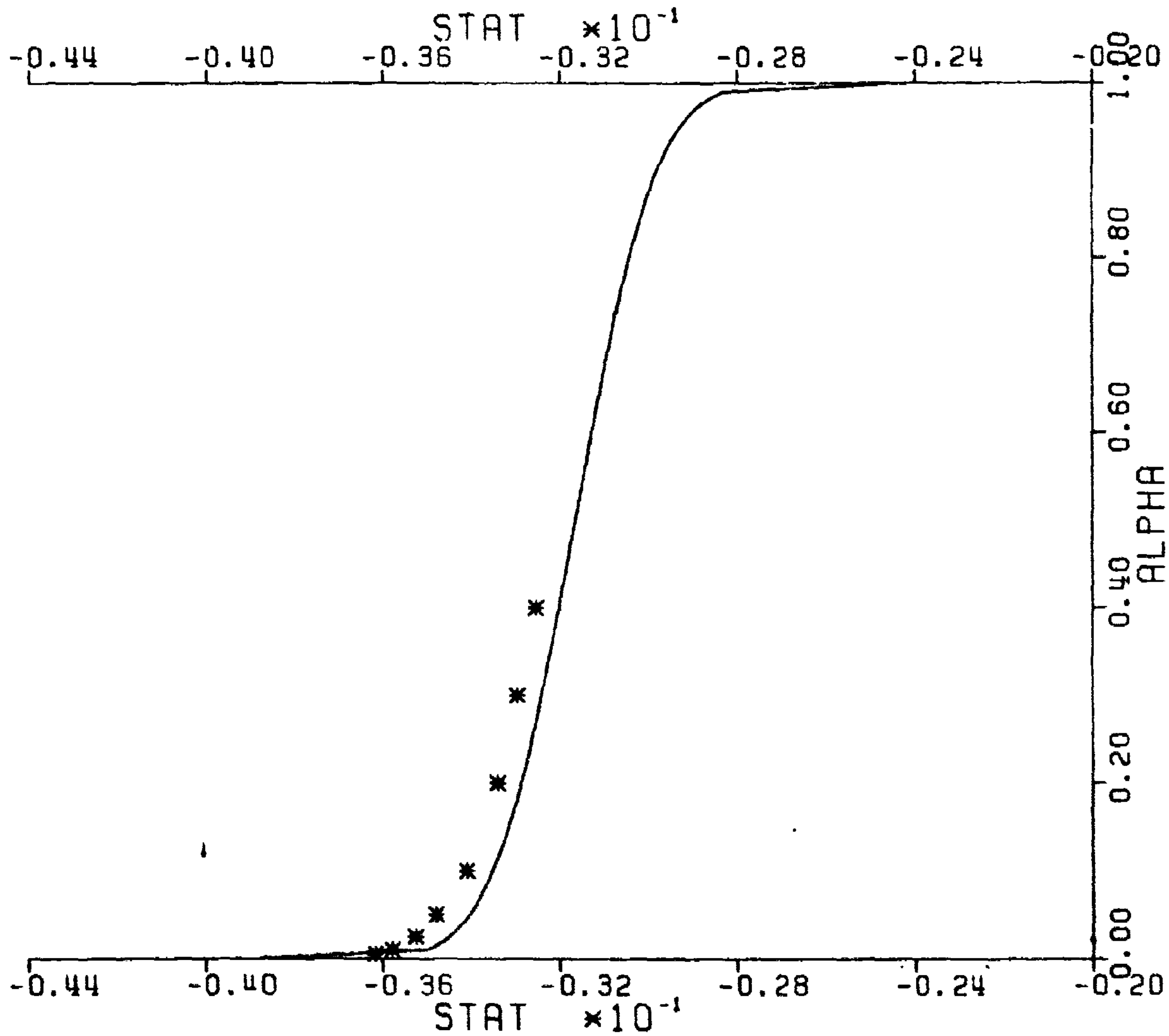

Figure 2i.

N= 10000

M= 9

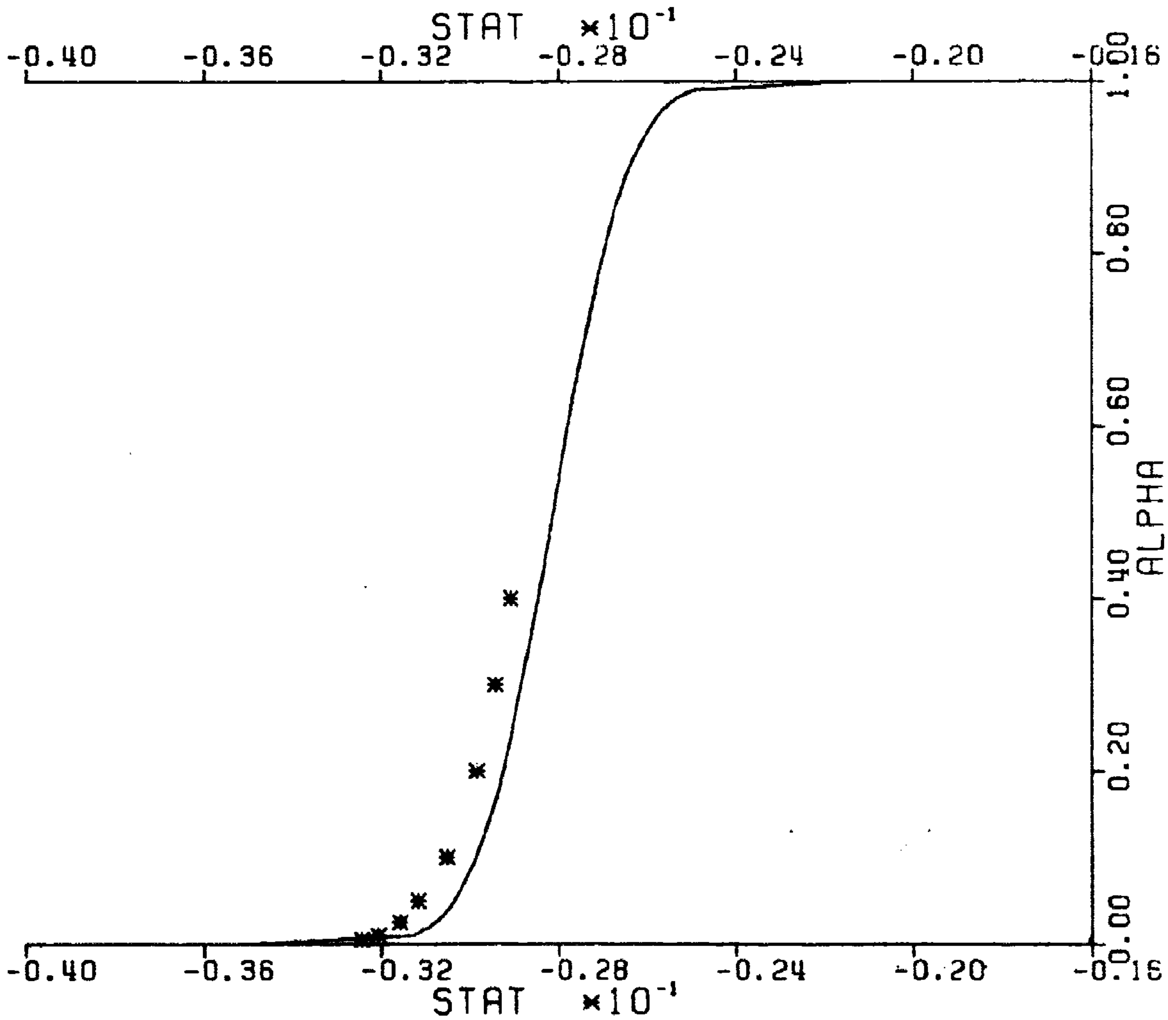

Figure 2j.

N= 10000

M= 10

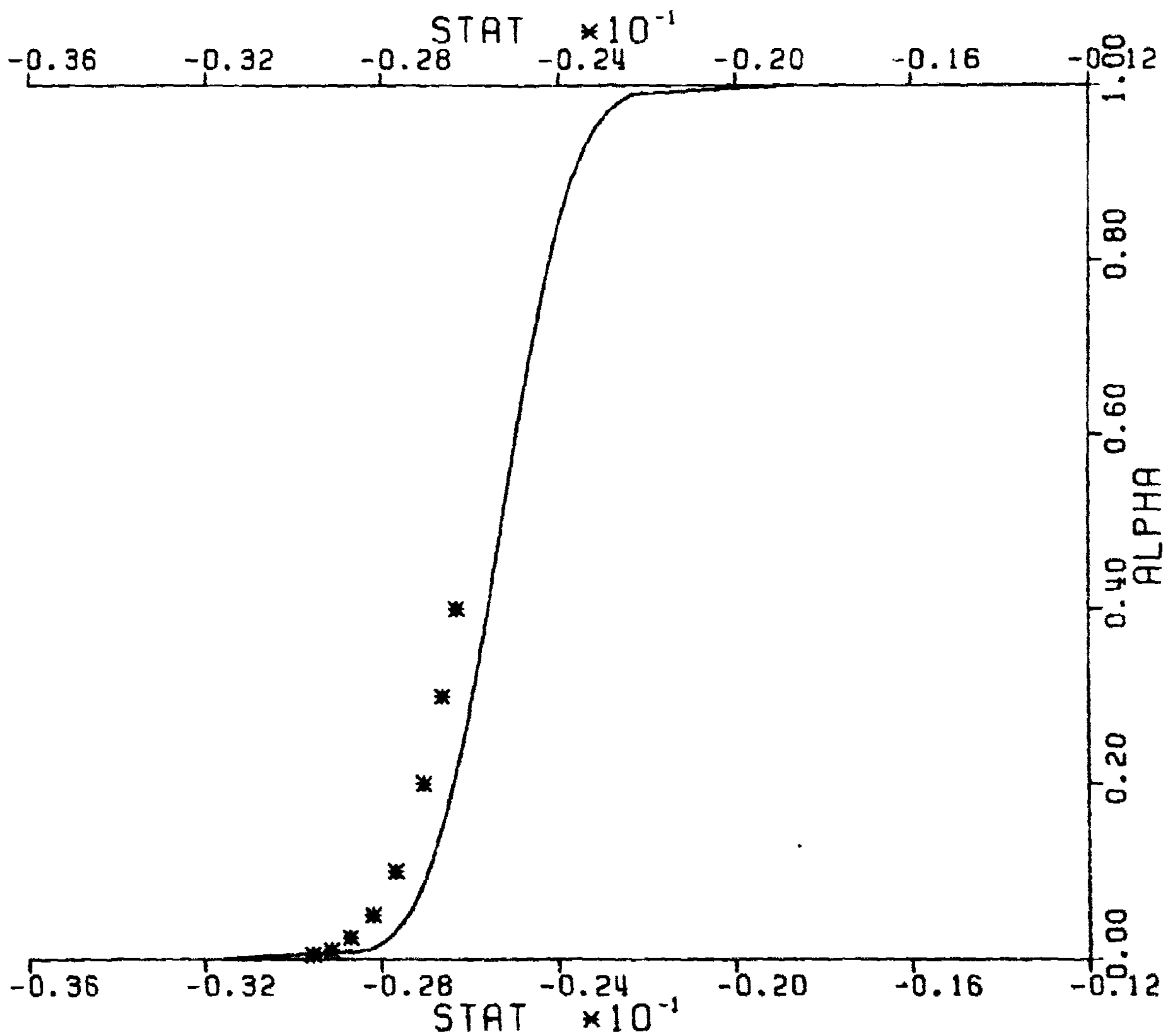

Figure 2k.

N= 10000

M= 15

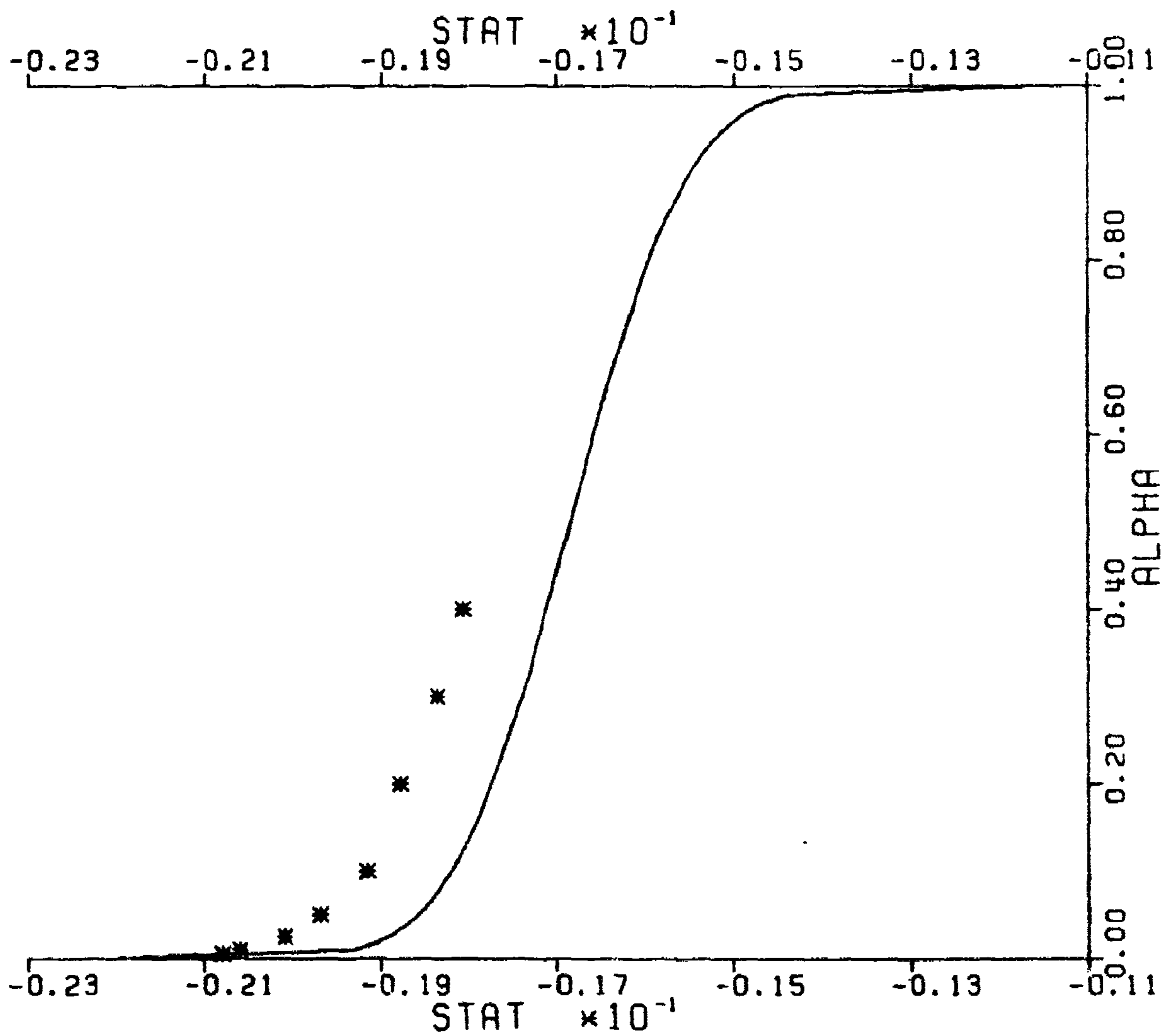

Figure 21.

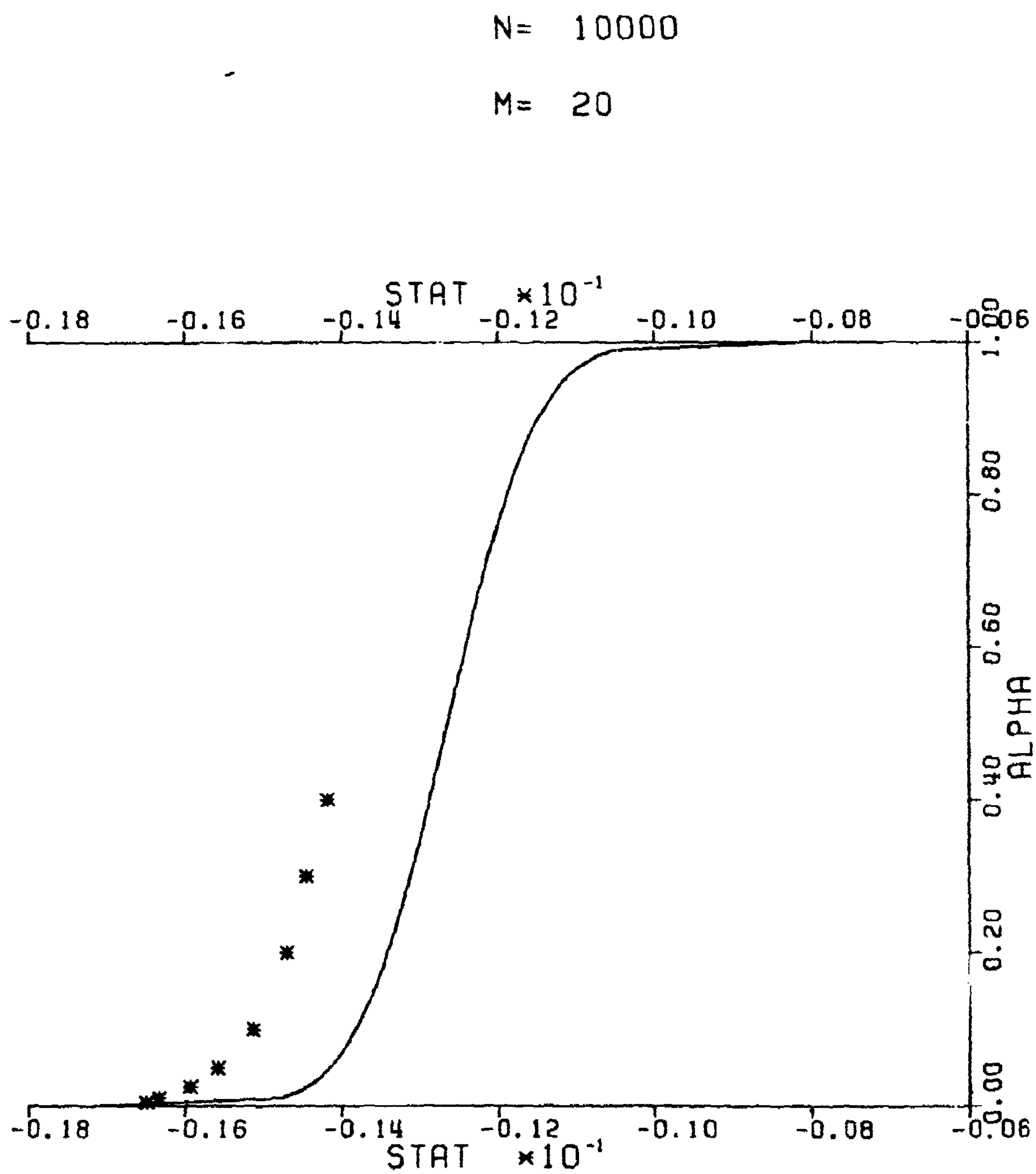
N= 10000
M= 20
STAT ×10^-1
-0.18
-0.16
-0.14
-0.12
-0.10
-0.08
-0.06
1.00
0.80
0.60
0.40
0.20
0.00
ALPHA
STAT ×10^-1

Figure 2m.

N= 10000

M= 30

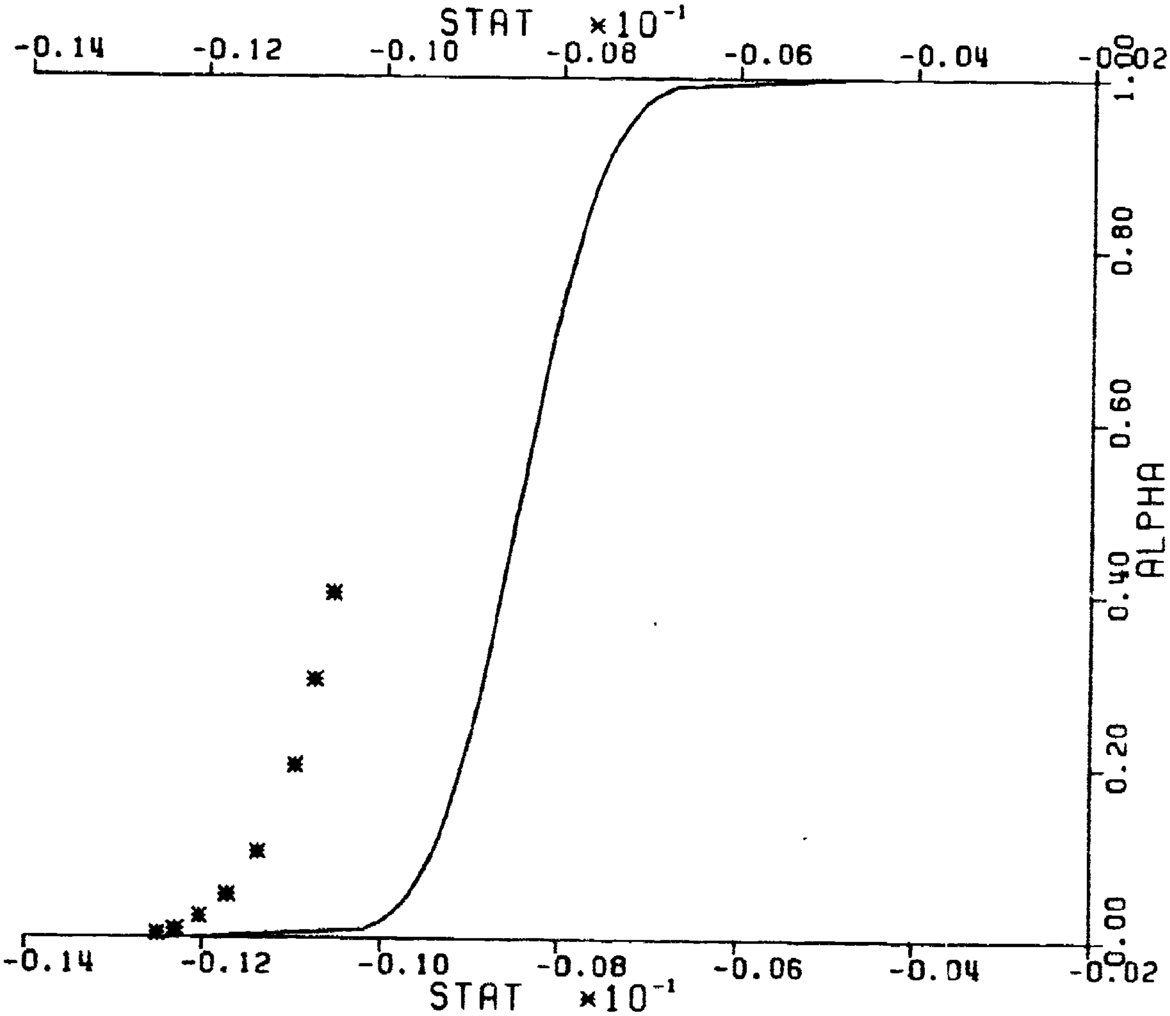

Figure 2n.

N= 10000

M= 40

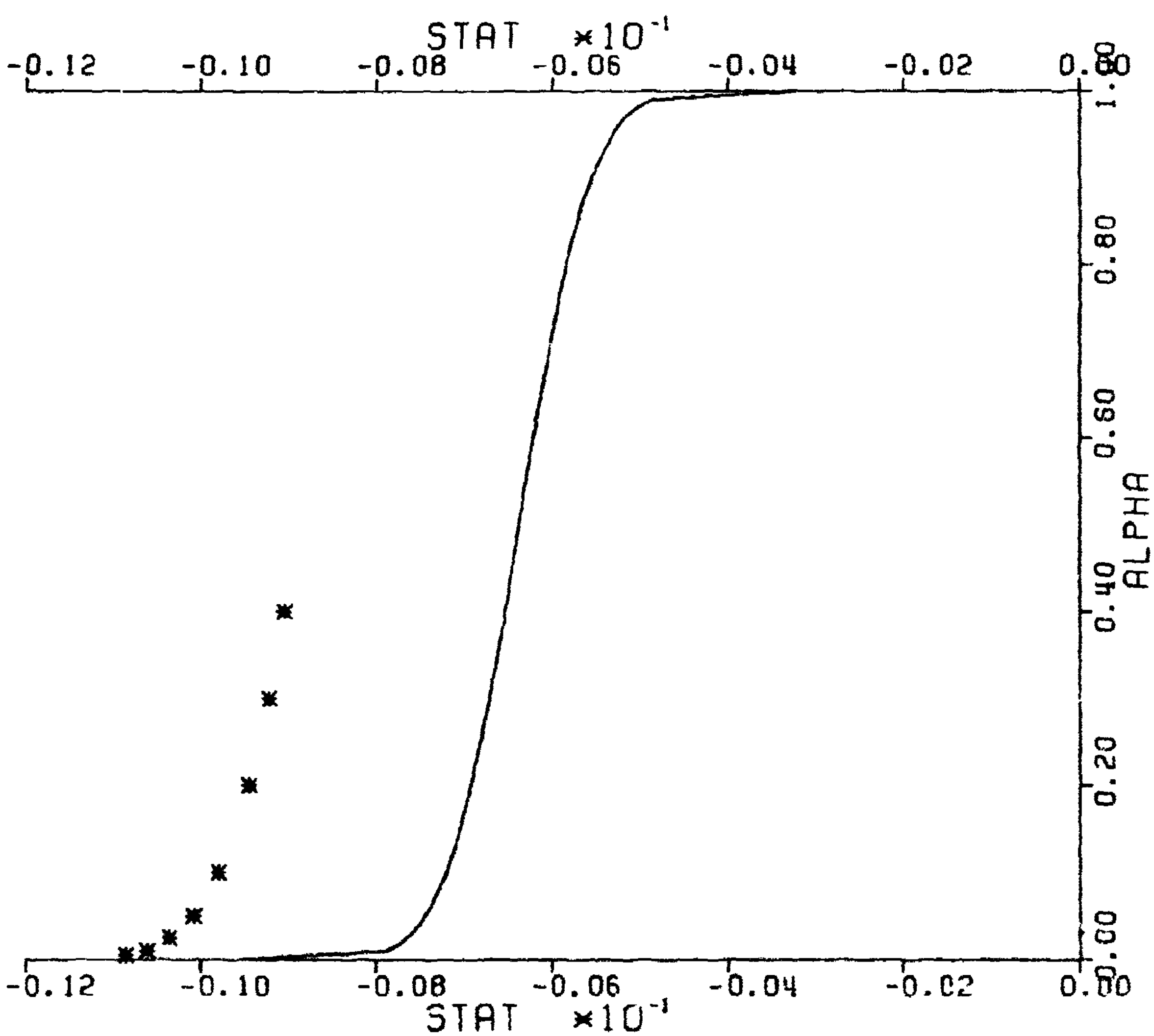

Figure 2o.

N= 10000

M= 50

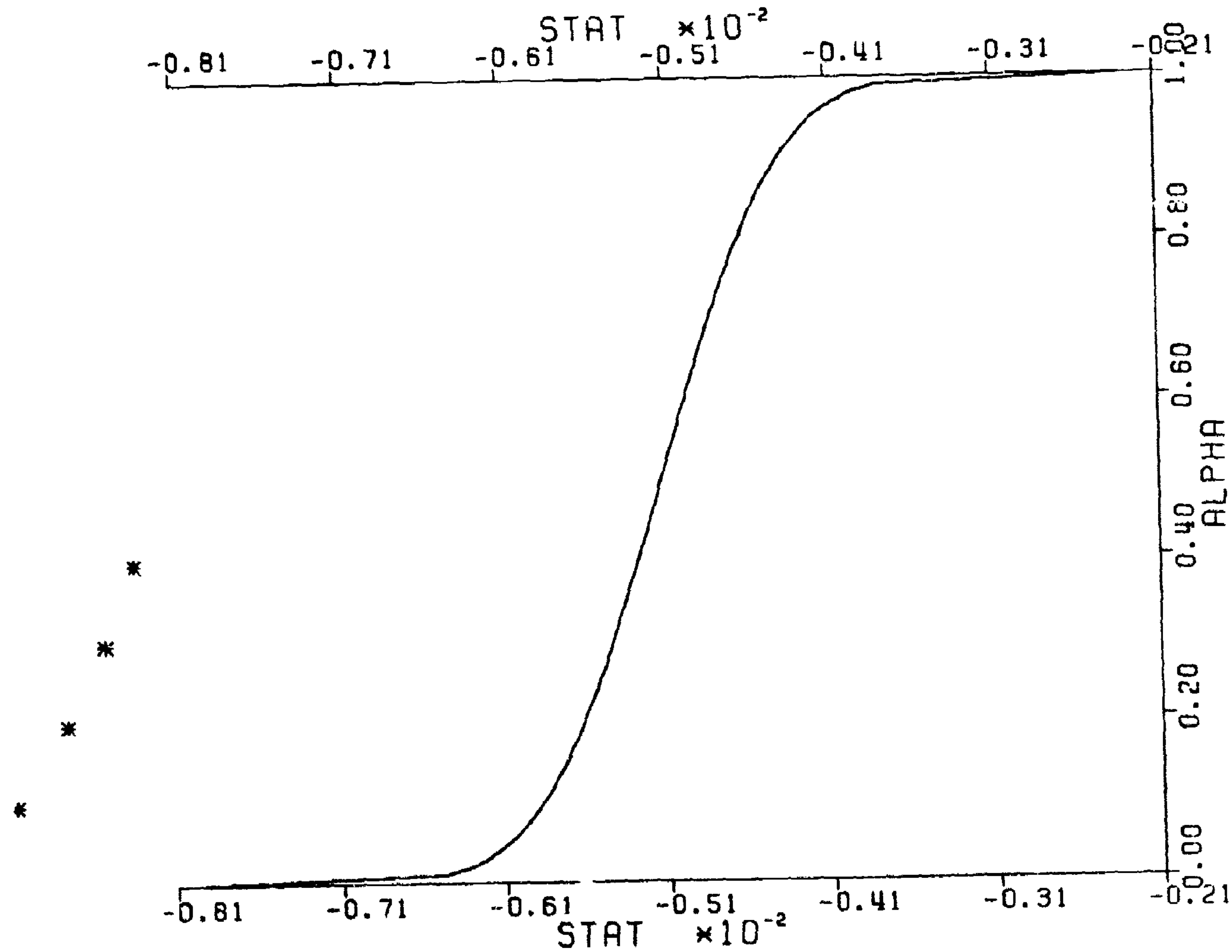

Figure 2p.

N= 10000

M= 250

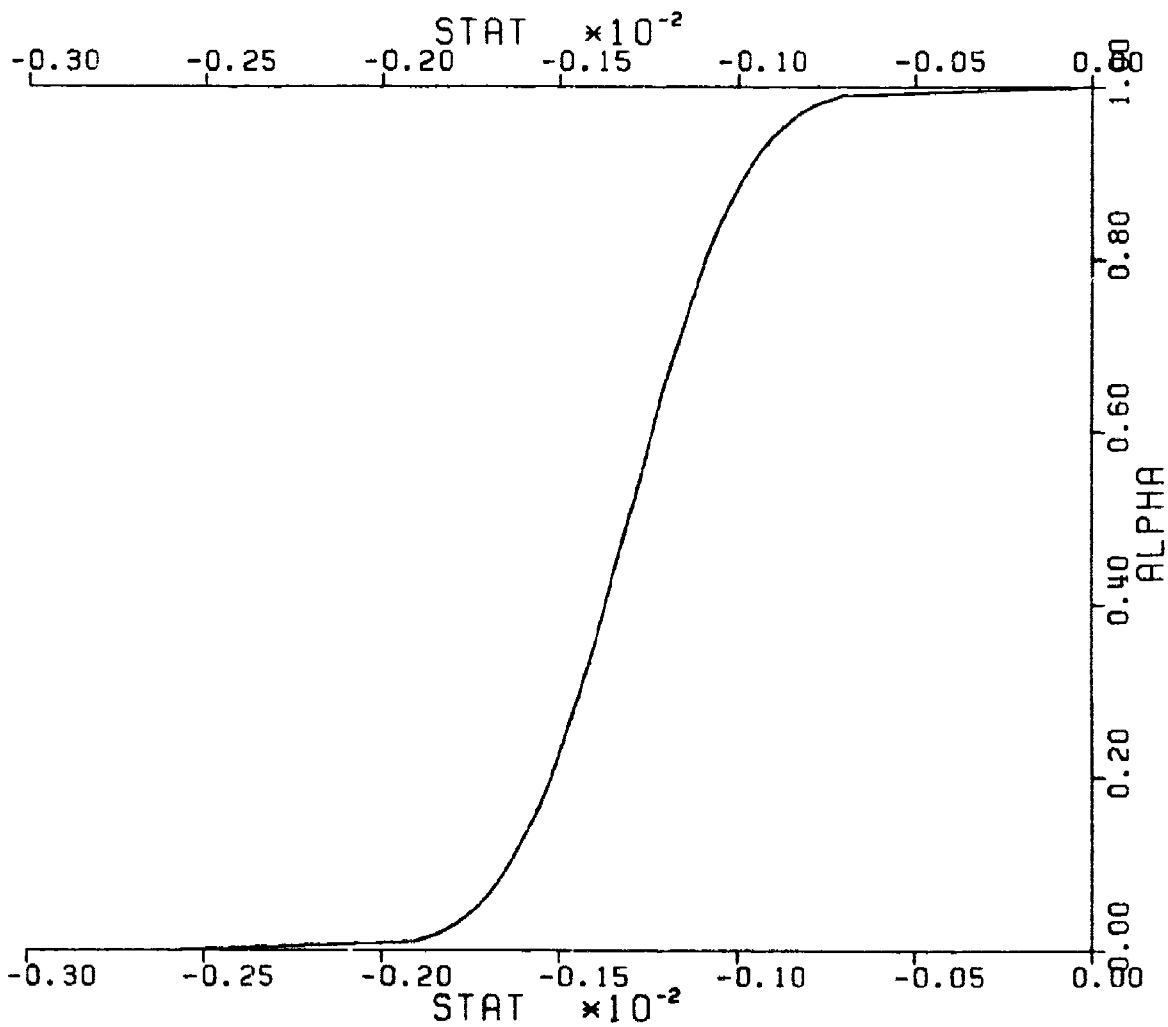

TABLE 1

COMPARISON OF MONTE CARLO(1000 REPLICATIONS,N=10000)

AND

ASYMPTOTIC CUTOFFS OF H(M,N),N=10000

M	ALPHA	H(M,N)	A-H(M,N)	ALPHA	H(M,N)	A-H(M,N)	ALPHA	H(M,N)	A-H(M,N)
1	0.4000	-0.271780	-0.271479	0.2000	-0.274281	-0.273881	0.0500	-0.278373	-0.277162
2	0.4000	-0.131208	-0.130990	0.2000	-0.132897	-0.132689	0.0500	-0.135420	-0.135009
3	0.4000	-0.086597	-0.086322	0.2000	-0.087994	-0.087708	0.0500	-0.089921	-0.089603
4	0.4000	-0.064700	-0.064401	0.2000	-0.065933	-0.065602	0.0500	-0.067693	-0.067243
5	0.4000	-0.051635	-0.051379	0.2000	-0.052815	-0.052453	0.0500	-0.054529	-0.053921
6	0.4000	-0.043078	-0.042751	0.2000	-0.044126	-0.043733	0.0500	-0.045701	-0.045073
7	0.4000	-0.037020	-0.036616	0.2000	-0.038011	-0.037524	0.0500	-0.039466	-0.038764
8	0.4000	-0.032545	-0.032027	0.2000	-0.033395	-0.032877	0.0500	-0.034794	-0.034037
9	0.4000	-0.029083	-0.028466	0.2000	-0.029844	-0.029267	0.0500	-0.031164	-0.030361
10	0.4000	-0.026313	-0.025623	0.2000	-0.027061	-0.026383	0.0500	-0.028201	-0.027421
15	0.4000	-0.018074	-0.017118	0.2000	-0.018766	-0.017739	0.0500	-0.019689	-0.018586
20	0.4000	-0.014175	-0.012878	0.2000	-0.014703	-0.013415	0.0500	-0.015572	-0.014149
30	0.4000	-0.010538	-0.008647	0.2000	-0.010956	-0.009086	0.0500	-0.011720	-0.009685
40	0.4000	-0.009040	-0.006537	0.2000	-0.009435	-0.006917	0.0500	-0.010072	-0.007436
1	0.3000	-0.272869	-0.272585	0.1000	-0.276558	-0.275679	0.0100	-0.282290	-0.279945
2	0.3000	-0.131953	-0.131773	0.1000	-0.134215	-0.133960	0.0100	-0.137528	-0.136977
3	0.3000	-0.087202	-0.086961	0.1000	-0.089042	-0.088746	0.0100	-0.091378	-0.091209
4	0.3000	-0.065211	-0.064934	0.1000	-0.066869	-0.066501	0.0100	-0.069138	-0.068634
5	0.3000	-0.052163	-0.051874	0.1000	-0.053676	-0.053257	0.0100	-0.055964	-0.055165
6	0.3000	-0.043560	-0.043203	0.1000	-0.044932	-0.044466	0.0100	-0.046938	-0.046208
7	0.3000	-0.037487	-0.037034	0.1000	-0.038720	-0.038203	0.0100	-0.040416	-0.039816
8	0.3000	-0.032958	-0.032419	0.1000	-0.034095	-0.033512	0.0100	-0.035784	-0.035021
9	0.3000	-0.029432	-0.028835	0.1000	-0.030529	-0.029866	0.0100	-0.032062	-0.031289
10	0.3000	-0.026648	-0.025973	0.1000	-0.027678	-0.026952	0.0100	-0.029140	-0.028300
15	0.3000	-0.018357	-0.017405	0.1000	-0.019164	-0.018203	0.0100	-0.020603	-0.019304
20	0.3000	-0.014436	-0.013125	0.1000	-0.015118	-0.013817	0.0100	-0.016334	-0.014771
30	0.3000	-0.010735	-0.008849	0.1000	-0.011366	-0.009414	0.0100	-0.012298	-0.010193
40	0.3000	-0.009206	-0.006712	0.1000	-0.009788	-0.007201	0.0100	-0.010610	-0.007875

TABLE 2

MONTE CARLO ESTIMATES OF H(M,N)-CUTOFFS,N=1000

NUMBER OF REPLICATIONS=10000

M	ALPHA	H(M,N)	ALPHA	H(M,N)	ALPHA	H(M,N)
1	0.4000	-0.276535	0.1000	-0.292290	0.0100	-0.309535
2	0.4000	-0.135315	0.1000	-0.146162	0.0100	-0.157613
3	0.4000	-0.090789	0.1000	-0.099620	0.0100	-0.109258
4	0.4000	-0.069154	0.1000	-0.076970	0.0100	-0.085471
5	0.4000	-0.056645	0.1000	-0.063591	0.0100	-0.071194
6	0.4000	-0.048560	0.1000	-0.054878	0.0100	-0.061655
7	0.4000	-0.042938	0.1000	-0.048749	0.0100	-0.055375
8	0.4000	-0.038886	0.1000	-0.044359	0.0100	-0.050749
9	0.4000	-0.035909	0.1000	-0.041058	0.0100	-0.047173
10	0.4000	-0.033625	0.1000	-0.038516	0.0100	-0.044410
15	0.4000	-0.028035	0.1000	-0.032257	0.0100	-0.037207
20	0.4000	-0.026820	0.1000	-0.030619	0.0100	-0.035135
30	0.4000	-0.028659	0.1000	-0.032112	0.0100	-0.036311
40	0.4000	-0.032706	0.1000	-0.035957	0.0100	-0.040148
50	0.4000	-0.037586	0.1000	-0.040721	0.0100	-0.044701
250	0.4000	-0.156811	0.1000	-0.161272	0.0100	-0.165923
1	0.3000	-0.280413	0.0500	-0.298241	0.0050	-0.313690
2	0.3000	-0.138010	0.0500	-0.149914	0.0050	-0.160344
3	0.3000	-0.093015	0.0500	-0.102708	0.0050	-0.111870
4	0.3000	-0.071126	0.0500	-0.079688	0.0050	-0.088101
5	0.3000	-0.058290	0.0500	-0.066112	0.0050	-0.073556
6	0.3000	-0.050087	0.0500	-0.057186	0.0050	-0.063835
7	0.3000	-0.044323	0.0500	-0.051053	0.0050	-0.057171
8	0.3000	-0.040243	0.0500	-0.046514	0.0050	-0.052227
9	0.3000	-0.037199	0.0500	-0.043125	0.0050	-0.048859
10	0.3000	-0.034807	0.0500	-0.040479	0.0050	-0.046037
15	0.3000	-0.029124	0.0500	-0.033953	0.0050	-0.038499
20	0.3000	-0.027781	0.0500	-0.032184	0.0050	-0.036544
30	0.3000	-0.029529	0.0500	-0.033442	0.0050	-0.037570
40	0.3000	-0.033477	0.0500	-0.037209	0.0050	-0.040946
50	0.3000	-0.038332	0.0500	-0.042046	0.0050	-0.045758
250	0.3000	-0.157944	0.0500	-0.162971	0.0050	-0.167206
1	0.2000	-0.285486	0.0250	-0.303572		
2	0.2000	-0.141417	0.0250	-0.153489		
3	0.2000	-0.095727	0.0250	-0.105685		
4	0.2000	-0.073571	0.0250	-0.082191		
5	0.2000	-0.060507	0.0250	-0.068317		
6	0.2000	-0.052066	0.0250	-0.059304		
7	0.2000	-0.046198	0.0250	-0.052992		
8	0.2000	-0.041991	0.0250	-0.048260		
9	0.2000	-0.038776	0.0250	-0.044829		
10	0.2000	-0.036416	0.0250	-0.042167		
15	0.2000	-0.030429	0.0250	-0.035420		
20	0.2000	-0.028920	0.0250	-0.033492		
30	0.2000	-0.030584	0.0250	-0.034678		
40	0.2000	-0.034461	0.0250	-0.038350		
50	0.2000	-0.039314	0.0250	-0.043144		
250	0.2000	-0.159343	0.0250	-0.164369		

TABLE 3

MONTE CARLO ESTIMATES OF H(M,N)-CUTOFFS,N=10000

NUMBER OF REPLICATIONS=1000

M	ALPHA	H(M,N)	ALPHA	H(M,N)	ALPHA	H(M,N)
1	0.4000	-0.271780	0.1000	-0.276558	0.0100	-0.282290
2	0.4000	-0.131208	0.1000	-0.134215	0.0100	-0.137528
3	0.4000	-0.086597	0.1000	-0.089042	0.0100	-0.091378
4	0.4000	-0.064700	0.1000	-0.066869	0.0100	-0.069138
5	0.4000	-0.051635	0.1000	-0.053676	0.0100	-0.055964
6	0.4000	-0.043078	0.1000	-0.044932	0.0100	-0.046938
7	0.4000	-0.037029	0.1000	-0.038720	0.0100	-0.040416
8	0.4000	-0.032545	0.1000	-0.034095	0.0100	-0.035784
9	0.4000	-0.029083	0.1000	-0.030529	0.0100	-0.032062
10	0.4000	-0.026313	0.1000	-0.027678	0.0100	-0.029140
15	0.4000	-0.018074	0.1000	-0.019164	0.0100	-0.020603
20	0.4000	-0.014175	0.1000	-0.015118	0.0100	-0.016334
30	0.4000	-0.010538	0.1000	-0.011366	0.0100	-0.012298
40	0.4000	-0.009040	0.1000	-0.009788	0.0100	-0.010610
50	0.4000	-0.008346	0.1000	-0.009084	0.0100	-0.009864
250	0.4000	-0.016589	0.1000	-0.017086	0.0100	-0.017630
500	0.4000	-0.031464	0.1000	-0.032089	0.0100	-0.032759
2500	0.4000	-0.153856	0.1000	-0.155276	0.0100	-0.156482
1	0.3000	-0.272869	0.0500	-0.278373	0.0050	-0.282546
2	0.3000	-0.131953	0.0500	-0.135420	0.0050	-0.137884
3	0.3000	-0.087202	0.0500	-0.089921	0.0050	-0.092385
4	0.3000	-0.065211	0.0500	-0.067693	0.0050	-0.069680
5	0.3000	-0.052163	0.0500	-0.054529	0.0050	-0.056263
6	0.3000	-0.043560	0.0500	-0.045701	0.0050	-0.047343
7	0.3000	-0.037487	0.0500	-0.039466	0.0050	-0.040877
8	0.3000	-0.032958	0.0500	-0.034794	0.0050	-0.036188
9	0.3000	-0.029432	0.0500	-0.031164	0.0050	-0.032433
10	0.3000	-0.026648	0.0500	-0.028201	0.0050	-0.029566
15	0.3000	-0.018357	0.0500	-0.019689	0.0050	-0.020795
20	0.3000	-0.014436	0.0500	-0.015572	0.0050	-0.016493
30	0.3000	-0.010735	0.0500	-0.011720	0.0050	-0.012498
40	0.3000	-0.009206	0.0500	-0.010072	0.0050	-0.010854
50	0.3000	-0.008526	0.0500	-0.009335	0.0050	-0.010183
250	0.3000	-0.016727	0.0500	-0.017324	0.0050	-0.017792
500	0.3000	-0.031602	0.0500	-0.032259	0.0050	-0.032962
2500	0.3000	-0.154178	0.0500	-0.155655	0.0050	-0.156909
1	0.2000	-0.274281	0.0250	-0.280814		
2	0.2000	-0.132897	0.0250	-0.136606		
3	0.2000	-0.087994	0.0250	-0.090697		
4	0.2000	-0.065933	0.0250	-0.068416		
5	0.2000	-0.052815	0.0250	-0.055066		
6	0.2000	-0.044126	0.0250	-0.046288		
7	0.2000	-0.038011	0.0250	-0.039990		
8	0.2000	-0.033395	0.0250	-0.033252		
9	0.2000	-0.029844	0.0250	-0.031588		
10	0.2000	-0.027061	0.0250	-0.028708		
15	0.2000	-0.018766	0.0250	-0.020088		
20	0.2000	-0.014703	0.0250	-0.015932		
30	0.2000	-0.010936	0.0250	-0.012028		
40	0.2000	-0.009435	0.0250	-0.010347		
50	0.2000	-0.008767	0.0250	-0.009539		
250	0.2000	-0.016889	0.0250	-0.017492		
500	0.2000	-0.031808	0.0250	-0.032462		
2500	0.2000	-0.154662	0.0250	-0.155928		

TABLE 4

ASYMPTOTIC POWER APPROXIMATION EVALUATION
(ALTERNATIVE : F, ALPHA = .05)

	10000 Replicates		10000 Replicates		1000 Replicates	
n	100		1000		10000	
m	Asymptotic Power Using Simulated Cutoffs	Simulated Power	Asymptotic Power Using Simulated Cutoffs	Simulated Power	Asymptotic Power Using Simulated Cutoffs	Simulated Power
1	0.009544	0.0496	0.031386	0.0499	0.045248	0.0560
2	0.001974	0.0500	0.021571	0.0519	0.044494	0.0540
3	0.000220	0.0501	0.014028	0.0521	0.040716	0.0590
4	0.000014	0.0508	0.008446	0.0507	0.032091	0.0620
5	0.0	0.0498	0.004879	0.0525	0.024144	0.0530
6	0.0	0.0501	0.002911	0.0523	0.022051	0.0560
7	0.0	0.0505	0.001660	0.0516	0.019433	0.0610
8	0.0	0.0497	0.001054	0.0528	0.018450	0.0590
9	0.0	0.0495	0.000643	0.0528	0.017837	0.0610
10	0.0	0.0501	0.000494	0.0537	0.021225	0.0680
15	0.0	0.0497	0.000766	0.0527	0.045716	0.0660
20	0.0	0.0500	0.003139	0.0518	0.095576	0.0650
30	0.0	0.0498	0.025842	0.0532	0.211254	0.0710
40	0.0	0.0499	0.05811	0.0531	0.278412	0.0800

Table 5

m	UNI	KERAND RN2 RN4	RNCG	RN1 SRAND	RN3
1	6.0000	11.00	16.40	12.20	7.40
2	4.8000	6.40	10.20	10.60	10.60
3	5.6000	6.00	6.60	8.00	11.60
4	14.4000	8.00	7.60	10.40	19.20*