

AD-A099 752

AERONAUTICAL RESEARCH ASSOCIATES OF PRINCETON INC NJ F/G 20/4
LONG INTERNAL WAVES OF MODERATE AMPLITUDE. II. VISCOUS DECAY OF--ETC(U)
MAY 81 C LEONE, H SEGUR N00014-80-C-0048

UNCLASSIFIED

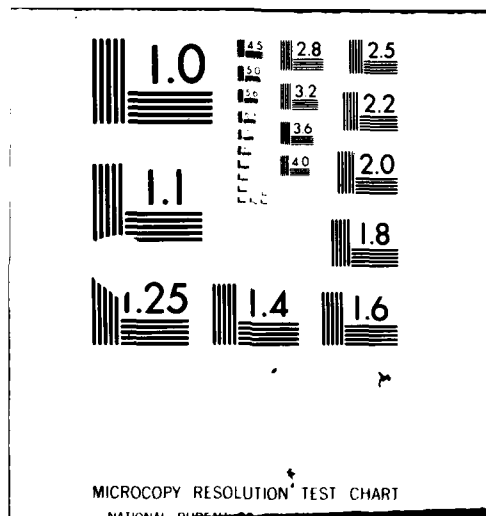
ARAP-TM-81-9

NL

1 of 1
A/A



END
DATE
FILMED
6-81
DTIC



AD A099752

A.R.A.P. Tech. Memo 81-9

Long Internal Waves of Moderate Amplitude
II. Viscous Decay of Solitary Waves

N00014-80-C-0048

by

C. Leone
Hingham High School
Hingham, Massachusetts

and

Harvey Segur
Aeronautical Research Associates of Princeton, Inc.
50 Washington Road, P.O. Box 2229
Princeton, New Jersey 08540

DTIC
ELECTE
S JUN 5 1981
A

May 1981

This document has been approved
for public release and sale; its
distribution is unlimited.

81 5 08 018

DTIC FILE COPY

ABSTRACT

We derive a formula for the viscous decay of long internal solitary waves that propagate into a quiescent fluid in a two-layer model. The result is analogous to Keulegan's (1948) formula for the viscous decay of long surface waves. The requirement that the fluid ahead of the wave be quiescent is important, and we show experimentally that the accuracy of the formula decreases significantly if the internal waves are preceded by faster-traveling surface waves.

Section For
 GRADE
 T.B.
 Inspected
 Little's file
 Special

1 Introduction and Analysis

Solitary water waves measured in laboratory wave tanks are known to exhibit appreciable decay due to viscous boundary layers on the walls of the tank. Keulegan (1948) developed an approximate theory to predict this viscous decay of surface solitary waves.

Long internal waves resemble long surface waves in many respects, and Figure 2 of the preceding article (hereafter called I) demonstrates that long internal waves also experience significant viscous dissipation. If the pycnocline is thin, then the dissipation of long internal waves is even stronger than that of long surface waves, because there is an additional viscous boundary layer at the interface of the two fluids. Thus, to account for the slow viscous dissipation of an internal KdV soliton, it is necessary not only to make relatively minor changes in Keulegan's analysis of the boundary layers on the walls of the wave tank, but also to account for the dissipation at the interface.

The purpose of this paper is to study the viscous dissipation of an internal KdV soliton as it propagates down a laboratory wave tank. The analysis is based on the original work of Keulegan (1948). The major assumptions are the following.

- 1) There are two fluids of different densities in a stably stratified configuration bounded below by a rigid horizontal bed and above by a free surface (see Figure 1 of I).
- 11) The inviscid flow is that of a two-dimensional internal wave in the form of a KdV soliton, traveling with a constant speed c . Consequently we may write $\partial/\partial t = -c \partial/\partial \xi$, where $\xi = x - ct$. Moreover,

the viscous time-scale is assumed to be slow in comparison with the KdV time-scale.

- iii) Nonlinear effects may be neglected in the thin viscous boundary layer, even though they are important in establishing the inviscid flow.
- iv) All streamlines may be considered flat and horizontal for the purpose of the viscous analysis.
- v) Only shear parallel to the interface (which is now considered flat) is of consequence.
- vi) Diffusion of species is neglected, as is compressibility.

With these assumptions, one may show that the larger and smaller roots of

$$(c^2/g)^2 - (c^2/g)(h_1 + h_2) + \Delta h_1 h_2 = 0 \quad (1)$$

define the propagation speeds of the long surface and internal waves, respectively, in the linear inviscid limit (see I or Lamb, 1932, section 231). Here g is the constant gravitational force, $\Delta = 1 - \rho_1/\rho_2$ is the dimensionless density difference, and h_1, h_2 are the upper and lower fluid depths, respectively. We assume that the two sets of roots are distinct, which is guaranteed if either $\Delta \neq 1$ or $h_1 \neq h_2$. Let c_i denote a solution of (1) corresponding to an internal wave.

The linear theory also defines the horizontal velocities in the two layers in terms of the displacement of the interface. In dimensional variables, the velocities of the lower and upper fluids, respectively, for a long internal wave are

$$u_2(x,t) \sim \frac{c_1}{h_2} \eta(x - c_1 t), \quad u_1 \sim - \frac{c_1}{h_1} \left(1 - \frac{c_1^2}{gh_1}\right)^{-1} \eta(x - c_1 t), \quad (2)$$

where η is the displacement of the interface. These provide the "outer" inviscid solution, which drives the viscous boundary layer at the interface.

The KdV equation governs the evolution on the next time-scale of the long internal wave traveling to the right. The dimensional equation is

$$\alpha c_1^{-1} \frac{\partial \eta}{\partial t} + \alpha \frac{\partial \eta}{\partial x} + \beta \eta \frac{\partial \eta}{\partial x} + \gamma \frac{\partial^3 \eta}{\partial x^3} = 0, \quad (3)$$

where

$$\alpha = \frac{h_1(\rho_2 h_1 - \rho_2 h_2 + 2\rho_1 h_2) + \rho_2(h_2 - h_1)(c_1^2/g)}{\rho_1 h_1(h_1 + h_2)}$$

$$\beta = \frac{3}{2h_1 h_2(h_1 + h_2) \rho_1 [h_1 - (c_1^2/g)]} \times$$

$$\times \left[h_1(h_1 - h_2)(\rho_2 h_1 - \rho_2 h_2 + 2\rho_1 h_2) - \rho_1 h_1 h_2 \left(\frac{c_1^2}{g}\right) - \rho_2(h_1 - h_2)^2 \left(\frac{c_1^2}{g}\right) \right]$$

$$\gamma = \frac{h_2}{4h_1(h_1+h_2)} \left[\frac{\rho_2 h_2}{\rho_1} \left(h_1 - \frac{c_f^2}{g} \right)^2 \left(\frac{g \Delta h_2}{c_f^2} - \frac{1}{3} \right) + \right. \\ \left. + \frac{h_1(2h_1^2 - 6\Delta h_1 h_2 - 3h_2^2)}{3} + h_2(2h_1 + h_2) \left(\frac{c_f^2}{g} \right) \right].$$

In the Boussinesq limit ($\Delta \rightarrow 0$, $g\Delta$ finite), $c_f^2 \rightarrow g\Delta h_1 h_2 / (h_1 + h_2)$, $\alpha \rightarrow 1$, $\beta \rightarrow \frac{3}{2} [h_2^{-1} - h_1^{-1}]$, $\gamma \rightarrow h_1 h_2 / 6$, and (3) reduces to equation (12) of I.

As a further check on the coefficients in (3), we note three limits in which (1) and (3) reduce to the dimensional form of the KdV equation for surface waves on a homogeneous fluid:

$$c^2 = gh,$$

$$(gh)^{-1/2} \frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} + \frac{3}{2} \frac{\zeta}{h} \frac{\partial \zeta}{\partial x} + \frac{h^2}{6} \frac{\partial^3 \zeta}{\partial x^3} = 0. \quad (4)$$

These limits are:

$$\begin{aligned} \text{i) } \Delta \rightarrow 0, \quad \frac{c^2}{g} &\rightarrow (h_1 + h_2), \quad \frac{n}{h_2} = \frac{\zeta}{h_1 + h_2}; \\ \text{ii) } \rho_1 \rightarrow 0, \quad \frac{c^2}{g} &\rightarrow h_2; \\ \text{iii) } h_1 \rightarrow 0, \quad \frac{c^2}{g} &\rightarrow h_2. \end{aligned} \quad (5)$$

Djordjevic and Redekopp (1978) also derived a KdV equation for long internal waves without the Boussinesq limit. Their results differ from those in (3) except in the limit $\Delta \rightarrow 0$, because they assumed a rigid lid on top, whereas (3) is based on having a free surface there. The boundary condition at the upper surface is unimportant only if $\Delta \ll 1$.

We turn now to the problem of the viscous boundary layer at the interface. In dimensional form, the viscous equations above and below the dividing streamline are

$$-c_1 \frac{\partial}{\partial \xi} (\hat{u}_1 - u_1) = \nu_1 \frac{\partial^2}{\partial z^2} (\hat{u}_1 - u_1), \quad z > 0, \quad (6)$$

$$-c_1 \frac{\partial}{\partial \xi} (\hat{u}_2 - u_2) = \nu_2 \frac{\partial^2}{\partial z^2} (\hat{u}_2 - u_2), \quad z < 0,$$

where $\xi = x - c_1 t$, $u_1(\xi)$, $u_2(\xi)$ are the inviscid horizontal velocities in the upper and lower fluid, $\hat{u}_1(\xi, z)$, $\hat{u}_2(\xi, z)$ are the solutions of the viscous equations in the two layers, and ν_1 , ν_2 are the two kinematic viscosities. The boundary conditions are that

$$\hat{u}_1 \rightarrow u_1 \quad \text{as} \quad z \rightarrow +\infty, \quad (7a)$$

$$\hat{u}_2 \rightarrow u_2 \quad \text{as} \quad z \rightarrow -\infty,$$

and at $z = 0$,

$$\hat{u}_1 = \hat{u}_2, \quad \mu_1 \frac{\partial \hat{u}_1}{\partial z} = \mu_2 \frac{\partial \hat{u}_2}{\partial z} \quad (7b)$$

where μ_1, μ_2 are the two dynamic viscosities. The solutions of (6) may be written in the form of the Duhamel integral solution of the heat equation:

$$\hat{u}_1 - u_1 = -\frac{2}{\sqrt{\pi}} \int_0^\infty F_1[\xi + (cz^2/4\nu_1 \lambda^2)] \exp(-\lambda^2) d\lambda \quad (8)$$

$$\hat{u}_2 - u_2 = -\frac{2}{\sqrt{\pi}} \int_0^\infty F_2[\xi + (cz^2/4\nu_2 \lambda^2)] \exp(-\lambda^2) d\lambda$$

where

$$F_1(\xi) = \frac{\mu_2(\nu_1)^{1/2}}{\mu_1(\nu_2)^{1/2} + \mu_2(\nu_1)^{1/2}} [u_1(\xi) - u_2(\xi)], \quad (9)$$

$$F_2(\xi) = -\frac{\mu_1(\nu_2)^{1/2}}{\mu_1(\nu_2)^{1/2} + \mu_2(\nu_1)^{1/2}} [u_1(\xi) - u_2(\xi)],$$

and we require that $u_1(\xi), u_2(\xi)$ be localized enough that the integrals in

(8) converge.

Next we need an expression for the rate of energy loss due to viscous effects at the interface. Consistent with assumption (v) above, the local dissipation function near the interface in each layer is

$$\phi_j(\xi, z) = \nu_j \left(\frac{\partial u_j}{\partial z} \right)^2, \quad j = 1, 2 \quad (10)$$

The total rate of dissipation due to the boundary layer at the interface is obtained by integrating ϕ_j over the volume of that boundary layer. Integrating once by parts and using (6), (7) shows that the rate of energy loss due to the boundary layer at the interface is

$$\frac{dE}{dt} = W \int_{-\infty}^{\infty} (u_1 - u_2) [\nu_j (\partial \hat{u}_j / \partial z)] \Big|_{z=0} d\xi, \quad j = 1 \text{ or } 2, \quad (11)$$

where W is the total width of the channel. Keulegan (1948) pointed out that (11) represents both energy dissipation in the boundary layer and kinetic energy left behind in the channel. We will discuss the latter effect again in Section 2.

Energy is also lost in the boundary layers on the walls of the wave tank. Keulegan (1948) showed that the rate of energy loss in each of these boundary layers is

$$\frac{dE}{dt} = (\Delta z) \int_{-\infty}^{\infty} u_0 [\nu (\partial u / \partial y)] d\xi, \quad (12)$$

where Δz is the width of the layer, u_0 is the inviscid velocity outside the layer, and the shear stress is evaluated at the wall. The total rate of energy loss is obtained by combining these separate contributions, including (11).

We now refer specifically to a soliton solution of (3) in order to define the inviscid velocities in (11) and (12). The expression for the total rate of energy loss per width of tank of a KdV internal soliton is complicated, but for $\Delta \rightarrow 0$,

$$\frac{dE}{dt} = \frac{4\sqrt{2} \rho_2 \Delta(g)^{5/4} v^{1/2} |\bar{\eta}|^{7/4}}{3^{1/4} \pi^{3/2} [h_1 h_2^2 |h_1^2 - h_2^2|]^{1/4}} \times$$

$$\times \left[\left(1 + \frac{2h_2}{W}\right) h_1^2 + \frac{2h_1 h_2^2}{W} + \frac{1}{2} (h_1 + h_2)^2 \right], \quad (13)$$

where $\bar{\eta}$ is the maximum wave amplitude, W is the tank width and we have assumed $v_1 = v_2 = v$ (because $\Delta \ll 1$). Of the three terms added together in (13), the first represents the energy loss from the lower fluid due to the boundary layers at the tank walls, the second represents a similar loss from the upper fluid, and the third represents the energy loss at the interface. In the particular set of experiments in which we will test this theory ($h_1 = 45$ cm, $h_2 = 5$ cm, $W = 39.4$ cm), about 1/3 of the total energy loss occurs at the interface.

The total energy per unit width of tank for an internal KdV soliton with $\Delta \ll 1$ is (from Keulegan, 1953)

$$E = \frac{8\rho_2 \Delta g h_1 h_2 |\bar{\eta}|^{3/2}}{3^{3/2} |h_1 - h_2|^{1/2}} \quad (14)$$

and the energy loss given by (13) comes entirely from this source. Between (13) and (14), one obtains an equation for the slow decay of a KdV soliton as it propagates over long distances:

$$\left(\frac{|h_1 - h_2| \bar{\eta}}{h_1 h_2} \right)^{-1/4} - \left(\frac{|h_1 - h_2| \bar{\eta}_0}{h_1 h_2} \right)^{-1/4} = K \frac{S}{(h_1 h_2)^{1/2}}. \quad (15a)$$

Here η_0 is the initial wave amplitude, and η is its amplitude after it has travelled a distance S . If $\Delta \ll 1$,

$$K = \frac{v^{1/2}}{12(g\Delta)^{1/4} (h_1 + h_2)^{3/4}} \left[\left(1 + \frac{2h_2}{W}\right) \frac{h_1}{h_2} + \frac{2h_2}{W} + \frac{(h_1 + h_2)^2}{2h_1 h_2} \right]; \quad (15b)$$

in the general case ($0 < \Delta < 1$),

$$K = \frac{(c_f^2/g)^{3/4} |\beta/\gamma|^{1/4} (h_1 h_2)^{3/4}}{12\sqrt{3} g^{1/4} |h_1 - h_2|^{1/4} [(1-\Delta)(c_f^2/g)^2 + \Delta(h_1 - c_f^2/g)^2]} \times$$

$$\times \left[\left(1 + \frac{2h_2}{W}\right) \frac{[h_1 - (c_f^2/g)]^2 v_2^{1/2}}{h_2^2} + \frac{2h_1(1-\Delta) v_1^{1/2}}{W} + \right]$$

$$+ \frac{\mu_1 \nu_2}{\mu_1 \nu_2^{1/2} + \mu_2 \nu_1^{1/2}} \frac{[h_1 + h_2 - (c_f^2/g)]^2}{h_2^2} \Bigg], \quad (15c)$$

where c_f^2/g is the smaller root of (1) and (β, γ) are given below (3). In the limit $\Delta \rightarrow 0$, $c_f^2/g \rightarrow (\Delta h_1 h_2)/(h_1 + h_2)$, and (15c) reduces to (15b). Moreover, in the three limits given in (5), (15) reduces to the results of Keulegan (1948) for the viscous decay of a surface solitary wave.

Koop and Butler (1981) used an approximate version of (15) that amounts to replacing (7b) with

$$\hat{u}_1 \rightarrow 0, \quad \hat{u}_2 \rightarrow 0 \quad \text{as} \quad z \rightarrow 0. \quad (16)$$

For $\Delta \ll 1$, their method gives the correct answer if $h_1 = h_2$, but overestimates the damping due to the interfacial boundary layer for any $h_1 \neq h_2$. The special case in which their method gives the correct decay rate ($h_1 = h_2$) happens to be the configuration in which the coefficient of the nonlinear term in (3) vanishes, so that there is no solitary wave.

2 Comparison with Experiments

The experimental data that we will use to test this theory are those shown in Figure 2 of I. Recall that in those experiments, a piston at one end of a long wave tank generated both surface and internal waves, which then propagated down the tank. A vertical plate 18.8 m from the downstream edge of the piston was carefully lowered into the water after the surface waves had passed, so that the much slower internal waves could be measured without disturbance. The fluid depths were $h_1 = 45$ cm, $h_2 = 5$ cm, and the density difference was $\Delta = 0.748$.

Based on Figure 3 of I, the lead waves measured at each of the last five stations may be considered a KdV soliton, to within experimental error. Consequently, we may use the lead wave at $x/(h_1 h_2)^{1/2} = 33.3$ as initial data for (15), which then predicts the amplitude of the lead wave at the next four stations. As shown in Figure 1a, (15) predicts far more decay than is observed. Whereas French (1969) found that Keulegan's (1948) formula for surface waves predicted the attenuation to within about 12%, errors up to 30% are shown in Figure 1a.

Another way to compare (15) with these data is shown in Figure 1b. Here we consider the lead waves measured at $x/(h_1 h_2)^{1/2} = 33.3, 60, 100$, and 151 to be initial data for (15) in four separate tests, and ask for the wave amplitude predicted at $x/(h_1 h_2)^{1/2} = 191$. In this graph, the observed wave amplitude is always the same ($\bar{\eta} = 0.47$ cm). Here (15) predicts too much decay for long propagation distances and too little for short distances! Clearly (15) by itself is inadequate to explain the observed attenuation of internal solitary waves, even though it is directly analogous to the much

more successful formula of Keulegan for surface solitary waves.

We propose the following conjecture to resolve this paradox. The lead wave in Figure 2 of I was not propagating into a quiescent fluid, as required by (15). It was preceded by the faster surface wave (not shown in the figure). As the surface wave propagated, it lost energy both by viscous dissipation and by leaving kinetic energy in the boundary layers it generated. This residual kinetic energy subsequently was lost by the boundary layers only on a (very slow) diffusive time scale. Therefore the fluid into which the internal wave propagates was not quiescent, but had a mean motion down the tank. Following this reasoning, the internal soliton should have decayed less than predicted by (15) while it propagated down the tank and more than predicted by (15) as it propagated back up the tank (and met not only the residual boundary layer of the faster surface wave but its own residual boundary layers as well). The relative position of the reflecting wall is shown explicitly in Figure 1, and all of the data are in qualitative agreement with this hypothesis.

Thus, we conjecture that (15), which requires that the internal soliton propagate into a quiescent fluid, does not predict the observed decay of the internal solitons in Figure 2 of I because the residual boundary layers from the faster surface wave are still active. Unfortunately, this conjecture leaves us without any means to predict internal soliton amplitude realistically. However, it does emphasize the importance of these residual boundary layers in wave tank experiments on solitary waves. In particular, attempts to measure experimentally the phase shifts of solitons due to their interaction probably should be regarded as inconclusive until these residual boundary layers have been taken into account properly.

The theoretical portion of this work constituted the M.S. thesis of the first author while at Clarkson College, Potsdam, NY. This work was supported in part by the Office of Naval Research and by the National Science Foundation.

REFERENCES

- Djordjevic, V. D., and L. G. Redekopp (1978); J. Phys. Ocean 8, 1016.
- French, J. A. (1969); "Wave Uplift Pressures on Horizontal Platforms", W. M. Keck Laboratory, Caltech Report KH-R-19, Pasadena, California.
- Hammack, J. L. (1980), "Baroclinic Tsunami Generation", J. Phys. Ocean., to appear.
- Keulegan, G. H. (1948); J. Res. Nat. Bur. Stand. 40, 487.
- Keulegan, G. H. (1953); J. Res. Nat. Bur. Stand. 51, 133.
- Koop, C. G., and G. Butler (1981?); "An Investigation of Internal Solitons in a Two Fluid System", J. Fluid Mech., to appear.
- Lamb, H. (1932); Hydrodynamics, Dover Press, New York.
- Leone, C. (1975); "Gradual Damping of Internal Solitary Waves", M.S. Thesis, Clarkson College, Potsdam, New York.
- Segur, H., and J. L. Hammack (1981); "Long Internal Waves of Moderate Amplitude, I. Solitons", preprint.

Figure Captions

1. Decay of amplitude of the lead soliton of Figure 2 of I as a function of propagation distances: (a) Distance measured from initial location at $x/(h_1 h_2)^{1/2} = 33.3$: Δ , measured; —, predicted by (15); (b) Distance measured from final location at $x/(h_1 h_2)^{1/2} = 191$: Δ , measured; 0, predicted by (15).

