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PROPAGATION PHENOMENA IN A BISTABLE REACTION DIFFUSION SYSTEM.(U)

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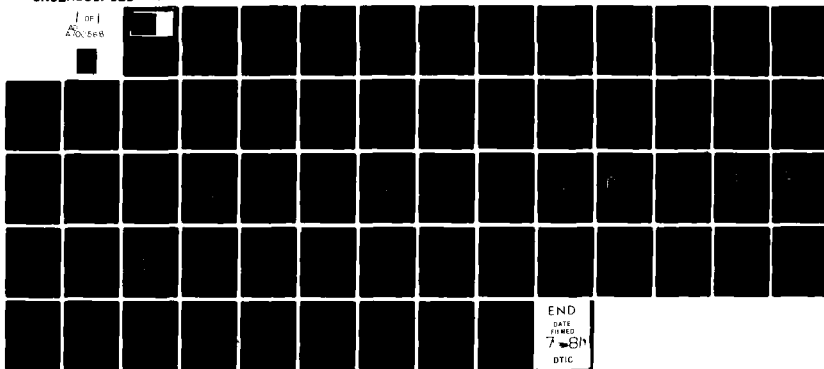
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REACTION DIFFUSION SYSTEM

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ABSTRACT

Consideration is given to a system of reaction diffusion equations which have qualitative significance for several applications including nerve conduction and distributed chemical/biochemical systems. These equations are of the FitzHugh-Nagumo type and contain three parameters. For certain ranges of the parameters the system exhibits two stable singular points. A singular perturbation construction is given to illustrate that there may exist both pulse type and transition type traveling waves. A complete, rigorous, description of which of these waves exist for a given set of parameters and how the velocities of the waves vary with the parameters is given for the case when the system has a piecewise linear nonlinearity. Numerical results of solutions to these equations are also presented. These calculations illustrate how waves are generated from initial data, how they interact during collisions, and how they are influenced by local disturbances and boundary conditions. In one example, a rightward moving "front" (rest-to-excited transition) slows down, stops, turns around, and develops into a leftward moving "back" (excited-to-rest transition) when it encounters appropriate spatially localized, transient perturbations in the dependent variables. In this case, the medium ultimately returns to the rest state. For the same parameter values we have sought, but have not found, analogous local disturbances sufficient to reflect a propagating "back". These observations are consistent with the notion that the rest state is in some sense dominant for a certain parameter range. It is conjectured that local disturbances or certain boundary conditions cannot cause the reflection of a nondominant to a dominant transition wave. Partial evidence for this conjecture is presented by means of an analytic argument.

AMS(MOS) Subject Classification - 35K65

Key Words - Traveling waves, pulse, front, back, bistable,
local disturbance, dominant state

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SIGNIFICANCE AND EXPLANATION

The mathematical equations studied here were originally introduced as a model for nerve conduction. They have also been considered as a model for chemical/biochemical reaction systems as well as electrical transmission lines. It has been demonstrated that electrical impulses in the nerve axon travel with constant shape and velocity. Mathematically this corresponds to a traveling wave solution. For a given mathematical model there may exist different types of traveling wave solutions each traveling with a different velocity. These include solitary pulses, multiple pulses, and traveling fronts.

In this report we describe which waves exist for a given set of parameters of the equations and how the velocities of the waves vary as functions of the parameters. Numerical results of solutions to these equations are presented to illustrate how the waves are generated from initial stimuli, how they interact during collisions, and how they are influenced by local disturbances.

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PROPAGATION PHENOMENA IN A BISTABLE REACTION DIFFUSION SYSTEM

John Rinzel and David Terman

1. Introduction.

The FitzHugh-Nagumo (FHN) equations

$$\begin{aligned} v_t &= v_{xx} + f(v) - w \\ w_t &= \epsilon(v - \gamma w) \end{aligned} \quad , \quad \epsilon, \gamma > 0 \quad (1.1)$$

has been studied extensively as a qualitative model for nerve conduction [7, 12, 18]. In this context the dynamics, (1.1) with $v_{xx} = 0$, are usually taken to be excitable with nullclines as in Figure 1A. The rest state $(0,0)$ is stable and globally attracting but an adequate initial displacement from rest leads to a large excursion before the eventual return to rest. Qualitatively similar dynamics are found for other excitable systems and study of the FHN equation has provided insight into excitability and propagation phenomena in several applications, e.g. see [18].

Here, we extend consideration to parameter ranges, e.g. γ large enough, for which the $v - w$ dynamics are bistable (Figure 1B). In this case there are two stable singular points: the rest state on the left branch of $f(v)$ and the excited state (E_v, E_w) on the right branch. Although this parameter range precludes direct applicability to most nerve or muscle membrane types there are examples of conditions which induce bistability [8, 11]. Dynamics similar to Figure 1B have also been considered theoretically in models for neuronal interactions at the population level [19]. For chemical/biochemical reaction systems [14], as well as electronic transmission lines [13], two variable systems with bistable behavior have been investigated. Sometimes in these studies, although not necessarily, the w -variable has much slower dynamics than v in which case $0 < \epsilon \ll 1$.

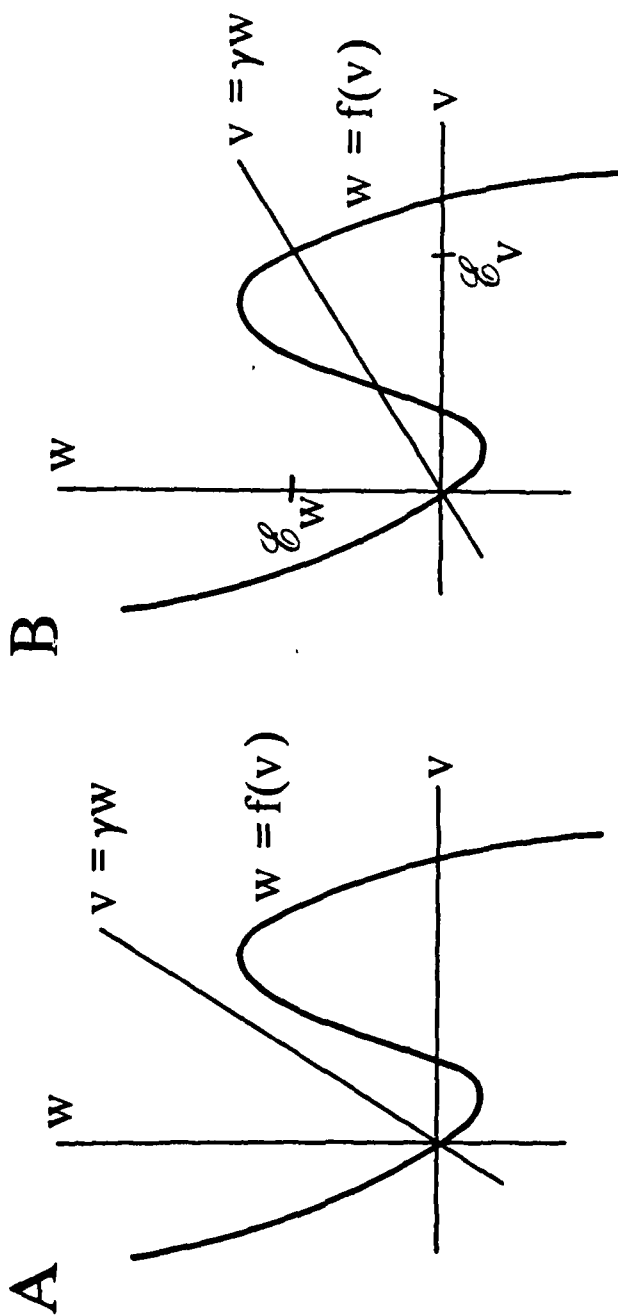


Figure 1. System (1.1) has only one equilibrium state if γ is small (A), and three equilibrium states, two of which are stable, if γ is sufficiently large (B).

In (1.1), ∂_x^2 operates only on v . This spatial coupling is physically natural for the electronic transmission line as well as the nerve fiber. For the neuronal population models, this corresponds to short-range coupling of one cell type (say, the inhibitory cells) relative to the length scale of coupling for the other cell type (the excitatory cells). For chemical systems, one might neglect w_{xx} if w , the species without the cubic-like instantaneous dynamics, diffuses much more slowly than the v -species. In [14], for an enzyme-mediated reaction scheme, the assumption of very rapid v -dynamics is used to scale the model equations and obtain a system qualitatively like (1.1) and Figure 1B with only v as diffusible.

In this paper we will investigate various propagation phenomena for (1.1). An example of Ortoleva (described by Pife [6]) offers preliminary motivation. Suppose that at $t = 0$, the medium is in the excited state for $x > 0$ and in the rest state for $x < 0$. Subsequently, a transition wave will develop which then propagates away from $x = 0$ so that the entire medium is ultimately brought to one of the two stable states, whichever is dominant (a descriptive term also used in [6, 14]). Hence (1.1) should have a traveling wave solution which corresponds to a steadily propagating transition either from the rest state to the excited state, or vice versa, (Figure 2 A,B) depending on which state dominates. As it turns out, for a certain parameter range, both types of transition waves exist, simultaneously with different speeds; we refer to these as traveling front and back solutions. Intriguingly, for a subset of the front-back parameter range, traveling pulse solutions are also found. A pulse represents an excursion either from the rest state back to itself or from the excited state back to itself (Figure 2 C,D). In addition, there may be solutions for steadily propagating trains of pulses although we will not treat trains here.

In Section 2 we will outline the singular perturbation construction [2, 3, 9] of the various traveling waves for $0 < \epsilon \ll 1$. This will develop our intuition for the parameter ranges in which the waves exist. Two special cases of (1.1) are treated explicitly: CUB, when $f(v) = v(v-a)(1-v)$ and PWL, when $f(v) = -v + H(v-a)$ where $H(\cdot)$ is a Heaviside step function; here, $0 < a < 1/2$. Next, in Section 3, we obtain analytically the front, back, and pulse solutions for PWL in a large parameter range which extends beyond the

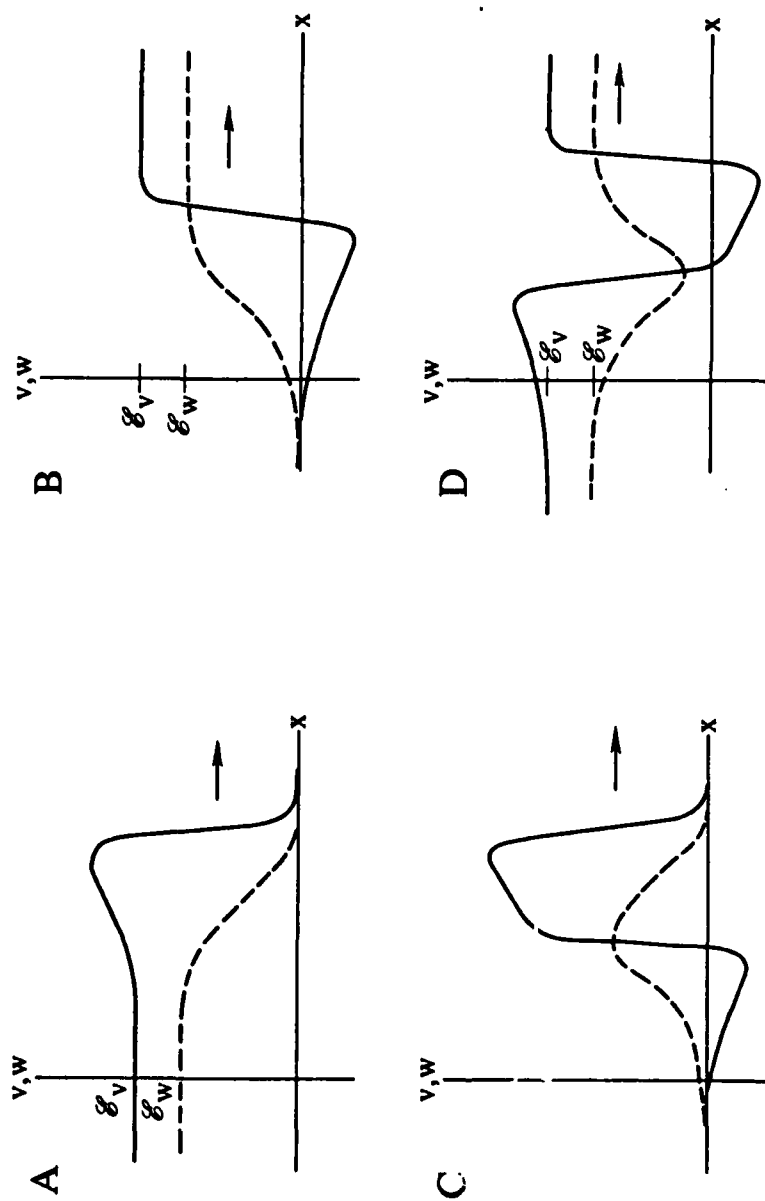


Figure 2. Traveling wave solutions for System (1.1): (A) front, (B) back, (C) pulse, and (D) E-pulse. The v -profiles are drawn with solid curves while the w -profiles are drawn with dashed curves.

region $0 < \varepsilon \ll 1$. Rigorous proofs for existence of these waves are given. We also calculate wave speeds as functions of the parameters. In addition to stable waves, we find unstable ones. A global picture (Figure 6) is provided for ε, γ -parameter regimes in which the various waves exist.

Finally, in Section 4, we present numerical results of solutions to (1.1), FWL, for given initial and boundary conditions. These calculations illustrate how waves are generated from initial data, how they interact during collisions, and how they are influenced by local disturbances. In one example, a rightward moving front (rest-to-excited transition) slows down, stops, turns around, and develops into a leftward moving back (excited-to-rest-transition) when it encounters appropriate spatially localized, transient perturbations in v and w . In this case, the medium ultimately returns to the rest state. For the same parameter values however, we have sought, but have not found, analogous local disturbances sufficient to reflect a propagating back. These observations are consistent with the notion that the rest state is in some sense dominant for these parameter values. We conjecture that, for a certain parameter range, local disturbances may lead to reflection of a front but cannot cause reflection of a back, or more generally, cannot cause reflection of a non-dominant to a dominant transition wave. We offer partial support for this conjecture with an analytic argument. Similarly we expect for such parameters that, when a boundary is held at rest, an approaching front will be reflected. On the other hand, a back which propagates toward a boundary held at the excited state will not reflect but rather may tend to a spatially nonuniform, t -independent, steady state, a solution which decays from the excited to the rest state with distance from the boundary. For a chemical system, these phenomena would be observed, for example, in a capillary tube which makes contact at the boundary with an infinite bath; where the bath maintains either the rest or excited state.

The influence of boundaries and initial disturbances on propagating transitions, and a notion of dominance, have also been discussed by Fife [6]. In his (as well as others) singular perturbation treatment, fronts are formed because of pseudo-steady state bistability; that is, the well-studied reduced equation, (1.1) with $\varepsilon = 0$ and w ,

constant, exhibits a transition wave. In such cases, only one type of stable transition wave exists and the definition of dominance follows naturally; the wave is always monotone. For the full system (1.1), fronts and backs are generally not monotone but rather have over or undershoots (Figure 2 A,B) or, in some cases, damped oscillatory approach to the steady state (also, see [15]). The bistability and (not yet rigorously defined) notion of dominance, which we discuss, is for the full system. Fronts, backs, and pulses, were also considered by Collins and Ross [4], Keener [9], and Nagumo, et al [13]; Ortoleva and Ross [14] describe fronts and backs, but not pulses, for a bistable system. Klaasen and Troy [10] also treat some behavior of two-variable, bistable systems.

2. The Singular Perturbation Construction of Traveling Waves.

In this section we describe ways of determining intuitively which traveling waves exist for a given set of parameters and how their velocities change as functions of parameters. Our description summarizes the results of [2], [3], [9]. If $(v_c(z), w_c(z))$, $z = x + ct$, is a traveling wave solution of system (1.1), then setting $u_c(z) = v'_c(z)$, it follows that $V_c(z) = (v_c(z), u_c(z), w_c(z))$ satisfies the first order system of ordinary differential equations

$$(2.1) \quad \begin{aligned} v'_c &= u_c \\ u'_c &= cu_c - f(v_c) + w_c \\ w'_c &= \frac{\epsilon}{c} (v_c - \gamma w_c) . \end{aligned}$$

The number of equilibrium states of system (2.1) will be either one, two, or three depending upon how the line $v_c = \gamma w_c$ intersects the cubic-like curve $w_c = f(v_c)$, $u_c \equiv 0$. If the parameters a and ϵ are fixed, then when γ is sufficiently small the origin will be the only equilibrium state of system (2.1). (Figure 1A). For γ large there will be three equilibrium states. In this case we shall refer to the origin, 0 , as the rest state and $E = (E_v, 0, E_w)$ as the excited state (Figure 1B). We shall assume $c > 0$ so V_c corresponds to a leftward moving wave; note that the spatial profiles in Figure 2 are for the rightward moving versions.

For a pulse we seek a value of c such that system (2.1) has a solution which satisfies $\lim_{|z| \rightarrow \infty} V_c(z) = 0$. A front is a solution of (2.1) which satisfies $\lim_{z \rightarrow \infty} V_c(z) = 0$ and $\lim_{z \rightarrow -\infty} V_c(z) = E$, while a back satisfies $\lim_{z \rightarrow -\infty} V_c(z) = E$ and $\lim_{z \rightarrow \infty} V_c(z) = 0$. In the case of multiple equilibrium states it is also possible to have solutions which satisfy $\lim_{|z| \rightarrow \infty} V(z) = E$. We shall refer to these as E-pulses.

We now sketch the singular perturbation construction of traveling waves. The solution of system (2.1) is sought in separate regions, labelled "inner" and "outer", and then joined appropriately. The inner region is defined to be those intervals over which rapid changes in v_c take place, while the outer region involves variations on a much larger z scale.

We first make the change of variables $\xi = \epsilon z$ in (2.1) to obtain:

$$(2.2) \quad \begin{cases} \epsilon v'_c = u_c \\ \epsilon u'_c = cu_c - f(v_c) + w_c \\ w'_c = \frac{1}{c} (v_c - \gamma w_c) \end{cases} \quad (') = \frac{d}{d\xi}.$$

Outer Region. Away from the locations of sharp transitions in v_c we obtain the lowest order approximation by setting $\epsilon = 0$ in (2.2) to obtain:

$$(2.3) \quad \begin{cases} u_c = 0 \\ w_c = f(v_c) \\ w'_c = \frac{1}{c} (v_c - \gamma w_c) \end{cases} \quad (') = \frac{d}{d\xi}.$$

This implies that in the outer region the solution lies on the slow manifold curve:

$w_c = f(v_c)$, $u_c \equiv 0$. We assume that the left branch of the cubic is given by $v = g^-(w)$ and the right branch is given by $v = g^+(w)$.

Inner Region. Here we use the stretched variable $z = \frac{1}{\epsilon}\xi$ and system (2.2) returns to system (2.1). With $\epsilon = 0$ in (2.1) we obtain the following lowest order approximation

$$(2.4) \quad \begin{cases} v'_c = u_c \\ u'_c = cu_c - f(v_c) + w_c \\ w'_c = 0 \end{cases} \quad (') = \frac{d}{dz}.$$

Therefore, in the inner region the solution must lie in a plane $w_c \equiv \text{constant}$. We shall see that this constant determines the speed of the traveling wave.

Matched Composite Solutions. To lowest order in ϵ , the full solution is obtained by matching the inner and outer solutions where their domains of definition overlap. We treat each wave separately.

Fronts. We assume that the parameters are chosen so that the line $v_c = \gamma w_c$ intersects the cubic as shown in Figure 1B. Since the independent variable ξ does not appear explicitly in Equation (2.2) we may also assume that the inner region lies in a small neighborhood about the origin, $\xi \approx 0$.

The outer region describes the front's plateau. For Equation (2.3) we take $v_c = g^+(w_c)$ and the initial condition:

$$(2.5) \quad w_c(0) = 0.$$

This means that $w_c \rightarrow E_w$ as $z \rightarrow \infty$.

In order to match the inner and outer solutions we take the following boundary conditions in Equation (2.4):

$$(2.6) \quad v_c(-\infty) = 0, \quad v_c(+\infty) = 1, \quad w_c(z) \equiv 0.$$

Equation (2.4) for cubic-like $f(v)$ with boundary conditions (2.6) was studied by Aronson and Weinberger [1]. They proved the existence of a unique solution for a unique value of c provided $A > 0$, where $A \equiv \int_0^1 f(v) dv$. Hence, these equations determine, up to lowest order in ϵ , the speed of the front. In the case $f(v)$ given by (PWL) one can give an explicit formula for the speed, c , of solutions of Equation (2.4) with boundary condition (2.6) as a function of the parameter a (see [16]). This formula is

$$c = (1 - 2a)[a(1 - a)]^{-1/2}.$$

Note that c is a decreasing function of a and $\lim_{a \rightarrow 1/2} c = 0$. This is similarly true

for $f(v)$ given by (CUB), in which case $c = \sqrt{2} (1/2 - a)$ (see [3]).

Note that the outer solution for a front lies on the right branch of the cubic while the inner solution lies in the plane $w_c \equiv 0$. The complete phase space trajectory for a lowest order front is shown in Figure 3.

Backs. In order for a back to exist the line $v_c = \gamma w_c$ must intersect the cubic as shown in Figure 1B. Again we may assume that the inner region lies in a small neighborhood about $\xi = 0$. For the outer region, we take $v_c = g^-(w_c)$ in Equation (2.3) and the initial condition:

$$(2.7) \quad w_c(0) = E_w.$$

In this case, $w_c' < 0$ and $w_c \rightarrow 0$ as $z \rightarrow \infty$. Then in Equation (2.4) we take the boundary conditions

$$(2.8) \quad v_c(-\infty) = E_v, \quad v_c(+\infty) = g^-(E_w), \quad w_c(z) \equiv E_w.$$

The results of Aronson and Weinberger imply that there exists a unique solution of Equation

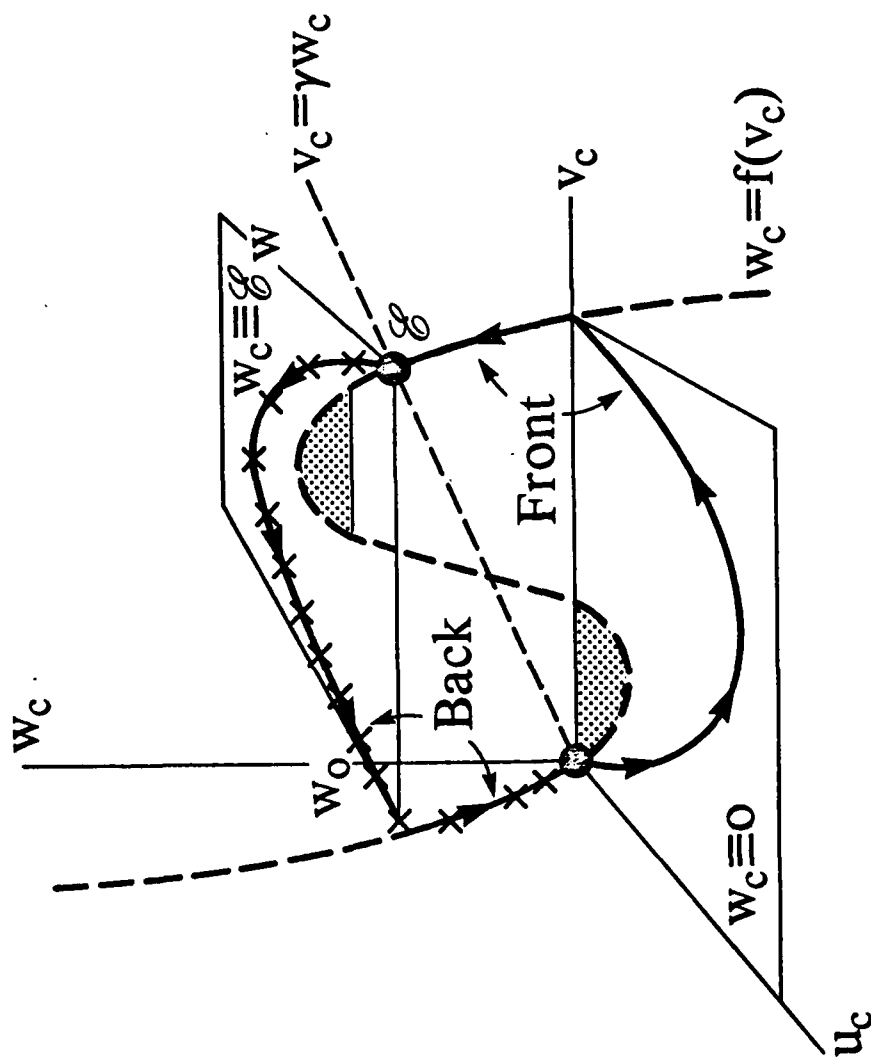


Figure 3. The phase space trajectories for lowest order singular perturbation approximations of a front (heavy solid) and a back (heavy dashed). The sharp transition from the right branch to the left branch of the curve $w_c = f(v_c)$ for the trajectory of a pulse occurs in the plane $w_c = w_0$. w_0 is chosen so that the shaded regions have equal areas.

(2.4) with boundary conditions (2.8) for a unique positive value of c if and only if

$$B = \int_{E_v}^{g^{-1}(E_w)} [f(v) - E_w] dv > 0. \quad \text{From this we conclude that the back (to lower order) does}$$

not exist if γ is too large. This is because the assumption $A > 0$ implies that there must exist a critical value of γ , $\gamma = \gamma_2$, such that $B < 0$ for $\gamma > \gamma_2$. Hence, the "singular" or lowest order back cannot exist for $\gamma > \gamma_2$. For $f(v)$ given by (PWL) a simple computation shows that $\gamma_2 = \frac{1+2a}{1-2a}$. In this case one can also give an explicit formula for the speed, c , of solutions of Equation (2.4) with boundary conditions (2.8) as a function of γ and a . This formula is

$$c = \left[2a - \frac{\gamma-1}{1+\gamma} \right] \left[\left(a - \frac{\gamma}{1+\gamma} \right) \left(a - \frac{1}{1+\gamma} \right) \right]^{1/2}.$$

We see that for the parameter a fixed the speed is a decreasing function of γ , and

$$\lim_{\gamma \rightarrow \gamma_2} c = 0. \quad \text{These properties also hold for } f(v) \text{ given by (CUB).}$$

Note that the outer solution of a back lies on the left branch of the cubic-like curve while the inner solution lies in the plane $w_c \equiv E_w$. The total phase space trajectory for a back is shown in Figure 3.

Pulses. A pulse can be thought of as a wave resembling a front followed by a back traveling at the same speed. The trajectory in the phase space for the front part of the pulse, or upstroke, begins at the origin and then makes a fast transition in the plane $w_c \equiv 0$ to the right branch of $w_c = f(v_c)$. It then slowly moves up the right branch of the cubic-like curve until it reaches some point $(v_0, 0, w_0)$; this forms the plateau of the pulse. The trajectory for the back part of the pulse, or downstroke, begins at the point $(v_0, 0, w_0)$, makes a fast transition in the plane $w \equiv w_0$ to the left branch of $w_c = f(v_c)$, and then slowly returns to the origin along this left branch to form the recovery phase.

From Equation (2.4) it follows that the velocity of the back part of the pulse is determined by the constant w_0 . We must choose w_0 so that the velocities of the front and back parts of the pulse are equal. If $f(v)$ is given by (CUB) or (PWL) then our previous discussions imply that w_0 must be chosen so that

$$\int_0^1 f(v) dv = \int_{g^+(w_0)}^{g^-(w_0)} [f(v) - w_0] dv .$$

Setting $\gamma_1 = \frac{g^+(w_0)}{w_0}$ it follows that a "singular" pulse does not exist for $\gamma > \gamma_1$. This is because if $\gamma > \gamma_1$ then $E_w < w_0$. The pulse cannot exist because when the trajectory for the front part of the pulse moves up the right branch of the cubic-like curve it must stop at the excited state. It will be unable to reach the point $(g^+(w_0), 0, w_0)$ where it would make the transition to the left branch of the cubic-like curve in order to form the back part of the pulse. If $f(v)$ is given by (PWL) then a simple computation shows that $\gamma_1 = \frac{2a}{1-2a}$.

We now summarize our results concerning the singular perturbation construction of traveling waves. These results are stated for $f(v)$ given by either (CUB) or (PWL), however similar results hold for some general cubic-like nonlinearities.

For $f(v)$ given by (CUB), choose γ_0 so that the line $v = \gamma_0 w$ intersects the local maximum of the cubic as shown in Figure 4. Choose γ_1 and γ_2 so that, respectively, the horizontally and vertically shaded regions in Figure 4 have equal areas. For $f(v)$ given by (PWL) choose $\gamma_0 = \frac{a}{1-a}$, $\gamma_1 = \frac{2a}{1-a}$, and $\gamma_2 = \frac{1+2a}{1-2a}$. The singular perturbation construction then suggests that the following proposition is true.

Proposition 2.1. For $0 < \epsilon \ll 1$ a pulse exists for $\gamma \in (0, \gamma_1)$, a front exists for $\gamma \in (\gamma_0, \infty)$, and a back exists for $\gamma \in (\gamma_0, \gamma_2)$.

Note that all three types of waves exist for $\gamma \in (\gamma_0, \gamma_1)$. We saw in the singular perturbation construction that the speed of the front is, to lowest order in ϵ , a decreasing function of $A = \int_0^1 f(v) dv$, while the speed of the back is a decreasing function of $B = \int_v^{g^-(E_w)} [f(v) - E_w] dv$. Note that if $\gamma \in (\gamma_0, \gamma_1)$ then $A > B$. Hence,

when all three types of waves exist, the back is faster than the front. For $\gamma \in (\gamma_1, \gamma_2)$, the front is faster than the back, and in this parameter range an E -pulse exists.

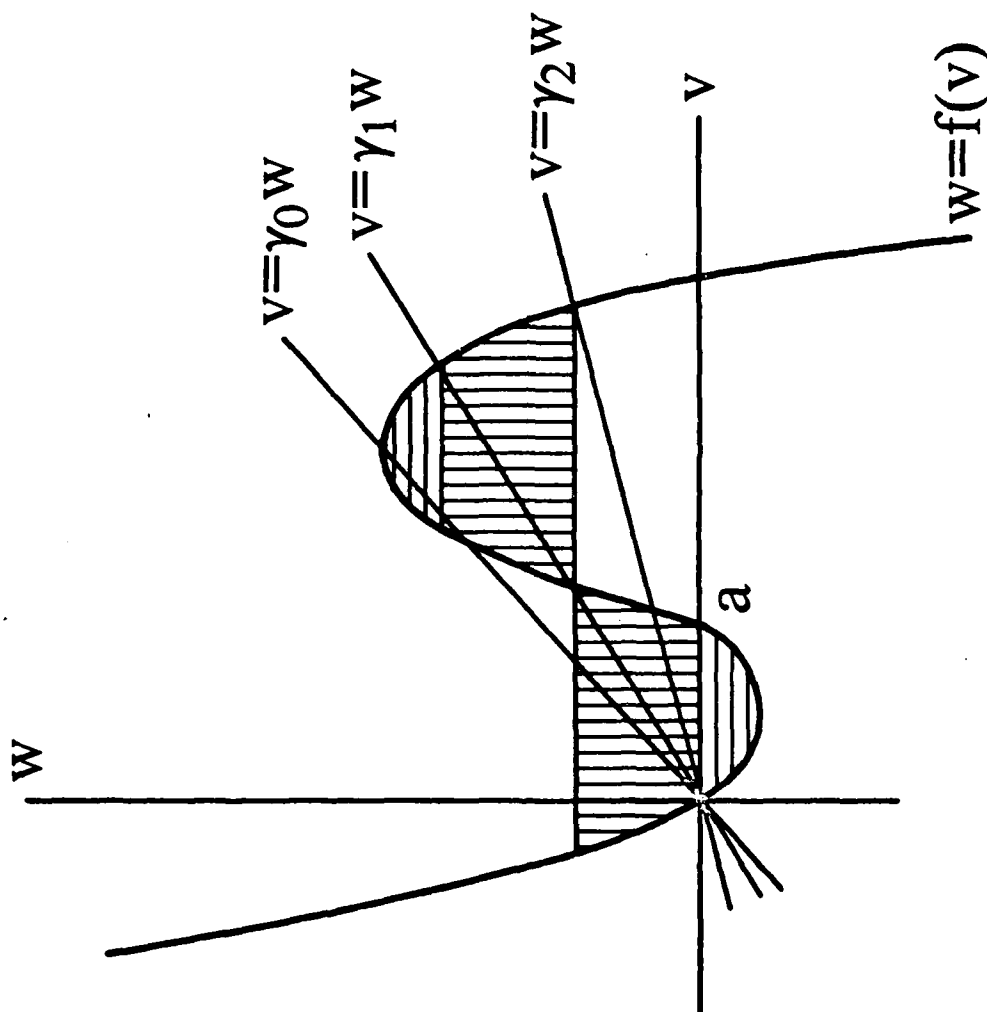


Figure 4. Critical values of γ for the singular perturbation constructions. To lowest order in ϵ , pulses exist for $\gamma \in (0, \gamma_1)$, fronts exist for $\gamma \in (\gamma_0, \infty)$, and backs exist for $\gamma \in (\gamma_0, \gamma_2)$.

It is of course possible for other waves to exist which do not arise from the singular perturbation construction. We shall see in the next chapter that there may exist two traveling waves of a particular type for a given set of parameters. The singular perturbation construction gives the faster wave which one expects, from numerical evidence, is stable. The slower wave is expected to be unstable.

3. Solutions of the Piecewise Linear Model.

In this section we consider system (1.1) with $f(v)$ defined by (PWL) and we give a complete, rigorous description of which traveling waves exist for a given set of parameters and how their speeds vary with the parameters. The construction we use here follows the work of Rinzel and Keller [16].

Pulses: We seek values of $c > 0$ for which system (2.1) has a pulse shaped solution. Because the independent variable, z , does not appear in the equation explicitly we can choose the origin, $z = 0$, so that $v_c(0) = a$. We also assume that $v_c(z_1) = a$ for some $z_1 > 0$. Therefore, the solution we seek is a leftward moving version of the form illustrated in Figure 2C, to be distinguished from an $\bar{E}$ -pulse (Figure 2D). It follows that along with Equation (1.1) the solution $V_c = (v_c, u_c, w_c)^T$ must satisfy the conditions

$$(3.1) \quad v_c \Big|_0^{0^+} = -1 \quad v_c \Big|_{z_1}^{z_1^+} = 1$$

$$(3.2) \quad v_c(0) = v_c(z_1) = a, \quad z_1 > 0.$$

This jump condition (3.1) results from the discontinuity of $f(v)$. To represent a pulse the solution must also satisfy the condition $v_c(z) \rightarrow 0$ as $|z| \rightarrow \infty$.

We express the solution as

$$(3.3) \quad v_c = \begin{cases} a X_1 e^{\alpha_1 z} & ; \quad z < 0 \\ \sum_{k=1}^3 B_k X_k e^{\alpha_k z} + \bar{v} & ; \quad 0 < z < z_1 \\ \sum_{k=1}^3 C_k X_k e^{\alpha_k (z-z_1)} & ; \quad z_1 < z \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3$ are the roots of the characteristic polynomial

$$(3.4) \quad p(x) = x^3 + \left(\frac{\epsilon\gamma - c^2}{c}\right)x^2 - (\epsilon\gamma + 1)x - \left(\frac{\epsilon\gamma + \epsilon}{c}\right),$$

and where

$$(3.5) \quad \bar{V} = \begin{pmatrix} \frac{\gamma}{1+\gamma} \\ 0 \\ \frac{1}{1+\gamma} \end{pmatrix} \quad \text{and} \quad X_k = \begin{pmatrix} 1 \\ \alpha_k \\ \varepsilon(c\alpha_k + \varepsilon\gamma)^{-1} \end{pmatrix}, \quad k = 1, 2, 3.$$

From (3.4) it follows that either

$$\alpha_1 > 0, \quad \alpha_3 < \alpha_2 < 0$$

or

$$\alpha_1 > 0, \quad \bar{\alpha}_3 = \bar{\alpha}_2, \quad \operatorname{Re} \alpha_2 < 0.$$

Because $v_c \rightarrow 0$ as $z \rightarrow \infty$ we must take $c_1 = 0$. It is this last condition which will give us a relationship for the speed, c , in terms of the parameters a, ε , and γ .

Using the continuity and jump conditions at $z = 0$ we obtain

$$(3.6) \quad B_1 = a - \frac{\delta_1}{p_1'}, \quad B_2 = \frac{-\delta_2}{p_2'}, \quad B_3 = \frac{-\delta_3}{p_3'}$$

where

$$(3.7) \quad \delta_k = \frac{\varepsilon\gamma + c\alpha_k}{c\alpha_k}$$

$$\text{and } p_k' = p'(\alpha_k).$$

Matching at $z = z_1$ yields

$$c_1 = ae^{\alpha_1 z_1} + \frac{\delta_1}{p_1'} (1 - e^{\alpha_1 z_1}).$$

Since $c_1 = 0$, we obtain

$$(3.8) \quad e^{-\alpha_1 z_1} = \left(1 - a \frac{p_1'}{\delta_1}\right) \equiv s.$$

Using (3.8) and the fact that $v_c(z_1) = a$ we are led to the following transcendental equation:

$$(3.9) \quad F(a, \epsilon, c, \gamma) \equiv 2-s + \frac{\delta_2}{\delta_1} \frac{p_1'}{p_2'} s^{-\alpha_2/\alpha_1} + \frac{\delta_3}{\delta_1} \frac{p_1'}{p_3'} s^{-\alpha_3/\alpha_1} - \frac{p_1'}{\sigma^2 \delta_1} = 0.$$

Here we set $\sigma^2 = \frac{1+\gamma}{\gamma}$. Note that as a function of s Equation (3.9) does not depend on the parameter a . Equation (3.8) gives a in terms of s .

The following theorem is proved in Appendix A by analyzing the roots of Equation (3.9).

Theorem 3.1: Assume that the parameters ϵ and γ are given.

- (i) There is a value of the parameter a for which a pulse exists if and only if $\epsilon\gamma^2 < 1$.
- (ii) For a given value of c at most one pulse can exist for a uniquely determined value of a .
- (iii) If $\epsilon\gamma^2 < 1$ then positive constants $c_{\min}, c_{\max}$ exist with $c_{\min} < c_{\max}$ and such that
 - (a) for each $c \in (c_{\min}, c_{\max})$ there is a unique value of a for which a pulse exists,
 - (b) $\lim_{c \rightarrow c_{\min}} a = 0$,
 - (c) $\lim_{c \rightarrow c_{\min}} z_1 = 0$ where z_1 was defined by $v(z_1) = a, z_1 > 0$,
 - (d) $\lim_{c \rightarrow c_{\max}} a = \frac{\gamma}{2(1+\gamma)}$,
 - (e) $\lim_{c \rightarrow c_{\max}} z_1 = \infty$,
 - (f) as $\epsilon\gamma^2 \rightarrow 1$ both $c_{\min}$ and $c_{\max} \rightarrow 0$.

In the proof of this result we will also see that a pulse cannot exist if c is either too small or too large.

We have solved Equation (3.9) numerically for fixed values of ϵ, c and γ . The corresponding value of a for each pulse is obtained from Equation (3.8). The solid curves in Figure 5 display pulse speed curves, c versus a , for different values of

γ and $\epsilon = 0.2$. Each curve has a well defined knee (a_v, c_v) characterized by $F(a_v, \epsilon, c_v, \gamma) = 0$ and $F_c(a_v, \epsilon, c_v, \gamma) = 0$. For each value of a , $\frac{\gamma}{2(1+\gamma)} < a < a_v$, we have two pulse solutions with speeds $c_f(a, \epsilon, \gamma)$ and $c_g(a, \epsilon, \gamma)$ with $c_f > c_g$. Rinzel and Keller [15] showed that in the case $\gamma = 0$ the lower branch, c_g , corresponds to unstable pulses while at the knee (a_v, c_v) the pulse is neutrally stable. Feroc [5] showed by numerically evaluating an analytic criterion that the upper branch, c_f , corresponds to stable pulses. For some values of ϵ and γ , other than those in Figure 5, the pulse speed curve has no upper branch or knee; in such a case, there is a unique (unstable) pulse for $a < \frac{\gamma}{2(1+\gamma)}$ and no pulse for larger values of a .

We observe that $c_{\min}(a)$, the minimum value of c on each speed curve, occurs at $a = 0$. The maximum value of c , $c_{\max}$, on each speed curve, occurs at $a = \frac{\gamma}{2(1+\gamma)}$ where the curve terminates (e.g., $\square$ for $\gamma = 1.0$) on the corresponding speed curve (dashed) for front solutions; here, $z_1 = \infty$. These results are predicted by Theorem 3.1, parts (iii, b and d).

Fronts: If in the construction of a solitary pulse, we take $B_1 = 0$ (see Equation (3.3)) then we will have a front as shown in Figure 2A. The front can be written explicitly as

$$v_c(z) = \begin{cases} aX_1 e^{\alpha_1 z}, & z < 0 \\ - \sum_{k=2}^3 \frac{\delta_k}{p_k'} X_k e^{\alpha_k z} + \bar{V}; & 0 < z \end{cases}$$

where $\bar{V}$ and X_k ($k = 1, 2, 3$) were defined in (3.5), δ_k and p_k were defined in (3.7), and the α_k are the roots of the characteristic polynomial $p(x)$ defined in (3.4). From Equation (3.6) we find that the condition $B_1 = 0$ is equivalent to:

$$(3.10) \quad a = \frac{\delta_1}{p_1'}.$$

In order to have a front we must impose the further restriction that

$$(3.11) \quad \frac{a}{1-a} < \gamma \quad \text{or} \quad a < \frac{\gamma}{1+\gamma}$$

to insure that there are multiple equilibrium states of system (2.1). By combining (3.10) and (3.11) we see that a front with speed c exists if and only if c satisfies

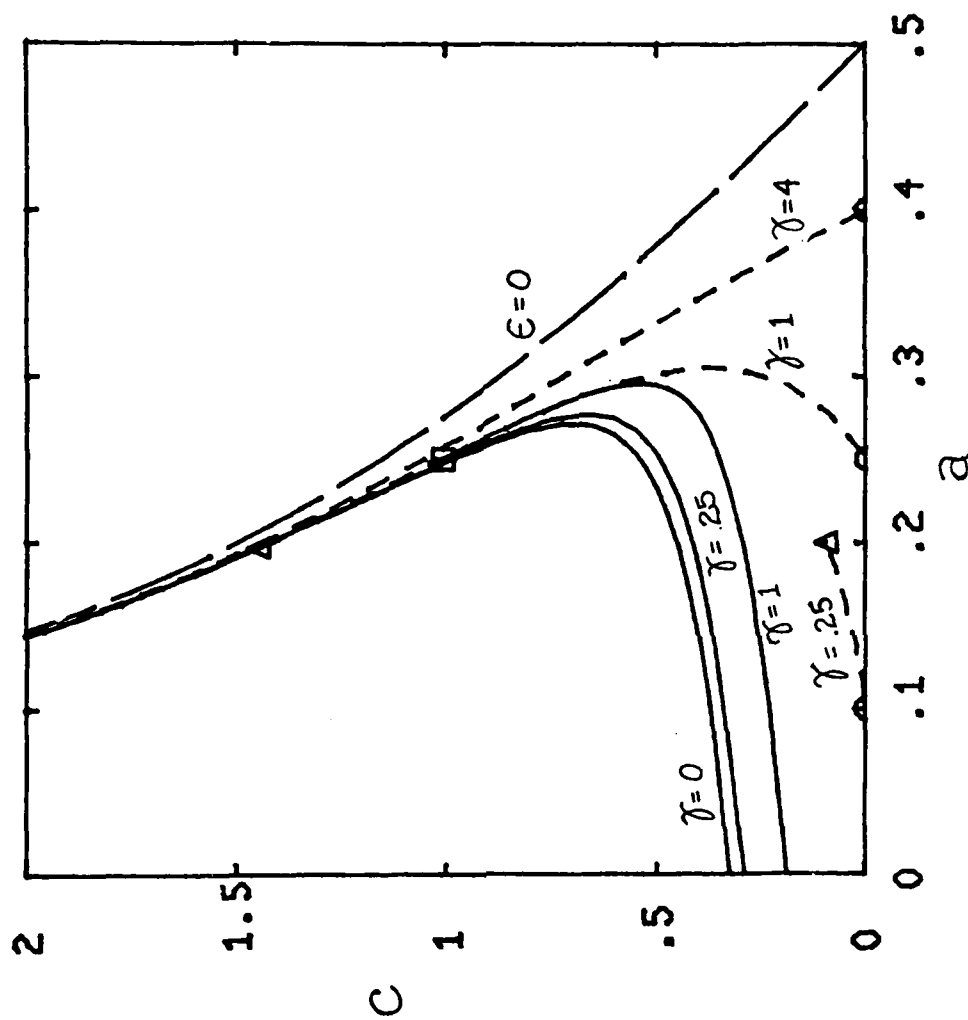


Figure 5. Pulse (solid curves) and front (short dashed curves) speed curves, c versus a , for different values of γ and $\epsilon = .2$. The long dashed curve corresponds to the front curve for the front when $\epsilon = 0$.

$$(3.12) \quad \frac{\delta_1}{p_1^*} < \frac{\gamma}{1+\gamma}$$

the corresponding value of a is given by (3.10). The following theorem shows that a front exists for c sufficiently small or large. It is proved in Appendix B.

Theorem 3.2: Assume that the parameters ϵ and γ are fixed. Then there exists positive constants c_1, c_2 with $c_1 < c_2$ such that if $0 < c < c_1$ or $c > c_2$ then (3.12) is satisfied, and hence a front exists. Furthermore, $\lim_{c \rightarrow 0} a = \frac{\gamma}{2(1+\gamma)}$ and

$$\left. \frac{\partial a}{\partial c} \right|_{c=0} \begin{cases} < 0 & \text{if } \epsilon \gamma^2 > 1 \\ = 0 & \text{if } \epsilon \gamma^2 = 1 \\ > 0 & \text{if } \epsilon \gamma^2 < 1. \end{cases}$$

The front for $0 < c \ll 1$ is unstable (or linearly stable) according as $\partial a / \partial c|_{c=0} > 0$ (or < 0).

We have solved Equation (3.10) numerically for fixed values of the parameters ϵ, c , and γ . The short dashed curves in Figure 5 display various front speed curves, c versus a , for different values of γ and $\epsilon = .2$. For $\gamma = .25$ and $\gamma = 1$, the speed curves have two branches. One expects, as in the case of solitary pulses, that the faster front corresponds to a stable wave while the slower front corresponds to an unstable wave. Notice that for $\gamma = .25$ the speed curve is not defined for all values of the speed c . This is because from Equation (3.11) it follows that a front cannot exist for $a > \frac{\gamma}{1+\gamma} = .2$. These cutoffs for the speed curve $\gamma = .25$ are illustrated by small triangles.

As expected from Theorem 3.2 we find that $\lim_{c \rightarrow 0} a = \frac{\gamma}{2(1+\gamma)}$ on each speed curve. This is indicated by the semicircles in Figure 5 which were placed at the points $(a, c) = (\frac{\gamma}{2(1+\gamma)}, 0)$. Note that $\left. \frac{\partial a}{\partial c} \right|_{c=0} > 0$ for $\gamma = .25$ and $\gamma = 1$, while $\left. \frac{\partial a}{\partial c} \right|_{c=0} < 0$ for $\gamma = 4$. This agrees with the last result stated in Theorem 3.2. The proof of Theorem 3.2 also generalizes the results on limiting behavior for c small, the bifurcation of a slow front from a standing front, to "cubic-like" $f(v)$.

Figure 5 demonstrates that the speed curves for pulses lie 'inside' those of fronts. That is, if the parameters ϵ, γ , and c are fixed then the value of a for which a pulse exists is always less than the value of a for which the front exists. This can be proven rigorously by considering Equations (3.8) and (3.10).

The long dashed curve in Figure 5 corresponds to the speed curve for the front when $\epsilon = 0$. It is given explicitly by $c = (1-2a)[a(1-a)]^{-1/2}$. As the singular perturbation construction, predicts for small ϵ , the stable branch of the speed curves for the fronts and the pulses lie very close to the speed curve for $\epsilon = 0$.

Backs: It is not necessary to present the analytic construction of a back because there is a one to one correspondence between fronts and backs under the change of variables

$(v, w) \rightarrow (\frac{\gamma}{1+\gamma} - v, \frac{1}{1+\gamma} - w)$. Hence a back with parameters (a, ϵ, c, γ) is identified, under the above change of variables, to a front with parameters $(\frac{\gamma}{1+\gamma} - a, \epsilon, c, \gamma)$. The speed curves, c versus a with ϵ and γ held fixed, for the fronts and backs are therefore symmetric with respect to the vertical line $a = \frac{\gamma}{2(1+\gamma)}$. The same correspondence applies to pulses and E-pulses.

Figure 6 offers a schematic summary of our numerical and analytic results concerning the existence of known pulse type and transition type traveling wave solutions of the piecewise linear model. It shows the first quadrant of the $\gamma - \epsilon$ plane divided into ten regions. In each region there are four numbers which indicate, respectively, the multiplicity of the different types of wave, pulses, fronts, backs, and E-pulses, which exist for those particular values of γ and ϵ . In this diagram we assume that the parameter a is fixed with $a \in (0, 1/2)$ and we only consider waves with positive speed. The constants γ_0, γ_1 , and γ_2 are chosen as before. That is, $\gamma_0 = \frac{a}{1-a}$, $\gamma_1 = \frac{2a}{1-2a}$, and $\gamma_2 = \frac{1+2a}{1-2a}$.

Figure 6 was constructed by analyzing various speed curves, c versus a and c versus γ , for different values of ϵ . We did not consider very large values of ϵ . It is, of course, possible that no waves exist if ϵ is too large.

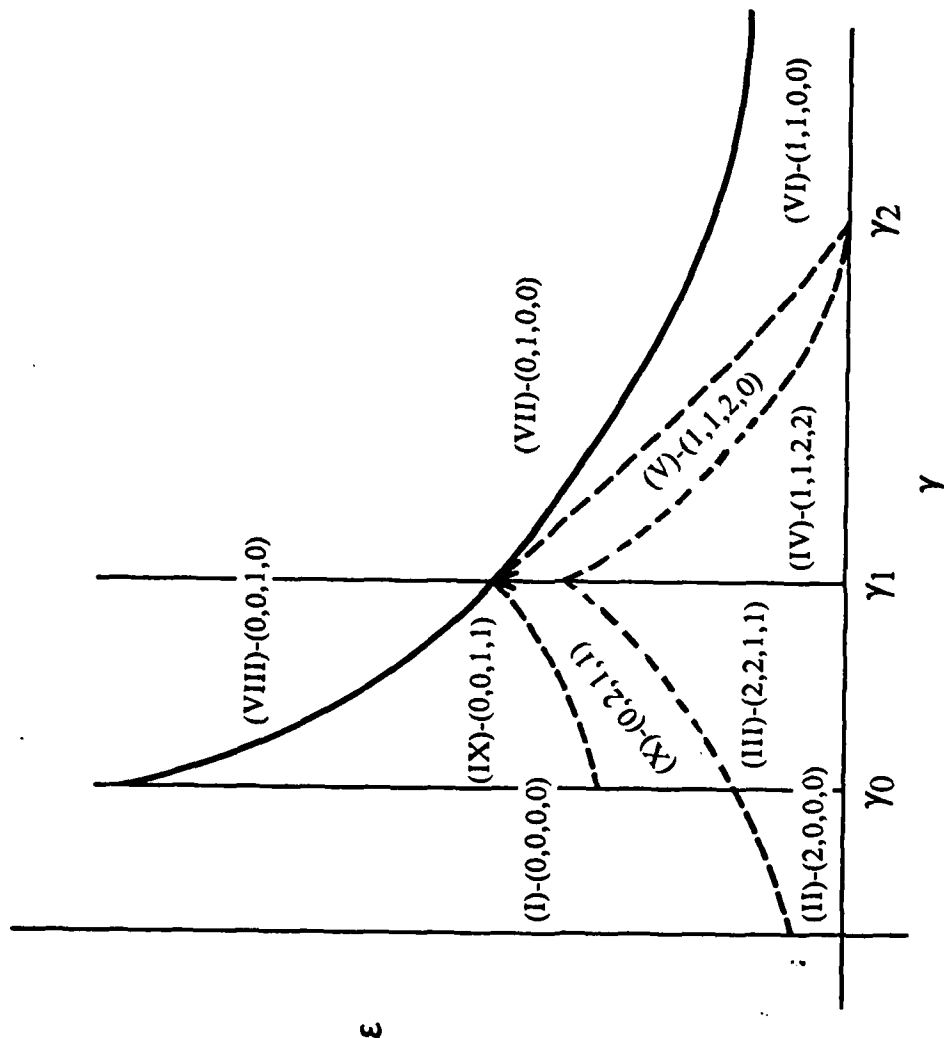


Figure 6. Global description of pulse type and front type traveling wave solutions of the piecewise linear model. The four numbers in each region indicate the multiplicity of the different types of waves: pulses, fronts, backs, and f-pulses, which exist for those particular values of γ and ϵ .

Whenever just one front or back exists in a particular region it corresponds to a wave lying on the upper branch of a speed curve. One, therefore, expects it to be stable.

Whenever just one pulse or E-pulse exists in a particular region it corresponds to a wave lying on the lower branch of a speed curve, and is, therefore, unstable.

In addition to the four types of waves already mentioned there also exist waves with zero speed, i.e. stationary waves. Note that these waves do not depend on the parameter

ϵ . For $\gamma \in (\gamma_0, \gamma_1)$ there exists a stationary E-pulse. That is a stationary wave, $V(x) = (v(x), u(x), w(x))$, such that $\lim_{|x| \rightarrow \infty} V(x) = E$. For $\gamma > \gamma_1$ there exists a stationary wave such that $\lim_{|x| \rightarrow \infty} V(x) = 0$. This will be called a stationary pulse. For $\gamma = \gamma_1$, two stationary transition waves exist. One satisfies $\lim_{x \rightarrow -\infty} V = E$ and $\lim_{x \rightarrow \infty} V = 0$ while the other satisfies $\lim_{x \rightarrow -\infty} V = 0$ and $\lim_{x \rightarrow \infty} V = E$. These represent limiting cases of the stationary pulse and stationary E-pulse with infinite pulse width.

Not only does Figure 6 illustrate how many traveling waves exist for a given set of parameters, but also how the waves appear or disappear through bifurcations as the parameters are varied. The dashed curves, for example, represent the existence of neutrally stable waves out of which bifurcate the stable and unstable waves. These correspond to the "knees" in the speed curves of Figure 5. The dashed curve separating regions (IX) from (X) corresponds to neutrally stable fronts, while the dashed curve separating regions (I) and (X) from regions (II) and (III) corresponds to neutrally stable pulses. Both curves are drawn as monotonically increasing functions of γ because the numerical results indicate, for fixed ϵ , that the minimum value of γ for which the "knees" of the speed curves in Figure 5 exist is an increasing function of ϵ . This makes intuitive sense for the following reason. Suppose the recovery damping constant γ is slightly increased from a critical value (for which the pulse is neutrally stable). This diminishes the recovery or inhibitory effect of w so the system would exhibit a stable pulse. Moreover, this pulse would persist even for a slightly greater recovery growth rate ϵ . Figure 6 also shows that for fixed ϵ the neutrally stable pulse occurs at a larger value of γ than the neutrally stable front. This follows from a preceding remark that

the speed curves for pulses lie "inside" those of fronts. It means that a stable pulse occurs simultaneously with a stable front only if w is adequately damped.

The dashed curve separating regions (IV) and (V) corresponds to a neutrally stable E -pulse, while the dashed curve separating regions (V) and (VI) corresponds to a neutrally stable back. Both curves are drawn as monotonically decreasing functions of γ and both approach $\gamma = \gamma_2$ as $\varepsilon \rightarrow 0$. This limiting case has the speeds of the E -pulse and back tending to zero and the width of the E -pulse tending to infinity.

The solid curve in Figure 6 is given explicitly as $\varepsilon\gamma^2 = 1$. The transition from region (VIII) into region (IX) corresponds to the bifurcation of a slow unstable E -pulse from the stationary E -pulse. The transition from region (VII) into region (VI) corresponds to the bifurcation of a slow unstable pulse from the stationary pulse. No pulses are shown for $\varepsilon\gamma^2 > 1$ because of Theorem 3.1(1).

The other bifurcations occur at $\gamma = \gamma_0$ and $\gamma = \gamma_1$. As γ increases past γ_0 System (1.1) picks up the equilibrium state E . For ε sufficiently small we therefore have the creation of the stable and unstable fronts, a stable back, and an unstable E -pulse. It is, of course, at $\gamma = \gamma_0$ that the discontinuity of f is reflected. For f given by (CUB) the creation of the fronts, backs, and E -pulses would not be so sudden. Since System (1.1) would have another equilibrium state besides 0 and E there may be other bounded solutions of System (1.1) which we have not considered.

For $\gamma < \gamma_1$ the speed of the stable back is greater than that of the stable front. The opposite is true for $\gamma > \gamma_1$. When $\gamma = \gamma_1$, the stable front and stable back have the same speed. As $\gamma \uparrow \gamma_1$ the width, z_1 , of the stable pulse approaches $+\infty$. In some sense, therefore, the stable pulse bifurcates from the front and back as γ decreases past $\gamma = \gamma_1$. Other bifurcations also occur at $\gamma = \gamma_1$. For example, as $\gamma \uparrow \gamma_1$ both the stationary E -pulse and the slow unstable front approach a stationary transition wave which connects the rest state with the excited state.

Finally, when both a stable front and stable back exist, a stable pulse also exists only if the back speed exceeds the front speed; this necessary condition is satisfied by

parameter values in region (III). The condition is not sufficient however as evidenced by region (X). The corresponding statements also hold for E-pulses in regions (IV) and (V) respectively.

4. Transient Interactions of Waves and Local Disturbances.

We performed a number of numerical experiments to determine various properties of traveling waves. In all of the results illustrated here we used the piecewise linear model, $f(v) = v - H(v - a)$, with parameter values $a = .25$, $\epsilon = .1$, and $\gamma = .5$; these parameter values correspond to region (III) in Figure 6. For numerical integration, we used a Crank-Nicholson scheme throughout with $\Delta x = .1$, $\Delta t = .05$, and a zero flux condition, $\partial v / \partial x = 0$, at both boundaries. For our choice of parameters the excited state is $(E_v, E_w) = (\frac{1}{3}, \frac{2}{3})$ and a stable front, a faster stable back, and a stable pulse exist.

In describing the numerical experiments it will be convenient to refer to different phases of the traveling waves. We shall refer to the sharp transition from the left branch to the right branch of $f(v)$ for the front or pulse as "the leading edge", and the sharp transition from the right branch to the left branch of $f(v)$ for the back or pulse as the "trailing edge". These, of course, correspond to the inner regions in the singular perturbation construction.

Formation of a Pulse from Square Initial Data. The formation of a pulse from specified initial conditions is illustrated in Figure 7. Spatial profiles $v(x,t)$ versus x (Figure 7A) and $w(x,t)$ versus x (Figure 7B) are shown for discrete $t: t_p = 5p$, $0 < p < 8$. For initial data we used square steps of unit height: $v(x,0) = \chi_{[5,35]}$ and $w(x,0) = \chi_{[2.5,7.5]}$ where $\chi_{[a,b]}$ is the characteristic step function of the interval $[a,b]$. Note the asymmetrical distribution of $v(x,0)$, $w(x,0)$ with respect to each other; the step in $w(x,0)$ overhangs that in $v(x,0)$. This helps ensure the formation of a pulse rather than two fronts traveling in opposite directions.

We see that the leading edge in the front part of the pulse is clearly formed at $t = t_1$ and proceeds to travel with constant velocity. The trailing edge in the back part of the pulse also forms quickly but begins by traveling very slowly ($t_0 < t < t_2$). The trailing edge then accelerates until it is actually moving faster than the leading edge ($t_1 < t < t_3$). The trailing edge then slows down and its speed approaches that of the leading edge. Then the leading and trailing edges travel with the same speed ($t_5 < t < t_8$) and a pulse is formed.

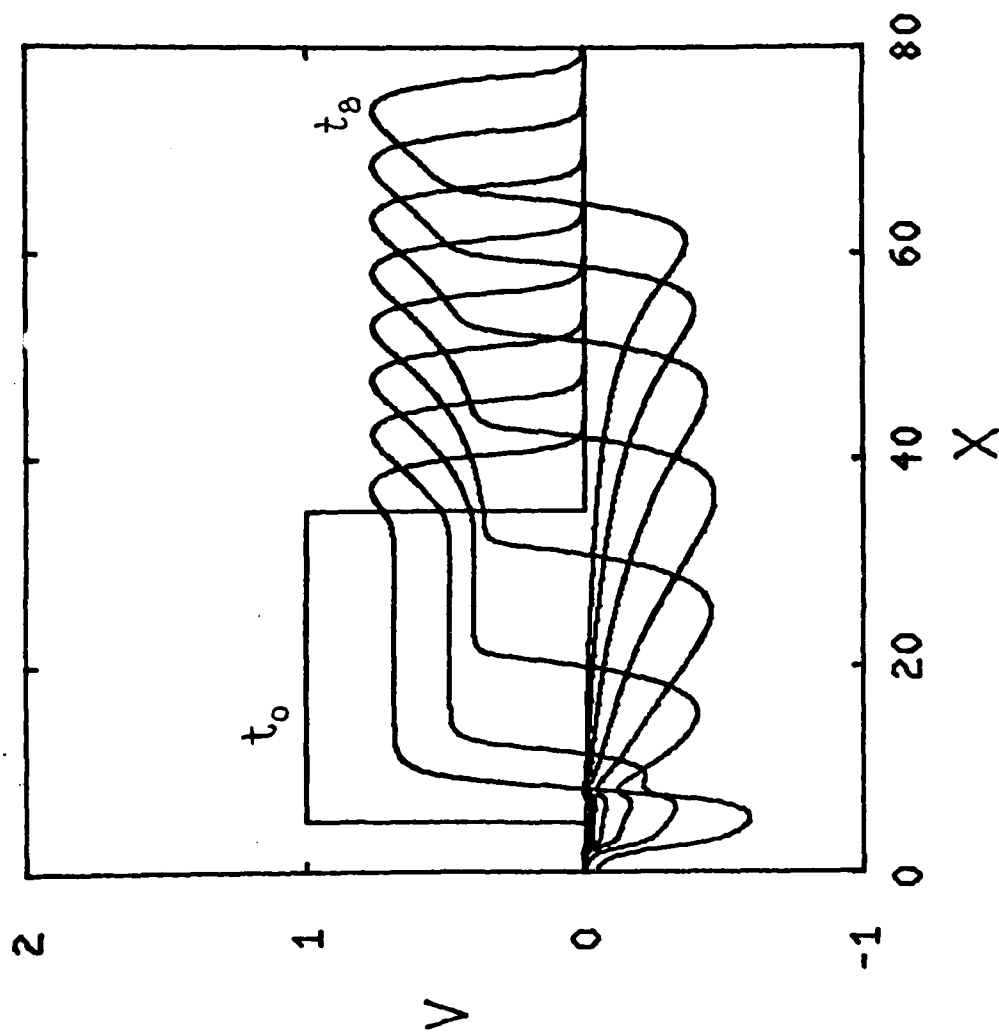


Figure 7A. Formation of a pulse from square initial data. Spatial profiles $v(x, t)$ versus x are shown for discrete $t : t_p = 5p, 0 \leq p \leq 8$.

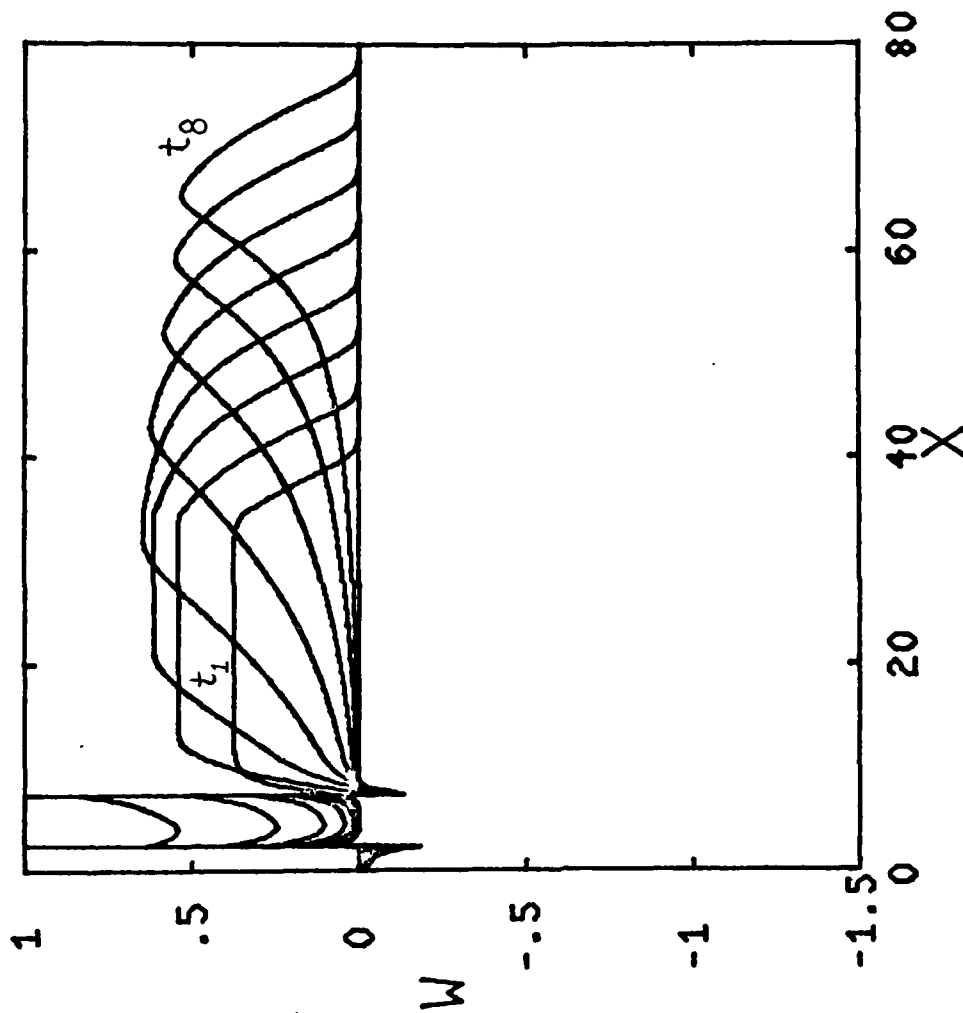


Figure 7B. Formation of a pulse from square initial data. Spatial profiles $w(x,t)$ versus x are shown for discrete $t : t_p = 5p, 0 \leq p \leq 8$.

In order to understand why the pulse is formed in this way recall that in the singular perturbation construction of a traveling wave the recovery variable, w , is assumed to be constant in the inner region. The speed of the leading edge is a decreasing function of this constant while the speed of the trailing edge is an increasing function of this constant. Ahead of the pulse w is always very close to zero, and hence the leading edge propagates with constant speed. The trailing edge, however, does not see a constant value of w , and therefore its speed varies.

In the region between the trailing and leading edges the spatial derivatives of v are very small, and hence the solution will follow very closely space clamped dynamics. That is, the solution (v, w) will evolve according to the ordinary differential equation:

$$\begin{cases} \bar{v}_t = f(\bar{v}) - \bar{w} \\ \bar{w}_t = \varepsilon(\bar{v} - \gamma\bar{w}) \end{cases}$$

with initial conditions $\bar{v}(0) = 1$ and $\bar{w}(0) = 0$. It follows that in the middle region the solution (v, w) becomes very close to the excited state (E_v, E_w) .

The velocity of the trailing edge initially increases until this edge sees a value of w very close to $w = E_w$. Because of our choice of parameters the trailing edge is now traveling faster than the leading edge. After the trailing edge passes $x = 35$ its speed decreases because then it advances into a region where w is decreasing. The pulse is formed when the trailing edge slows to the same speed as the leading edge. Then, because both edges have the same speed, the trailing one always sees a constant value of w .

For a singular perturbation construction of how traveling waves are formed from initial data see Keener [9].

Collision of a Pulse and a Front. In Figure 8 we take for initial data (dashed curves) a pulse moving to the right and a front moving to the left. Figure 8A shows profiles, $v(x, t)$ versus x , and Figure 8B shows profiles $w(x, t)$ versus x , for discrete t : $t_p = 5p$, $0 \leq p \leq 7$. When the two leading edges reach each other the dynamics are very similar to the collision of two fronts. This brings the region near the fronts to the excited

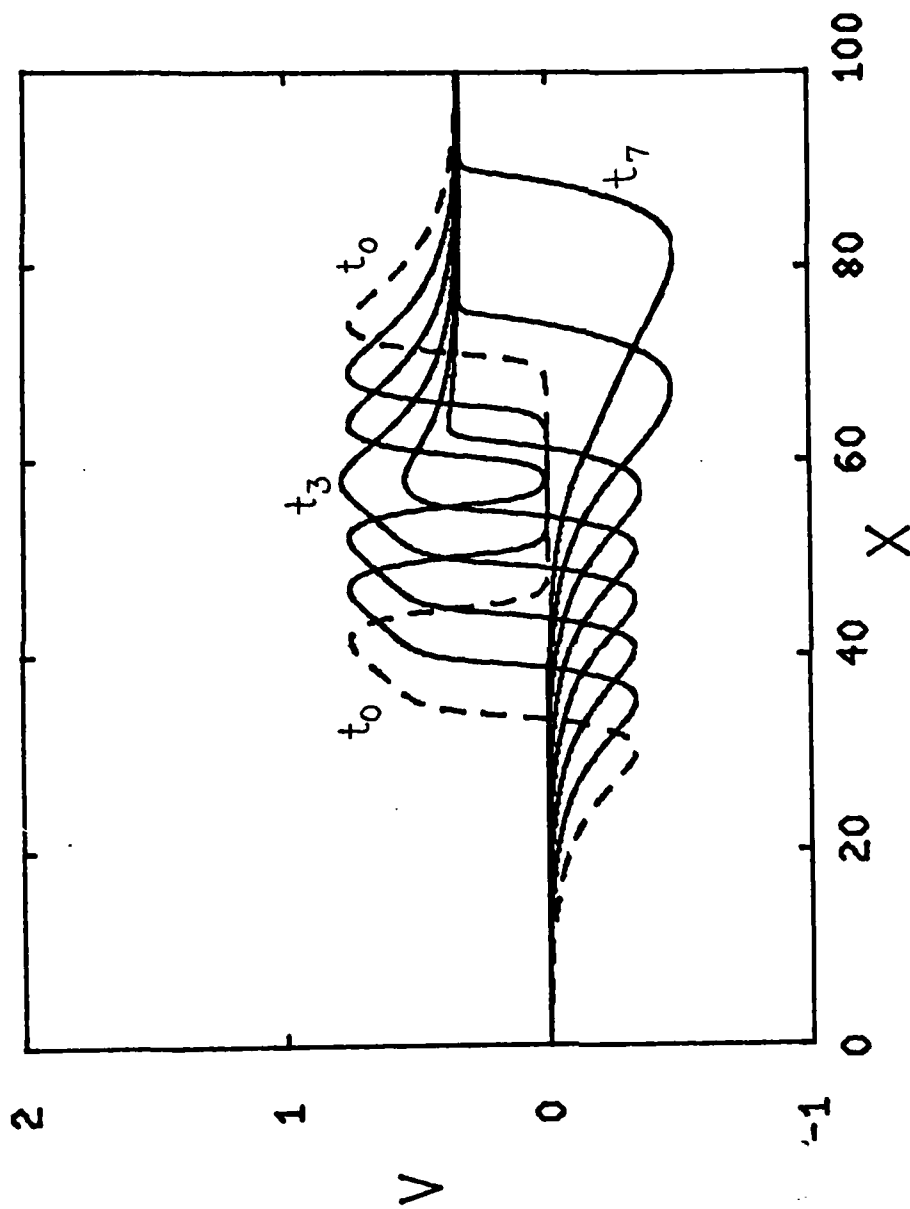


Figure 8A. Collision of a pulse and a front. Spatial profiles $v(x, t)$ versus x are shown for discrete $t: t = 5, 6, 7$. The initial data is drawn with dashed curves.

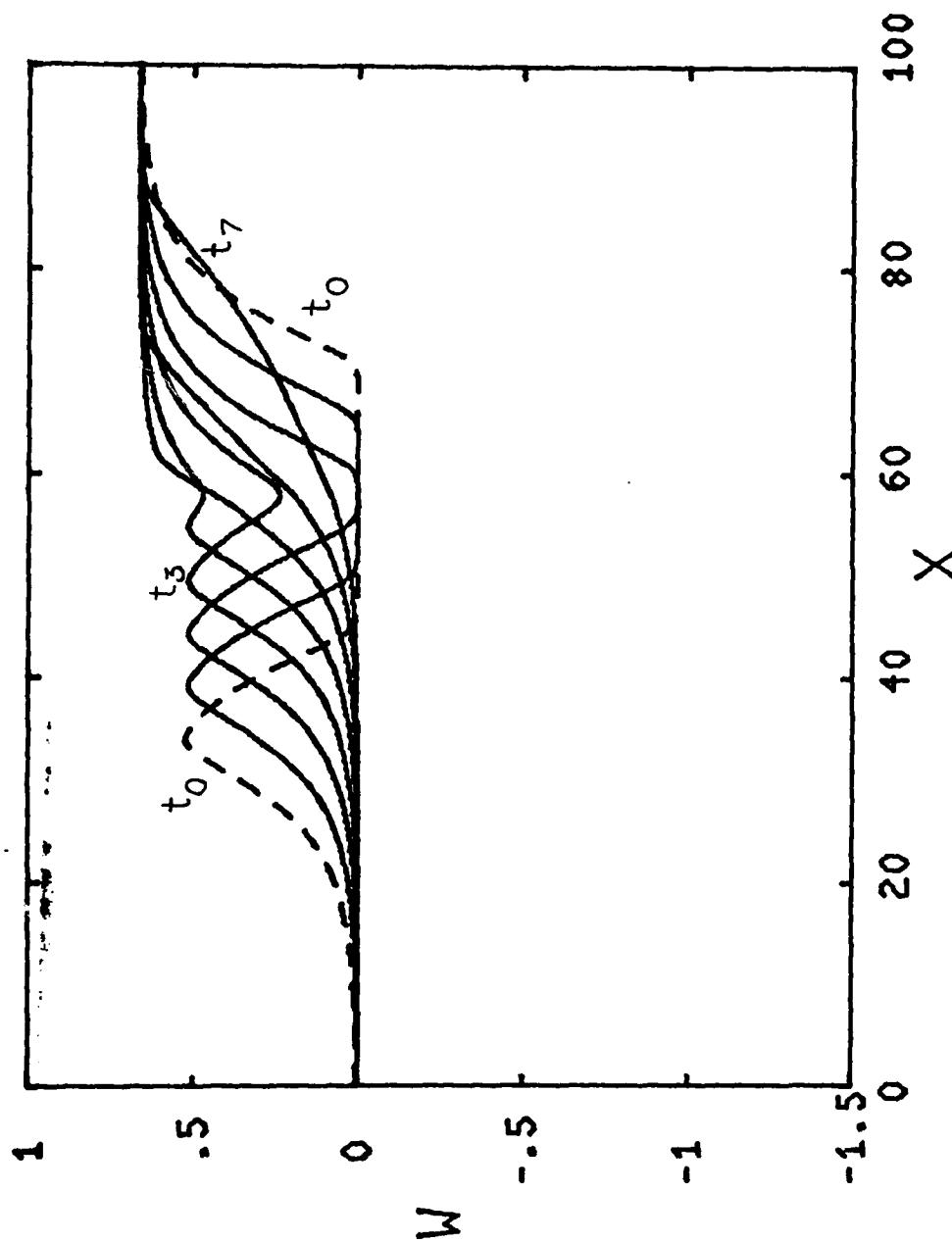


Figure 88. Collision of a pulse and a front. Spatial profiles $w(x, t)$ versus x are shown for discrete $t : t_p = 5, 0 \leq p \leq 7$. The initial data is drawn with dashed curves.

state ($t_3 < t < t_4$). When the trailing edge of the pulse enters this excited region it increases in velocity and evolves into a back moving to the right ($t_5 < t < t_7$). The reason that the trailing edge increases in velocity is that the value of w at the excited state is greater than the value of w at the trailing edge of a pulse. The singular perturbation construction showed that the velocity of the trailing edge of the pulse is an increasing function of the value of w at which the trailing edge takes place.

Reflections of a Traveling Wave. Suppose that $(\alpha(x,t), \beta(x,t))$ is the solution of System (1.1) with initial data $(\eta_1(x), \eta_2(x))$. We define $(\eta_1(x), \eta_2(x))$ to be a local disturbance if $\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} (|\alpha(x,t)| + |\beta(x,t)|) = 0$. This is a local disturbance with respect to the steady state $(v,w) = (0,0)$; one may also define analogously a local disturbance with respect to (E_v, E_w) . We say $(\bar{v}(x,t), \bar{w}(x,t))$ is a reflection of the traveling wave solution $(v(x,t), w(x,t))$, a reflection caused by a local disturbance, if

- (i) For some time $T > 0$ and some local disturbance $(\eta_1(x), \eta_2(x))$, $(\bar{v}(x,t), \bar{w}(x,t))$ is the solution of systems (1.1) with initial data $(v(x,T) + \eta_1(x), w(x,T) + \eta_2(x))$, and
- (ii) There exists a traveling wave solution $(v_1(x,t), w_1(x,t))$ moving in the opposite direction of (v,w) such that

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} (|\bar{v}(x,t) - v_1(x,t)| + |\bar{w}(x,t) - w_1(x,t)|) = 0.$$

We say that it is possible to reflect the traveling wave (v,w) if there exists a reflection of (v,w) .

Our numerical experiments indicate that for given values of the parameters a , ϵ , and γ it may be possible to reflect some traveling waves but not others. Figure 9 shows a front being reflected into a back with profiles $v(x,t)$ versus x (Figure 9A) and profiles $w(x,t)$ versus x (Figure 9B) for discrete $t: t_p = 2.5p$, $0 < p < 17$. The first three profiles show a front moving to the right. At $t = t_3$ (dashed curves) we increased the values of w over a short interval just ahead of the leading edge. This certainly represents a local disturbance. In fact one sees that in the disturbed region the solution slowly returns to rest. As the leading edge enters the region of increased w it slows down ($t_5 < t < t_{10}$) and then collapses to become a back moving towards the

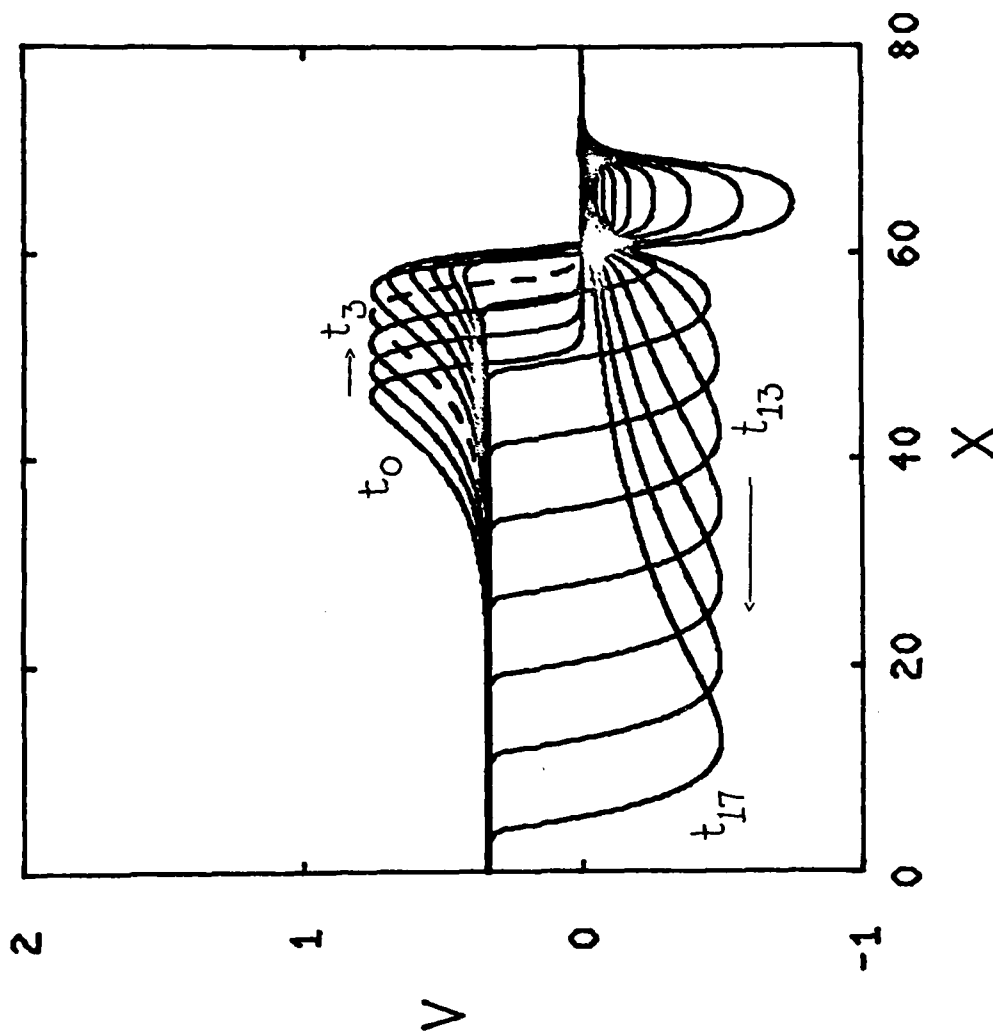


Figure 9A. Reflection of a front. Spatial profiles $v(x,t)$ versus x are shown for discrete $t : t_p = 2.5p, 0 \leq p \leq 17$. The local disturbance in w occurs at $t = t_3$ (dashed curves).

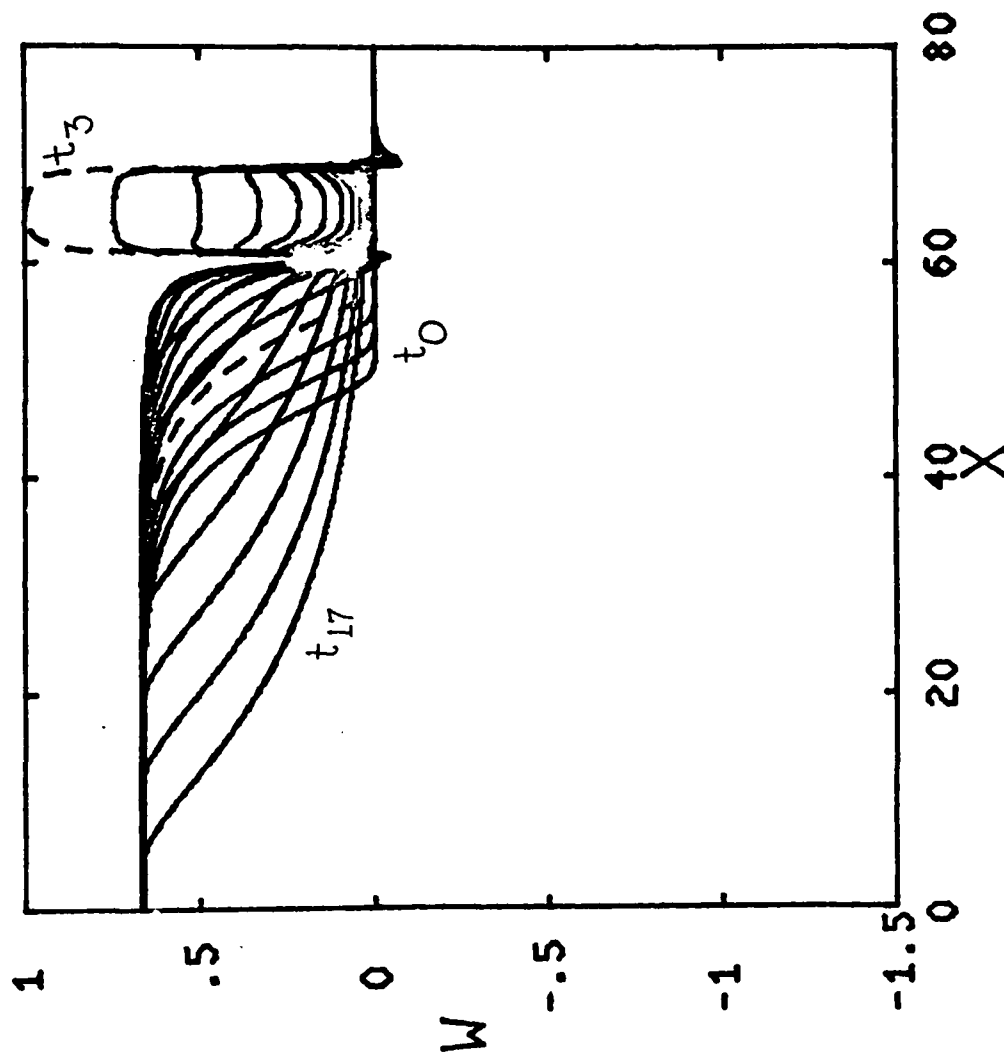


Figure 9B. Reflection of a front. Spatial profiles $w(x, t)$ versus x are shown for a discrete $t : t_p = 2.5p, 0 \leq p \leq 17$. The local disturbance in w occurs at $t = t_3$ (dashed curves).

left ($t_{10} < t < t_{17}$). Notice that the back travels with a speed greater than the front, as indeed it should for parameter values from region (III) of Figure 6. This is also consistent with the singular perturbation prediction which we note gives reasonably accurate estimates for the back and front speed if ϵ is small.

It may be surprising to discover that under the corresponding circumstances a back will not reflect to become a front. This is shown in Figure 10 with Figure 10A showing profiles $v(x,t)$ versus x and Figure 10B showing profiles $w(x,t)$ versus x for discrete t : $t_p = 2.5p$, $0 < p < 14$. The first two profiles show a back moving towards the right. At $t = t_2$ (dashed lines) we sharply decreased w over an interval directly ahead of the trailing edge of the back. This is certainly a local disturbance (with respect to (E_v, E_w)). In fact, one sees that in the disturbed region the solution slowly returns to the excited state. Upon entering the disturbed region the trailing edge slows down and then apparently stops, waiting for the disturbance ahead of it to return to the excited state ($t_4 < t < t_9$). The back then continues to move in its original direction ($t_9 < t < t_{15}$).

Our numerical experiments suggest that if the parameters a , ϵ , and γ lie in a certain subset of values for which the back speed exceeds the front speed then it is possible to reflect a front but impossible to reflect a back by local disturbances. To understand this behavior we formulate and study an idealized model problem. We consider first the case of a back moving rightward toward a local disturbance of decreased w (Figure 10). As one recalls from the singular perturbation construction, the back speed is determined by the value of w directly ahead of it and, moreover, this speed decreases with decreasing w . Hence one expects intuitively that if w falls abruptly in the disturbed region then the back will decelerate suddenly and virtually halt. An appropriate formal perturbation calculation for this transient stage, which includes the deceleration, temporary halt, and initiation of either reflected or continued propagation, is yet to be worked out. Our approach is to observe from the numerical results during this transient state, $T < t < T_1$ say, that v and w at the "halting location" x_0 have values close to E_v and E_w . Thus we consider the model problem in which v, w satisfy the equations

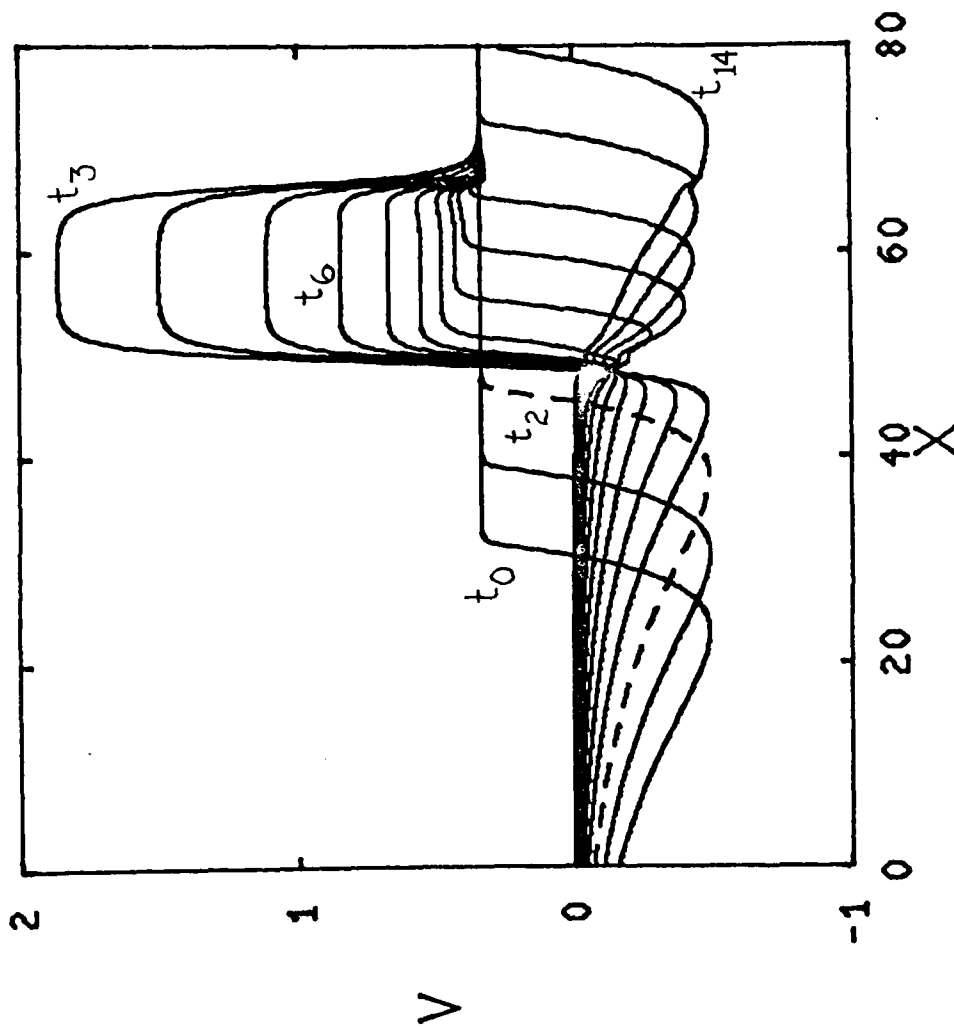


Figure 10A. A back encountering a local disturbance. Spatial profiles $v(x, t)$ versus x are shown for discrete $t : t_p = 2.5p, 0 \leq p \leq 14$. The local disturbance in w occurs at $t = t_2$ (dashed curves).

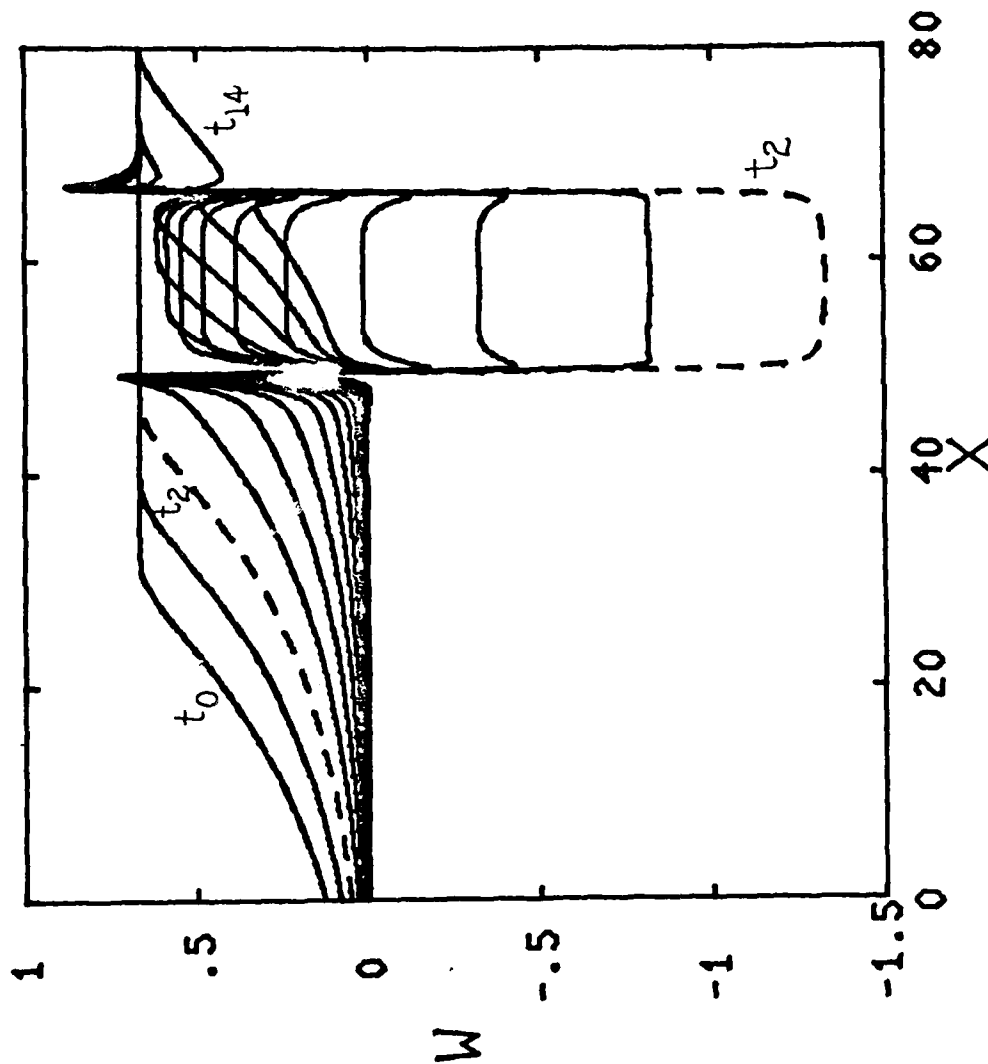


Figure 10B. A back encountering a local disturbance. Spatial profiles $w(x, t)$ versus x are shown for discrete $t : t_p = 2.5p, 0 \leq p \leq 14$. The local disturbance in w occurs at $t = t_2$ (dashed curves).

$$v_t = v_{xx} + f(v) - w \quad (4.1a)$$

$$w_t = \epsilon(v - \gamma w) \quad \text{for } x < x_0, \quad t \in (T, T_1)$$

with the boundary conditions:

$$(v(x_0, t), w(x_0, t)) = (E_v, E_w) \quad (4.1b)$$

$$(v, w) \rightarrow (0, 0) \quad \text{as } x \rightarrow -\infty \quad (4.1c)$$

and with the approaching back profile as initial conditions.

We remark that one can also motivate the boundary condition (4.1b) as follows.

Let $\mathcal{D}$ denote the disturbed region in Figure 10. Shortly after the disturbance occurs, $v > E_v$, $w < E_w$ in $\mathcal{D}$ and so v diffuses outward across $\partial\mathcal{D}$. Hence, just outside $\mathcal{D}$, $w_t > 0$ and one observes a local increase in w there. The local spread of v also means that w_t is less positive just inside $\partial\mathcal{D}$ than it is toward the middle of $\mathcal{D}$. This explains the under/overshoots in the w -profile (Figure 10B) that one sees in crossing $\partial\mathcal{D}$ outward. From this it also follows that between the under and overshoot, $w \approx E_w$ (at $x = x_0$, as in the preceding paragraph), at least for a while. Finally, if the back approaches soon then $v > E_v$ just inside and $v < E_v$ just to the left of $\partial\mathcal{D}$ so that $v \approx E_v$ near x_0 . The above argument does not apply if the local disturbance is not fairly sharp. But of course, one can probably not induce reflections by disturbances which are not sharp. At the same time, the singular perturbation treatment would not predict halting if the local disturbance were slowly varying in x ; in such a case, the wave could decelerate sufficiently without stopping as the disturbance wears off.

One expects that the temporary halt of a back may be viewed as the approach of the solution of (4.1) to a time-independent steady state solution. The steady state solution $(q(x), p(x))$ must satisfy

$$q'' + f(q) - p = 0$$

$$q - \gamma p = 0$$

with boundary conditions

$$(q(x_0), p(x_0)) = (E_v, E_w) \quad \text{and} \quad (q(-\infty), p(-\infty)) = (0, 0).$$

Since $p(x) = q(x)/\gamma$ in $(-\infty, x_0)$ we seek a function $q(x)$ which satisfies the boundary value problem:

$$(4.2) \quad \begin{aligned} q'' + f(q) - q/\gamma &= 0 \quad \text{in} \quad (-\infty, x_0) \\ q(-\infty) &= 0 ; \quad q(x_0) = E_v . \end{aligned}$$

For $\gamma \in (\gamma_0, \gamma_1)$ problem (4.2) has a unique solution. This can be seen by considering the phase plane portrait for the equation $q'' + f(q) - q/\gamma = 0$ as shown in Figure 11.

Finally, the steady state will be approached only if it is stable as a solution to the System (4.1). In Theorem 4.1 we demonstrate (see Appendix C) the stability or instability of (q, p) .

Theorem 4.1.

Suppose $\gamma_0 < \gamma < \gamma_1$, or equivalently $1 < 2a/E_v < 2$. If $\epsilon\gamma^2 > 1$, then $(q(x), p(x))$ is linearly stable as a solution to (4.1). For $\epsilon\gamma^2 < 1$, $(q(x), p(x))$ is linearly stable when

$$(4.3) \quad 2a/E_v > 1.278\dots$$

but unstable, for given ϵ, γ , if

$$(4.4) \quad 0 < 2a/E_v - 1 < 1 .$$

We conjecture that when the back is faster than the front and $(q(x), p(x))$ is stable then the back cannot be reflected by local disturbances.

A few comments on Theorem 4.1 are in order. The sufficient condition (4.4) which guarantees instability is an asymptotic condition. Precise conditions are those for which inequality (C.11) of Appendix C is satisfied. As the proof demonstrates, the loss or gain of stability as parameters are varied leads to Hopf bifurcation of periodic solutions. This applies to more general and related problems for bistable systems which we will pursue in further work. Finally, note that the statement in Theorem 4.1 for $\epsilon\gamma^2 > 1$ holds for any "cubic-like" $f(v)$; see Appendix C.

Our computed examples are for parameter values which satisfy inequality (4.3). Hence, corresponding to our model problem (4.1), we identify the halted back in Figure 10 with the steady state solution (q, p) of (4.1). The back waits at this steady state for the local

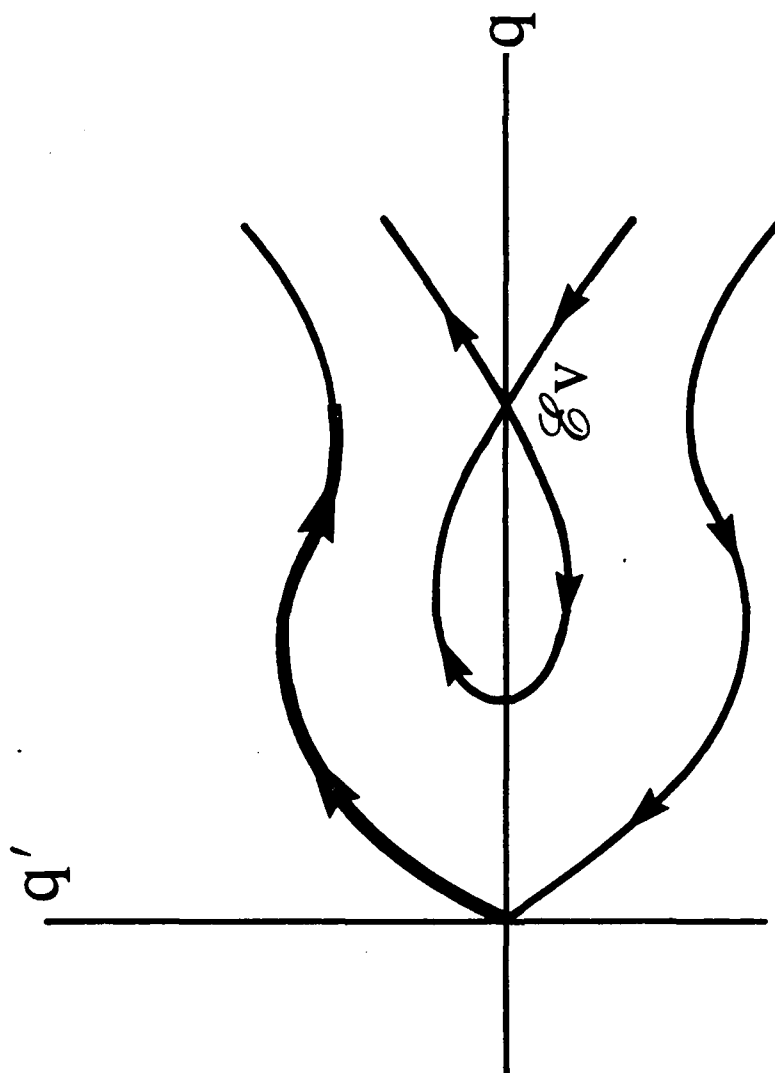


Figure 11. The phase plane diagram of the equation $q'' + f(q) - q/\gamma = 0$
for $\gamma \in (\gamma_0, \gamma_1)$.

disturbance to decay sufficiently. When this has happened the back can proceed through the disturbed region. For parameter values such that the steady state $(q(x), p(x))$ exists but is unstable we would not predict that it is impossible to reflect the back.

For the analogous problem (Figure 9), the front does reflect, rather than stop and wait for the local disturbance to decay, because the corresponding steady state solution does not exist. To be more precise, the model problem, in this case, is the system of Equations (4.1a) with the boundary conditions

$$(4.5a) \quad (v(x_0, t), w(x_0, t)) = (0, 0),$$

$$(4.5b) \quad (v, w) \rightarrow (E_v, E_w) \text{ as } x \rightarrow -\infty$$

rather than (4.1b,c). We would seek a steady state $q(x)$, with $p(x) = q(x)/\gamma$, as a solution to the boundary value problem

$$(4.6) \quad q'' + f(q) - q/\gamma = 0$$

$$q(-\infty) = E_v, \quad q(x_0) = 0.$$

From the phase plane portrait (Figure 11), we see that this problem does not have a solution.

Let us examine further the transient phase of the reflection in Figure 9. Observe that the leading edge is stopped by the local disturbance at $t = t_4$ and does not move for $t_4 < t < t_{10}$. It cannot move to the right because of the local disturbance in w . It also cannot move to the left. If it did it would resemble the trailing edge of a back moving to the left. Since the value of w immediately to the left of this trailing edge is very close to zero this would be impossible. The leading edge of the front must therefore wait at $x = x_0$ for either the values of w directly ahead of it to decrease sufficiently so that it can move to the right, or for the values of w directly behind it to increase sufficiently so that it can move to the left. In Figure 9B we see that the value of w steadily increases behind the leading edge for $t_5 < t < t_9$. When these values of w are large enough the leading edge collapses to form a back moving to the left. Hence, for a front to reflect the local disturbance must be sufficiently large so that the leading edge is stopped long enough for the values of w behind it to increase

sufficiently. Note that the values of w behind the leading edge would be unable to increase sufficiently if there existed a steady solution of (4.5).

The analysis we have presented so far holds for $\gamma \in (\gamma_0, \gamma_1)$. Only for these values of γ does the back speed exceed the front speed. On the other hand, the front is faster than the back if $\gamma \in (\gamma_1, \gamma_2)$. In this case there exists a solution of the boundary value problem (4.6) but there does not exist a solution of Equation (4.2). Hence we conjecture that, for a subset of parameter values for which the front speed exceeds the back speed, it is always possible to reflect a back but not a front.

These results indicate that for $\gamma < \gamma_1$, the rest state in some sense dominates the excited state. For $\gamma < \gamma_1$ the only type of stable traveling wave which brings the system to the excited state is a front which can be reflected by means of a local disturbance. For a subset of this range of parameters, when $(q(x), p(x))$ is stable, the back and pulse cannot be reflected. Hence, if the system is being switched, as $t \rightarrow \infty$, towards the excited state then it is possible, by means of a local disturbance, to return the system to rest. However, if the system is switching towards the rest state then it is impossible, by means of a local disturbance, to bring the system to the excited state. For $\gamma > \gamma_1$ the corresponding opposite statement is true and the excited state is dominant.

As mentioned in the introduction, our consideration of reflections by local disturbances also leads to expectations about the outcome of a front or back which approaches a boundary where v, w are held fixed at either $(0, 0)$ or (E_v, E_w) . This is because our model problems (4.1a,b,c) and (4.1a, 4.5a,b) incorporate Dirichlet boundary conditions explicitly. Suppose parameters are such that the back speed exceeds the front speed. Then, because (4.6) has no solution, we would expect a front to be reflected from a boundary where $(v, w) = (0, 0)$. Next consider a back approaching a boundary where $(v, w) = (E_v, E_w)$. If the steady state solution $(q(x), p(x))$ to (4.1) is stable, then we expect the back will tend to $(q(x), p(x))$ and not be reflected. On the other hand, if $(q(x), p(x))$ is unstable ($\epsilon\gamma^2 < 1$ and inequality (4.11), cf. Theorem 4.1) then we expect that the back might be reflected. Indeed, we have obtained numerical solutions of the model problems, for parameter values in this latter regime, which illustrate that

reflection may take place for a back approaching an excited boundary as well as for a front approaching a resting boundary.

Appendix A - Proof of Theorem 3.1.

Throughout this discussion we assume that the parameters ϵ and γ are fixed and set $h(s,c) = F(a,\epsilon,c,\gamma)$. We determine for which values of c there exists a root, s , of the equation

$$h(s,c) = 0$$

such that $s \in (0,1)$.

Note that,

$$(A.1) \quad h(1,c) = \frac{\partial h}{\partial s}(1,c) = 0$$

for all values of c . Furthermore, for any fixed value of c there can exist at most one root of the equation $\frac{\partial^2 h}{\partial s^2}(s^*,c) = 0$ for $s^* > 0$. If a root exists then it is given by

$$s^* = \left[\frac{-(\alpha_3 + \alpha_1)(\gamma + c\alpha_3)}{(\alpha_2 + \alpha_1)(\gamma + c\alpha_2)} \frac{p_2'}{p_3'} \right] \alpha_1 / (\alpha_3 - \alpha_2).$$

In the case $\alpha_2 = \bar{\alpha}_3$ we must choose a branch of the logarithm to determine s^* . This, along with (A.1), implies that for a given value of c at most one pulse can exist for a uniquely determined value of s and, therefore, from (3.8), for a uniquely determined value of a . Furthermore,

Lemma A.1. Assume that the parameters ϵ, γ , and c are fixed. Then a necessary and sufficient condition for pulses to exist is that either

$$(A.2) \quad \begin{array}{ll} \text{(a)} & h(0,c) > 0 \quad \text{and} \quad \frac{\partial^2}{\partial s^2} h(1,c) < 0 \\ \text{or} & \\ \text{(b)} & h(0,c) < 0 \quad \text{and} \quad \frac{\partial^2}{\partial s^2} h(1,c) > 0. \end{array}$$

In Lemmas A.3 and A.4 we determine for which values of c the functions $h(0,c)$ and $\frac{\partial^2}{\partial s^2} h(1,c)$ are either positive or negative. We first prove some properties of α_1 , the positive root of the characteristic polynomial $p(x)$ defined in (3.4). Recall that

$$\sigma^2 \equiv \frac{1+\gamma}{\gamma}.$$

Lemma A.2. Assume that the parameters ϵ and γ are fixed. Then

$$(a) \quad \lim_{c \rightarrow 0} \alpha_1 = \sigma$$

$$(b) \quad \left. \frac{\partial \alpha_1}{\partial c} \right|_{c=0} = \frac{\epsilon \gamma^2 - 1}{2 \epsilon \gamma^2}$$

$$(c) \quad \alpha_1 > c \text{ for all } c > 0$$

$$(d) \quad \text{if } \epsilon \gamma^2 > 1 \text{ then } \alpha_1 > \sigma \text{ for all } c > 0$$

$$(e) \quad \text{if } \epsilon \gamma^2 < 1 \text{ then, } \alpha_1 \begin{cases} < \sigma \text{ for } 0 < c < \hat{\sigma} = \frac{1 - \epsilon \gamma^2}{\sqrt{\gamma(1 + \epsilon \gamma)}} \\ = \sigma \text{ for } c = \hat{\sigma} \\ > \sigma \text{ for } c > \hat{\sigma}. \end{cases}$$

Proof. Parts (a) and (b) follow from substituting an expansion of α_1 in powers of c into the characteristic polynomial (3.4). For (c) note that $p(c) = -c - \frac{\epsilon \gamma^2 c}{c} < 0$.

Since $p(x) > 0$ for $x > \alpha_1$, the result follows.

We prove (d) by showing that $p(\sigma) < 0$ for all $c > 0$. Now, $p(\sigma) = \sigma^{1/2}(\sigma^2 - c\sigma - (1 + \epsilon \gamma))$. Hence, $p(\sigma) = 0$ if $c = \hat{\sigma} = \frac{1 - \epsilon \gamma^2}{\sqrt{\gamma(1 + \epsilon \gamma)}}$. If $\epsilon \gamma^2 > 1$ then $p(\sigma)$ is never equal to zero for any positive value of c . Therefore, either $p(\sigma) < 0$ or $p(\sigma) > 0$ for all $c > 0$. It follows that either $\alpha_1 > \sigma$ or $\alpha_1 < \sigma$ for all $c > 0$. But, by part (c) $\alpha_1 > c$ for all c . Choosing $c > \sigma$ the result follows.

To prove (e) assume that $\epsilon \gamma^2 < 1$. Hence $\hat{\sigma} > 0$. By inspection, $p(\sigma) < 0$ for $c > \hat{\sigma}$. Hence, $\sigma < \alpha_1$ because $p(x) > 0$ for any $x > \alpha_1$. Similarly, $p(\sigma) > 0$ for $c < \hat{\sigma}$ so, in this case, $\sigma > \alpha_1$. //

We now study the function $h(0, c)$ which can be written explicitly as:

$$(A.3) \quad h(0, c) = 2 - \frac{p_1'}{\sigma^2 \delta_1}.$$

First note that the condition $h(0, c) > 0$ is equivalent to

$$(A.4) \quad H(c) = (c^2 - \epsilon \gamma) \alpha_1^2 + \left(\frac{\epsilon \gamma^2 - 1}{\gamma} \right) 2c \alpha_1 + (\epsilon + \epsilon \gamma) < 0.$$

(A.4) is obtained by substituting into (A.3) the explicit formulas for p_1' , σ^2 , and δ_1 and then using the fact that $p(\alpha_1) = 0$.

Lemma A.3: Assume that the parameters ϵ and γ are fixed.

(a) There exists a positive constant $\bar{c}_1$ such that if $\epsilon \gamma^2 < 1$ and $c \in (0, \bar{c}_1)$ then $h(0, c) > 0$.

(b) There exists a positive constant $\bar{c}_2$ such that if $\epsilon\gamma^2 < 1$ and $c > \bar{c}_2$ then

$$h(0, c) < 0.$$

(c) If $\epsilon\gamma^2 > 1$ then $h(0, c) < 0$ for all $c > 0$.

Proof:

From Lemma A.2(a) and (b) it follows that $H(0) = 0$ and $H'(0) = \alpha_1 \left(\frac{\epsilon\gamma^2 - 1}{\gamma} \right) < 0$.

Part (a) now follows immediately. (b) is true because $\alpha_1 \sim c$ for $c \gg 1$ so that

$\lim_{c \rightarrow \infty} H(c) = +\infty$. We prove (c) by showing that $H(c) > 0$ for all $c > 0$. From

$p(\alpha_1) = 0$, we replace the first and third terms of $H(c)$ by $c\alpha_1^3 - c(\epsilon\gamma + 1)\alpha_1$. Now

$H(c)$ may be written as

$$H(c) = c\alpha_1 \left(\alpha_1^2 - \sigma^2 + \frac{\epsilon\gamma^2 - 1}{\gamma} \right).$$

Hence, from Lemma A.2(d), $H(c) > 0$. //

We now study $\frac{\partial^2}{\partial s^2} h(1, c)$ which can be written explicitly as $\frac{\partial^2}{\partial s^2} h(1, c) = \frac{p'_1}{\alpha_1^2 \delta_1} - 2$.

Lemma A.4. A necessary and sufficient condition for $\frac{\partial^2}{\partial s^2} h(1, c) < 0$ is that

$$(A.5) \quad (\epsilon\gamma + \epsilon)^{1/2} < c(\epsilon\gamma + c)^{1/2} + \sqrt{(c^2 + \epsilon\gamma)^2 + c^2 + \epsilon\gamma}.$$

Proof: If we substitute the explicit expressions for p'_1 and δ_1 into that of $\frac{\partial^2}{\partial s^2} h(1, c)$ and use the fact that $p(\alpha_1) = 0$ we find that $\frac{\partial^2}{\partial s^2} h(1, c) < 0$ is equivalent to

$$(A.6) \quad \left(\frac{\epsilon\gamma + \epsilon}{\epsilon\gamma + c^2} \right)^{1/2} < \alpha_1$$

which is equivalent to

$$p \left[\left(\frac{\epsilon\gamma + \epsilon}{\epsilon\gamma + c^2} \right)^{1/2} \right] < 0.$$

This can be written as

$$(A.7) \quad \Gamma^2 - 2c\Gamma - (\epsilon\gamma + 1) < 0$$

where

$$\Gamma = \left(\frac{\epsilon\gamma + \epsilon}{\epsilon\gamma + c^2} \right)^{1/2}.$$

Finally, (A.7) is equivalent to

$$\left(\frac{\epsilon\gamma + \epsilon}{\epsilon\gamma + c^2} \right)^{1/2} = \Gamma < c + \sqrt{c^2 + \epsilon\gamma + 1}$$

or,

$$(\epsilon\gamma + \epsilon)^{1/2} < c(\epsilon\gamma + c^2)^{1/2} + \sqrt{(c^2 + \epsilon\gamma)^2 + c^2 + \epsilon\gamma}. //$$

Lemma A.5: Assume that the parameters ϵ and γ are fixed.

(a) If $\epsilon\gamma^2 > 1$ then $\frac{\partial^2}{\partial s^2} h(1,c) < 0$ for all $c > 0$.

(b) If $\epsilon\gamma^2 < 1$ then there exists a positive constant $c_{\min}$ such that

$$\frac{\partial^2}{\partial s^2} h(1,c) > 0 \text{ for } c \in (0, c_{\min}) \text{ and}$$

$$\frac{\partial^2}{\partial s^2} h(1,c) < 0 \text{ for } c > c_{\min}.$$

Proof. (a) Note that (A.5) is true for $\epsilon\gamma^2 > 1$ at $c = 0$. The result follows because the right hand side of (A.5) is an increasing function of c .

(b) This is true because (A.5) fails to be true for $\epsilon\gamma^2 < 1$ at $c = 0$, and the right hand side of (A.5) is a monotonically increasing function which tends to $+\infty$ as $c \rightarrow \infty$. //

With these preliminary lemmas the proof of Theorem 3.1 now follows easily. From Lemmas A.1, A.3(c), and A.5(a) we conclude that a pulse cannot exist if $\epsilon\gamma^2 > 1$. From Lemmas A.1, A.3(a), and A.5(b) it follows that a pulse cannot exist if $\epsilon\gamma^2 < 1$ and $c < \min(\bar{c}_1, c_{\min})$. Similarly a pulse cannot exist for $c > \max(\bar{c}_2, c_{\min})$. We assume that $\epsilon\gamma^2 < 1$, unless stated otherwise, throughout the rest of Appendix A. We claim that pulses exist for values of c close to, but greater than $c_{\min}$. From Lemma A.5(b) we have that $\frac{\partial^2}{\partial s^2} h(1,c) < 0$ for $c > c_{\min}$. Our claim follows from Lemma A.1 if we can show that $h(0, c_{\min}) > 0$. But, from (A.6) we have that

$$\alpha_1(c_{\min}) = \left(\frac{\epsilon\gamma + \epsilon}{\epsilon\gamma + c_{\min}} \right)^{1/2} < \sigma.$$

Furthermore, $\frac{\partial^2}{\partial s^2} h(1, c_{\min}) = 0$, and therefore $\left(\frac{p_1'}{\delta_1'} \right)_{c_{\min}} = 2\alpha_1^2 < 2\sigma^2$. Since

$$h(0, c_{\min}) = 2 - \frac{p_1'}{\sigma^2 \delta_1'} \text{ the result follows.}$$

We next show that $\lim_{c \rightarrow c_{\min}} a = \lim_{c \rightarrow c_{\min}} z_1 = 0$. From (3.8) it suffices to show that at

$c = c_{\min}$, $s = 1$. From A.1 and Lemma A.5(b), it follows that

$h(1, c_{\min}) = \frac{\partial}{\partial s} h(1, c_{\min}) = \frac{\partial^2}{\partial s^2} h(1, c_{\min}) = 0$. Therefore, $h(s, c_{\min})$ has a triple root at $s = 1$ and the result follows.

We now wish to define $c_{\max}$. Recall that $h(0, c_{\min}) > 0$ and, from Lemma A.3(b), $h(0, c) < 0$ for $c > \bar{c}_2$. Let $c_{\max} = \inf\{c > c_{\min}, h(0, c) = 0\}$. Our previous discussion shows that pulses exist for $c \in (c_{\min}, c_{\max})$. Note that the condition $h(0, c_{\max}) = 0$ implies that $\frac{p_1'}{\delta_1} = 2\sigma^2$ at $c_{\max}$. Furthermore, from (3.8), it follows that at $c = c_{\max}$,

$$0 = s = 1 - a \frac{p_1'}{\delta_1} \quad \text{or,}$$

$$a = \frac{\delta_1}{p_1'} = \frac{1}{2\sigma^2} = \frac{\gamma}{2(1+\gamma)}.$$

That is, $\lim_{c \uparrow c_{\max}} a = \frac{\gamma}{2(1+\gamma)}$. Since $s = e^{-\alpha_1 z_1}$ we also conclude that $\lim_{c \uparrow c_{\max}} z_1 = \infty$.

It remains to prove that as $\epsilon\gamma^2 \uparrow 1$ both $c_{\min}$ and $c_{\max} \downarrow 0$. Recall that $h(0, c_{\max}) = 0$. Therefore, $H(c_{\max}) = 0$ where $H(c)$ has been defined in formula (A.4). Moreover, at $\epsilon\gamma^2 = 1$, $H(0) = H'(0) = 0$ and, for $\epsilon\gamma^2 > 1$, the proof of Lemma A.3(c) shows that $H(c) > 0$ for all $c > 0$. Hence, $c_{\max} = 0$ must be a double root of the equation $H(c) = 0$ at $\epsilon\gamma^2 = 1$ and the result follows.

Appendix B - Proof of Theorem 3.2.

Assume that the parameters ϵ and γ are fixed. Recall that a necessary and sufficient condition for a front to exist is that the speed, c , satisfies the inequality

$$\frac{p'_1}{\delta_1} > \sigma^2.$$

From the proof of Lemma A.3(a), we conclude that $\lim_{c \rightarrow 0} h(0, c) = \lim_{c \rightarrow 0} a - \frac{p'_1}{\sigma^2 \delta_1} = 0$. That

is, $\lim_{c \rightarrow 0} \frac{p'_1}{\delta_1} = 2\sigma^2 > \sigma^2$. Therefore, a front exists for c sufficiently small.

Furthermore, if c is sufficiently large, then $\frac{p'_1}{\delta_1} > 2\sigma^2 > \sigma^2$ and a front exists. This is an immediate consequence of Lemma A.3(b) and (c).

We next show that $\lim_{c \rightarrow 0} a = \frac{1}{2\sigma^2} = \frac{\gamma}{2(1+\gamma)}$. In the preceding paragraph we saw that

$\lim_{c \rightarrow 0} \frac{p'_1}{\delta_1} = 2\sigma^2$. The result now follows from Equation (3.10).

It remains to prove that

$$(B.1) \quad \left. \frac{\partial a}{\partial c} \right|_{c=0} \begin{cases} > 0 & \text{if } \epsilon \gamma^2 < 1 \\ = 0 & \text{if } \epsilon \gamma^2 = 1 \\ < 0 & \text{if } \epsilon \gamma^2 > 1. \end{cases}$$

It will be easier to consider c as a function of γ . Note, for PWL, that at $c = 0$,

$\gamma = \gamma_1 = \frac{2a}{1-2a}$ which implies that $\partial \gamma / \partial a|_{c=0} > 0$. Hence, (B.1) is equivalent to

$$(B.2) \quad \left. \frac{\partial \gamma}{\partial c} \right|_{c=0} \begin{cases} > 0 & \text{if } \epsilon \gamma_1^2 < 1 \\ = 0 & \text{if } \epsilon \gamma_1^2 = 1 \\ < 0 & \text{if } \epsilon \gamma_1^2 > 1. \end{cases}$$

The result (B.2) can be verified algebraically for PWL. In the following proof we require only that f is "cubic-like" as shown in Figure 4. For PWL and CUB, γ_1 is defined in Section 2. More generally, when f may not have the symmetry properties of PWL or CUB, then γ_1 is the value of γ such that

$$0 = \int_0^{E_V} [f(v) - v/\gamma_1] dv.$$

In this more general case, one should remember that $\partial a/\partial c|_{c=0}$ and $\partial \gamma/\partial c|_{c=0}$ will have opposite sign if $\partial \gamma/\partial a|_{c=0} < 0$.

We first write γ , v_c , w_c and $f(v_c)$ as power series in c

$$\begin{aligned} \gamma(c) &= \gamma_1 + c\bar{\gamma}_1 + c^2\bar{\gamma}_2 + \dots \\ v_c(z) &= \varphi_0(z) + c\varphi_1(z) + \dots \\ w_c(z) &= \psi_0(z) + c\psi_1(z) + \dots \\ f(v_c) &= f(\varphi_0) + c\varphi_1 f'(\varphi_0) + \dots \end{aligned} \quad (B.3)$$

Recall that (v_c, w_c) solves

$$(B.4) \quad \begin{cases} cv'_c = v''_c + f(v_c) - w_c \\ cw'_c = \varepsilon(v_c - \gamma w_c) \end{cases}$$

with boundary conditions

$$(v_c(-\infty), w_c(-\infty)) = (0, 0)$$

$$(v_c(+\infty), w_c(+\infty)) = (E_v, E_w).$$

Substituting (B.3) into (B.4) and equating like order terms we find to the zeroth order that:

$$\begin{cases} 0 = \varphi_0'' + f(\varphi_0) - \psi_0 \\ 0 = \varepsilon(\varphi_0 - \gamma_1 \psi_0) \end{cases}$$

$$(\varphi_0(-\infty), \psi_0(-\infty)) = (0, 0) \quad \text{and} \quad (\varphi_0(+\infty), \psi_0(+\infty)) = (E_v, E_w).$$

Hence, $\psi_0(z) = \varphi_0(x)/\gamma_1$. Therefore, $\varphi_0(z)$ is a solution of the ordinary differential equation

$$0 = \varphi_0'' + f(\varphi_0) - \frac{1}{\gamma_1} \varphi_0$$

with boundary conditions

$$\varphi_0(-\infty) = 0, \quad \varphi_0(+\infty) = E_v.$$

It is not hard to show that such a solution exists and is a monotonically increasing function.

We now equate first order terms to obtain

$$(B.5) \quad \begin{cases} \varphi_0' = \varphi_1'' + f'(\varphi_0)\varphi_1 - \psi_1 \\ \psi_0' = \epsilon(\varphi_1 - (\gamma_1\psi_1 + \bar{\gamma}_1\psi_0)) \end{cases}$$

This implies that

$$\psi_1 = \frac{\varphi_1 - \bar{\gamma}_1\psi_0}{\gamma_1} - \frac{\psi_0'}{\epsilon\gamma_1}$$

and, hence, φ_1 is a solution of the equation

$$(B.6) \quad \varphi_1'' + f'(\varphi_0)\varphi_1 - \frac{\varphi_1}{\gamma_1} = \varphi_0' \left(1 - \frac{1}{\epsilon\gamma_1^2}\right) - \frac{\bar{\gamma}_1}{\gamma_1^2} \varphi_0.$$

Since φ_0' is a solution of the homogeneous equation:

$$(\varphi_0')'' + (\varphi_0')' f'(\varphi_0) - \frac{\varphi_0'}{\gamma_1} = 0,$$

a necessary and sufficient condition for (B.6) to have a solution, φ_1 , is that

$$(B.7) \quad \langle \varphi_0', \varphi_0' \left(1 - \frac{1}{\epsilon\gamma_1^2}\right) - \frac{\bar{\gamma}_1}{\gamma_1^2} \varphi_0 \rangle = 0.$$

Here $\langle u, v \rangle \equiv \int_{-\infty}^{\infty} u v dx$. Now $\langle \varphi_0', \varphi_0' \rangle = \frac{E^2}{2}$. Therefore, (B.7) becomes

$$\left(1 - \frac{1}{\epsilon\gamma_1^2}\right) \langle \varphi_0', \varphi_0' \rangle - \frac{\bar{\gamma}_1}{2\gamma_1^2} E_v^2 = 0$$

or

$$\bar{\gamma}_1 = 2 \left[1 - \frac{1}{\epsilon \gamma_1^2} \right] \frac{\gamma_1^2}{\epsilon^2} \langle \varphi_0', \varphi_0' \rangle$$

from which the result follows.

We note that the same proof applies for the emergence of the slow back solution from $c = 0$ except, in that case, $\bar{\gamma}_1$ takes the opposite sign because $\langle \varphi_0', \varphi_0' \rangle = -\epsilon^2/2$.

To treat stability we introduce a traveling coordinate frame (z, t) , $z = x + ct$, and seek exponential solutions $(e^{\lambda t} V(z), e^{\lambda t} W(z))$ to the linear variational equation. Hence, λ, V, W , satisfy

$$\begin{aligned} \lambda V &= V'' - cV' + f'(v_c(z))V - W \\ \lambda W &= -cW' + \epsilon(V - \gamma W). \end{aligned} \quad (\text{B.8})$$

As in the proof above, we follow a perturbation argument for small c and write

$$\begin{aligned} V(z) &= V_0(z) + c V_1(z) + \dots \\ W(z) &= W_0(z) + c W_1(z) + \dots \\ \lambda &= \lambda_0 + c \lambda_1 + \dots \end{aligned}$$

To lowest order we find for $\gamma = \gamma_1 + O(c)$ that $W_0 = \frac{\epsilon}{\lambda_0 + \epsilon \gamma_1} V_0$, and

$$(\text{B.9}) \quad V_0'' + [\mu_0 + f'(\varphi_0(z))] V_0 = 0, \quad z \in \mathbb{R}^+$$

where

$$(\text{B.10}) \quad \mu_0 = -\lambda_0 - \epsilon/(\lambda_0 + \epsilon \gamma_1).$$

The perturbation procedure is straightforward: λ_1 is obtained from the solvability condition for the next order equation, etc. Therefore we concentrate on (B.9), a Schroedinger eigenvalue problem. If (B.9) has a bounded solution with $\text{Re } \lambda_0 > 0$ then (v_c, w_c) is unstable for $0 < c \ll 1$. On the other hand, if (B.9) has no bounded solution with $\text{Re } \lambda_0 > 0$ then (v_c, w_c) is linearly stable for $0 < c \ll 1$. We remark that eigenvalue problems of the form (B.9), (B.10) were also encountered in [16].

Equation (B.9) always has the solution $\lambda_0 = 0$, i.e., $\mu_0 = -1/\gamma_1$, and $V_0 = \varphi_0'$. This corresponds to the translation invariance of $\varphi_0(z)$ and does not determine

stability. Because of the quadratic relation between μ_0 and λ_0 , a second solution to (B.9) with $\mu_0 = -1/\gamma_1$, $v_0 = \varphi'_0$ is $\lambda_0 = -\epsilon\gamma_1 + 1/\gamma_1$. If $\epsilon\gamma^2 < 1$ then this second solution satisfies $\lambda_0 > 0$ so that $v_c(z), w_c(z)$ is unstable. On the other hand, suppose $\epsilon\gamma^2 > 1$. Then the second solution satisfies $\lambda_0 < 0$ and does not lead to an unstable mode. Moreover, since the eigenfunction φ'_0 is of one sign it corresponds to the lowest value of μ_0 in the discrete spectrum of (B.9). Therefore, any other μ_0 must be greater than $-1/\gamma_1$ and, from (B.10), the corresponding values of λ_0 satisfy $\text{Re } \lambda_0 < 0$. Hence, if $\epsilon\gamma^2 > 1$ we have linear stability for small c .

Appendix C - Proof of Theorem 4.1.

Without loss of generality we take $x_0 = 0$. If $\gamma_0 < \gamma < \gamma_1$, i.e. $1 < 2a/E_v < 2$, then for PWL the steady state solution to (4.2) is

$$\begin{aligned} q(x) &= a e^{\sigma(x-\xi)} , & x < \xi \\ (C.1) \quad & & \\ &= E_v + (a - E_v/2)e^{\sigma(x-\xi)} - (E_v/2)e^{-\sigma(x-\xi)} , & \xi < x < 0 \end{aligned}$$

where

$$(C.2) \quad \xi = \frac{1}{2\sigma} \ln (2a/E_v - 1)$$

and, as in Section 3, $\sigma^2 = 1 + 1/\gamma$. For stability, we consider exponential solutions $e^{\lambda t} V(x) e^{\lambda x} W(x)$ to the linear variational equation. As in Appendix B, W satisfies $W(x) = \varepsilon V(x)/(\lambda + \varepsilon\gamma)$ and $\lambda, V(x)$ solve the Schrodinger eigenvalue problem:

$$(C.3) \quad V'' + [\mu + f'(q(x))]V = 0 \quad -\infty < x < 0$$

where

$$(C.4) \quad \mu = \mu(\lambda) = -\lambda - \varepsilon/(\lambda + \varepsilon\gamma)$$

and

$$(C.5) \quad V(0) = 0, \quad |V(x)| \rightarrow 0 \text{ as } x \rightarrow -\infty.$$

For PWL,

$$(C.6) \quad f'(q(x)) = -1 + \delta(\xi)/q'(\xi).$$

Here $\delta(\xi)$ is the Dirac delta function.

If (C.3) - (C.6) has a solution with $\text{Re } \lambda > 0$, then we say that $q(x)$ is unstable. On the other hand, if there is no solution with $\text{Re } \lambda > 0$ then $q(x)$ is linearly stable. Indeed, it is possible that there may be no values of λ for which (C.3)-(C.6) has a solution; i.e., there may be no eigenvalue in the discrete spectrum for which $V(0) = 0$. In such a case we would conclude that $q(x)$ is linearly stable. We note in passing that there is always a solution $\mu_0, V_0(x)$ which satisfies (C.3) and for which $V_0'(0) = 0$; the lowest such value for μ_0 constitutes the "ground state". For the solutions we seek, the relation (C.4) implies that the largest $\text{Re } \lambda$ corresponds to the algebraically smallest μ ; for this smallest μ , V is of one sign for $x < 0$.

First we show, for "cubic-like" f , that $q(x)$ is linearly stable if $\varepsilon\gamma^2 > 1$. If there is an eigensolution, it satisfies (C.3) rewritten as

$$V'' + [\mu + 1/\gamma + f'(q(x)) - 1/\gamma]V = 0.$$

Now multiply this equation by $q'(x)$ and (4.2) by $V'(x)$, add the two equations, and then integrate from $x = -\infty$ to $x = 0$ to obtain:

$$q'(0)V'(0) = \int_{-\infty}^0 \frac{d}{dx} [-Vf(q) + Vq/\gamma] dx - (\mu + 1/\gamma) \int_{-\infty}^0 q' V dx.$$

Since $V(0) = 0$ and $f(E_V) = E_V/\gamma$ we arrive at

$$\mu + 1/\gamma = -q'(0)V'(0) / \int_{-\infty}^0 q' V dx.$$

But $q'(x)/q'(0) > 0$ and $V(x)/V'(0) < 0$ for $x < 0$. Therefore $\mu > -1/\gamma$, and from (C.4) we conclude that $\text{Re } \lambda < 0$ when $\varepsilon\gamma^2 > 1$.

Next we consider $\varepsilon\gamma^2 < 1$ and restrict attention to PWL. If we normalize V so that $V(\xi) = 1$, then

$$\begin{aligned} V(x) &= e^{\alpha(x-\xi)}, & x < \xi \\ (C.7) \quad & & \\ &= \sinh \alpha x / \sinh \alpha \xi, & \xi < x < 0 \end{aligned}$$

where

$$\alpha = \sqrt{1-\mu}.$$

Since V must satisfy the jump condition

$$V' \Big|_{\xi}^{\xi^+} = -V(\xi)/q'(\xi)$$

we find that μ solves the "characteristic equation"

$$(C.8) \quad G(\mu, a, \varepsilon, \gamma) = 0$$

where

$$(C.9) \quad G(\mu, a, \varepsilon, \gamma) \equiv 1 - \coth \alpha \xi - 1/(\alpha a \sigma).$$

Equation (C.8) can be analyzed graphically. For all values of the parameters a, ϵ, γ , we have that $G > 0$ for $\mu \ll 0$. Next, we find parameter ranges such that $G > 0$ for $0 < 1 - \mu \ll 1$. In such cases we conclude that G is of one sign for $\mu < 1$. (If not, then G would have two zeroes for $\mu < 1$. This would imply that (C.3)-(C.6) has two different eigenvalues μ , each with $V(x)$ non-zero for $x < 0$, which contradicts the known spectral properties of Schroedinger problems.) From this it follows that there is no solution to (C.3)-(C.6) of the assumed form so that $q(x)$ is linearly stable for such parameter ranges.

To find the parameter range, suppose μ is slightly less than one. Then $G > 0$ is equivalent to $-\xi < a\sigma$ or, using (C.2),

$$(C.10) \quad -\ln(2a/E_v - 1) < 2a/E_v.$$

Hence, if $\epsilon\gamma^2 < 1$ and

$$2a/E_v > 1.278... ,$$

the inequality (C.10) is satisfied, there is no eigensolution of the desired form, and therefore, $q(x)$ is linearly stable.

To see when $q(x)$ is unstable, observe from (C.4) that $\mu < -\epsilon\gamma$ implies that at least one of the two corresponding λ values has $\text{Re } \lambda > 0$. Hence, if $G < 0$ for $\mu = -\epsilon\gamma$, then G has a zero for $\mu < -\epsilon\gamma$ and $q(x)$ is unstable. For fixed ϵ, γ and $\mu = -\epsilon\gamma$, we write $G < 0$ as

$$(C.11) \quad 1 - \coth[\sqrt{1 + \epsilon\gamma} \ln(2a/E_v - 1)/2\sigma] < 2\sigma/[\sqrt{1 + \epsilon\gamma} 2a/E_v].$$

If $0 < 2a/E_v - 1 \ll 1$, (C.11) becomes $1 < \sigma/\sqrt{1 + \epsilon\gamma}$ which is satisfied when $\epsilon\gamma^2 < 1$ so that $q(x)$ is unstable. Finally, observe that at critical values of the parameters

a, ϵ, γ for which $G(-\epsilon\gamma, a, \epsilon, \gamma) = 0$, the corresponding λ values are a complex pair with $\text{Re } \lambda = 0$, $\text{Im } \lambda = \pm i \sqrt{\epsilon(1 - \epsilon\gamma^2)}$ so that a branch of periodic solutions emerges through Hopf bifurcation.

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waves vary with the parameters is given for the case when the system has a piecewise linear nonlinearity. Numerical results of solutions to these equations are also presented. These calculations illustrate how waves are generated from initial data, how they interact during collisions, and how they are influenced by local disturbances and boundary conditions. In one example, a rightward moving "front" (rest-to-excited transition) slows down, stops, turns around, and develops into a leftward moving "back" (excited-to-rest transition) when it encounters appropriate spatially localized, transient perturbations in the dependent variables. In this case, the medium ultimately returns to the rest state. For the same parameter values we have sought, but have not found, analogous local disturbances sufficient to reflect a propagating "back". These observations are consistent with the notion that the rest state is in some sense dominant for a certain parameter range. It is conjectured that local disturbances or certain boundary conditions cannot cause the reflection of a nondominant to a dominant transition wave. Partial evidence for this conjecture is presented by means of an analytic argument.

