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A THEORY OF INTERVAL ITERATION.(U)
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A THEORY OF INTERVAL ITERATION.

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A THEORY OF INTERVAL ITERATION

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ABSTRACT

A theory of interval iteration, based on a few simple assumptions, is given for the fixed point problem for operators in partially ordered topological spaces. A comparison of interval with ordinary iteration is made which shows that their properties are converse in a certain sense with respect to existence or nonexistence of fixed points. The theory of interval iteration is shown to hold without modification if the computation is restricted to a finite set of points, as in actual practice. In this latter case, interval iteration is shown to converge or diverge in a finite number of steps, for which an upper bound is given. By the introduction of a suitable iteration operator, the method of interval iteration is extended to the problem of solution of equations in linear spaces.
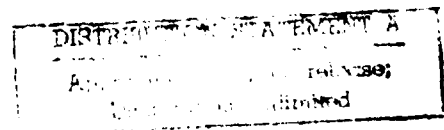
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             and upper bounds, Finite Convergence, Solution of equations

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SIGNIFICANCE AND EXPLANATION

Interval iteration is a powerful computational method with applications
to the solution of linear and nonlinear systems of equations, integral
equations (see MRC TSR #2128), and differential equaitons. Because of this
diversity, it is helpful to have a general theory to act as a guide to
specific applications. This report gives such a general theory, based on only
a few simple assumptions to give it as wide a scope as possible. The notions
of convergence and divergence of interval iteration are defined and their
implications with regard to existence or nonexistence of solutions are
presented. It turns out that convergence of interval iteration is assured if
the initial interval contains a solution, and the iteration in this case can
provide improved lower and upper bounds for the solution. Divergence of
interval iteration, on the other hand, shows that the initial interval did not
contain a solution, and thus may be discarded. These properties are compared
with the well-known results for ordinary iteration, and shown to be converse
in a certain sense. Remarks on how to use interval iteration in applications
are made, and the theory is also shown to hold if the computation is
restricted to a finite set of elements, as on an ordinary computer.
Construction of interval extensions which are exactly representable in the
finite case is explained, and it is observed that interval iteration will
converge or diverge in a finite number of steps in actual computation, and an
upper bound is obtained for this number which depends only on the initial
interval chosen. Now that microprogrammed interval operations can be
performed with speeds comparable to floating point arithmetic, interval
iteration can be applied to a wider class of problems to obtain solutions of
known reliability.

A THEORY OF INTERVAL ITERATION

L. B. Rall

1. **A fixed point problem.** Suppose that ϕ is an operator which maps a partially ordered topological space S into itself. A <u>fixed point</u> $y^* \in S$ of ϕ satisfies the equation

(1.1)
$$y = \phi(y);$$

solving this equation is called the <u>fixed point problem</u> for ϕ in S.

The partial ordering relation in S will be denoted, as usual, by "$\leq$". Elements $\underline{y}$, $\overline{y}$ of S such that $\underline{y} \leq \overline{y}$ define an <u>interval</u> $Y = [\underline{y}, \overline{y}]$ in S, which is the nonempty set

(1.2)
$$Y = [\underline{y}, \overline{y}] = \{y \mid \underline{y} \leq y \leq \overline{y},\ y \in S\}.$$

The elements $\underline{y}$, $\overline{y}$ are called respectively the <u>lower</u> and <u>upper</u> <u>endpoints</u> of Y. The set of all intervals in S is denoted by IS. The elements y of S are identified with the corresponding <u>degenerate</u> intervals $y = [y, y]$ which have equal endpoints.

The following assumptions will be made concerning the topology of S:

    (i) Intervals are closed subsets of S;

    (ii) Each nondegenerate interval contains a limit point of countable order.

The last assumption means that if Y is nondegenerate, then it contains a limit point ℓ such that each neighborhood of ℓ contains at least a countable number of points of Y different from ℓ. This technical property is called $\aleph_0$-compactness by Sierpiński [10].

2. **Interval extensions and interval iteration.** An operator which maps the set IS of intervals in S into itself is called an <u>interval operator</u> in S. An interval operator Φ is said to be an <u>interval extension</u> of an operator ϕ in S if (i) it is an <u>extension</u> in the sense that

(2.1)
$$\{\phi(y) \mid y \in Y\} \subset \Phi(Y), \quad Y \in IS,$$

and (ii) ϕ is <u>inclusion</u> <u>monotone</u>, that is,

(2.2) $Y \subset Z \;\Rightarrow\; \phi(Y) \subset \phi(Z), \quad Y, Z \in IS.$

ϕ is called an interval extension of an interval operator F if $F(Y) \subset \phi(Y)$ for

$Y \in IS$ and (2.2) holds.

Definition 2.1. Given an <u>initial</u> <u>interval</u> Y_0 and an interval operator ϕ, the

sequence $\{Y_n\}$ defined by

(2.3) $Y_{n+1} = Y_n \cap \phi(Y_n), \quad n = 0,1,2,\ldots,$

is said to be generated by <u>interval</u> <u>iteration</u> starting from Y_0.

Note that, from (2.3),

(2.4) $Y_0 \supset Y_1 \supset Y_2 \supset \ldots ;$

hence, interval iteration generates a <u>nested</u> (or <u>descending</u>) sequence of closed sets.

Definition 2.2. The interval iteration process (2.3) is said to <u>diverge</u> if

(2.5) $Y_N = \emptyset \quad (\text{empty})$

for some positive integer N; otherwise,

(2.6) $Y^* = \bigcap_{n=0}^{\infty} Y_n$

is nonempty by the Cantor theorem ([10], pp. 34-35), and the interval iteration is

said to <u>converge</u> to the <u>limit</u> $Y^* = \lim\limits_{n \to \infty} \{Y_n\}$ given by (2.6).

Thus, according to this definition, convergence and divergence of interval

iteration have a converse relationship, as one would expect.

Theorem 2.1. If ϕ is an interval extension of the operator ϕ in S, and the

initial interval Y_0 contains a fixed point y^* of ϕ, then the interval iteration (2.3)

converges; furthermore,

(2.7) $y^* \in Y^* = \lim\limits_{n \to \infty} \{Y_n\} = \bigcap_{n=0}^{\infty} Y_n .$

Proof. This follows, as is well known [3], [6], from (2.3) and mathematical

induction, since $y^* \in Y_k \Rightarrow \phi(y^*) = y^* \in \phi(Y_k)$, and thus $y^* \in Y_{k+1} = Y_k \cap \phi(Y_k) \neq \emptyset$. QED

The contrapositive of Theorem 2.1 is the following result, which is more incisive.

Theorem 2.2. If Φ is an interval extension of ϕ, and the interval iteration (2.3) diverges, then the initial interval Y_0 contains no fixed points y^* of ϕ.

This assertion was noted by Nickel [8] in connection with an interval version of Newton's method.

Observe that convergence of interval iteration is a necessary, not sufficient, condition for the existence of a fixed point $y^* \in Y_0$ of ϕ; divergence, on the other hand, is a sufficient condition for nonexistence of fixed points of ϕ in Y_0.

3. __Comparison with ordinary iteration.__ The ordinary iteration method

$$(3.1) \qquad y_{n+1} = \phi(y_n), \quad n = 0,1,2,\ldots,$$

is often used to attempt to generate a sequence $\{y_n\}$ which converges to a fixed point y^* of ϕ, starting from some initial point y_0. If ϕ is continuous in the topology for S, in which also $\lim_{n\to\infty}\{y_n\} = y^* \in S$, then y^* will be a fixed point of ϕ, and if Y_0 is a closed subset of S such that $\{y_n\} \subset Y_0$, then $y^* \in Y_0$. On the other hand, if Y_0 is a subset of S which does not contain a fixed point of the continuous operator ϕ, then the sequence generated by the iteration (3.1) cannot converge to a point of Y_0. The first alternative will be called __convergence__ __into__ Y_0, and the second __divergence__ __from__ Y_0. On the basis of these definitions and the corresponding concepts for interval iteration given in Definition 2.2, a comparison of ordinary and interval iteration is shown in Figure 3.1.

Ordinary Iteration $y_{n+1} = \phi(y_n)$.	Interval Iteration $Y_{n+1} = Y_n \cap \Phi Y_n$.
Convergence (into Y_0) $\Rightarrow$ Existence ($y^* \in Y_0$)	Existence ($y^* \in Y_0$) $\Rightarrow$ Convergence ($Y = \bigcap_{n=0}^{\infty} Y_n \neq \emptyset$)
Nonexistence ($y^* \notin Y_0$) $\Rightarrow$ Divergence (from Y_0)	Divergence (some $Y_N = \emptyset$) $\Rightarrow$ Nonexistence ($y^* \notin Y_0$)

Figure 3.1. Ordinary and interval Iteration Compared.

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Thus, interval iteration stands in a converse relationship to ordinary iteration under the above assumptions. It is worth noting that in metric spaces S, the convergence of the ordinary iteration process (3.1) often depends on being able to choose the initial point y_0 "close" to the fixed point y^*, and that some operators ϕ have fixed points y^* which repel the iteration sequence $\{y_n\}$ for all $y_0 \neq y^*$. The convergence of interval iteration, on the other hand, follows if the initial interval Y_0 is "large enough" to contain a fixed point y^* of ϕ.

4. **Applications of interval iteration.** Interval iteration may be applied in several ways to the fixed point problem (1.1).

1°. Suppose that the interval Y_0 is known to contain a fixed point y^* of ϕ, perhaps on the basis of a nonconstructive fixed point theorem. In this case, the interval iteration (2.3) will converge, and may be used to obtain lower and upper bounds for y^*, namely,

$$(4.1) \qquad \underline{y}_n \leq y^* \leq \overline{y}_n, \quad n = 0,1,2,\ldots,$$

where $Y_n = [\underline{y}_n, \overline{y}_n]$, and in the limit,

$$(4.2) \qquad \underline{y}^* \leq y^* \leq \overline{y}^*,$$

where $Y^* = [\underline{y}^*, \overline{y}^*]$ is the limit (2.6) of $\{Y_n\}$.

The bounds (4.1) give improved results as long as the inclusions (2.4) are strict. However, if $Y_{N+1} = Y_N$ for some positive integer N, then

$$(4.3) \qquad Y^* = \bigcap_{n=0}^{\infty} Y_n = Y_N ;$$

this is called <u>finite convergence</u> of the interval iteration (2.3). If finite convergence takes place, then the best lower and upper bounds obtainable for y^*, starting from $Y_0 = [\underline{y}_0, \overline{y}_0]$, are

$$(4.4) \qquad \underline{y}^* = \underline{y}_N \leq y^* \leq \overline{y}_N = \overline{y}^* .$$

2°. If it is not known whether or not Y_0 contains a fixed point y^* of ϕ, interval iteration may still be useful in one of the following ways.

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(i) As long as the interval iteration is producing intervals which de-
crease at each step (strict inclusion holds in (2.4)), then an existence test that
fails because Y_0, Y_1, ..., Y_{N-1} are "too large" may succeed for Y_N. This Y_N may then
be taken as the initial interval Y_0, and one has the favorable case 1° discussed above.

(ii) If the iteration produces an empty intersection (divergence), then
this establishes conclusively that the initial interval Y_0 does <u>not</u> contain a fixed
point $y*$ of ϕ, so that this interval may be excluded from further consideration.

There is, of course, a third possibility:

(iii) The interval iteration (2.3) leads only to an interval Y° in
which no conclusive assertion about existence or nonexistence of a fixed point $y*$
of ϕ is available. (Y° may be the limit Y* if obtained in a finite number of steps,
or otherwise.)

Possible alternatives in this situation include partition of the resulting in-
terval Y° into subintervals for further examination, a strategy developed by Moore
and Jones [7], or acceptance of Y° as a "generalized" or "pseudosolution" (relative
to the initial interval Y_0) of the fixed point problem. This latter choice may be
useful in the development of an interval version of regularization of solutions of
ill-posed problems.

5. <u>Interval iteration on a grid</u>. The ordinary iteration process (3.1) is a poor
model of what actually occurs in computation, since it is usually impossible to car-
ry out the indicated transformations exactly. Interval iteration, on the other hand,
is readily adaptable to actual machine computation, and its theory can be preserved
intact.

Suppose that G (called a <u>grid</u>) is a finite subset of the space S. Here, one
may think of the set of numbers which have exact representations on a given computer,
and finite Cartesian products of such a set. The subset of IS consisting of intervals
with endpoints in G will be denoted by IG, that is,

(5.1) $IG = \{[a, b] \mid a, b \in G\}$.

The union of all intervals in IG, considered as subsets of S, defines a closed

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subset D of S, since IG is a finite collection of closed sets [10]. The operation of _directed_ _rounding_ will now be defined in ID, the set of all intervals in S having endpoints in D.

Definition 5.1. For $x \in D$, the _upward_ _rounding_ _operator_ ∇ to G is defined by

$$(5.2) \qquad \nabla x = \min \{b \mid b \geq x, b \in G\} ,$$

and the _downward_ _rounding_ _operator_ Δ to G by

$$(5.3) \qquad \Delta x = \max \{a \mid a \leq x, a \in G\} .$$

For $X = [\underline{x}, \overline{x}] \in ID$, the _directed_ _rounding_ _operator_ $\square$ to IG is defined by

$$(5.4) \qquad \square X = \square[\underline{x}, \overline{x}] = [\Delta\underline{x}, \nabla\overline{x}], \quad X = [\underline{x}, \overline{x}] \in ID.$$

It follows immediately from this definition that $\square$ is an inclusion monotone interval operator. Furthermore, if ϕ maps D into itself and Φ is an interval extension of ϕ, then $\square\Phi$ will be an interval extension of ϕ which maps IG into IG. Thus, for actual computation, it may be assumed that the interval extension Φ of ϕ in (2.3) has been constructed to map IG into itself. This means that the transformed intervals $\Phi(Y_n)$ will be _exactly_ _representable_ in terms of elements of G for $Y_n \in IG$. The theory of interval iteration given above applies to operators of this type without modification. Furthermore, under the following reasonable assumption, the entire interval iteration (2.3) may be carried out _exactly_, using only elements of the grid G.

Assumption 5.1 (Intersection Property). If $X, Z \in IG$, then $X \cap Z = \emptyset$, the empty set, or $X \cap Z \in IG$.

Thus, if IG has the intersection property, and the interval extension Φ of ϕ has been constructed to map IG into itself, then the selection of an initial interval $Y_0 \in IG$ will assure that the intervals $Y_1, Y_2, \ldots$ generated by the interval iteration (2.3) also belong to IG. Furthermore, this interval iteration on IG will always converge or diverge in a finite number of steps. To see this, let $G\#X = G\#[\underline{x}, \overline{x}]$ denote the number of _grid_ _points_ (elements of G) contained in the interval $X \in IG$. Then, the following result holds.

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Theorem 5.1. On IG, the interval iteration (2.3) will converge or diverge in at most $G\#(Y_0)$ steps.

Proof. If the interval iteration (2.3) has not terminated in n steps by divergence or convergence, then it will have generated distinct intervals Y_0, Y_1, ..., $Y_n \in IG$, including the initial interval Y_0. Since (2.4) holds, one has

$$(5.5) \qquad G\#(Y_n) \leq G\#(Y_{n-1}) - 1 \leq G\#(Y_{n-2}) - 2 \leq \ldots \leq G\#(Y_0) - n.$$

Thus, the maximum length of a sequence of distinct nested intervals in IG is $\nu = G\#(Y_0) - 1$, since each interval in IG must contain at least one point of G. If the interval iteration has not terminated by the νth step, then

$$(5.6) \qquad\qquad G\#(Y_\nu) = 1$$

by (5.5), which implies that Y_ν is a degenerate interval. Now, one has either $Y_\nu \subset \Phi(Y_\nu)$, in which case $Y_{\nu+1} = Y_\nu$ (convergence in ν steps), or $Y_\nu \cap \Phi(Y_\nu) = \emptyset$ (divergence in $\nu + 1 = G\#(Y_0)$ steps). QED

Thus, in actual computation, convergence or divergence of an interval iteration is an observable event in principle, since one works on a grid of machine numbers. Of course, $G\#(Y_0)$, although finite, could be prohibitively large; however, termination of interval iteration is usually observed in far fewer steps. The construction of the interval operator Φ is crucial to the success of interval iteration [3], but depends heavily on the nature of S, ϕ, and the grid G available.

6. Solution of equations. In many applications, S is a linear space, and the problem of interest is to find a solution x* of the equation

$$(6.1) \qquad\qquad f(x) = 0.$$

This may be transformed into a fixed point problem (1.1) in many ways by the introduction of suitable iteration operators ϕ. For example, one may take

$$(6.2) \qquad\qquad \phi(x) = x - Yf(x)$$

to define ϕ, where Y is an invertible linear operator in S. Given an interval extension F of f, the corresponding interval extension Φ of ϕ is

(6.3) $\quad\quad\quad\quad\quad\quad \phi(X) = X - YF(X),$

using interval arithmetic [4], [6]. If S is a Banach space, and f has a Fréchet derivative f', then

(6.4) $\quad\quad\quad\quad\quad\quad \phi'(x) = I - Yf'(x),$

where I denotes the identity operator, and a more accurate interval extension of ϕ can be constructed on the basis of its mean-value form [1]. Let F' be an interval extension of f', then the corresponding interval extension ϕ' of ϕ' is given by

(6.5) $\quad\quad\quad\quad\quad\quad \phi'(X) = I - YF'(X),$

and the mean-value form of (6.2) is, for $y \in X$,

(6.6) $\quad\quad\quad \phi(X) = y - Yf(y) + \{I - YF'(X)\}(X - y),$

and this interval extension ϕ of ϕ is called the Krawczyk iteration operator. It arose from consideration of an interval version of Newton's method [2], and has many useful properties [3], [5], [9]. In actual practice, computation would be done with a rounded version of (6.6). Suppose that F, F' are interval extensions in IG of f, f', then

(6.7) $\quad\quad\quad \phi(X) = \square\{y - YF(y) + \{I - YF'(X)\}(X - y)\}$

will have values in IG for $y \in G$, $X \in IG$. An even more rounded interval extension of (6.6) to IG is

(6.8) $\quad\quad\quad \phi(X) = \square\{\square\{y - \square YF(y)\} + \square\{\square\{I - \square YF'(X)\}\cdot\square(X - y)\}\},$

which models a realistic computational interval operator.

<div align="center">REFERENCES</div>

1. O. Caprani and K. Madsen, Mean value forms in interval analysis, Computing 25 (1980), 147-154.

2. R. Krawczyk, Newton-Algorithmen zur Bestimmung von Nullstellen mit Fehler-schranken, Computing 4 (1969), 187-201.

3. R. Krawczyk, Interval extensions and interval iterations, Computing $\underline{24}$ (1980), 119-129.

4. R. E. Moore, Interval Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1966.

5. R. E. Moore, A test for existence of solutions to nonlinear systems, SIAM J. Numer. Anal. $\underline{14}$ (1977), 611-615.

6. R. E. Moore, Methods and Applications of Interval Analysis, SIAM Studies in Applied Mathematics 2, Society for Industrial and Applied Mathematics, Philadelphia, 1979.

7. R. E. Moore and S. T. Jones, Safe starting regions for iterative methods, SIAM J. Numer. Anal. $\underline{14}$ (1977), 1051-1065.

8. K. Nickel, On the Newton method in interval analysis, MRC Tech. Summary Rept. 1136, Univ. of Wisconsin-Madison, 1971.

9. L. B. Rall, A comparison of the existence theorems of Kantorovich and Moore, SIAM J. Numer. Anal. $\underline{17}$ (1980), 148-161.

10. W. Sierpiński, General Topology, Tr. by C. Cecilia Krieger, Univ. of Toronto Press, Toronto, 1952.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A theory of interval iteration, based on a few simple assumptions, is
given for the fixed point problem for operators in partially ordered topo-
logical spaces. A comparison of interval with ordinary iteration is made
which shows that their properties are converse in a certain sense with respect
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to a finite set of points, as in actual practice. In this latter case,
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ABSTRACT (continued)

interval iteration is shown to converge or diverge in a finite number of steps, for which an upper bound is given. By the introduction of a suitable iteration operator, the method of interval iteration is extended to the problem of solution of equations in linear spaces.

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