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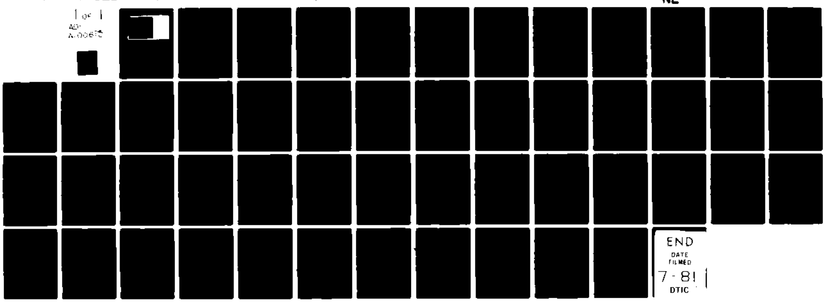

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ON THE EXISTENCE OF POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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UNIVERSITY OF WISCONSIN - MADISON
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ABSTRACT

In this paper we study the existence of positive solutions of semilinear elliptic equations. Various possible behaviors of the nonlinearity are considered and in each case nearly optimal multiplicity results are obtained. The results are also interpreted in terms of bifurcation diagrams.

AMS (MOS) Subject Classifications: 35J65, 35P30

Key Words: Semilinear equations, positive solutions, topological degree, bifurcation

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SIGNIFICANCE AND EXPLANATION

In this paper semilinear elliptic equations are considered: these are Laplace equations perturbed by a nonlinearity depending only on the solution u. This type of problem arises in many situation (theory of nonlinear diffusion generated by nonlinear sources, theory of thermal ignition of gases, quantum field theory, theory of gravitational equilibrium of stars, population genetics ....). Since the solution represents a temperature, or a concentration, or a density ... it is reasonable to restrict our attention to positive solutions of such equations. Here we consider all possible types of nonlinearities, and in each case we give nearly optimal existence results. These results are also explained using bifurcation diagram representing the set of possible solutions.

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ON THE EXISTENCE OF POSITIVE SOLUTIONS
OF SEMILINEAR ELLIPTIC EQUATIONS

P. L. Lions*

Introduction:

The goal of this paper is to give a survey concerning the problem of the existence of positive solutions for semilinear elliptic problems: that is, we consider the following problem

$$
\begin{cases} -\Delta u = f(u) \quad \text{in } \Omega,\ u \in C^2(\bar{\Omega}) \\ u > 0 \quad \text{in } \Omega,\ u = 0 \quad \text{on } \partial\Omega \ ; \end{cases} \tag{0.1}
$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, and $f(t)$ is some given nonlinearity.

We study all the possible behaviors of f and prove - or recall when these results are known - not only the eventual existence of a solution of (0.1) but we also give multiplicity results. In many case, we consider parametrized versions of (0.1) (take $\lambda f(u)$ instead of $f(u)$ in (0.1)), and we give "bifurcation diagrams" for the set of solutions of (0.1).

Such problems arise in a variety of situations - in the theory of nonlinear diffusion generated by nonlinear sources, in the theory of thermal ignition of gases, (see D. D. Joseph and T. S. Lundgren [36], I. M. Gelfand [31]), in quantum field theory and in mechanical statistics (see W. Strauss [55], Coleman, Glazer and A. Martin [23], H. Berestycki and P. L. Lions [7]), in the theory of gravitational equilibrium of stars (see D. D. Joseph and T. S. Lundgren [36], P. L. Lions [42]).

Our main tools for proving existence and multiplicity results are topological degree arguments (we shall also use variational techniques due to A. Ambrosetti and P. H. Rabinowitz [4], P. H. Rabinowitz [52]); we also refer to H. Amann [1], [2] for multiplicity results which are useful in the context of (0.1).

Of course the existence of a solution (or of multiple solutions) depends significantly on the assumptions made on f: we will first distinguish between two cases, the case when $f(0) > 0$ and the case when $f(0) = 0$.

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In addition in these two cases we have to consider different possibilities whether f is superlinear or sublinear at $+\infty$.

The plan is as follows:

I. The case when $f(0) = 0$:

I.1: Superlinear nonlinearities

I.2: Sublinear nonlinearities

II. The case when $f(0) > 0$:

II.1: Superlinear nonlinearities

II.2: Sublinear nonlinearities

III. The shape of the nonlinearity

III.1: Buckles and multiplicity results

III.2: Bumps and the shape of the nonlinearity

IV. Variants and open questions:

IV.1: Unbounded domains

IV.2: Open questions

We would like to point out that all the results which follow can be extended to more general second-order elliptice operators than $-\Delta$ and that nonlinearities which depend on $x(f(x,u))$ may be treated as well, together with different boundary conditions.

Finally in this paper we do not consider the question of asymptotically linear functions f even if some of the techniques described below give results.

I. The case when $f(0) = 0$

I.1 Superlinear nonlinearities:

In all that follows, we will assume that f is locally Lipschitz continuous from $\mathbb{R}$ into $\mathbb{R}$ and in this section we assume $f(0) = 0$. By superlinearity, we mean the following condition:

$$\varliminf_{t\to+\infty} f(t)t^{-1} > \lambda_1 \quad , \tag{1}$$

where λ_1 is the first eigenvalue of $(-\Delta)$ with Dirichlet boundary conditions.

Our first existence result is due to D. G. de Figueiredo, P. L. Lions and R. D. Nussbaum [29], [30]:

Theorem I.1: *Let us assume that we have in addition to* (1):

$$\varlimsup_{t\to 0_+} f(t)t^{-1} < \lambda_1 \quad , \tag{2}$$

$$\lim_{t\to+\infty} f(t)t^{-\ell} = 0, \quad \text{with } \ell = \frac{N+2}{N-2} \text{ if } N \geq 3,\ \ell < \infty \text{ if } N = 1,2 \quad ; \tag{3}$$

and either

(4) $\partial\Omega = \Gamma_1 \cup \Gamma_2$ *such that if* Γ_1 *is closed and there exists* $\alpha > 0$ *such that at all points of* Γ_1 *all the sectional curvatures of* $\partial\Omega$ *are bounded below by* α *and there exists* $x_0 \in \mathbb{R}^N$ *such that* $(x-x_0,n(x)) \leq 0$, *for* $x \in \Gamma_2$ (*where* $n(x)$ *is the unit outward normal vector to* $\partial\Omega$ *at* x)

or

(5) $f(t)t^{-\ell}$ *is nonincreasing for* $t > 0$ ($\ell = \frac{N+2}{N-2}$, *if* $N \geq 3$; *if* $N = 1,2$ *this condition is not necessary*).

Then, under these assumptions, there exists a solution u *of*

$$\begin{cases} -\Delta u = f(u) \ \text{in } \Omega,\ u \in C^2(\bar{\Omega}) \\ u > 0 \ \text{in } \Omega,\ u = 0 \ \text{on } \partial\Omega \quad . \end{cases} \tag{0.1}$$

Remark I.1: Assumptions (2) and (3) will be justified below; we believe that (4) or (5) are purely technical. Let us add that if $N = 1$, , we do not need (4) or (5), and if $N = 1$ (3) is not needed, and if $N = 2$ (3) may be replaced by

(3')
$$\lim_{t\to\infty} f(t)e^{-t^{\alpha}} = 0, \quad \text{for some } \alpha < 2 .$$

Remark I.2: There are other general existence results: we refer for example to A. Ambrosetti and P. H. Rabinowitz [4], H. Brézis and R. E. L. Turner [19].

In [4], it is assumed (1) - (3) (actually stronger forms of (1), (2) are used) and

(6) $\exists\theta \in (0,\frac{1}{2})$, $\exists t_0 > 0$ such that $\theta t f(t) - F(t) \geq 0$, for $t \geq t_0$ where

$$F(t) = \int_0^t f(s)ds.$$

In [19], it is assume (1), (2) and

(7)
$$\lim_{t\to\infty} f(t)t^{-\sigma} = 0, \quad \text{with } \sigma = \frac{N+1}{N-1} \ (N \geq 2) .$$

Compared to these results Theorem I.1 appears to be the most general but the proof (see [30]) uses the symmetries of Δ and it is not clear to extend it to more general elliptic operators. On the other hand, the method used in [4] is valid for general equations of the type (0.1) but with a variational structure. Finally, even if [19] appears to be the weakest result, it extends to general equations of the type (0.1) even without a variational structure.

We now comment on assumptions (2) and (3): first, the fact that $\frac{N+2}{N-2}$ is, in general, the best exponent is well-known (see, for example, S. I. Pohozaev [51], D. D. Joseph and T. S. Lundgren [36]). Of course we cannot say it is necessary since if u is a solution of (0.1) for some f, we may change $f(t)$ as we want for $t \geq \|u\|_{L^\infty}$ and u will remain a solution of (0.1). (More interesting examples may be found in P. H. Rabinowitz [53], H. Brézis and L. Nirenberg [18], J. Hempel [35]).

Now concerning (2) we just remark that if $f(t) = \lambda t + g(t)$ with $\lambda \geq \lambda_1$, $g(t) > 0$ for $t > 0$, then (0.1) has no solution: indeed let v_1 be a positive eigen-function associated to λ_1 that is

$$-\Delta v_1 = \lambda_1 v_1 \text{ in } \Omega,\ v_1 \in C^2(\bar{\Omega}),\ v_1 > 0 \text{ in } \Omega,\ v_1 = 0 \text{ on } \partial\Omega .$$

Then, we multiply (0.1) by v_1 and we obtain after two integrations by parts:

$$\lambda_1 \int_\Omega uv_1 \, dx = \lambda \int_\Omega uv_1 \, dx + \int_\Omega g(u) v_1 \, dx > \lambda \int_\Omega uv_1 \, dx$$

and this contradicts $\lambda \geq \lambda_1$.

Nevertheless, something can be said when (2) is not satisfied (see Theorem II.2 below).

We will need some assumptions which insure that the solutions of (0.1) are a priori bounded in $L^\infty(\Omega)$: we will use the results of D. G. de Figueiredo, P. L. Lions and R. D. Nussbaum [30].

Theorem I.2: Let us assume that Ω is convex and that f satisfies (3) and

(1') $$\lim_{t\to+\infty} f(t)t^{-1} = +\infty$$

(7) $$\overline{\lim_{t\to+\infty}} \frac{tf(t)-\theta F(t)}{t^2 f(t)^{2/N}} \leq 0, \quad \text{for some } 0 < \theta < \frac{2N}{N-2} \quad (\text{if } N \geq 3)$$

(8) f is differentiable near 0 and $f'(0) = 1$.

(9) $f(t) > 0$, for all $t > 0$.

Let λ^* be the supremum of all $\lambda > 0$ such that there exists a solution of $(0.1 - \lambda)$

$(0.1 - \lambda)$ $\quad -\Delta u = \lambda f(u)$ in Ω, $u \in C^2(\bar{\Omega})$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$.

i) Then $+\infty > \lambda \geq \lambda_1$ and for $0 < \lambda < \lambda^*$, there exists at least one solution of $(0.1 - \lambda)$.

ii) If $\lambda^* > \lambda_1$, then for $\lambda_1 < \lambda < \lambda^*$, there exist at least two solutions u_1, u_2 of $(0.1 - \lambda)$ which are ordered that is: $u_1 < u_2$ on Ω. In addition, there exists at least one solution of $(0.1 - \lambda^*)$.

We will give below some conditions which insure that $\lambda^* > \lambda_1$.

Remark 1.3: The convexity of the domain and the assumption (7) are just technical assumptions which imply a priori bounds of the solutions of $(0.1 - \lambda)$ in view of the results of [30]. Of course the theorem still holds under other assumptions which imply a priori bounds, as in [30] or in [19]. The theorem is still true without the assumption (9) but if (9) is not satisfied, f falls into a class of nonlinearities which will be investigated later on.

Before going into the proof of Theorem 1.2, we give a condition which implies $\lambda^* > \lambda_1$.

<u>Corollary 1.1</u>: <u>Under the assumptions of Theorem</u> 1.2 <u>and if</u> $f(t) - t < 0$ <u>for</u> $t > 0$, t <u>small</u>, <u>then</u> $\lambda^* > \lambda_1$.

<u>Proof of Corollary 1.1</u>:

In view of Crandall-Rabinowitz result on bifurcation from a simple eigenvalue (see [25]) there exists a connected component C in (λ, u) emanating from $(\lambda_1, 0)$. Let $t_1 > 0$ be such that

$$f(t) < t \quad \text{for} \quad t \in (0, t_1) \quad .$$

We thus have solutions of $(0.1 - \lambda)$ for $|\lambda - \lambda_1|$ small and with $\|u\|_{C(\bar\Omega)} < t_1$. Now, if $\lambda \leq \lambda_1$, $(\lambda, u) \in C$, we would have

$$-\Delta u = \lambda f(u) < \lambda_1 u \quad \text{in} \quad \Omega, \ u > 0 \quad \text{in} \quad \Omega, \ u = 0 \quad \text{on} \quad \partial\Omega \quad .$$

This is not possible since λ_1 is the first eigenvalue. This contradiction proves that $\lambda^* > \lambda_1$.

<u>Remark 1.4</u>: If $f(t) > t$, for t small, $t > 0$; then locally near $(\lambda_1, 0)$ the only possible solutions (λ, u) of $(0.1 - \lambda)$ are for $\lambda < \lambda_1$.

<u>Proof of Theorem 1.2</u>: Let us first remark that in view of assumptions (1'), (7) we know that for all $\alpha, B > 0$ there exists $C > 0$ such that, if (λ, u) is a solution of $(0.1 - \lambda)$ with $\alpha \leq \lambda \leq B$, then $\|u\|_{C(\bar\Omega)} \leq C$.

In addition in view of Theorem 1.1, we just need to prove that if $\lambda^* > \lambda_1$, then for $\lambda_1 < \lambda < \lambda^*$, there exist at least two solutions u_1, u_2 of $(0.1 - \lambda)$. To this end let $\lambda_0 \in (\lambda_1, \lambda^*)$ be such that there exists a solution $\bar u$ of $(0.1 - \lambda_0)$.

Let us prove that for $\lambda \in (\lambda_1, \lambda_0)$, there exist two ordered solutions of $(0.1 - \lambda)$. We are going to use a topological degree argument (see J. Leray and J. Schauder [41] or L. Nirenberg [49] for the definition and basic properties of the topological degree). We first define a compact map associated with the problem $(0.1 - \lambda)$ for $u \in C^1_0(\bar\Omega)$ $(^*)$, we set: $v = F_\lambda u$ is the solution of

---

$(^*)$ $C^1_0(\bar\Omega) = \{u \in C^1(\bar\Omega), u = 0 \text{ on } \partial\Omega\}$.

$$-\Delta v = \lambda f(u) \quad \text{in} \quad \Omega,\ v \in W^{2,p}(\Omega)\,(p < \infty),\ v = 0 \quad \text{on} \quad \partial\Omega \quad ;$$

where we take $f(t) = 0$ if $t \leq 0$. Let $C > 0$ be such that all solutions of $(0.1 - \lambda)$ (that is all nonzero fixed points of K_0) are bounded by C and such that $\|\bar{u}\|_{C^1(\bar{\Omega})} < C$. Since f is locally Lipschitz, there exists μ such that

$$\lambda f(t) + \mu t \text{ is nondecreasing for } t \in [0,C] \quad .$$

Finally we set: $v = Ku$ is the solution of

$$-\Delta v + \mu v = \lambda f(u) + \mu u \quad \text{in} \quad \Omega,\ v \in W^{2,p}(\Omega)\,(p < \infty),\ v = 0 \quad \text{on} \quad \partial\Omega \quad .$$

Let us now define some open sets in $C_0^1(\bar{\Omega})$:

$$B = \{u \in C_0^1(\bar{\Omega}),\ \|u\|_{C^1(\bar{\Omega})} < C,\ u > \varepsilon_0 v_1 \quad \text{in} \quad \Omega,\ \frac{\partial u}{\partial n} < \varepsilon_0 \frac{\partial v_1}{\partial n} \quad \text{on } \partial\Omega\}$$

where n denotes the outward unit normal, and where v_1 as in Corollary 1.1 is a positive eigenfunction associated with λ_1. The constant ε_0 will be determined later on (we already impose ε_0 to be small enough such that $\bar{u} \in B$, and such that $\lambda f(\varepsilon_0 v_1) > \lambda_1 \varepsilon_0 v_1$ in Ω). We also set: $\mathcal{O} = \{u \in B,\ u < \bar{u}$ in $\Omega,\ \frac{\partial u}{\partial n} > -\frac{\partial u}{\partial n}$ on $\partial\Omega\}$.

We are going to prove that the topological degree of $I - K$ is well defined on B and on $\mathcal{O}$ (with respect to 0) and that its values are:

$$d(I-K, B, 0) = 0,\ d(I-K, \mathcal{O}, 0) = +1 \quad .$$

This will imply: $d(I-K, B-\bar{\mathcal{O}}, 0) = -1$, therefore there exists u_2 solution of $(0.1 - \lambda)$ and $u_2 \in B-\bar{\mathcal{O}}$. In particular $u_2 \wedge \bar{u}$ is not a solution of $(0.1 - \lambda)$ and a straightforward computation shows that, actually, $u_2 \wedge \bar{u}$ satisfies:

$$-\Delta(u_2 \wedge \bar{u}) \geq \lambda f(u_2 \wedge \bar{u}) \quad \text{in} \quad D'(\Omega),\ u_2 \wedge \bar{u} \in W_0^{1,\infty}(\Omega) \quad .$$

To conclude, that is to prove the existence of u_1, we just have to notice that for ε small enough, εv_1 (with the same notations as in Corollary 1.1) is a subsolution of $(0.1 - \lambda)$ and $\varepsilon v_1 \leq u_2 \wedge \bar{u}$. And thus there exists a solution u_1 of $(0.1 - \lambda)$ satisfying:

$$0 < u_1 \leq u_2 \wedge \bar{u} \quad \text{in} \quad \Omega \quad .$$

Now, we prove the claims on the topological degree. Indeed, in view of [30], let us first remark that one may choose C such that, if K_θ denotes the following compact operator: $\tilde{K}_\theta u = v$ defined by

$$\begin{cases} -\Delta v + \theta\mu v = \theta(\lambda f(u) + \mu u) + (1-\theta)(\nu u^+ + 1) \\ v \in W^{2,p}(\Omega)(p < \infty),\ v = 0 \quad \text{on} \quad \partial\Omega \ ; \end{cases}$$

where $\nu > \lambda_1$, then all fixed points of $\tilde{K}_\theta$ (for $0 \le \theta \le 1$) are bounded in $C_0^1(\bar{\Omega})$ norm by C. In addition an easy argument shows that one may choose ε_0 small enough such that all fixed points u of $\tilde{K}_\theta$ (for $0 \le \theta \le 1$) (distinct from 0 if $\theta = 1$) satisfy: $u \geq \varepsilon_0 v_1$. This is due to the fact that both λ and ν are greater than λ_1. Therefore $d(I-\tilde{K}_\theta, B, 0)$ is well defined and independent of $\theta \in [0,1]$. But it is easy to see that $\tilde{K}_0$ cannot have any fixed point because of the choice of ν and thus

$$d(I-K, B, 0) = d(I-\tilde{K}_0, B, 0) = 0 \quad .$$

Next, we prove that $d(I-K, \mathcal{O}, 0) = 1$. Indeed, let us choose φ in $\mathcal{O}$, and let us remark that we may choose C such that K maps φ into $\mathcal{O}$. Remark that if $u \in \bar{\mathcal{O}}$ and if $v = Ku$, we have

$$-\Delta v + \mu v = \lambda f(u) + \mu u \le \lambda f(\bar{u}) + \mu\bar{u}$$

and by the strong maximum principle we conclude: $v \in \mathcal{O}$. Therefore $d(I - (\theta K + (1-\theta)\varphi), \mathcal{O}, 0)$ is well defined and is independent of $\theta \in [0,1)]$. In particular:

$$d(I-K, \mathcal{O}, 0) = d(I-\varphi, \mathcal{O}, 0) = 1, \quad \text{since } \varphi \in \mathcal{O} \quad .$$

In order to complete the proof of Theorem 1.2, the last thing we have to check is that $\lambda^* < +\infty$. And this is an easy consequence of (1') and (9): Indeed these assumptions imply that there exists $\alpha > 0$ such that $f(t) \geq \alpha t$ for all $t \geq 0$. Now multiplying $(0.1 - \lambda)$ by v_1 and integrating by parts, this yields

$$\lambda_1 \int_\Omega u v_1 dx = \int_\Omega \lambda f(u) v_1 dx \geq \alpha\lambda \int_\Omega u v_1 dx$$

and therefore $\lambda \leq \frac{\lambda_1}{\alpha}$, and thus $\lambda^* < \infty$.

Remark I.5: In the case where we have

$$\lim_{t \to 0_+} f(t)t^{-1} = +\infty \quad .$$

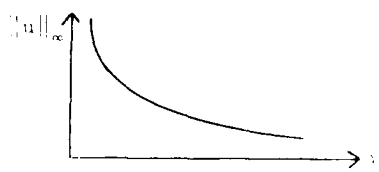
and under the same assumptions as in Theorem I.2 (except (8) of course), then, by a similar proof to the preceeding one, there exists $\lambda^* < \infty$ such that for $0 < \lambda < \lambda^*$, there exist at least two solutions u_1, u_2 of $(0.1 - \lambda)$ which are ordered that is: $u_1 < u_2$ in Ω and for $\lambda = \lambda^*$, there exists at least one solution of $(0.1 - \lambda^*)$. This is the case for example, when $f(t) = (t^\theta + t^p)(t \geq 0)$ with $0 < \theta < 1$, $1 < p < \frac{N+2}{N-2}$ (if $N > 2$).

Remark 1.5: We conjecture that Theorems I.1 and I.2 are not optimal in the sense that for example in Theorem I.1, assumptions (4) or (5) should not be needed, or Theorem 1.2 should be true just under assumptions (1') - (3) - (8) and (9). These extensions depend only on extensions of results implying a priori bounds for solutions of (0.1) or $(0.1 - \lambda)$ (see also section IV.2 below).

Remark 1.6: To summarize the results of this section, we are going to give a few "bifurcation diagrams". Let us emphasize that these diagrams are formal and in some sense are "minimal" diagrams: we will see in section III below that the set of solutions may be a lot more complicated. The curves below represent the maximum norm of u as a function of λ, whenever (u,λ) is a solution of $(0.1 - \lambda)$. In all these diagrams we assume at least that f is superlinear (and satisfies (1')) and f satisfies (3), (9) (and other precise assumption can be found in Theorem 1.1 and 1.2).

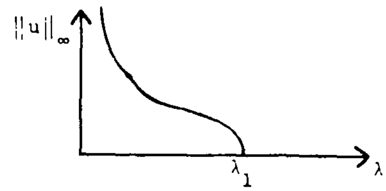
Case 1: $f'(0) = 0$

Ex: $f(t) = t^p (1 < p < \frac{N+2}{N-2})$



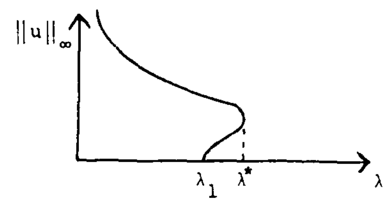
Case 2: $f'(0) = 1$, $f(t) > t$ for $t > 0$, t small

Ex: $f(t) = t + t^p (1 < p < \frac{N+2}{N-2})$



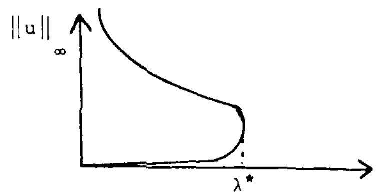
Case 3: $f'(0) = 1$, $f(t) < t$ for $t > 0$, t small

Ex: $f(t) = t(1 - \sin t) + t^p (1 < p < \frac{N+2}{N-2})$



Case 4: $\lim\limits_{t \to 0_+} f(t)t^{-1} = +\infty$

Ex: $f(t) = \sqrt{t} + t^p (1 < p < \frac{N+2}{N-2})$



1.2. Sublinear nonlinearities.

In this section we still assume that f is locally Lipschitz continuous from R into R and that $f(0) = 0$. In addition f will be assumed to be sublinear that is

(10) $$\overline{\lim_{t \to +\infty}} f(t)t^{-1} < \lambda_1 .$$

Our first existence result is well-known (see for example H. Amann [1], [2] or H. Berestycki and P. L. Lions [8]) and thus we will skip its proof:

Theorem 1.3: Let us assume, in addition to (10), that f satisfies:

(11) $$\underline{\lim_{t \to 0_+}} f(t)t^{-1} > \lambda_1 .$$

Then there exists a maximum positive solution of (0.1).

Remark 1.8: Of course, if f satisfies (11) and

(12) $$f(\beta) = 0, \quad \text{for some} \quad \beta > 0 \quad ;$$

then the conclusion of Theorem 1.3 holds, where maximum solution is replaced by maximum solution among all solutions less than β. Indeed by the maximum principle a solution of (0.1) with f replaced by $\tilde{f}(t) = f(t \wedge \beta)$ is a solution of (0.1) which is less than β and let us remark that $\tilde{f}$ now satisfies (10).

Remark 1.9: In the case where $f(t)$ satifies (10, (11) and

(13) $$f(t)t^{-1} \text{ is strictly decreasing for } t > 0$$

then it is well-known (see H. Berestycki [6] for an elegant proof) that (0.1) has a unique positive solution. In particular (13) is satisfied if f is strictly concave.

We now turn to the parametrized version of (0.1), namely $(0.1 - \lambda)$.

Theorem 1.4: We assume that f satisfies:

(10') $$\lim_{t \to +\infty} f(t)t^{-1} = 0 \quad (\text{resp.} \quad f(\beta) = 0 \quad \text{for some} \quad \beta > 0) \quad .$$

(8) $$f \quad \text{is differentiable near} \quad 0 \quad \text{and} \quad f'(0) = 1 \quad .$$

(9) $$f(t) > 0, \quad \text{for all} \quad t > 0 \quad (\text{resp. for all} \quad \beta > t > 0) \quad .$$

Let λ^* be the infinum of all $\lambda > 0$ such that there exists a solution of $(0.1 - \lambda)$ (resp. less than β). Then we have

i) $0 < \lambda^* \leq \lambda_1$ and for $\lambda^* < \lambda$, there exists a maximum positive solution u_1 of $(0.1 - \lambda)$ (resp. maximum among all positive solutions of $(0.1 - \lambda)$ less than β).

ii) If $\lambda^* < \lambda_1$, then for $\lambda^* < \lambda < \lambda_1$, there exists a second solution u_2 of $(0.1 - \lambda)$ which thus satisfies: $0 < u_2 < u_1$ in Ω. In addition, there exists a maximum positive solution of $(0.1 - \lambda^*)$.

Let us give immediately some condition which implies $\lambda^* < \lambda_1$.

Corollary 1.2: Under the assumptions of Theorem 1.4 and if we have

$$f(t) > t \quad \text{for} \quad t > 0, \ t \quad \text{small} \quad ;$$

then $\lambda^* < \lambda_1$.

Remark 1.10: If $f(t) < t$, for t small, $t > 0$; then locally near $(\lambda_1, 0)$ the only possible solutions (λ, u) of $(0.1 - \lambda)$ are for $\lambda > \lambda_1$.

Since the proofs of Theorem 1.4 and Corollary 1.2 are somewhat similar to those of Theorem 1.2 and Corollary 1.1, we will skip them.

We want now to discuss the case where (8) is replaced by

$$\overline{\lim_{t\to 0_+}} f(t)t^{-1} < 0 \quad . \tag{8'}$$

Theorme 1.5: We assume that f satisfies (8') and

$$\alpha = \inf(t > 0,\ f(t) > 0) \ \text{exists and} \ \alpha > 0 \ ; \tag{14}$$

$$\lim_{t\to+\infty} f(t)t^{-1} = 0 \ \ (\text{resp.} \ f(\beta) = 0 \ \text{for some} \ \beta > \alpha) \ ; \tag{15}$$

$$\exists\, \zeta > 0 \ (\text{resp.}\ \beta > \zeta > 0),\ F(\zeta) > 0, \ \text{where} \ F(t) = \int_0^t f(s)ds.$$

Let λ^* be the infimum of all $\lambda > 0$ such that there exists a solution of $(0.1 - \lambda)$ (resp. less than β). Then λ^* is finite and positive and we have, if Ω is star shaped

i) For $\lambda > \lambda^*$, there exists a maximum positive solution u_1 of $(0.1 - \lambda)$ (resp. maximum among all psotive solutions of $(0.1 - \lambda)$ less than β).

ii) For $\lambda > \lambda^*$, there exists a second solution u_2 of $(0.1 - \lambda)$ which satisfies: $0 < u_2 < u_1$ in Ω.

If Ω is not star shaped, i) is true for λ large enough and ii) is replaced by

iii) If there exists a solution of $(0.1 - \lambda)$, then there exists a maximum positive solution u_1 of $(0.1 - \lambda)$ (resp. maximum among all positive solutions of $(0.1 - \lambda)$ less than β) and there exists a second solution u_2 of $(0.1 - \lambda)$.

We do not know if the assumption on Ω is necessary.

Remark I.11: This result is essentially contained in H. Berestycki and P. L. Lions [8] (see also P. H. Rabinowitz [52]). Remark that the fact that λ^* is finite (i.e. there exist solutions for λ large) is easily obtained, for example, by a variational argument. Indeed (replacing if necessary f by $\tilde{f}(t) = f(t \wedge \beta)$) if we consider the minimization problem:

$$I_\lambda = \min_{u \in H_0^1(\Omega)} \int_\Omega \frac{1}{2} |\nabla u|^2 - \lambda F(u)dx, \quad \text{where} \quad F(t) = \int_0^t f(s)ds \quad ;$$

then this problem has always a solution $\tilde{u}_1$ and for λ large enough $\tilde{u}_1 > 0$ in Ω, since $I_\lambda < 0$. We will see below that in general $\tilde{u}_1$ and u_1 do not coincide for $\lambda > \lambda^*$.

Remark I.12: In a very special case ($\Omega = [0,1]$, $f(t) = -\mu t + \nu t^2 - t^3$ with $\nu > 0$ and $\nu^2 > 4\mu > 0$) it is proved in Conley and Smoller [24] that for $\lambda > \lambda^*$ there are exactly two solutions (u_1, u_2) of $(0.1 - \lambda)$ while for $\lambda = \lambda^*$ there is a unique solution u_1 of $(0.1 - \lambda^*)$ (in addition a precise description of the stability of these solutions is given). Remark that (8'), (14), (15) are satisfied in this case (with $\alpha = \frac{1}{2}(\nu - \sqrt{\nu^2 - 4\mu})$, $\beta = \frac{1}{2}(\nu + \sqrt{\nu^2 - 4\mu})$. This shows that Theorem I.5 is optimal in this case.

We claim that this example shows that $I_\lambda > 0$ for $\lambda \in [0, \bar{\lambda}^*)$ where $\bar{\lambda}^* > \lambda^*$ and that $u_1 \not\equiv \tilde{u}_1$ if $\lambda \in [\lambda^*, \bar{\lambda}^*)$ - we use here the notation of Remark I.11 above. Indeed we just need to show that $I_{\lambda^*} > 0$: if one had $I_{\lambda^*} < 0$, then using the main result of A. Ambrosetti and P. H. Rabinowitz [4] we would deduce that $(0.1 - \lambda^*)$ should have a solution u_2 with

$$\int_\Omega \frac{1}{2} |\nabla u_2|^2 - \lambda^* F(u_2)dx > 0 \quad .$$

In addition, one would have $I_\lambda < 0$ for $\lambda > \lambda^*$ and therefore this would imply $\tilde{u}_1 > 0$ in Ω. Now because of the stability properties proved in [24], this would imply $\tilde{u}_1 = u_1$ for $\lambda > \lambda^*$. Now by continuity one would have

$$\int_\Omega \frac{1}{2} |\nabla u_1|^2 - \lambda^* F(u_1)dx \leq 0 \quad .$$

And this would contradict the uniqueness of the solution of $(0.1 - \lambda^*)$.

Proof of Theorem I.5: We already know that λ^* is finite and we will just prove that 1) λ^* is positive, 2) if there exists a solution of $(0.1 - \lambda_0)$ and if Ω is star shaped that for all $\lambda > \lambda_0$, there exists a solution of $(0.1 - \lambda)$. The remaining statement of the Theorem is about the existence of u_2 and we refer to H. Berestycki and P. L. Lions [8], [9] (the method uses a topological degree argument somewhat similar to the one used in the proof of Theorem I.2).

1) λ^* is positive:

indeed if $(0.1 - \lambda)$ had a solution u for λ arbitrarily small, for λ small enough we would have

$$-\Delta u = \lambda f(u) < \frac{\lambda_1}{2} u \quad \text{in} \quad \Omega, \; u > 0 \quad \text{in} \quad \Omega, \; u = 0 \quad \text{on} \quad \partial\Omega \quad .$$

And this is clearly impossible.

2) Let us suppose that there exists a solution u_1 of $(0.1 - \lambda_0)$ and let $\lambda > \lambda_0$ - we may assume that Ω is star shaped with respect to 0. Then the existence of a solution of $(0.1 - \lambda)$ is equivalent to the existence of a solution of

$$(16) \qquad -\Delta u = \lambda_0 f(u) \quad \text{in} \quad \left(\frac{\lambda}{\lambda_0}\right)^{1/2}\Omega = \Omega', \; u > 0 \quad \text{in} \quad \Omega', \; u = 0 \quad \text{on} \quad \partial\Omega' \; .$$

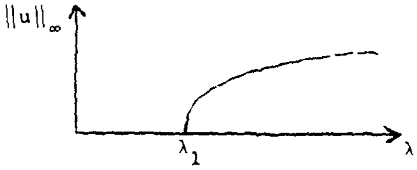
$$(\Omega' = \{(\frac{\lambda}{\lambda_0})^{1/2} x, \; x \in \Omega\}) \quad .$$

Remark that $\Omega \subset \Omega'$. Therefore using the general results of [7], if we extend u_1 by 0 to Ω', we obtain a weak subsolution of (16). It is easy to build a supersolution above u_1, using the assumption (15) and we conclude involving classical results on sub and supersolutions.

Remark I.13: Again we summarize the results of this section with a few "bifurcation diagrams" (all the remarks made on those diagrams in Remark I.7 are still valid here). In all these diagrams we assume at least that f is sublinear (and satisfies (10') or even (15) - the other precise assumptions can be found in Theorems I.3 - 5).

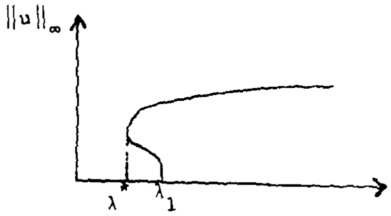
Case 1: $f'(0) = 1$, $f(t) < t$ for $t > 0$, t small

Ex.: $f(t) = t - t^p (1 < p < \infty)$



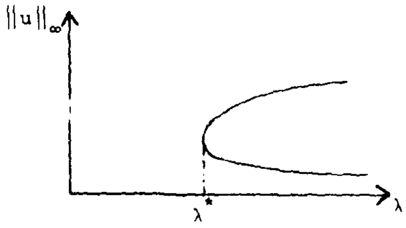
Case 2: $f'(0) = 1$, $f(t) > t$ for $t > 0$, t small

Ex.: $f(t) = t + \alpha t^2 - \beta t^3$, $\alpha, \beta > 0$.



Case 3: $f'(0) < 0$

Ex.: $f(t) = -\mu t + \nu t^2 - t^3$, $\nu > 0$, $\nu^2 > 4\mu > 0$.



Remark: *The case where*

$$\lim_{t \to 0_+} f(t)t^{-1} = +\infty$$

is nearly included in Theorem I.3. In this case one has a maximum positive solution of $(0.1 - \lambda)$ for all $\lambda > 0$.

Case 4: $\lim_{t \to 0_+} f(t)t^{-1} = +\infty$

Ex.: $f(t) = t^p$, $0 < p < 1$.

II. The case when $f(0) > 0$:

II.1. Superlinear nonlinearities

In this section we still assume that f is locally Lipschitz continuous from $\mathbf{R}$ into $\mathbf{R}$. We assume now that

(17) $$f(0) > 0 \quad .$$

We will restrict our attention to the case when f satisfies

(9) $$f(t) > 0, \quad \text{for all} \quad t > 0$$

(again let us indicate that the case when f vanishes for some reduces to the sublinear case).

Our first existence result is the following:

Theorem II.1: Let us assume that f satisfies (17), (9), (1') and (3), (7); and suppose in addition that Ω is convex. Then there exists $\lambda^* > 0$ such that

i) for $0 < \lambda \leq \lambda^*$, there exists a minimum positive solution $\underline{u}_\lambda$ of $(0.1 - \lambda)$
$-\Delta u = \lambda f(u)$ in Ω, $u \in C^2(\bar{\Omega})$, $u > 0$ in Ω, $u = 0$ in $\partial\Omega$.

ii) If $\lambda > \lambda^*$, *there exists no positive solution of* $(0.1 - \lambda)$.

iii) If $0 < \lambda < \lambda^*$, then there exists at least one positive solution u_λ of $(0.1 - \lambda)$ distinct from $\underline{u}_\lambda$ i.e. satisfying: $u_\lambda > \underline{u}_\lambda$ in Ω.

Remark II.1: Again the assumption (7) (and the convexity of Ω) is purely technical and we believe it is not necessary: this assumption is used only in iii) in order to insure some a priori estimate and the Theorem (and its proof) remains true with any assumption (replacing (7)) insuring a priori bounds of the solutions of $(0.1 - \lambda)$ (such results can be found in [30] and in [19]).

This result is essentially due to D. G. de Figueiredo, P. L. Lions and R. D. Nussbaum [30]. Statements i) and ii) are somewhat classical and easy to prove since once we know a positive solution u_{λ_0} of $(0.1 - \lambda_0)$ for some $\lambda_0 > 0$, u_{λ_0} is a supersolution of $(0.1 - \lambda)$ while 0 is always a subsolution. Therefore i) and ii) follow easily from the general results of H. Amann [1].

We will see that the proof of iii) shows a little more than statement iii): if $0 < \mu_0 < \lambda_0 < \lambda^*$ are fixed, we will prove that there exists a connected component C in $\mathbb{R} \times C_0^1(\bar{\Omega})$ such that for all $(\lambda,u) \in C$ with $\lambda \in [\mu_0,\lambda_0]$ then u solves $(0.1 - \lambda)$ and is distinct from $\underline{u}_\lambda$. In the convex case a lot more can be said (see [30] and Theorem II.2 below).

<u>Remark II.2</u> The example $f(t) = \alpha + \beta t$ with $\alpha, \beta > 0$ shows that assumption (1') is necessary in order to have

i) a solution for $\lambda = \lambda^*$

ii] at least two solutions for $0 < \lambda < \lambda^*$.

Indeed in this case it is well-known that $(0.1 - \lambda)$ has a solution if and only if $\lambda < \lambda^* = \lambda_1 \beta^{-1}$ *and the solution is unique.*

<u>Proof of Theorem II.1:</u> Let $\lambda < \lambda^*$, we know there exists a solution $\underline{u}_{\lambda^*}$ of $(0.1 - \lambda^*)$. Thus $\underline{u}_{\lambda^*}$ is a supersolution of $(0.1 - \lambda)$ and $\underline{u}_\lambda < \underline{u}_{\lambda^*}$ in Ω. We are going to prove there exists a solution u_λ of $(0.1 - \lambda)$ such that

$$u_\lambda \nleq \underline{u}_{\lambda^*} \quad \text{in } \Omega \ .$$

To this end we will use a topological degree argument (somewhat similar to the one used in the proof of Theorem I.2).

In all that follows, C will denote a positive constant such that all solutions u of $(0.1 - \lambda)$ satisfy: $\|u\|_{C^1(\bar{\Omega})} < C$. In addition since f is locally Lipschitz, there exists μ such that

$$\lambda f(t) + \mu t \quad \text{is nondecreasing for } t \in [0,C] \ .$$

Finally we define a compact map K from $C_0^1(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$: $v = Ku$ is the solution of

$$-\Delta v + \mu v = \lambda f(u) + \mu u \quad \text{in } \Omega,\ v \in C^2(\bar{\Omega}),\ v = 0 \quad \text{on } \partial\Omega \ .$$

Let B and $\mathcal{O}$ be the following bounded open sets in $C_0^1(\bar{\Omega})$:

$$B = \{u \in C_0^1(\bar{\Omega}),\ \|u\|_{C^1(\bar{\Omega})} < C,\ u > 0 \ \text{ in } \Omega,\ \frac{\partial u}{\partial n} < 0 \ \text{ on } \partial\Omega\} \ ;$$

$$\mathcal{O} = \{u \in B,\ u < \underline{u}_{\lambda^*} \ \text{ in } \Omega,\ \frac{\partial u}{\partial n} > \frac{\partial}{\partial n}\left(\underline{u}_{\lambda^*}\right) \ \text{ on } \partial\Omega\} \ .$$

We claim now that the degrees of $I - K$ on $B, \mathcal{O}$ (with respect to 0) are well defined and are equal to:

$$d(I-K, B, 0) = 0, \ d(I-K, \mathcal{O}, 0) = 1 \quad .$$

The fact that these degrees are well defined is obvious from the definitions of B and $\mathcal{O}$. In addition the computation made in the proof of Theorem I.2 is easily adapted to this case and it shows:

$$d(I-K, B, 0) = 0 \quad .$$

Finally 0 and $\underline{u}_{\lambda^*}$ being respectively sub and supersolution of $(0.1 - \lambda)$ it is clear that K maps $\overline{\ }$ into , and since is convex this implies

$$d(I-K, \mathcal{O}, 0) = 1 \quad .$$

Indeed take $\varphi \in \mathcal{O}$ and define $K_t = (1-t)\varphi + tK$ (for $0 \le t \le 1$), K_t maps $\overline{\mathcal{O}}$ into $\mathcal{O}$ and thus

$$d(I-K, \mathcal{O}, 0) = d(I-K_t, \mathcal{O}, 0) = d(I-\varphi, \mathcal{O}, 0) = 1 \quad ,$$

since $\varphi \in \mathcal{O}$.

We are now able to conclude since by the additivity of the Leray-Schauder degree, we have:

$$d(I-K, B-\overline{\mathcal{O}}, 0) = -1$$

and this means there exists a solution u of $(0.1 - \lambda)$ in $B - \overline{\mathcal{O}}$; and this proves the theorem.

We now consider the case when f is convex; this case has been studied by many authors: some simple interesting cases were discovered by H. B. Keller and D. S. Cohen [37], H. B. Keller and J. P. Keener [38]. More general results were obtained in T. Laetsch [39], M. G. Crandall and P. H. Rabinowitz [26], C. Bandle [5], F. Mignot and J. P. Puel [47], D. G. de Figueiredo, P. L. Lions and R. D. Nussbaum [30]. Some particular results (in the case when Ω is a ball) are described in J. Leray [40], I. M. Guelfand [31], D. D. Joseph and J. S. Lundgren [36], C. M. Brauner and B. Nicolaenko [17] (we will come back on these particular results).

We give now a new existence result (some properties of the solutions may be found in the references listed above).

Theorem II.2: Let us assume that f satisfies (17), (9), (1') and that f is strictly convex on R_+. Then there exists $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$ there exists a minimum positive solution $\underline{u}_\lambda$ of $(0.1 - \lambda)$ (in addition $\underline{u}_\lambda$ is of class C^1 with respect to λ) and such that for $\lambda > \lambda^*$ there exist no positive solutions of $(0.1 - \lambda)$.

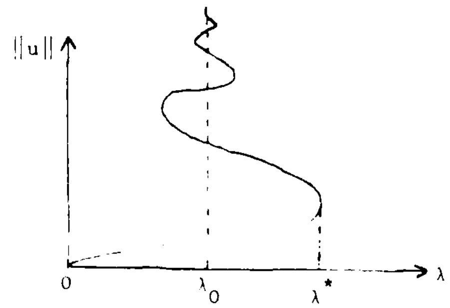
If we assume in addition (3), (7) and that Ω is convex, there exists a connected component $\mathcal{C}$ in $[0,\lambda^*] \times C^2(\bar{\Omega})$ such that i) for all $(\lambda,u) \in \mathcal{C}$, u is a positive solution of $(0.1 - \lambda)$ (except if $\lambda = 0$, then $u = 0$), ii) for all $\lambda \in (0,\lambda^*)$ there exists u distinct from $\underline{u}_\lambda$ such that $(\lambda,u) \in \mathcal{C}$; iii) $(\lambda,\underline{u}_\lambda) \in \mathcal{C}$ iv) If $(\lambda,u) \in \mathcal{C}$ and $\lambda \to 0_+$, $u \neq \underline{u}_\lambda$ then $\|u\|_\infty \to +\infty$. Finally if we assume (4) instead of (7) and if the convexity of Ω then for all $\lambda \in (0,\lambda^*)$ there exists a solution distinct from $\underline{u}_\lambda$ of $(0.1 - \lambda)$.

Remark II.3: The results of [36] show that the last part of the theorem is nearly optimal. We conjecture that the fact solutions of $(0.1 - \lambda)$ distinct from $\underline{u}_\lambda$ lie on a connected component $\mathcal{C}$ is still true without assumption (7).

Let us finally indicate that the question of the existence of a (unique) solution of $(0.1 - \lambda^*)$ is investigated in [26] and [47].

Again assumptions (7) and of the convexity of Ω are technical and are used in order to insure a priori bounds of the solutions of $(0.1 - \lambda)$ (as in Theorem II.1 - see Remark II.1).

Remark II.4: It is shown in [36], that the set of solutions of $(0.1 - \lambda)$ may be rather complicated since in some examples ($f(u) = e^u$, $3 \leq N \leq 9$ and Ω being a ball) we have the following "bifurcation diagram":



(in particular $(0.1 - \lambda_0)$ has infinitely many solutions).

For the second part of Theorem II.2 we refer to [30] and we are going to prove only the last statement (the method used here is very similar to the one used to prove Theorem II.3 in [30]).

Proof of Theorem II.2: Let $t_n \geq 1$, $t_n \uparrow +\infty$ such that f is differentiable at f_n. Let us define

$$f_n(t) = f(t_n) + f'(t_n)(t - t_n) + (t - t_n)^\gamma \quad \text{for} \quad t \geq t_n$$

$$f_n(t) = f(t) \quad \text{if} \quad t \leq t_n$$

where γ is any constant in $\left(1, \frac{N+2}{N-2}\right)$.

Since f_n is convex and $f_n = f$ for $t \leq t_n$, we claim that for n large enough $\underline{u}_\lambda$ is still the minimum solution of the problem $(0.1 - \lambda)$ corresponding to f_n. Indeed by [47], we have for n large enough that $\underline{u}_\lambda$ is solution and

$$\lambda_1^n = \lambda_1 > 0$$

where λ_1^n (resp. λ_1) is the first eigenvalue on $H_0^1(\Omega)$ of the operator

$$-\Delta - \lambda f_n'(\underline{u}_\lambda) \quad (\text{resp.} \quad -\Delta - \lambda f'(\underline{u}_\lambda)) \quad .$$

And this implies (see [47]) that $\underline{u}_\lambda$ is the minimum solution (for f_n). Now, by [26], we know that $(0.1 - \lambda)$ for f_n has a second positive solution $u_n : u_n > \underline{u}_\lambda$ in Ω. Moreover, from an easy inspection of the proof in [26], we see that

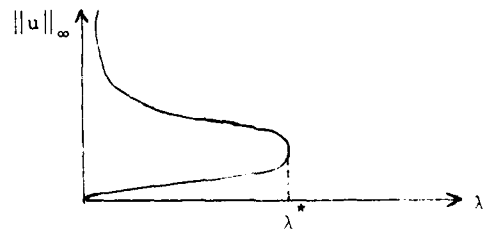
$$\left|\int_\Omega \frac{1}{2}|\nabla u_n|^2 - F_n(u_n)dx\right| \leq \text{Const.}, \text{ where } F_n(t) = \int_0^t f_n(s)ds.$$

Since $\lambda_1 = \lambda_1^n > 0$, it is easy to prove that $\|u_n - \underline{u}_\lambda\|_\infty$ remains bounded away from 0.

Now exactly as in the proof of Theorem II.3 in [30], we derive a H^1 bound on u_n, which implies a C^2 bound on u_n. It is then straightforward to pass to the limit.

Remark II.5: In the last statement of Theorem II.2, we may as well replace (4) by (5) (the construction of f_n is a little more technical in this case).

We now summarize the results of this section by a bifurcation diagrm representing the set of solutions of $(0.1 - \lambda)$ (under the assumptions of Theorems II.1 or II.2)



Of course let us recall that when f has a supercritical growth (ex: $f(t) = \alpha + \beta t^p$, α, $\beta > 0$, $p > \frac{N+2}{N-2}$) various modifications of this diagram may happen (see [40], [36]).

II.2. Sublinear nonlinearities:

We still assume that f is locally Lipschitz continuous from $\mathbf{R}$ into $\mathbf{R}$ and that f satisfies (17).

We have the following easy existence result:

Theorem II.3: *Let us assume that* f *satisfies* (17) *and either* (9) *and*

(10') $$\lim_{t\to+\infty} f(t)t^{-1} = 0$$

or that there exists $\beta > 0$ *such that* $f(\beta) = 0$. *Then for all* $\lambda > 0$, *there exists a minimum solution* $\underline{u}_\lambda$ *of* $(0.1 - \lambda)$.

Remark II.6: If we assume instead of (10'), $\overline{\lim}_{t\to+\infty} f(t)t^{-1} = K < +\infty$, then the existence result is valid for all $\lambda \in (0, \lambda_1|K)$.

The proof is a straightforward application of order arguments since 0 is a subsolution and it is easy to build a supersolution (for example β is a supersolution in the case when $f(\beta) = 0$).

The following result shows that for some class of nonlinearities f, there is uniqueness (but we will see in Section III.2 that this is not true in general).

Proposition II.1: *Under the assumptions of* Theorem II.3 *and if we assume in addtion that* f *is concave then* $\underline{u}_\lambda$ *is the unique solution of* $(0.1 - \lambda)$.

We believe this result is well-known but we were unable to find a precise reference (see H. Berestycki [6] for a related result). The proof is an easy adaptation of an argument from [6]. We will denote by $\lambda_1(-\Delta - c(x))$ the first eigenvalue of the operator $-\Delta - c(x)$ for some $c(x) \in L^\infty(\Omega)$.

Since we have: $-\Delta\underline{u}_\lambda = \{\frac{f(\underline{u}_\lambda) - f(0)}{\underline{u}_\lambda}\}\underline{u}_\lambda + f(0)$, $\underline{u}_\lambda > 0$ in Ω and since $f(0) > 0$, necessarily we have:

$$\lambda_1(-\Delta - \{f(\underline{u}_\lambda) - f(0)\}\underline{u}_\lambda^{-1}) > 0$$

(if $\underline{u}_\lambda = 0$ this is to be understood as $f'(0_+)$ - which exists since f is concave).

On the other hand, if v is a solution of $(0.1-\lambda)$ distinct from u_λ, we have

$$-\Delta(v-\underline{v}_\lambda) = \left(\frac{f(v)-f(\underline{u}_\lambda)}{v-\underline{u}_\lambda}\right)(v-\underline{u}_\lambda), \quad v-\underline{u}_\lambda > 0 \text{ in } \Omega$$

and thus

$$\lambda_1\left(-\Delta - \frac{f(v)-f(\underline{u}_\lambda)}{v-\underline{u}_\lambda}\right) = 0$$

(again $(f(v) - f(\underline{u}_\lambda))(v-\underline{u}_\lambda)^{-1}$ is to be understood as $f'(\underline{u}_\lambda+)$ on the set $v = \underline{u}_\lambda$ - that is $\partial\Omega$). But since f is concave, we have

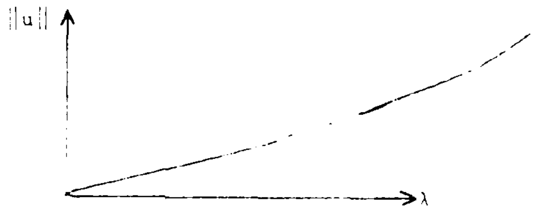
$$(f(v) - f(\underline{u}_\lambda))(v-\underline{u}_\lambda)^{-1} \leq (f(\underline{u}_\lambda) - f(v))\underline{u}_\lambda^{-1} .$$

This inequality contradicts the spectral informations given above in view of well-known comparison principles of first eigenvalues.

The bifurcation diagram of this section looks like

Ex.: $f(t) = \alpha + \beta t^\theta$

$\alpha > 0$, $\beta \geq 0$, $\theta \in (0,1)$.



III. The shape of the nonlinearity and multiplicity results.

III.1. Buckles and multiplicity results.

Again we assume that f is locally Lipschitz continuous from $\mathbb{R}$ into $\mathbb{R}$. In this section we will consider the case when f changes sign and we will show how this type of behavior may affect multiplicity results. The results of this section are adaptations or extend results of H. Berestycki and P. L. Lions [10].

The first type of results we want to discuss concerns the case when f changes sign once (or more) between 0 and some β such that $f(\beta) = 0$ and when $f(t)$ is positive and superlinear for $t > \beta$. By sections I.2 and II.2, we have existence results for solutions whose maximum is less than β. We want to prove here that under general assumptions there exists another solution whose maximum is larger than β.

More precisely we assume:

(18) $$\exists \beta > 0,\ f(\beta) = 0 \text{ and } f(t) > 0, \text{ for } t > \beta \ ;$$

(1) $$\varliminf_{t\to+\infty} f(t)t^{-1} > \lambda_1 \ ,$$

and of course $f(0) > 0$.

Theorem III.1: We assume that f satisfies (18), (1), (3), (7) and $f(0) > 0$; and that Ω is convex. Then there exists u solution of (0.1) satisfying:

$$\max_{\Omega} u > \beta \ .$$

Remark III.1: Again the convexity of Ω and the assumption (7) are purely technical conditions, which are used here in order to ensure a priori bounds on solutions of (0.1). In particular we could replace these conditions by other ones which imply a priori bounds (see [30] and [19]).

Remark III.2: Theorem III.1 can be combined with the results given in sections I.2 and II.2 to state general existence results. We give below some examples.

Example III.1: Assume that f satisfies

i) $f(t) > 0$, for $t \in [0,\alpha)$

ii) $f(t) < 0$, for $t \in (\alpha,\beta)$ [if $\alpha = \beta$ we just assume $f(\alpha) = 0$]

iii) $f(t) > 0$ for $t \in (\beta,+\infty)$

and that f satisfies (1), (3) and (7).

Then there exists two positive solutions u_1, u_2 (distinct) of (0.1) such that

$$0 < u_1(x) < \beta < \max_{\Omega} u_2, \quad \text{for all } x \text{ in } \Omega .$$

Indeed we just need to combine Theorem III.1 with Theorem II.3 (remark that since u_1 may be chosen to be the minimum solution and therefore to satisfy: $u_1(x) < u_2(x)$ in Ω) .

Example III.2: Assume that f satisfies:

i) $f(0) = 0$, $f'(0) = 1$, $f(t) > 0$ for $t \in (0,\alpha)$

ii) $f(t) < 0$, for $t \in (\alpha,\beta)$ [if $\alpha = \beta$ we just assume $f(\alpha) = 0$]

iii) $f(t) > 0$, for $t \in (\beta,+\infty)$

and that f satisfies (1'), (3) and (7).

Then if λ^* is the infimum of all $\lambda > 0$ such that there exists a solution of (0.1-λ) less than α. Then we have:

*) $0 < \lambda^* \leq \lambda_1$ and for $\lambda^* < \lambda$, there exists a positive solution u_1 of (0.1-λ) maximum among all positive solutions of (0.1-λ) less than α;

*) If $\lambda^* < \lambda_1$, then for $\lambda^* < \lambda < \lambda_1$, there exists a second solution u_2 of (0.1-λ) satisfying: $0 < u_2 < u_1$ in Ω. In addition, there exists a maximum positive solution of (0.1-λ^*);

*) For all $\lambda > 0$, there exists a positive solution u_3 of (0.1-λ) satisfying

$$\max_{\Omega} u_3 > \beta .$$

Indeed this is just the combination of Theorem I.4 and Theorem III.1. (We could also combine Theorem I.5 and Theorem III.1 - see also the bifurcation diagrams below.)

Proof of Theorem III.1: We are going to use a topological degree argument and we will again use the notations of the proofs of Theorems II.1 (or I.2). Since we are interested only in positive solutions we may assume

$$f(t) = f(t^+) \quad \text{for all} \quad t \in \mathbf{R} .$$

We choose $C > 0$, large enough, $\mu > 0$, the ball B and the compact map K as in the proof of Theorem II.1.

We already know that $d(I-K, B, 0) = 0$.

We finally define $\mathcal{O} = \{u \in B, u(x) < \beta \text{ in } \Omega\}$. Since $f(\beta) = 0$ it is clear that K maps $\bar{\mathcal{O}}$ into $\mathcal{O}$ and since $\mathcal{O}$ is convex, in the same way as in the proof of Theorem II.1, this implies: $d(I-K, \mathcal{O}, 0) = 1$. Therefore we have

$$d(I-K, B-\bar{\mathcal{O}}, 0) = 1$$

and this means that there exists a solution of (0.1) (with $f(t)$ replaced by $f(t^+)$ thus positive satisfying: $\max_\Omega u > \beta$ - indeed remark that the case $\max_\Omega u = \beta$ is ruled out by the strong maximum principle.

We now turn to another type of results: we consider the case when f satisfies:

(19) $\gamma \geq \beta > 0$ such that $f(\beta) = 0$; $f(t) < 0$, for $t \in (\beta,\gamma)$ (if $\beta = \gamma$, this assumption is not needed);

(15') $\lim_{t\to+\infty} f(t)t^{-1} = 0$ and $f(t) > 0$ for $t > \gamma$ (resp. $f(\delta) = 0$ for some $\delta > \gamma$) ;

(20) $$\int_\beta^\zeta f(s)ds > 0, \quad \text{for some } \zeta > \beta \text{ (resp. for some } \delta > \zeta > \beta);$$

and of course $f(0) \geq 0$.

We want to show how this implies the existence for λ large enough of two solutions u_i of $(0.1-\lambda)$ such that $0 < u_1 < u_2$ in Ω and $\max_\Omega u_i > \gamma$ (resp. $\delta > \max_\Omega u_i > \gamma$).

<u>Theorem III.2:</u> <u>We assume that</u> f <u>satisfies</u> (19), (15'), (20) <u>and</u> $f(0) \geq 0$. <u>Let</u> λ^* <u>be the infimum of all</u> $\lambda > 0$ <u>such that there exits a solution</u> u <u>of</u> $(0.1-\lambda)$ <u>satisfying</u> $\max_\Omega u > \beta$ (<u>resp.</u> $\delta > \max_\Omega u > \beta$). <u>Then</u> λ^* <u>is finite and positive. If</u> Ω <u>is star-shaped we have</u>

i) <u>for</u> $\lambda > \lambda^*$, <u>there exists a maximum positive solution</u> u_1 <u>of</u> $(0.1-\lambda)$ (<u>resp. maximum among all positive solutions of</u> $(0.1-\lambda)$ <u>less than</u> δ) <u>such that</u> $\max_\Omega u_1 > \gamma$.

ii) *for* $\lambda > \lambda^*$, *there exists a second solution* u_2 *of* (0.1-λ) *which satisfies:* $0 < u_2 < u_1$ *in* Ω, $\max_\Omega u_2 > \gamma$.

If Ω *is not star-shaped,* i) *is true for* λ *large enough and* ii) *is replaced by*

iii) *if there exists a solution of* (0.1-λ) *such that* $\max_\Omega u > \gamma$ (*resp.* $\gamma < \max_\Omega u < \delta$) *then there exists a maximum positive solution* u_1 *of* (0.1-λ) (*resp. maximum among all positive solutions of* (0.1-λ) *less than* δ) *such that* $\max_\Omega u_1 > \gamma$; *and there exists a second solution* u_2 *of* (0.1-λ) *which satisfies:* $0 < u_2 < u_1$ *in* Ω, $\max_\Omega u_2 > \gamma$.

Again, as in Theorem I.5, we do not know if the assumption that Ω is star-shaped is necessary.

Remark III.3: Theorm III.2 can be combined with the results given in sections I.2 and II.2 to state general existence results: indeed we already know existence results for solutions less than β and Theorem III.2 give existence results for solutions whose maximum are larger than β. Instead of giving examples like Examples III. 1-3, we refer the reader to the bifurcation diagrams below.

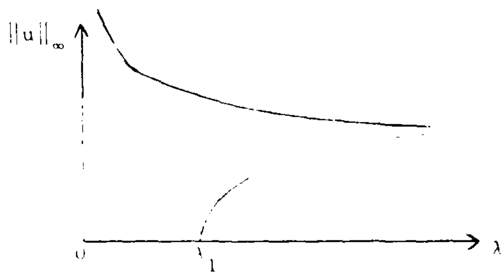
Remark III.4: This result is given in [10] (and extends a more particular, previous result of K. J. Brown and H. Budin [21]), where also some estimates of λ^* are also obtained. Part i) of the Theorem is proved by a straightforward variational argument, while the proof of part ii) requires a topological degree argument somewhat similar to those made above.

We now conclude again the section by a brief list of "bifurcation diagrams": let us recall that the remarks made on the validity and optimality of these diagrams are still valid for those below.

Case 1: $\begin{cases} f(0) = 0;\ f'(0) = 1;\ f(t) < t \text{ for } t > 0,\ t \text{ small};\ f(t) > 0 \text{ for } t \in (0,\alpha) \\ f(t) < 0 \text{ for } t \in (\alpha,\beta);\ f(t) > 0 \text{ for } t > \beta;\ f \text{ superlinear.} \end{cases}$

Ex.: $f(t) = t - t^p + t^q$

$\left(1 < p < q < \frac{N+2}{N-2}\right)$

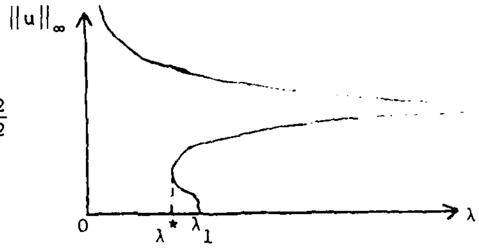


Case 2: $f(0) = 0$; $f'(0) = 1$; $f(t) > t$ for $t > 0$, t small; $f(t) > 0$ for $t \in (0,\alpha)$

$f(t) < 0$ for $t \in (\alpha,\beta)$; $f(t) > 0$ for $t > \beta$; f superlinear.

Ex.: $f(t) = t + t^p - \beta t^q + t^q$

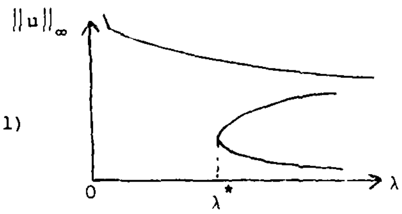
(β large, $1 < p < q < r < \frac{N+2}{N-2}$



Case 3: $f(0) = 0$; $f'(0) < 0$; $f(\beta) = 0$, $f(t) > 0$ for $t > \beta > 0$; f superlinear.

Ex.: $f(t) = -t + t^p - \mu t^q + \nu t^r$

($1 < p < q < r < \frac{N+2}{N-2}$, $0 < \nu << \mu << 1$)

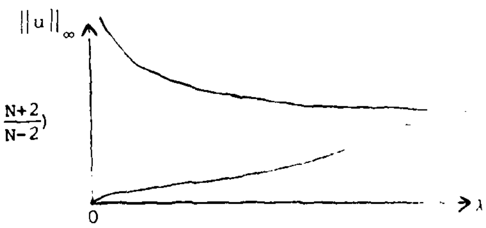


Case 4: $f(0) > 0$; $f(t) > 0$, for $t \in [0,\alpha)$; $f(t) < 0$, for $t \in (\alpha,\beta)$; $f(t) > 0$,

for $t \in (\beta,+\infty)$; f superlinear.

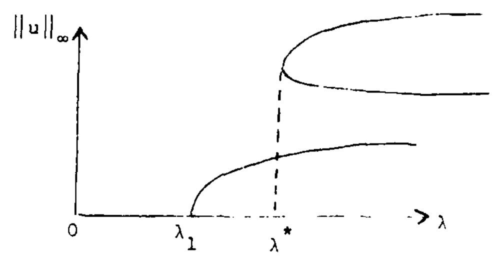
Ex.: $f(t) = \lambda - \mu t^p + \nu t^q$

($\lambda,\mu,\nu > 0$, ν small, $1 < p < q < \frac{N+2}{N-2}$)



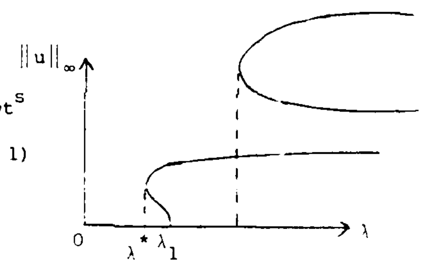
Case 5: $f(0) = 0$; $f'(0) = 1$; $f(t) < t$ for $t > 0$, t small; $f(t) > 0$ for $t \in (0,\alpha)$
$f(t) < 0$ for $t \in (\alpha,\beta)$, $f(\beta) = 0$; f sublinear for $t > \beta$.

Ex.: $f(t) = t - t^p + \mu t^q - \nu t^r$
$(1 < p < q < r,\ 0 < \nu << \mu << 1)$



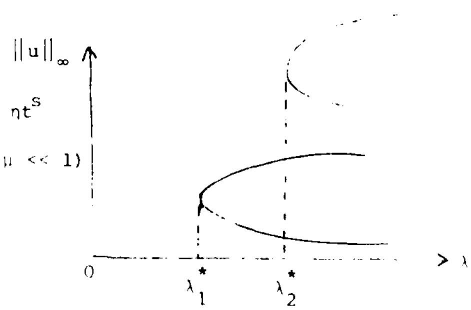
Case 6: $f(0) = 0$; $f'(0) = 1$; $f(t) > t$ for $t > 0$, t small; $f(t) > 0$ for $t \in (0,\alpha)$
$f(t) < 0$ for $t \in (\alpha,\beta)$, $f(\beta) = 0$; f sublinear for $t > \beta$.

Ex.: $f(t) = t + t^p - t^q + \mu t^r - \nu t^s$
$(1 < p < q < r < s,\ 0 < \nu << \mu << 1)$



Case 7: $f(0) = 0$; $f'(0) \leq 0$; $f(\alpha) = 0$, $f(t) < 0$ for $t \in (\alpha,\beta)$, $f(\beta) = 0$; f sublinear for $t > \beta$.

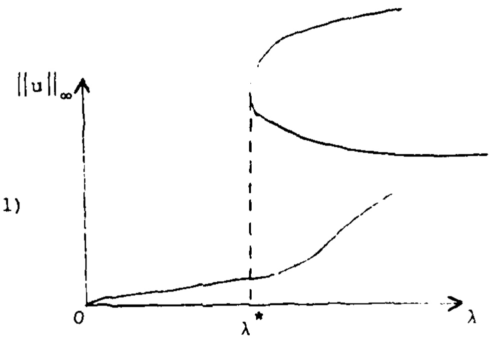
Ex.: $f(t) = -t + t^p - \mu t^q + \nu t^r - \eta t^s$
$(1 < p < q < r < s,\ 0 < \eta << \nu << \mu << 1)$



Case 8: $f(0) > 0$; $f(t) > 0$ for $t \in [0,\alpha)$; $f(t) < 0$, for $t \in (\alpha,\beta)$, $f(\beta) = 0$; f sub-linear for $t > \beta$.

Ex.: $f(t) = 1 - t^p + \mu t^q - \nu t^r$

$(0 < p < q < r,\ 0 < \nu << \mu << 1)$



III.2. Bumps and the shape of the nonlinearity:

We want, in this section, to show how bumps or some "slightly oscillatory" shape of the function f may affect the "bifurcation diagrams" or the multiplicity results. Let us describe in some unprecise way two results we prove here: the nonlinearity $f_\varepsilon(t)$ will depend on a parameter $\varepsilon \in [0,1]$ and in the figures below we present both the shape of f_ε and the associated (minimal) "bifurcation diagram."

Example 1:

Figure 1;

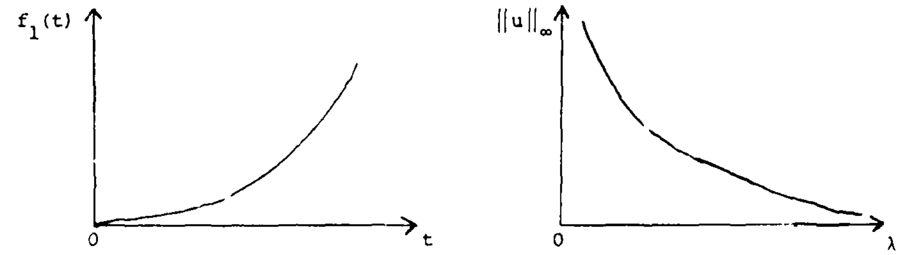


Figure 2:

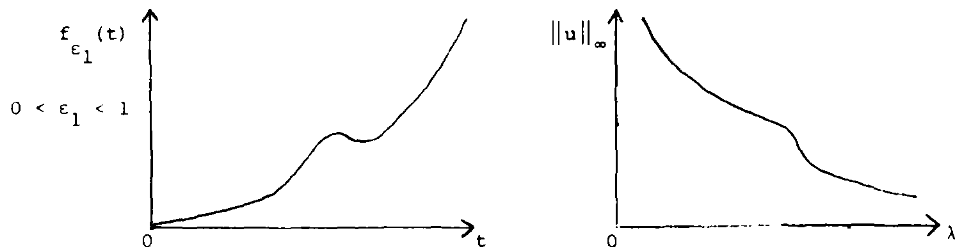


Figure 3:

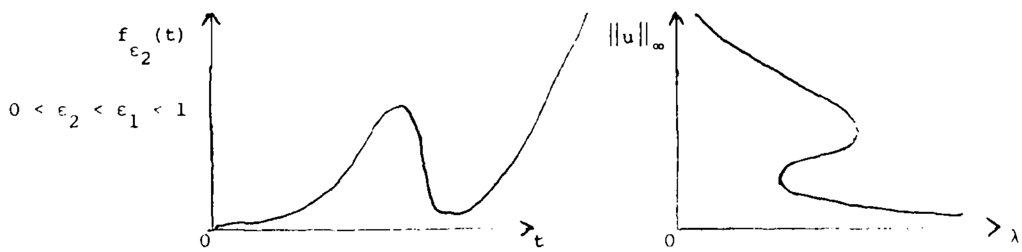
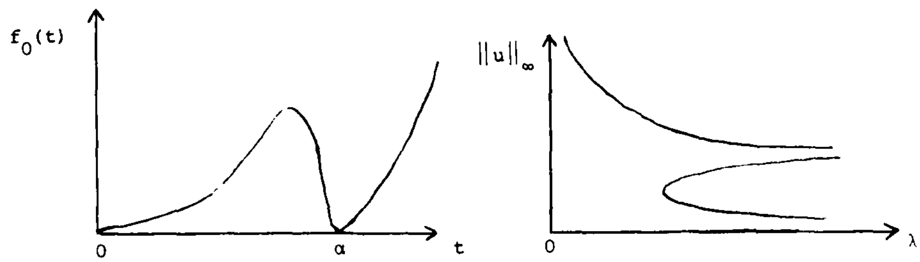


Figure 4:



Example 2:

Figure 1:

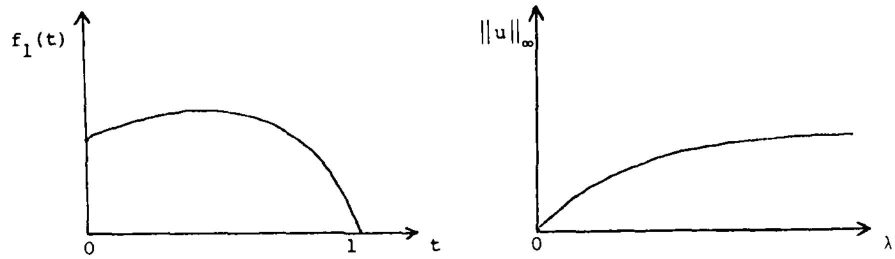


Figure 2:

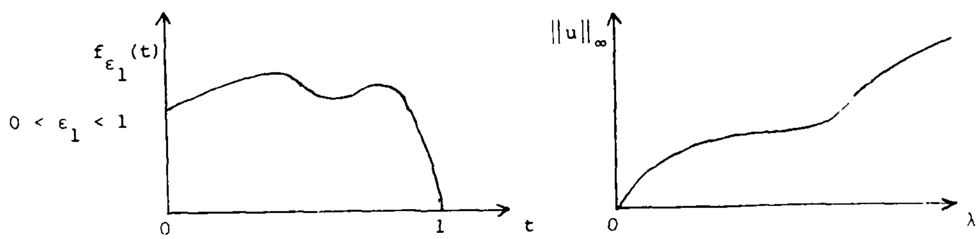


Figure 3:

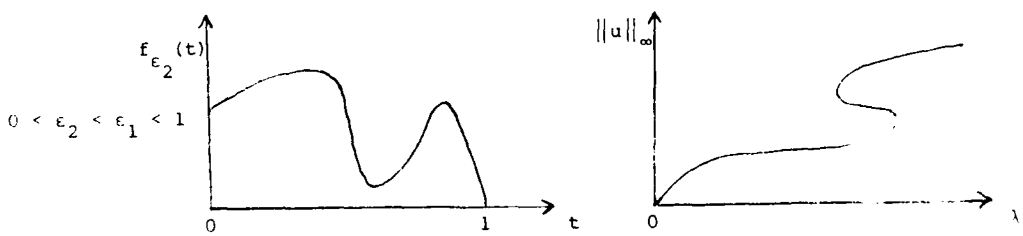
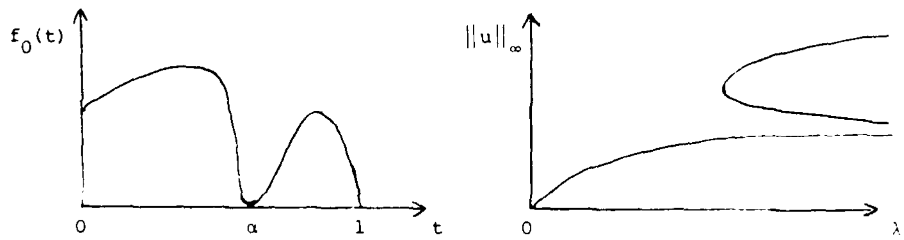


Figure 4:



We begin by explaining what Example 2 means: this will be clear in view of the two results above:

Theorem III.3: Let $f(t)$ be a locally Lipschitz function from $\mathbf{R}$ into $\mathbf{R}_+$ satisfying:

(21) $$f(t) > 0 \ \text{for} \ t > 0 \ (\text{resp. for} \ t \in (0,\alpha)), \ \lim_{t\to+\infty} f(t)t^{-1} = 0$$

$$(\text{resp.} \ f(\alpha) = 0) \ .$$

Then there exists a maximum positive solution $\bar{u}_\lambda$ of $(0.1\text{-}\lambda)$ (resp. maximum among all positive solutions less than α) and a minimum positive solution $\underline{u}_\lambda$ thus satisfying: $0 < \underline{u}_\lambda \le \bar{u}_\lambda$ in Ω. Now, if we assume there exist $0 < \lambda_1 < \lambda_2 < +\infty$ such that for $\lambda \in [\lambda_1,\lambda_2]$ $\bar{u}_\lambda \ne \underline{u}_\lambda$, and $\lim_{\mu\uparrow\lambda} \uparrow \bar{u}_\mu \nleq \lim_{\mu\downarrow\lambda} \downarrow \underline{u}_\mu$, then, for all $\lambda \in (\lambda_1,\lambda_2)$ there exists a third solution u_λ of $(0.1\text{-}\lambda)$ such that:

$$0 < \underline{u}_\lambda < u_\lambda < \bar{u}_\lambda \quad \text{in} \quad \Omega \ .$$

We now apply this theorem on the setting of Example 2.

Corollary III.1: Let $(f_\varepsilon(t))_{\varepsilon\in[0,1]}$ be a family of locally Lipschitz functions from $\mathbf{R}$ into $\mathbf{R}_+$ satisfying

$$0 < f_\varepsilon(t) \ \text{for} \ t \in [0,1), \ f_\varepsilon(1) = 0 \ ;$$

$$f_0(t) > 0 \ \text{for} \ t \in [0,\alpha) \ \text{and} \ t \in (\alpha,1], \ f_0(\alpha) = 0; \ f_\varepsilon(t) = f_0(t) \ \text{on} \ [0,(\alpha-\varepsilon)^+] \ ;$$

$$f_\varepsilon(t) > f_0(t) \ \text{for} \ t \in [0,1] \ .$$

Then there exists $\lambda^* > 0$, *and* $\overline{u}_\lambda$ (*resp.* $\underline{u}_\lambda$) *for* $\lambda > \lambda^*$ *satisfying*: $\overline{u}_\lambda$ (*resp.* $\underline{u}_\lambda$) *is the maximum (resp. minimum) solution less than* 1 *of* $(0.1-\lambda)$ - *for* $f_0(t)$ - *and*

$$0 < \underline{u}_\lambda < \alpha < \max_\Omega \overline{u}_\lambda, \quad 0 < \underline{u}_\lambda < \overline{u}_\lambda < 1 \quad \text{in } \Omega .$$

On the other hand, for all $\lambda > 0$, *there exists* $\overline{u}_{\lambda,\varepsilon}$ (*resp.* $\underline{u}_{\lambda,\varepsilon}$) *satisfying*: $\overline{u}_{\lambda,\varepsilon}$ (*resp.* $\underline{u}_{\lambda,\varepsilon}$) *is the maximum (resp. minimum) solution less than* 1 *of* $(0.1-\lambda)$ - *for* $f_\varepsilon(t)$ - *and* $0 < \underline{u}_{\lambda,\varepsilon} \leq \overline{u}_{\lambda,\varepsilon} < 1$ *in* Ω.

Now let $\overline{\lambda} > \lambda^*$. *there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \leq \varepsilon_0$ *we have*: $\overline{u}_{\lambda,\varepsilon} \neq \underline{u}_{\lambda,\varepsilon}$ *for all* $\lambda \in [\lambda^*,\overline{\lambda}]$. *And for such* ε *and* λ *there exists a third solution* $u_{\lambda,\varepsilon}$ *of* $(0.1-\lambda)$ - *for* $f_\varepsilon(t)$ - *satisfying*

$$0 < \underline{u}_{\lambda,\varepsilon} < u_{\lambda,\varepsilon} < \overline{u}_{\lambda,\varepsilon} \quad \text{in } \Omega .$$

The first part of Theorem III.3 and Corollary III.1 is deduced from the results of the preceding sections: indeed the existence of $\underline{u}_\lambda$, $\overline{u}_\lambda$ (in Theorem III.3) and of $\overline{u}_{\lambda,\varepsilon}$, $\underline{u}_{\lambda,\varepsilon}$ (in Corollary III.1) is deduced from section II.2 (this is a standard application of super and subsolutions arguments since 0 is a subsolution and 1 is a supersolution, for example, in the setting of Corollary III.1). And the existence of λ^*, $\underline{u}_\lambda$, $\overline{u}_\lambda$ (in Corollary III.1) is deduced from section III.1 (Theorem III.2).

Before proving Theorem III.3 (that is the existence of u_λ), we derive Corollary III.1 from Theorem III.3.

Proof of Corollary III.1: We first remark that since f_ε are nonnegative we have obviously:

$$\overline{u}_\lambda,\ \underline{u}_\lambda,\ \overline{u}_{\lambda,\varepsilon},\ \underline{u}_{\lambda,\varepsilon} \quad \text{are nonincreasing with } \lambda ,$$

in addition, since $f_\varepsilon \geq f_0$, we have:

$$\underline{u}_{\lambda,\varepsilon} \geq \underline{u}_\lambda \quad \text{for } \lambda > 0; \quad \overline{u}_{\lambda,\varepsilon} \geq \overline{u}_\lambda \quad \text{for } \lambda \geq \lambda^* .$$

Now, to conclude, we just need to remark that if $\lambda \leq \overline{\lambda}$, for ε small enough we have $\underline{u}_{\lambda,\varepsilon} = \underline{u}_\lambda$. Indeed, if $\lambda \leq \overline{\lambda}$

$$\max_\Omega \underline{u}_\lambda \leq \max_\Omega \underline{u}_{\overline{\lambda}} = \alpha_{\overline{\lambda}} < \alpha < \max_M \overline{u}_{\lambda^*} .$$

Therefore for $\varepsilon \leq \alpha - \alpha_{\lambda}$, $\underline{u}_{\lambda}$ is also a solution of $(0.1-\lambda)$ for $f_{\varepsilon}(t)$ but, since we already know that $\underline{u}_{\lambda,\varepsilon} \geq \underline{u}_{\lambda}$ and that $\underline{u}_{\lambda,\varepsilon}$ is the minimum solution, we conclude: $\underline{u}_{\lambda,\varepsilon} = \underline{u}_{\lambda}$ for $\lambda \leq \bar{\lambda}$, ε small enough.

Proof of Theorem III.3: As remarked above, we already know the existence of $\underline{u}_{\lambda}$ and of $\bar{u}_{\lambda}$; in addition we know that $\underline{u}_{\lambda}$ and $\bar{u}_{\lambda}$ increase with λ. We now assume that for $\lambda \in [\lambda_1,\lambda_2]$ $\bar{u}_{\lambda} \neq \underline{u}_{\lambda}$ and let $\lambda \in (\lambda_1,\lambda_2)$. To prove the existence of the third solution we are going to use a topological degree argument. To simplify, we will assume that there exists $\alpha > 0$ such that $f(\alpha) = 0$.

Let us first introduce a few notations: first we replace $f(t)$ by $\tilde{f}(t)$ defined by: $\tilde{f}(t) = f(0)$ if $t < 0$, $\tilde{f}(t) = f(t)$ for $t \in [0,\alpha]$, $\tilde{f}(t) = 0$ for $t > \alpha$. Let $\mu > 0$ be such that $\lambda\tilde{f}(t) + \mu t$ is nondecreasing, for $\lambda \in [\lambda_1,\lambda_2]$. We now introduce a cmpact map K from $C_0^1(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$: for u in $C_0^1(\bar{\Omega})$, $v = Ku$ is defined by

$$-\Delta v + \mu v = \lambda\tilde{f}(u) + \mu u \quad \text{in} \quad \Omega, \quad v \in C^2(\bar{\Omega}), \quad v = 0 \quad \text{on} \quad \partial\Omega \quad .$$

Remark there exists $C > 0$ such that $\|Ku\|_{C^1(\bar{\Omega})} < C$, for all u in $C_0^1(\bar{\Omega})$ satisfying $0 \leq u \leq \alpha$ in Ω.

We next define three open sets:

$$I = \{u \in C_0^1(\bar{\Omega}),\ 0 < u < \alpha \ \text{in} \ \Omega,\ \frac{\partial u}{\partial n} < 0 \ \text{on} \ \partial\Omega,\ \|u\|_{C^1(\bar{\Omega})} < C\}$$

$$J = \{u \in C_0^1(\bar{\Omega}),\ \bar{u}_{\lambda_1} < u < \alpha \ \text{in} \ \Omega,\ \frac{\partial u}{\partial n} < \frac{\partial}{\partial n}\left(\underline{u}_{\lambda_1}\right) \ \text{on} \ \partial\Omega,\ \|u\|_{C^1(\bar{\Omega})} < C\}$$

$$\tilde{J} = \{u \in C_0^1(\bar{\Omega}),\ 0 < u < \underline{u}_{\lambda_2} \ \text{in} \ \Omega,\ 0 > \frac{\partial u}{\partial n} > \frac{\partial}{\partial n}\left(\underline{u}_{\lambda_2}\right) \ \text{on} \ \partial\Omega,\ \|u\|_{C^1(\bar{\Omega})} < C\}.$$

If $\bar{J} \cap \bar{\tilde{J}} \neq \emptyset$, then necessarily $\bar{u}_{\lambda_1} \leq \underline{u}_{\lambda_2}$ in Ω. Since we assume $\lim\limits_{\mu\uparrow\lambda} \uparrow \bar{u}_{\lambda} \not\leq \lim\limits_{\mu\downarrow\lambda} \downarrow \underline{u}_{\mu}$ and since we may take λ_1,λ_2 as near λ as we want, we see that we may assume $\bar{J} \cap \bar{\tilde{J}} = \emptyset$.

We are going to prove that the following degreees exist and are equal to $d(I-K,I,0) = d(I-K,J,0) = d(I-K,\tilde{J},0) = 1$. This is quite obvious since K maps $\bar{I}$, $\bar{J}$, $\bar{\tilde{J}}$ into I, J, $\tilde{J}$ and since I, J, $\tilde{J}$ are convex. Therefore we deduce

$$d(I-K,\ I - (\bar{J} \cup \bar{\tilde{J}}),\ 0) = d(I-K,I,0) - d(I-K,J,0) - d(I-K,\tilde{J},0) = -1$$

and this implies there exists a solution u_λ of $(0.1\text{-}\lambda)$ which lies in $I - (\bar{J} \cap \bar{\tilde{J}})$. Since $\bar{u}_\lambda$ lies in J and $\underline{u}_\lambda$ lies in $\tilde{J}$, we obtain a third solution as stated in Theorem III.3.

Remark III.5: In Corollary III.1, we may replace the assumption $f_\varepsilon(t) = f_0(t)$ on $[0,(\alpha-\varepsilon)^+]$ by:

$$f_\varepsilon(t) \to f_0(t) \quad \text{uniformly for} \quad t \in [0,\alpha] \quad .$$

Then in the proof, we need to use a variational argument in order to prove that $\max_\Omega \underline{u}_{\lambda,\varepsilon} < \alpha$ for $\lambda \in [\lambda^*,\bar{\lambda}]$ and for $\varepsilon < \varepsilon_0$.

We now turn to a result concerning the Example III.1: *for the sake of simplicity*, we are not going to state an abstract result like Theorem III.1, but instead we give directly a result similar to Corollary III.1. Let us first state a few assumptions we are going to need: let $(f_\varepsilon(t))_{\varepsilon\in[0,1]}$ be a family uniformly *(in* ε*) locally Lipschitz functions* satisfying

$$\text{(21)} \quad \begin{cases} f_\varepsilon(0) = 0,\ f'_\varepsilon(0) \text{ exists and } f'_\varepsilon(0) = 0,\ f_\varepsilon(t) > 0 \text{ for } t > 0, \\ \qquad f_\varepsilon(t) \geq f_0(t) \text{ for } t > 0 \quad ; \\ f_0(t) > 0 \text{ for } t \in (0,\alpha) \text{ and } t \in (\alpha,+\infty),\ f_0(\alpha) = 0; \\ \qquad f_\varepsilon(t) = f_0(t) \text{ for } t \in [0,(\alpha-\varepsilon)^+] \quad ; \end{cases}$$

and

$$\text{(22)} \qquad \lim_{t\to+\infty} f_0(t)t^{-1} = +\infty \quad ;$$

$$\text{(23)} \qquad \lim_{t\to+\infty} f_\varepsilon(t)t^{-(N+1)(N-1)} = 0, \quad \text{uniformly in } \varepsilon \in [0,1]$$

(if $N = 1$, we do not need this assumption).

We assume here (23) only to simplify our setting (this assumption is used in order to obtain a priori estimates). We may now state our result.

Proposition III.1: *Under assumptions* (21) - (23) on $(f_\varepsilon(t))_{\varepsilon\in[0,1]}$, there exists $\lambda^* > 0$, and u^1_λ, u^2_λ (for $\lambda > \lambda^*$) satisfying: u^i_λ is a positive solution of (0.1-λ) (for $f_0(t)$) less than α and u^2_λ is the maximum solution among all solutions less than α. In addition, let $\bar\lambda > \lambda^*$, then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \le \varepsilon_0$ and for any $\lambda \in (\lambda^*,\bar\lambda)$, u^i_λ $(i = 1,2)$ are also solutions of (0.1-λ) for $f_\varepsilon(t)$ and in addition there exists a third solution u_λ of (0.1-λ) (for $f_\varepsilon(t)$) satisfying: $\max_\Omega u_\lambda > \alpha$.

Proof of Proposition III.1: The first part of Proposition III.1 is deduced from Theorem I.5. In addition we know that u^2_λ is increasing with respect to λ. Now let $\bar\lambda > \lambda^*$ be fixed, we have for $\lambda \in [\lambda^*,\bar\lambda]$

$$\max_{\bar\Omega} u^2_\lambda \le \max_{\bar\Omega} u^2_{\bar\lambda} = \alpha' < \alpha \quad .$$

Now, let us choose $\varepsilon_0 = \frac{\alpha-\alpha'}{2} \wedge 1$. It is clear that for $\varepsilon \le \varepsilon_0$ and for $\lambda \in (\lambda^*,\bar\lambda)$, u^i_λ $(i = 1,2)$ are solutions of (0.1-λ) for $f_\varepsilon(t)$. And we are going to prove the existence of a third solution u_λ by a topological *degree argument*.

We first introduce a few notations: we replace $f_\varepsilon(t)$ by $f_\varepsilon(t^+) = \tilde f_\varepsilon(t)$ and we choose $\mu > 0$ such that $\lambda \tilde f_\varepsilon(t) + \mu t$ is nondecreasing for $t \le \alpha$ and for $\lambda \in [\lambda^*,\bar\lambda]$. We next define a compact map K_ε from $C^1_0(\bar\Omega)$ into $C^1_0(\bar\Omega)$: for u in $C^1_0(\bar\Omega)$, $K_\varepsilon u = v$ is defined by

$$-\Delta v + \mu v = \lambda \tilde f_\varepsilon(u) + \mu u \quad \text{in } \Omega,\ v \in C^2(\bar\Omega),\ v = 0 \quad \text{on } \partial\Omega \quad .$$

We finally choose C large enough such that, in particular, all solutions of (0.1-λ) for $f_\varepsilon(t)$ $(\varepsilon \in [0,1])$ are bounded by C in the space $C^1(\bar\Omega)$ - this is possible because of the assumptions (22) - (23) which imply, in view of [19], the a priori bounds.

We now define two open sets:

$$B = \{u \in C^1_0(\bar\Omega),\ \|u\|_{C^1(\bar\Omega)} < C\}$$

$$\mathcal{O} = \{u \in B,\ -1 < u < \frac{\alpha+\alpha'}{2} \text{ in } \bar\Omega\} \quad .$$

As in the proof of Theorem I.2, we can prove that $d(I-K,B,0)$ exists and is equal to: $d(I-K_{\varepsilon},B,0) = 0$.

We want to prove now that $d(I-K_{\varepsilon},\mathcal{O},0)$ is well-defined and is equal to 1. Indeed if it were not defined, this would imply the existence of a solution u of $(0.1-\lambda)$ for $f_{\varepsilon}(t)$ satisfying: $0 < u \leq \frac{\alpha+\alpha'}{2}$ in Ω and $\max_{\Omega} u = \frac{\alpha+\alpha'}{2}$. Since $f_{\varepsilon}(t) = f(t)$ for $t \in [0, \frac{\alpha+\alpha'}{2}]$, if $\varepsilon \leq \varepsilon_0$, u is also a solution of $(0.1-\lambda)$ for $f_0(t)$ but $\max_{\Omega} u > \max_{\Omega} u_{\lambda}^2$ and this is impossible since u_{λ}^2 is the maximum solution (among solutions less than α). Therefore $d(I-K_{\varepsilon},\mathcal{O},0)$ is well-defined for $\varepsilon \leq \varepsilon_0$ and the argument given above shows: $d(I-K_{\varepsilon},\mathcal{O},0) = d(I-K_0,\mathcal{O},0)$, for $0 \leq \varepsilon \leq \varepsilon_0$. Next, we remark that all solutions less than α of $(0.1-\lambda)$ for $f_0(t)$ satisfy

$$\max_{\Omega} u \leq \max_{\Omega} u_{\lambda}^2 = \alpha' < \frac{\alpha+\alpha'}{2} < \alpha \quad .$$

Therefore if $J = \{u \in B, -1 < u < \alpha \text{ in } \Omega\}$, we have:

$$d(I-K_0,J-\overline{\mathcal{O}},0) = 0.$$

And we deduce $d(I-K_{\varepsilon},\mathcal{O},0) = d(I-K_0,J,0)$, for $0 \leq \varepsilon \leq \varepsilon_0$. But K_0 maps $\overline{J}$ into J and J is convex and this implies (see similar arguments given in the preceding sections): $d(I-K_0,J,0) = +1$.

Hence

$$d(I-K_{\varepsilon},B-\overline{\mathcal{O}},0) = d(I-K_{\varepsilon},B,0) - d(I-K_{\varepsilon},\mathcal{O},0)$$

$$= -d(I-K_0,J,0) = -1 \quad .$$

And we are able to conclude.

Remark III.6: We want to conclude this section by remarking the two examples given above and the results we have proved are only examples of a general feature: in particular one can build other examples by changing the behavior of f near 0. The general feature is that large bumps in the shape of the nonlinearity may create two bending points (as in Figure 3 in Examples III.1 and III.2) and we could combine the arguments given above with the results of sections I, II and III.1.

IV. Variants and open questions:

IV.1: Unbounded domains.

We are still concerned with the problem (0.1) but we assume that Ω is unbounded and hence we consider the problem:

$$(24) \qquad \begin{aligned} &-\Delta u = f(u) \quad \text{in} \quad \Omega, \; u \in C^2(\bar{\Omega}), \; u > 0 \quad \text{in} \quad \Omega \\ &u = 0 \quad \text{on} \quad \partial\Omega, \; u(x) \to 0 \quad \text{as} \quad |x| \to +\infty, \; x \in \Omega \quad . \end{aligned}$$

In some cases this problem is more complicated than the one considered in the preceding sections since in view of the results of M. J. Esteban and P. L. Lions [27], [28] the geometry of Ω seems to play an essential role. Let us recall a result taken from [27] (the proof is based on an extension of a powerful identity due to S. Pohozaev [51]):

Theorem IV.1: Let $f(t)$ be locally Lipschitz from $\mathbf{R}$ into $\mathbf{R}$ and suppose that $f(0) = 0$. If we assume that Ω satisfies

$$(25) \qquad \exists\, \chi \in \mathbf{R}^N, \; |\chi| = 1 \quad \text{such that} \quad n(x) \cdot \chi \geq 0 \quad \text{on} \quad \partial\Omega \quad \text{and}$$
$$n(x) \cdot \chi \not\equiv 0 \quad \text{on} \quad \partial\Omega \quad .$$

Then, if u satisfies

$$(24') \qquad \begin{cases} -\Delta u = f(u) \quad \text{in} \quad \Omega, \; u = 0 \quad \text{on} \quad \partial\Omega, \; u \in L^\infty(\Omega \cap B_R) \quad R < \infty \\ \nabla u \in L^2(\Omega), \; F(u) \in L^1(\Omega) \end{cases}$$

where $F(t) = \int_0^t f(s)ds$, necessarily we have

$$u \equiv 0 \quad .$$

Very few existence results for (24) are known except in the case when $\Omega = \mathbf{R}^N$ (or when Ω is a band [16]): in this case a nearly optimal existence result is given in H. Berestycki and P. L. Lions [7] (see also [11], [12], [14]) which we recall here:

Theorem IV.2: Let $N \geq 3$ and let f be a continuous function from $\mathbf{R}_+$ into $\mathbf{R}$ satisfying $f(0) = 0$ and

(26) $\overline{\lim_{t \to 0_+, +\infty}} F(t)t^{-\ell} \leq 0$, <u>with</u> $\ell = \frac{2N}{N-2}$, $\overline{\lim_{t \to +\infty}} f(t)t^{-\frac{n+2}{n-2}} \leq 0$

(27) $F(\zeta) > 0$, <u>for some</u> $\zeta > 0$.

<u>Then there exists</u> $u \in C_b^2(\mathbb{R}^N)$, <u>spherically symmetric, positive, decreasing with respect to</u> $r = |x|$, <u>satisfying</u>

(28) $-\Delta u = f(u)$ in $\mathbb{R}^N$, $\nabla u \in L^2(\mathbb{R}^N)$, $F(u) \in L^1(\mathbb{R}^N)$, $u \in L^{2N/(N-2)}(\mathbb{R}^N)$.

<u>Remark IV.1</u>: Under quite general assumptions it is known (see B. Gidas, Wei-Ming Ni, L. Nirenberg [32], [33]) that any positive solution of (29) is necessarily radial (up to a translation, of course). In addition some results conerning the uniqueness or the non uniqueness of the positive radial solution are known (see [50], [46], [13]).

<u>Remark IV. 2</u>: Theorem IV.2 extends some particular results proved by C. V. Coffman [22], Ryder [54], Nehari [48], M. S. Berger [15], W. Strauss [55] and Coleman, Glazer and Martin [23].

IV.2 Open questions:

We now conclude with a list of open questions

(a) the main open question concerning the problem considered here is the proof of a priori bounds of solutions u of (0.1) where f satisfies only

$$\underline{\lim}_{t\to+\infty} f(t)t^{-1} > \lambda_1, \ \lim_{t\to+\infty} f(t)t^{-(N+2)/(N-2)} = 0 \quad (\text{for } N \geq 3) \ .$$

The best result in this direction is given in [30] (see also [19] and B. Gidas and J. Spruck [34]).

(b) it is known that, if Ω is star-shaped, there is no solution of (0.1) if $f(t) = t^p$ and $p \geq \frac{N+2}{N-2}$ (for $N \geq 3$). On the other hand, it is very easy to realize that if Ω is ring shaped ($\Omega = \{|x| \in (a,b)\}$) then (0.1) has a solution for $f(t) = t^p$ and $1 < p < \infty$. It would be interesting to understand the relation between the geometry of Ω and the existence of solutions of (0.1) for supercritical nonlinearities (some related results are given in J. Hempel [35] and in H. Brezis and L. Nirenberg [18]).

(c) an important question for applications is to extend the results concerning (0.1) to systems of the type

$$-\Delta u_i = f_i(u_1,\ldots,u_m) \ \text{ in } \ \Omega, \ u_i \in C^2(\bar{\Omega}), \ u_i > 0 \ \text{ in } \ \Omega, \ u_i = 0 \ \text{ on } \ \partial\Omega \ .$$

Very few results are known for such systems.

(d) a few qualitative properties of solutions (0.1) are known: symmetry properties (B. Gidas, Wei-Ming Ni and L. Nirenberg [32], [33]), behavior in the neighborhood of isolated singularities (C. Loewner and L. Nirenberg [49], L. Veron [56], H. Brezis and L. Veron [20], P. L. Lions [43], B. Gidas and J. Spruck [34]). A natural question which remains open is to deduce whether solution u of (0.1) in convex domains have convex level sets (i.e. $\{u > t\}$ is convex). In a very special case ($f(t) = \lambda t - \mu t^p$, $\lambda,\mu > 0$, $p > 1$) this is proved in P. L. Lions [44] (actually it is proved that u is Log concave, which implies in particular the convexity of level sets).

(e) it would be interesting to understand the relation between the geometry of Ω and the existence (or non existence) of solutions of (0.1) when Ω is unbounded (see section IV.1 above).

(f) a difficult question concerns the uniqueness problem for (0.1) or to prove exact multiplicity results: very few are known (see the references in the sections above)

(g) related problems concern the existence of solutions u of

$$-\Delta u = f(x,u) \quad \text{in} \quad \Omega,\ u \in C^2(\bar{\Omega}),\ u = 0 \quad \text{on} \quad \partial\Omega \ ;$$

with non constant sign and where $f(x,0) \not\equiv 0$. A lot of partial results are known and we do not want to give any references, but at the moment there is no general understanding of this question. Let us also remark that this seems to be more a challenging mathematical question than a question important for applications since in most of the applications where such problems arise, u represents a concentration or a temperature or a density and thus has to be nonnegative (and thus the problem reduces to (0.1)).

(h) a totally formal way of guessing how looks the bifurcation diagram of solutions of $(0.1-\lambda)$ is to replace the operator $-\Delta u$ by $\lambda_1 u$ (and $(0.1-\lambda)$ reduces to a simple equation $\lambda_1 t = \lambda f(t)$). This often gives a good qualitative account of the solutions set but it may be completely false as the following example show: take $f(u) = e^u$, $\Omega = B_1$(ball), $3 \leq N \leq 9$ since in this case there exists $\lambda_0 > 0$ such that $(0.1-\lambda_0)$ has infinitely many solutions (see [36]) while the equation $\lambda_1 t = \lambda e^t$ has at most two solutions.

It would be interesting to show a more rigorous connection between $(0.1 - \lambda)$ and the equation $\lambda_1 t = \lambda f(t)$. Remark also that in the case of Neumann boundary conditions $(\lambda_1 = 0)$ this simple equation just gives all constant solutions of $(0.1 - \lambda)$.

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In this paper we study the existence of positive solutions of semilinear elliptic equations. Various possible behaviors of the nonlinearity are considered and in each case nearly optimal multiplicity results are obtained. The results are also interpreted in terms of bifurcation diagrams.

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