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A NESTED-DECOMPOSITION APPROACH FOR SOLVING STAIRCASE-STRUCTURE--ETC(U)

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Philip G. Abrahamson

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Abstract

The algorithm solves a T-period staircase-structured linear program by applying a compact basis-inverse scheme for the Simplex Method in conjunction with a choice mechanism which uses the dual of the Nested Decomposition Principle of Manne and Ho to determine the incoming basic column. A sequence of one-period problems is solved in which, typically, information is provided to period t from previous and subsequent periods in the form of surrogate columns and modified right-hand side, and surrogate rows and modified cost coefficients, respectively.

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I. Introduction

This paper describes preliminary work on an algorithm for solving staircase-structured linear programs. Such problems often arise in the modeling of phenomena which are naturally described as evolving over a sequence of temporal or spatial "periods". A pure nested decomposition algorithm transforms the problem into an ordered set of smaller problems, one for each period, which are coordinated only through price and activity communication between adjacent periods. The process of achieving an optimal coordination involves repeated solution of the individual problems. Preliminary experience has indicated that the convergence of the pure algorithm may be slow. To accelerate this convergence, the algorithm is modified to enable the individual problems to communicate in an implicit fashion whenever possible. This modification involves the generation of surrogate columns which are passed to subsequent periods, allowing these period to parametrically adjust solutions to earlier periods. A compact basis-inverse scheme is used to represent these parametric variations.

Section 2 states the problem of interest and describes the pure nested decomposition algorithm. Section 3 outlines the modified approach and discusses some details of implementation.

2. Nested Decomposition of the Staircase Structure

2.1 The Staircase Structure

The problem of interest is

$$\text{minimize} \quad \sum_{t=1}^{T} c_t x_t$$

$$\begin{aligned} \text{subject to} \quad & A_1 x_1 && = b_1 && \\ & -B_{t-1} x_{t-1} + A_t x_t && = b_t \,, && t = 2, \ldots, T \\ & x_t \geq 0 && , && t = 1, \ldots, T \end{aligned}$$

where x_t is $n_t \times 1$, A_t is $m_t \times n_t$, and all other vectors and matrices are of conformable dimension. This linear program is said to be staircase-structured because the constraint matrix has the form:

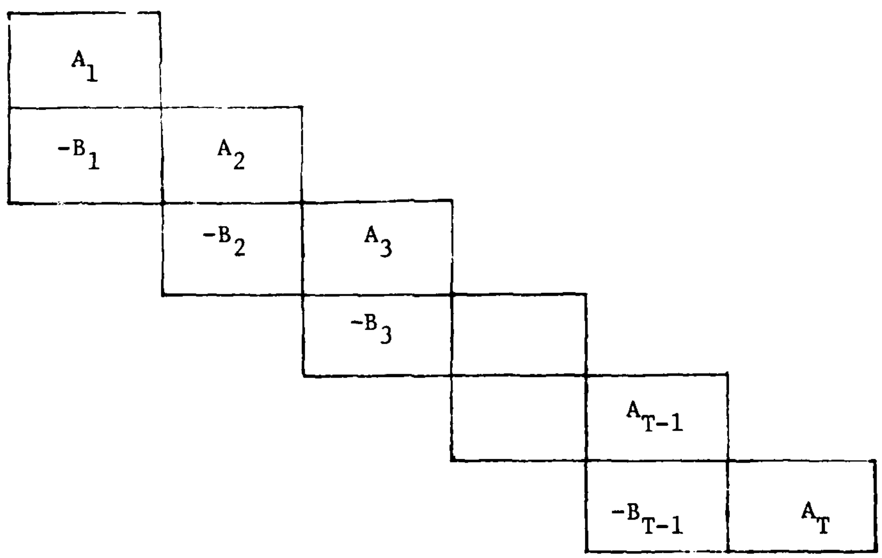


Figure 1: Staircase-Structured Constraint Matrix

Activities in period t are represented by the matrix A_t . The inventory provided by period t to period $t + 1$ is described by the matrix $-B_t$. Otherwise, the activities in period t have no direct effect on either previous or subsequent periods.

The dual of (P) is

$$(D) \qquad \text{maximize} \quad \sum_{t=1}^{T} \pi_t b_t$$

$$\text{subject to} \quad \pi_t A_t - \pi_{t+1} B_t \leq c_t \ , \qquad t = 1, \ldots, T - 1$$

$$\pi_T A_T \leq c_T \ .$$

2.2 Nested Decomposition

Manne and Ho [7] and Glassey [5] repeatedly applied the decomposition principle of linear programming developed by Dantzig and Wolfe [3] to achieve a nested decomposition of (P) . A sequence of applications, modifications, and improvements has led to advanced implementations by Ho and Loute [6], who have solved some large-scale problems more rapidly in this fashion than by directly applying commercial linear programming to (P) .

Van Slyke and Wets [8] describe an algorithm for solving (P) in the case $T = 2$ which is equivalent to applying the decomposition algorithm to (D) . Dantzig [2] outlined an algorithm consisting of a nested decomposition of (D) . This paper represents preliminary work on the development of a technique which is based on his approach.

2.3 A Nested Decomposition Algorithm for (D)

The algorithm of Dantzig [2] consists of the following nested decomposition:

$$
\begin{array}{llll}
\text{maximize} & \pi_1 b_1 + \pi_2 b_2 + \pi_3 b_3 + \pi_4 b_4 + \cdots + \pi_T b_T & & \\
 & & & \text{master}_1 \\
\text{subject to} & \pi_1 A_1 - \pi_2 B_1 & \leq c_1 & \\
 & \qquad \pi_2 A_2 - \pi_3 B_2 & \leq c_2 & \text{master}_2 \\
 & \qquad\qquad \pi_3 A_3 - \pi_4 B_3 & \leq c_3 & \\
 & \qquad\qquad\qquad \ddots & & \text{sub}_1 \quad \text{sub}_2 \quad \ddots \\
 & \qquad\qquad\qquad\quad \pi_{T-1} A_{T-1} - \pi_T B_{T-1} & \leq c_{T-1} & \text{master}_{T-1} \\
 & \qquad\qquad\qquad\qquad\qquad \pi_T A_T & \leq c_T & \text{sub}_{T-1}
\end{array}
$$

Figure 2: Nested Decomposition of (D)

Let x_{t-1}^0 be multipliers for master_{t-1}, and let

$$\bar{b}_t = b_t + B_{t-1} x_{t-1}^0 \quad . \tag{1}$$

Then the t-th period master problem has the form

$$
\begin{array}{llll}
(D_t) & \text{maximize} & \pi_t \bar{b}_t + \sigma_t f_t + \rho_t \ell_t & \\
 & \text{subject to} & \pi_t A_t + \sigma_t F_t + \rho_t L_t \leq c_t & : x_t \\
 & & \rho_t e \leq 1 & : \theta_t \\
 & & \sigma_t \, , \; \rho_t \geq 0 & ,
\end{array}
$$

where (F_t, f_t) and (L_t, ℓ_t) are, respectively, the extreme ray and extreme point proposals generated by period $t + 1$. For $t = 1$, $\bar{b}_1 = b_1$, and for $t = T$, only the terms involving π_T are present. Note that each proposal is a row vector.

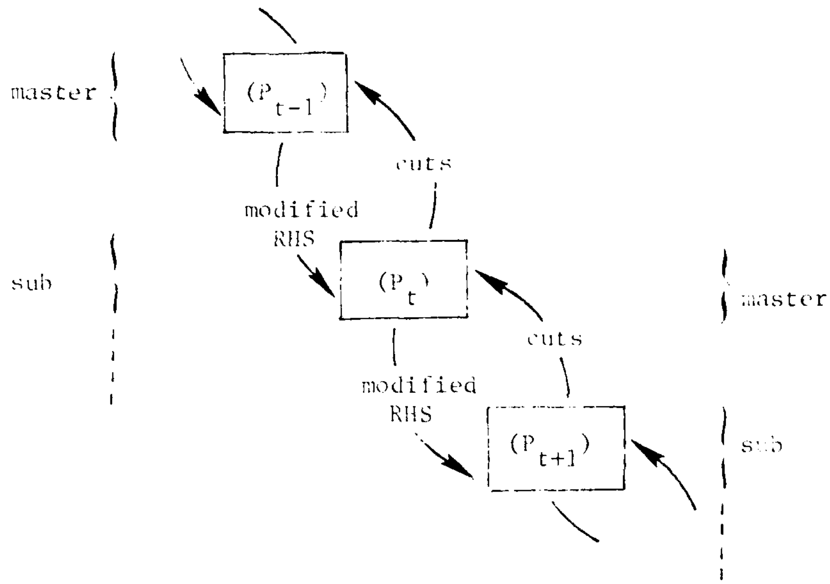
The dual of (D_t) is

$$
\begin{aligned}
(P_t) \quad \text{minimize} \quad & z_t = c_t x_t + \theta_t \\
\text{subject to} \quad & A_t x_t = b_t + B_{t-1} x_{t-1}^0 & : \pi_t \\
& F_t x_t \geq f_t & : \sigma_t \\
& L_t x_t + e\theta_t \geq \ell_t & : \rho_t \\
& x_t \geq 0 \ .
\end{aligned}
$$

In these problems, $\bar{b}_t$ represents the original term b_t from (P) as modified by the inventory $B_{t-1} x_{t-1}^0$ supplied by the current period $t - 1$ solution x_{t-1}^0. The dependence of $\bar{b}_t$ on x_{t-1}^0 will usually be suppressed for clarity. The constraints $F_t x_t \geq f_t$ and $L_t x_t + e\theta_t \geq \ell_t$ are necessary conditions for a solution x_t to not lead to future infeasibilities or non-optimal solutions. Accordingly, they are called feasibility and look-ahead (optimality) cuts, respectively. The constraints $A_t x_t = b_t + B_{t-1}^0 x_{t-1}^0$ are called the body of the constraints.

The full master problem (D_1), which includes all possible proposals from the future, is equivalent to (P). Computationally, a restricted master is maintained at each period by including proposals as they are generated and occasionally purging old proposals which are no longer basic in (D_t). It will be clear from context which sets of proposals are indicated by (F_t, f_t) and (L_t, ℓ_t).

While nested decomposition of (D) produces the problems (D_t) , it is usually more instructive and more convenient computationally to work with their respective duals (P_t) . In general, the problem at period t acts as a subproblem with respect to periods $1, \ldots, t-1$, and as a restricted master problem with respect to periods $t+1, \ldots, T$. The communication between the problems (P_t) can be represented schematically:



Note that the form of the look-ahead cuts is such that they essentially modify the objective function in each (P_t) .

The act of solving one of the restricted master problems (P_t) is called a step. If, at some step, (P_t) is infeasible, a feasibility cut is generated and imposed on (P_{t-1}) . If a step terminates with a finite optimal solution to (P_t) , a modified right-hand side for (P_{t+1}) is generated. If in the latter case it is also found that

$$z_t^* > \theta_{t-1}^0 , \qquad (2)$$

where z_t^* is the new optimal objective for (P_t) and θ_{t-1}^0 corresponds to the most recent solution to (P_{t-1}), a look-ahead cut is generated which may be imposed on (P_{t-1}). If (P_t) is found to have a class of solutions with objective unbounded below, a solution x_t^0 and a homogeneous solution h_t^0 are generated in the usual fashion. The desired effect of providing (P_{t+1}) with a right-hand side of the form

$$b_{t+1} + B_t x_t^0 + \alpha B_t h_t^0 \quad , \quad \alpha \geq 0 \tag{3}$$

is achieved by introducing into (P_{t+1}) a surrogate activity, with level $\alpha \geq 0$, represented by the column

$$-B_t h_t^0 \tag{4}$$

with cost coefficient $c_t h_t^0$. If (P) has a class of solutions with objective unbounded below, eventually a ray indicating this will be generated in (P_T). Otherwise, a look-ahead cut will be generated in $(P_{t'})$, for some $t' > t$, which "cuts off" the ray successively in $(P_{t'-1})$, ..., (P_t).

A wide variety of computational strategies may be employed within the framework described above. There is freedom both in the order in which the problems (P_t) are solved and in when to pass information between problems in the form of cuts and modified right-hand sides. Computational experience has indicated that the rate of convergence of the algorithm can vary significantly when different strategies are employed. This experience has also indicated that in order to attain the ease of solution initially envisioned for this approach, signficant modifications to the algorithm described are necessary.

3. A Modified Nested Decomposition Approach

3.1 Passing Surrogate Columns Forward

Suppose, at some step in the course of executing the algorithm described in Section 2.3, a finite optimal solution to (P_t) is obtained. The optimal basis must include θ_t and, assuming A_t is of full rank, at least m_t of the variables x_t. Some slack variables corresponding to cuts which have been imposed may also be basic. Let $k_1 \geq 0$ and $k_2 \geq 1$ be the number of feasibility and look-ahead cuts, respectively, whose slack variables are not basic. Then $m_t + k_1 + k_2 - 1$ of the variables x_t are in the optimal basis. Ordinarily, the optimal solution x_t^0 is used to form a right-hand side

$$b_{t+1} + B_t x_t^0 \tag{5}$$

for the body of the constraints in (P_{t+1}).

A modification to this technique is outlined as follows. Let

$$k = k_1 + k_2 - 1 \tag{6}$$

be the number of "surplus" variables, and let β_t^0 be the optimal basis, excluding slacks from cuts. Partition

$$\beta_t^0 = LB_t \cup NB_t \cup \{\theta_t\} \quad , \tag{7}$$

where $|LB_t| = m_t$, $|NB_t| = k$, and $A_{LB_t}^{-1}$ exists, where A_{LB_t} is a convenient abuse of the proper notation $(A_t)_{.(LB_t)}$. The variables x_{LB_t} are called the local basis. Solve the body of the constraints in (P_t) versus the right-hand side $\bar{b}_t$ to yield the locally basic solution

$$(x_{LB_t})^0 = A_{LB_t}^{-1}\bar{b}_t \quad . \tag{8}$$

Represent the remaining basic variables in terms of the local basis,

$$Y_{NB_t} = A_{LB_t}^{-1}A_{NB_t} \quad , \tag{9}$$

so that

$$x_t^0 = \begin{pmatrix} x_{LB_t}^0 \\ x_{NB_t}^0 \\ 0 \end{pmatrix} = \begin{pmatrix} (x_{LB_t})^0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -Y_{NB_t} \\ I \\ 0 \end{pmatrix} x_{NB_t}^0 \quad . \tag{10}$$

Let

$$H_t = \begin{pmatrix} -Y_{NB_t} \\ I \\ 0 \end{pmatrix} \tag{11}$$

be the set of homogeneous solutions to the body of the constraints which are generated by the representations Y_{NB_t} . In contrast to (5), use the locally basic solution $(x_{LB_t})^0$ *to form a* right-hand side

$$b_{t+1} + B_t(x_{LB_t})^0 \tag{12}$$

for (P_{t+1}) . In addition, introduce into (P_{t+1}) a set of k surrogate activities, with levels $\alpha_{t+1} \geq 0$, represented by columns with

$$S_{t+1} = -B_tH_t \tag{13}$$

in the body of the constraints, zero coefficients in the cuts on (P_{t+1}), and cost coefficients $s_{t+1} = c_tH_t$. The modified problem is

$(\tilde{P}_{t+1})$

$$\begin{array}{llll} \text{minimize} & \tilde{z}_{t+1} = s_{t+1}\alpha_{t+1} + c_{t+1}x_{t+1} + \theta_{t+1} & & \\ \text{subject to} & S_{t+1}\alpha_{t+1} + A_{t+1}x_{t+1} & = \bar{b}_{t+1} & : \pi_{t+1} \\ & F_{t+1}x_{t+1} & \geq f_{t+1} & : \sigma_{t+1} \\ & L_{t+1}x_{t+1} + \theta_{t+1} & \geq \ell_{t+1} & : \rho_{t+1} \\ & \alpha_{t+1},\ x_{t+1} \geq 0 \quad . & & \end{array}$$

This structure, in essence, endows $(\tilde{P}_{t+1})$ with a right-hand side which is parametrized in terms of the k variables x_{NB_t} , and allows $(\tilde{P}_{t+1})$ to select the coefficients α_{t+1} of the parametrization. Setting $\alpha_{t+1} = x^0_{NB_t}$ corresponds to the right-hand side (5) obtained in the pure nested decomposition framework. Passing activities down the staircase and allowing α_{t+1} to vary from these values provides an avenue of implicit communication between $(\tilde{P}_{t+1})$ and $(\tilde{P}_t)$, ..., $(\tilde{P}_1)$ which decreases the number of steps needed to obtain an optimal coordination of the single-period problems.

There are two potential disadvantages inherent in this approach. First, it seems that forcing $(\tilde{P}_{t+1})$ to have some or all of the variables α_{t+1} in its basis would unacceptably limit the number of basic variables chosen from x_{t+1} . Second, the framework sketched above allows $(\tilde{P}_{t+1})$ to use the surrogate activities at any nonnegative levels. In order to satisfy the body of the constraints in $(\tilde{P}_t)$, the relationship

$$x_{LB_t}(\alpha_{t+1}) = (x_{LB_t})^0 - (Y_{NB_t})\alpha_{t+1} \tag{14}$$

must be maintained. Since $(\tilde{P}_t)$ may include surrogate activities inherited from $(\tilde{P}_{t-1})$, in general a parametrization of all variables which are locally basic in $(\tilde{P}_t), \ldots, (\tilde{P}_1)$ is obtained in terms of the surrogate activities received by $(\tilde{P}_{t+1})$. Clearly, nonnegative levels of α_{t+1} could cause the parametrized values of variables in earlier periods to have negative components. These two points are addressed in the next sections.

3.2 Degeneracy in Optimal Solutions Generated by Nested Decomposition of (D)

It is well-known that bases for (P) inherit the staircase structure of the general problem and exhibit a "surplus-shortage" property which generalizes the fact that a basis which contains $m_1 + k$ of the variables x_1 , with $k > 0$, must exhibit a "shortage" of k , relative to the number of remaining constraints, $\sum_{t=2}^{T} m_t$, in the number of basic variables selected from $x_2, \ldots, x_T$. Fourer [4] has developed a set of bounds on the magnitudes of the surpluses and shortages which each period of a basis may possess.

The following result describes a manifestation of this property in the setting of the algorithm developed in Section 2.3.

Theorem. Suppose an optimal coordination of the single-period problems (P_t) has been obtained. If $t < T$ and (P_t) has $k + 1$ cuts whose slack variables are nonbasic, then the basic solution to (P_{t+1}) has at least k degenerate variables.

Under the modification outlined in Section 3.1, surrogate columns passed from period t replace degenerate variables in the optimal basis in period $t + 1$. These degenerate variables may include basic slack variables for the cuts in period $t + 1$. The fact that the surrogate columns replace degenerate variables guarantees that no solutions of interest are excluded in $(\tilde{P}_{t+1})$.

3.3 Maintaining Feasibility of Locally Basic Solutions

As indicated at the end of Section 3.1, nonnegativity of α_{t+1} does not imply nonnegativity of the parametrized values of the variables which are locally basic in periods $1, \ldots, t$. Two general approaches to surmounting this difficulty are possible. First, the feasibility of earlier periods can be ignored during the optimization of the modified problem $(\tilde{P}_{t+1})$ and restored in a subsequent procedure which would minimize an appropriate infeasibility form. The alternative is to employ, while optimizing $(\tilde{P}_{t+1})$, an extended minimum-ratio test which follows the parametric variation of locally basic variables in earlier periods and indicates when such a variable blocks the increase of an incoming column in $(\tilde{P}_{t+1})$.

The latter approach is adopted here. The representation of an incoming column in $(\tilde{P}_{t+1})$ includes weights on any surrogate columns which are in the local basis. Using these weights and repeatedly applying (14) and (9) yields a representation of the incoming column in terms of the variables which are locally basic in periods $1, \ldots, t+1$. This representation is used to implement the extended minimum-ratio test. This scheme of local inverses linked by representations of surrogate columns can be viewed as a compact basis-inverse technique which maintains a nearly block-angular inverse of the columns which are locally basic in periods $1, \ldots, t+1$.

When the extended minimum-ratio test reveals that a variable in a period prior to $t+1$ blocks the increase of an incoming column in $(\tilde{P}_{t+1})$ several strategies may be employed. The choice in this work is to "shuffle" the structure of the surrogate columns by exchanging the roles of some of the activities which are and are not locally basic. This process is designed to take the blocking variable and pass it down to $(\tilde{P}_{t+1})$ as a surrogate column. Primal and dual solutions in each period are unchanged, and when optimization of $(\tilde{P}_{t+1})$ is resumed, the same incoming variable is blocked by this surrogate column. The indicated pivot in $(\tilde{P}_{t+1})$ is made and maintains the nonnegativity of locally basic solutions in periods prior to $t + 1$.

3.4 Computational Strategies and Further Work

Again, a wide variety of computational strategies may be employed within the framework described above. There is freedom in the order in which the problem $(\tilde{P}_t)$ are solved and in when to pass information between problems in the form of cuts, modified right-hand sides, and surrogate columns. The presence of the surrogate columns opens additional options, including variations of the "shuffle" described in Section 3.3.

Currently, computational experience is being obtained with a code written by Wittrock in Mathematical Programming Language at Stanford University. This language facilitates experimentation of the type needed at this stage of the work. The thrust of this experimentation is to devise computational strategies which tend to minimize the computational effort needed to obtain an optimal solution to (P). Since the manipulations of data structures, and the form and frequency of updates to the local inverses, depend heavily upon the computational strategies which are employed, decisions about these factors have not yet been made.

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