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ON (n-1)-WISE AND JOINT INDEPENDENCE AND NORMALITY OF n RANDOM VARIABLES: AN EXAMPLE*

by

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ON (n-1)-WISE AND JOINT INDEPENDENCE AND NORMALITY OF n RANDOM VARIABLES: AN EXAMPLE

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ABSTRACT

An example is given of a vector of n random variables such that any (n-1)-dimensional subvector consists of n-1 independent standard normal variables. The whole vector however is neither independent nor normal.

1. INTRODUCTION

When discussing stochastic independence in a course on probability theory, it is customary to give an example of three identically distributed random variables X, Y and Z which are pairwise independent but not mutually independent. As Driscoll (1978) has pointed out, the standard examples (Feller (1957); Gnedenko (1963); DeGroot (1975); Hogg and Craig (1970)) can be reduced to consideration of a random triple (X, Y, Z) which takes the values (0,0,0), (0,1,1), (1,0,1) and (1,1,0) each with probability one-fourth.

Driscoll gave a more interesting example: $\tilde{X}$, $\tilde{Y}$ independent each with the rectangular distribution on the unit interval and

$\tilde{Z} = \tilde{X} + \tilde{Y} \bmod 1$. This example also yielded a characterization of the rectangular distribution.

Our example shares with Driscoll's the fact of being more interesting than the standard ones and at the same time illustrates a point concerning the multi-dimensional normal distribution.

It is well known that the whole distribution of an n-dimensional normal vector $(X_1,X_2,\ldots,X_n)$ is determined if the distribution of each pair (X_i,X_j) is known. In a different context one of the authors (KJM) raised the question whether $(X_1,X_2,\ldots,X_n)$ is necessarily normal if all the pairs (X_i,X_j) are two-dimensional normal vectors. The following example shows that even joint normality of all (n-1)-tuples does not suffice.

2. THE EXAMPLE

Let $n \geq 3$ and let $(Y_1,Y_2,\ldots,Y_n)$ be a random vector of signs, i.e. with components +1 or -1 such that any particular sign vector $(y_1,y_2,\ldots,y_n)$ is taken with probability a if $\prod_{i=1}^{n} y_i = +1$ and with probability $b = 2^{-(n-1)} - a$ if $\prod_{i=1}^{n} y_i = -1$. Here $0 \leq a \leq 2^{-(n-1)}$.

Proposition: The random variables $Y_1,Y_2,\ldots,Y_n$ are (n-1)-wise independent. If $a \neq 2^{-n}$ they are not mutually independent.

Proof: Let $1 \leq k \leq n-1$. Any vector $(y_{i_1},y_{i_2},\ldots,y_{i_k})$ can then be extended in 2^{n-k-1} ways to a vector $(y_1,y_2,\ldots,y_n)$ with $\prod_{i=1}^{n} y_i = +1$ and in as many ways to one for which the product of its components is -1. Thus $P(Y_{i_1} = y_{i_1}, Y_{i_2} = y_{i_2},\ldots,Y_{i_k} = y_{i_k}) = 2^{n-k-1}a + 2^{n-k-1}b = 2^{-k}$ for all $k \leq n-1$. This is the (n-1)-wise independence. However the relation $P(Y_1 = +1, Y_2 = +1,\ldots,Y_n = +1) = a$ contradicts the total independence unless $a = 2^{-n}$.

Now let $Z_1,Z_2,\ldots,Z_n$ be standard normal variables mutually independent and independent of the random vector $(Y_1,Y_2,\ldots,Y_n)$

and define $X_i = Y_i|Z_i|$, $i = 1,\ldots,n$. Then clearly the X_i are again standard normal. Also the independence of the Z_i together with the proposition imply that $X_1,X_2,\ldots,X_n$ are $(n-1)$-wise independent. Thus any $(n-1)$-tuple out of $X_1,X_2,\ldots,X_n$ is also $(n-1)$-dimensional normal. However $P(X_1 > 0, X_2 > 0,\ldots,X_n > 0) = P(Y_1 = Y_2 =\ldots= Y_n = +1) = a$ which, if $a \neq 2^{-n}$, contradicts the mutual independence of $X_1,X_2,\ldots,X_n$ and thus also their joint normality, where mutual independence would be equivalent to all covariances being zero.

3. REMARKS

The example does in no way characterize the normal distribution. In fact we can replace the normal distribution of the Z_i by any other distribution symmetric around zero to obtain a similar example where all subvectors of $(Z_1,\ldots,Z_n)$ except $(Z_1,\ldots,Z_n)$ itself consist of mutually independent identically distributed random variables. With $n = 3$ and $a = 0$ the vector $2^{-1}(Y_1 + 1, Y_2 + 1, Y_3 + 1)$ is the random triple (X, Y, Z) mentioned in the introduction.

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BIBLIOGRAPHY

DeGroot, Morris H. (1975). *Probability and Statistics*. Addison-Wesley Publishing Co., Reading, Massachusetts.

Driscoll, Michael F. (1978). On pairwise and mutual independence: Characterizations of rectangular distributions. *Journal of the American Statistical Association*, 73, 432-3.

Feller, William (1957). *Introduction to Probability Theory and Its Applications*. 2nd ed., John Wiley & Sons, New York.

Gnedenko, B. V. (1963). *The Theory of Probability*. 2nd ed., Chelsea Publishing Co., New York.

Hogg, Robert V. and Craig, Allen T. (1970). *Introduction to Mathematical Statistics*. 3rd ed., The MacMillan Co., New York.

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