TECHNICAL REPORT ARLCB-TR-81031

NUMERICAL SOLUTION TO BEAM VIBRATIONS UNDER A MOVING COUPLE

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August 1981

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
LARGE CALIBER WEAPON SYSTEMS LABORATORY
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AMCMS No. 611102H600011

PRON No. 1A1283121A1A

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER ARLCB-TR-81031	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) NUMERICAL SOLUTION TO BEAM VIBRATIONS UNDER A MOVING COUPLE		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) J. J. WU		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS US Army Armament Research & Development Command Benet Weapons Laboratory, DRDAR-LCB-TL Watervliet, NY 12189		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS AMCMS No. 611102H600011 PRON No. 1A1283121A1A
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Armament Research & Development Command Large Caliber Weapon Systems Laboratory Dover, NJ 07801		12. REPORT DATE August 1981 13. NUMBER OF PAGES 29
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Presented at the U.S. Army Numerical Analysis and Computer Conference, Huntsville, AL, 26-27 Feb 81.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Moving Load
Beam Vibrations
Finite Element
Gun Dynamics
Projectile Eccentricity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The finite element solution formulation in time- and space-coordinates is extended to beam vibrations effected by a moving couple. This problem has direct application to gun motions analysis with an unbalanced moving projectile. The moving load, instead of being a time-dependent Dirac delta function as for the case of a moving concentrated force, is now the derivative of this Dirac delta function. This singular function does not present any difficulty due to

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20. ABSTRACT (CONT'D)

the variational process employed. This solution procedure is described together with results of beam motions subjected to a couple moving with various speeds.

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INTRODUCTION

In a previous report,[1] this writer presented a finite element-variational formulation which discretizes the spatial and time variable in the same manner. The method was applied to a problem of beam motion subjected to moving concentrated forces. Results were shown to be in excellent agreement with known solutions. This same formulation is now applied to the problem of a moving couple, i.e., a concentrated bending moment.

A recent investigation by S. H. Chu[2] on the interacting forces between a projectile and the cannon tube indicates that the couple produced by the eccentricity of the projectile as it moves down the tube may be of such a magnitude that its effect on the tube motion becomes significant. It is then important that the problem associated with moving moments can be analyzed adequately. The purpose of this note is to present the modification necessary to the previous formulations so that the solutions of a beam motion problem under a moving bending moment can be obtained routinely. Results of a cantilevered beam subjected to such a load are also present.

DIFFERENTIAL EQUATION AND NONDIMENSIONALIZATION

Consider a Euler-Bernoulli beam subjected to a moving couple M. The equation differential can be written as

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[1] J. J. Wu, "Beam Motions Under Moving Loads Solved by Finite Element Method Consistent in Spatial and Time Coordinates," ARLCB-TR-80046, USA ARRADCOM, Large Caliber Weapon Systems Laboratory, Benet Weapons Laboratory, Watervliet, NY, November 1980.

[2] S. H. Chu, "In-Bore Motion Analysis of the 155 mm XM712 Projectile When Fired in the M198 Howitzer," Proceedings of the Army Symposium on Solid Mechanics, AMMRC MS 80-4, Army Materials and Mechanics Research Center, Watertown, MA, pp. 270-288, 1980.

$$EI y'''' + \rho A \ddot{y} = -M\bar{\delta}'(x-\bar{x}) \tag{1}$$

where y(x,t) denotes the beam deflection as a function of spatial coordinate x and time t. E, I, A, ρ denote elastic modulus, second moment of inertia area and material density respectively. A dirac function is denoted by $\bar{\delta}$, $\bar{x} = \bar{x}(t)$ is the location of M, a prime (') denotes differentiation with respect to x and a dot (˙), differentiation with respect to t. Note that the right hand side of Eq. (1) has a dimension of force due to the fact that

$$\bar{\delta}'(x-\bar{x}) = \frac{d}{dx}[\bar{\delta}(x-\bar{x})]$$

and it has a dimension of $(\text{length})^{-1}$.

Introducing nondimensional quantities

$$\hat{y} = y/\ell \ , \quad \hat{x} = x/\ell \ , \quad \hat{t} \tag{2}$$

where ℓ is the length of the beam and T is a finite time, within $0 \leq t \leq T$, the problem is of interest, Eq. (1) can be written as

$$y'''' + \gamma^2 \ddot{y} = -Q\bar{\delta}'(x-\bar{x}) \tag{3}$$

The hats (ˆ) have been omitted in Eq. (3) and

$$\gamma = \frac{c}{T}$$

$$Q = \frac{M\ell}{EI} \tag{4}$$

with

$$c^2 = \frac{\rho A \ell^4}{EI}$$

Boundary conditions associated with Eqs. (1) or (2) will now be introduced in conjunction of a variational problem. Consider

$$\delta I = 0 \tag{5a}$$

with

$$I = \int_0^1 \int_0^1 [y''y^{*''} - \gamma^2 \dot{y}\dot{y}^* + Q\delta(x-\bar{x})y^*]dxdt$$

$$+ \int_0^1 dt\{k_1 y(0,t)y^*(0,t) + k_2 y'(0,t)y^{*'}(0,t)$$

$$+ k_3(y(1,t)y^*(1,t) + k_4 y'(1,t)y^{*'}(1,t)\}$$

$$+ \gamma^2 \int_0^1 dx\{k_5[y(x,0) - Y(x)]y^*(x,1)\} \tag{5b}$$

where $y^*(x,t)$ is the adjoint variable of $y(x,t)$. If one takes the first variation of I considering $y(x,t)$ to be fixed:

$$(\delta I)_{\delta y=0} = 0 \tag{5a'}$$

and consider δy^* to be completely arbitrary, it is easy to see that Eqs. (5) is equivalent to the differential Eq. (3) and the following boundary and initial conditions.

$$\left.\begin{aligned} y'''(0,t) + k_1 y(0,t) &= 0 \\ y''(0,t) - k_2 y'(0,t) &= 0 \\ y'''(1,t) - k_3 y(1,t) &= 0 \\ y''(1,t) + k_4 y'(1,t) &= 0 \end{aligned}\right\} \quad 0 \leq t \leq 1 \tag{6a}$$

and

$$\left.\begin{aligned} \dot{y}(x.0) &= 0 \\ \dot{y}(x,1) - k_5[y(x,0) - Y(x)] &= 0 \end{aligned}\right\} \quad 0 \leq x \leq 1 \tag{6b}$$

Taking appropriate values for k_1, k_2, k_3, and k_4, problems with a wide range of boundary conditions can be realized. The initial conditions in Eqs. (6b) are that the beam has zero initial velocity, and, if one takes k_5 to be ∞ (or larger number compared with unity),

$$y(x,0) = Y(x)$$

The meaning for cases where k_5 is not so need not be our concern here.

To derive the finite element matrix equations, one begins with Eq. (5a') and write

$$(\delta I)_{\delta y=0} = 0 \tag{7a}$$

$$= \int_0^1 \int_0^1 [y''\delta y^{*''} - \gamma^2 \dot{y}\delta \dot{y}^* + Q\delta'(x-\bar{x})\delta y^*]dxdt$$

$$+ \int_0^1 dt[k_1 y(0,t)\delta y^*(0,t) + k_2 y'(0,t)\delta y^{*'}(0,t)$$

$$+ k_3 y(1,t)\delta y^*(1,t) + k_4 y'(1,t)\delta y^{*'}(1,t)]$$

$$+ \int_0^1 dx[\gamma^2 k_5 y(x,0) - Y(x)]\delta y^*(x,1) \tag{7b}$$

Introducing element local variables

$$\begin{aligned} \xi &= \xi^{(i)} = Kx-i+1 \\ \eta &= \eta^{(i)} = Lt-j+1 \end{aligned} \tag{8a}$$

or

$$\begin{aligned} x &= \frac{1}{K}(\xi+i-1) \\ t &= \frac{1}{L}(\xi+j-1) \end{aligned} \tag{8b}$$

where K is the number of divisions in x and L, in t. (A typical grid scheme is shown in Figure 1). Equation (7b) can now be written as

$$\sum_{i=1}^{K} \sum_{j=1}^{L} \int_0^1 \int_0^1 [\frac{K^3}{L} y''(ij) \delta y^{*''}(ij) - \frac{\gamma^2 L}{K} y(ij) \delta y^*(ij)] d\xi d\eta$$

$$+ \sum_{j=1}^{L} \int_0^1 d\eta \; [\frac{k_1}{L} y(ij)(0,\eta) \delta y^*(ij)(0,\eta) + k_2 \frac{K^2}{L} y'(ij)(0,\eta) \delta y^{*'}(ij)(0,\eta)$$

$$+ \sum_{i=1}^{K} \int_0^1 \frac{d\xi}{K} [\gamma^2 k_5 (y(ij)(\xi,0) \delta y^*(ij)(\xi,1))]$$

$$= - \sum_{i=1}^{K} \sum_{j=1}^{L} \frac{Q}{L} \int_0^1 \int_0^1 \bar{\delta}'(x-\bar{x}) \delta y^*(ij)(\xi,\eta) d\xi d\eta$$

$$+ \sum_{i=1}^{K} \frac{\gamma^2 k_5}{K} \int_0^1 d\xi \; [Y(i)(\xi) \delta y^*(iL)(\xi,1)] \qquad (9)$$

The shape function vector is now introduced. Let

$$y(ij)(\xi,\eta) = \underset{\sim}{a}^T(\xi,\eta) Y(ij)$$

$$y^*(ij)(\xi,\eta) = \underset{\sim}{a}^T(\xi,\eta) \underset{\sim}{Y}^*(ij) = \underset{\sim}{Y}^{*T}(ij) \underset{\sim}{a}(\xi,\eta) \qquad (10)$$

Equation (9) then becomes

$$\sum_{i=1}^{K}\sum_{j=1}^{L} \delta \underset{\sim}{Y}^{*T}(ij) \left\{\frac{K^3}{L} \underset{\sim}{A} - \frac{\gamma^2 L}{K} \underset{\sim}{B}\right\} \underset{\sim}{Y}(ij)$$

$$+ \sum_{i=1}^{L} \delta \underset{\sim}{Y}^{*T}(1j) \left\{\frac{k_1}{L} \underset{\sim}{B}_1 + \frac{k_2 K^2}{L} \underset{\sim}{B}_2\right\} \underset{\sim}{Y}(1j)$$

$$+ \sum_{i=1}^{L} \delta \underset{\sim}{Y}^{*T}(Kj) \left\{\frac{k_3}{L} \underset{\sim}{B}_3 + \frac{k_4 K^2}{L} \underset{\sim}{B}_4\right\} \underset{\sim}{Y}(1j)$$

$$+ \sum_{i=1}^{K} \delta \underset{\sim}{Y}^{*T}(iL) \left\{\frac{\gamma^2 k_5}{K} \underset{\sim}{B}_5\right\} \underset{\sim}{Y}(iL)$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{L} \delta \underset{\sim}{Y}^{*T}(ij) \frac{Q}{L} F(ij) + \sum_{i=1}^{K} \delta \underset{\sim}{Y}^{*T}(iL) \frac{\gamma^2 k_5}{K} \underset{\sim}{G}(i) \tag{11}$$

where, as it can be seen easily, that

$$\underset{\sim}{A} = \int_0^1 \int_0^1 \underset{\sim}{a}_{,\xi\xi}\, \underset{\sim}{a}^T_{,\xi\xi}\, d\xi d\eta$$

$$\underset{\sim}{B} = \int_0^1 \int_0^1 \underset{\sim}{a}_{,\eta}\, \underset{\sim}{a}^T_{,\eta}\, d\xi d\eta$$

$$\underset{\sim}{B}_1 = \int_0^1 \underset{\sim}{a}(0,\eta)\underset{\sim}{a}^T(0,\eta)d\eta \quad , \quad \underset{\sim}{B}_2 = \int_0^1 \underset{\sim}{a}_{,\xi}(0,\eta)\underset{\sim}{a}^T_{,\xi}(0,\eta)d\eta$$

$$\underset{\sim}{B}_3 = \int_0^1 \underset{\sim}{a}(1,\eta)\underset{\sim}{a}^T(1,\eta)d\eta \quad , \quad \underset{\sim}{B}_4 = \int_0^1 \underset{\sim}{a}_{,\xi}(1,\eta)\underset{\sim}{a}^T_{,\xi}(1,\eta)d\eta \tag{12}$$

$$\underset{\sim}{B}_5 = \int_0^1 \underset{\sim}{a}(\xi,1)\underset{\sim}{a}^T(\xi,0)d\xi$$

and

$$\underset{\sim}{F}(ij) = \int_0^1 \int_0^1 a(\xi,\eta)\bar{\delta}'(ij)(\xi-\bar{\xi})d\xi d\eta \quad , \quad \underset{\sim}{G}(i) = \int_0^1 \underset{\sim}{a}(\xi,1)Y(i)(\xi)d\xi$$

where

$$\bar{\delta}'_{(ij)}(\xi-\bar{\xi}) = \frac{d}{dx}\,\bar{\delta}_{(ij)}(\xi-\bar{\xi})$$

is the local version of the function $\bar{\delta}(x-\bar{x})$ appeared in Eq. (9). The specific form of $\bar{\delta}(y)(\xi-\bar{\xi})$ will be described later in a paragraph prior to Eqs. (18).

Now Eq. (11) can be assembled in a global matrix equation

$$\delta \underset{\sim}{Y}^{*T}\, \underset{\sim}{K}\, \underset{\sim}{Y} = \delta \underset{\sim}{Y}^{*}\, \underset{\sim}{F} \tag{13}$$

By virtue of the fact that $\delta\underset{\sim}{Y}^*$ is not subjected to any constrained conditions, one has

$$\underset{\sim}{K}\, \underset{\sim}{Y} = \underset{\sim}{F} \tag{14}$$

which can be solved routinely. Numerical results of several problems in this class will be presented in a later section.

FORCE VECTOR DUE TO A MOVING COUPLE

We shall describe here the procedures involved to arrive at the force vector contributed by a moving couple. This force vector has appeared in Eq. (12) as

$$\underset{\sim}{F}_{(ij)} = \int_0^1 \int_0^1 \underset{\sim}{a}(\xi,\eta)\,\bar{\delta}'_{(ij)}(\xi-\bar{\xi})\,d\xi d\eta \tag{15a}$$

Perform integration-by-parts once. Equation (15a) can be written as

$$F_{(ij)} = -\int_0^1 \int_0^1 \underset{\sim}{a}_{,\xi}(\xi,\eta)\,\bar{\delta}_{(ij)}(\xi-\bar{\xi})\,d\xi d\eta \tag{15b}$$

The shape function $a(\xi,\eta)$ is a vector of 16 in dimension. In the present formulation we have chosen the form:

$$a_k(\xi,\eta) = \bar{b}_i(\xi)\bar{b}_j(\eta) \quad , \qquad \begin{matrix} k = 1,2,3,\ldots 16 \\ i,j = 1,2,3,4 \end{matrix} \tag{16a}$$

and

$$a_{k,\xi}(\xi,\eta) = \bar{b}_i'(\xi)\bar{b}_j(\xi) \tag{16b}$$

The relations between k and i,j are given in Table I. These are the consequences of the choice of the shape function such that $Y_{(ij)}$, the generalized coordinates of the (ij)th element, represent the displacement, slope, velocity, and angular velocity at the local nodal points. Thus

$$\bar{b}_i(\xi) = \sum_{p=1}^{4} \bar{b}_{ip}\xi^{p-1} \quad ; \quad \bar{b}_i'(\xi) = \sum_{p=1}^{4} \bar{b}'_{ip}\xi^{p-1} \tag{17}$$

The values of $\bar{b}_{ip}$ are given in Tables II and III.

TABLE I. RELATIONSHIP BETWEEN (i,j) AND k IN EQUATION (16)

k	(i,j)	k	(i,j)
1	(1,1)	9	(1,3)
2	(2,1)	10	(2,3)
3	(1,2)	11	(1,4)
4	(2,2)	12	(2,4)
5	(3,1)	13	(3,3)
6	(4,1)	14	(4,3)
7	(3,2)	15	(3,4)
8	(4,2)	16	(4,4)

TABLE II. VALUES OF $\bar{b}_{ip}$ IN EQUATION (17)

i \ p	1	2	3	4
1	1	0	-3	1
2	0	1	-2	1
3	0	0	3	-2
4	0	0	-1	1

TABLE III. VALUES OF $\bar{b}'_{ip}$ IN EQUATION (17)

i \ p	1	2	3	4
1	0	-6	6	0
2	1	-4	3	0
3	0	6	-6	0
4	0	-2	3	0

Now, let us consider $\bar{\delta}_{(ij)}(\xi-\bar{\xi})$. This "function" represents the effect of the Dirac delta function $\delta(x-\bar{x})$ on the (ij)th element. If the curve of travel $\bar{x} = \bar{x}(t)$ does not go through the element (i,j), $\bar{\delta}_{(ij)}(\xi-\bar{\xi}) = 0$. If it passes through that element, one has

$$\bar{\delta}_{(ij)}(\xi-\bar{\xi}) = \bar{\delta}(x-\bar{x}) = K\bar{\delta}(\xi-\bar{\xi}) \tag{18a}$$

with

$$\bar{\xi} = \bar{\xi}(\eta) \tag{18b}$$

The function $\bar{\xi}(\eta)$ is derived from $\bar{x} = \bar{x}(t)$. For example, if the force moves with a constant velocity, one has

$$\bar{x} = \bar{x}(t) = vt \tag{19a}$$

it follows from Eqs. (8) that

$$\bar{\xi} = \bar{\xi}(\eta) = -i+1 + \frac{vK}{L}(\eta+j-1) \tag{19b}$$

With Eqs. (16), (17), (18), and (19), one writes (15) as

$$F_{(ij)k} = K\int_0^1 \int_0^1 a_{k},\xi(\xi,\eta)\bar{\delta}(\xi-\bar{\xi})d\xi d\eta \tag{20a}$$

$$F_{(ij)k} = K\int_0^1 \int_0^1 \bar{b}'_{p}\bar{b}_{jq}\ \xi^{p-1}\eta^{q-1}\bar{\delta}(\xi-\bar{\xi})d\xi d\eta \tag{20b}$$

Equation (20) can then be evaluated easily once the exact form of $\bar{\xi}$ is written. For example, if $\bar{\xi} = \eta$, Eq. (20) reduces to

$$F_{(ij)k} = \sum_{p=1}^{4}\sum_{q=1}^{4} k\,\bar{b}'_{ip}\,\bar{b}_{iq}\int_0^1 \xi^{p+q-2}d\xi$$

$$= \sum_{p=1}^{4}\sum_{q=1}^{4} \frac{k\bar{b}'_{ip}\bar{b}_{jq}}{p+q-1} \tag{21}$$

TABLE IV. DEFLECTION $y(x,t)/\ell$ OF A CANTILEVERED BEAM UNDER A MOVING CONCENTRATED MOMENT ($T = 10^{10}$ sec.)

t/T \ x/ℓ	0	0.25	0.50	0.75	1.00
0.	0.	0.	0.	0.	0.
0.25	0.	.03125	.09375	0.15625	0.21875
0.50	0.	.03125	.12500	0.25000	0.37500
0.75	0.	.03125	.12500	0.28125	0.46875
1.00	0.	.03125	.12500	0.28125	0.50000

TABLE V. DEFLECTION $y'(x,t)/\ell$ OF A CANTILEVERED BEAM UNDER A MOVING CONCENTRATED MOMENT ($T = 10^{10}$ sec.)

t/T \ x/ℓ	0	0.25	0.50	0.75	1.00
0.	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	(0.0072)	0.2366	0.2481	0.2496	0.2500
0.50	(0.0021)	0.2481	0.4856	0.4981	0.5000
0.75	0.0005	0.2496	0.4981	0.7366	0.7500
1.00	0.0000	0.2505	0.5021	0.7572	1.0000

TABLE VI. DEFLECTION $y(x,t)/\ell$ OF A CANTILEVERED BEAM UNDER A MOVING CONCENTRATED MOMENT

(T = 10.0 sec.)

t/T \ x/ℓ	0.	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.25	0.0000	0.0051	0.0313	0.0564	0.0891	0.1134	0.1453	0.1700	0.2021
0.50	0.0000	-0.0023	0.0333	0.0484	0.1126	0.1488	0.2711	0.2560	0.3256
0.75	0.0000	-0.0319	0.0462	-0.0054	0.0966	0.0686	0.1920	0.1983	0.3320
1.00	0.0000	-0.4458	0.3255	-0.6155	0.1702	-0.7785	-0.0488	-0.8189	-0.0162

TABLE VII. DEFLECTION $y(x,t)/\ell$ OF A CANTILEVERED BEAM UNDER A MOVING CONCENTRATED MOMENT

(T = 1.0 sec.)

t/T \ x/ℓ	0.	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.25	0.0000	0.0028	0.0131	0.0238	0.0283	0.0277	0.0233	0.0168	0.0095
0.50	0.0000	0.0029	0.0148	0.0370	0.0709	0.1075	0.1396	0.1688	0.1969
0.75	0.0000	0.0052	0.0304	0.0776	0.1473	0.2307	0.3263	0.4253	0.5223
1.00	0.0000	0.0289	-0.0109	0.0734	0.2351	0.3871	0.5186	0.6218	0.7262

TABLE VIII. DEFLECTION $y(x,t)/\ell$ OF A CANTILEVERED BEAM UNDER A MOVING CONCENTRATED MOMENT

(T = 0.10 sec.)

$y(x,t)/\ell$ [x 10^{-1}]

t/T \ x/ℓ	0.	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.25	0.0000	0.0051	0.0445	0.0354	-0.0089	-0.0103	0.0037	0.0032	0.0058
0.50	0.0000	-0.0056	0.0260	0.0437	0.0970	0.0760	-0.0291	-0.0388	0.0558
0.75	0.0000	-0.0115	0.0564	-0.0019	0.0296	0.1196	0.1806	0.0559	-0.2438
1.00	0.0000	-0.1861	0.2165	-0.1210	0.2015	0.0879	0.0013	-0.0888	0.3299

NUMERICAL DEMONSTRATIONS

Some numerical results obtained will now be presented. Let us consider a cantilevered beam subjected to a unit moving couple with a constant velocity

$$v = \frac{\ell}{T}$$

As T varies from ∞ to 0, the velocity varies from 0 to ∞.

It will be helpful to compare v with some reference velocity which is a characteristic of the given beam. It is known that for a cantilevered beam, the first mode of vibration has a frequency (see, for example, reference 3)

$$f_1 = \frac{\omega}{2\pi} = \frac{1}{2\pi}\left(\frac{1.875^2}{c}\right) = \frac{0.560}{c} \quad \text{(cycles per seconds)}$$

and the period,

$$T_1 = 1.786\ c$$

where

$$c^2 = \frac{\rho A \ell^4}{EI}$$

Consider the vibration as standing waves. They travel at a speed

$$v_1 = 2\ell f_1 = \frac{1.12\ell}{c}$$

Hence, the relative velocity

$$\bar{v} = \frac{v}{v_1} = \frac{T_1}{2T} = 0.893\ \frac{c}{T}$$

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[3]K. N. Tong, Theory of Mechanical Vibration, John Wiley, New York, 1960, p. 257; p. 256.

We shall take $c = 1.0$ for the moving force problems. Thus, $f_1 = 0.560$ Hz. $T_1 = 1.786$ sec. and

$$\bar{v} = 0.893/T$$

Using grid schemes of 4x4 (i.e., four segments in spatial and four in time coordinates) and 8x4. Tables IV through VIII show the beam deflections (and slopes) as the concentrated moment $Q = 1.0$ moves from the left to the right end. Since we have defined T as the time required for the load to travel from one end to another, $t = 0.5T$, for example, indicates the point in time when the load is at the midspan of the beam.

In Tables IV and V, T is set to 10^{10} sec. which is extremely large compared with the beam characteristic time of $T_1 = 1.786$ sec. The solution should reduce to the static problem. This is certainly the case as shown in these two tables. These results are obtained using a grid scheme of 4x4.

For results shown in Tables VI through VIII an 8x4 grid scheme has been used. The beam deflections for T = 10, 1.0, and 0.1 seconds are shown in Table VI, VII, and VIII respectively.

Finally, these deflection curves are also plotted in Figures 2 through 10. From these figures and the tabulated results, one observes that while some of the results are extremely good, others are changing so rapidly with respect to time or space variable that an assessment on their accuracy is very difficult. Hence, further investigations on numerical convergence of these data is necessary.

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2. S. H. Chu, "In-Bore Motion Analysis of the 155 MM XM712 Projectile When Fired in the M198 Howitzer," Proceedings of the Army Symposium on Solid Mechanics, AMMRC MS 80-4, Army Materials and Mechanics Research Center, Watertown, MA, pp. 270-288, 1980.
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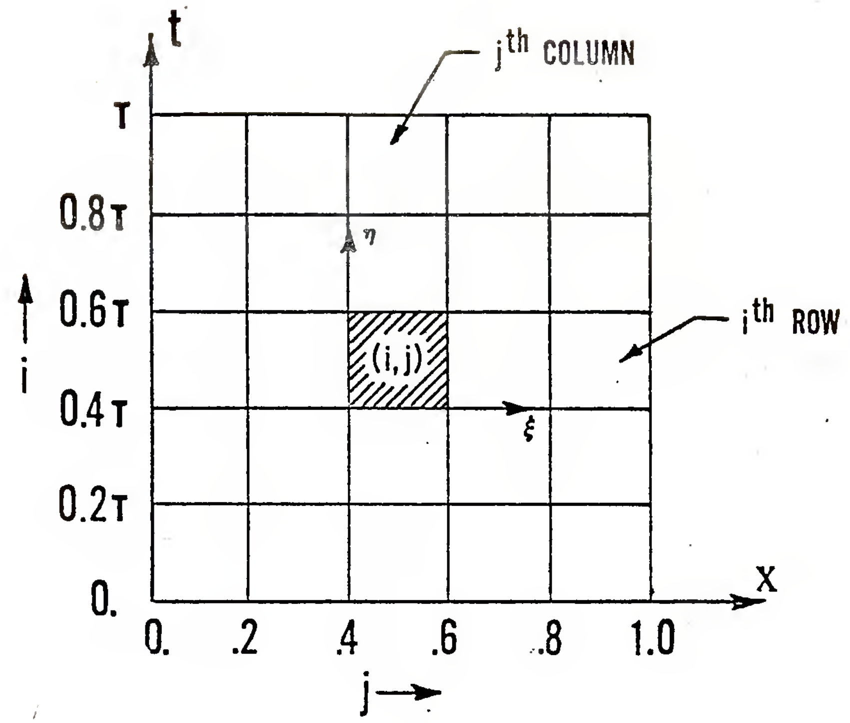

Figure 1. A Typical Finite Element Grid Scheme Showing the $(i,j)^{th}$ Element and the Global, Local Coordinates.

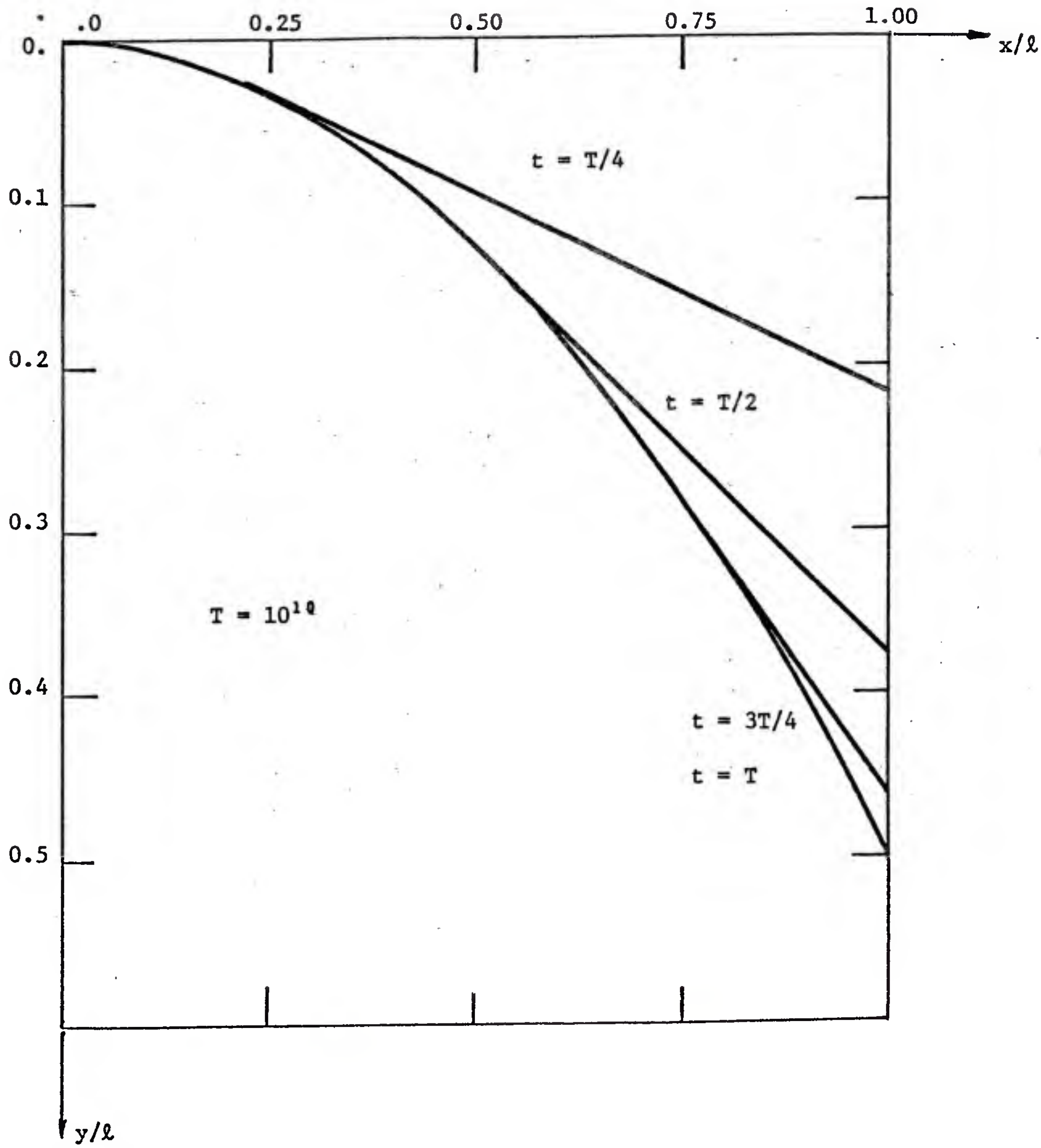

Figure 2. Deflection(s) of a Cantilevered Beam Under a Moving Couple.

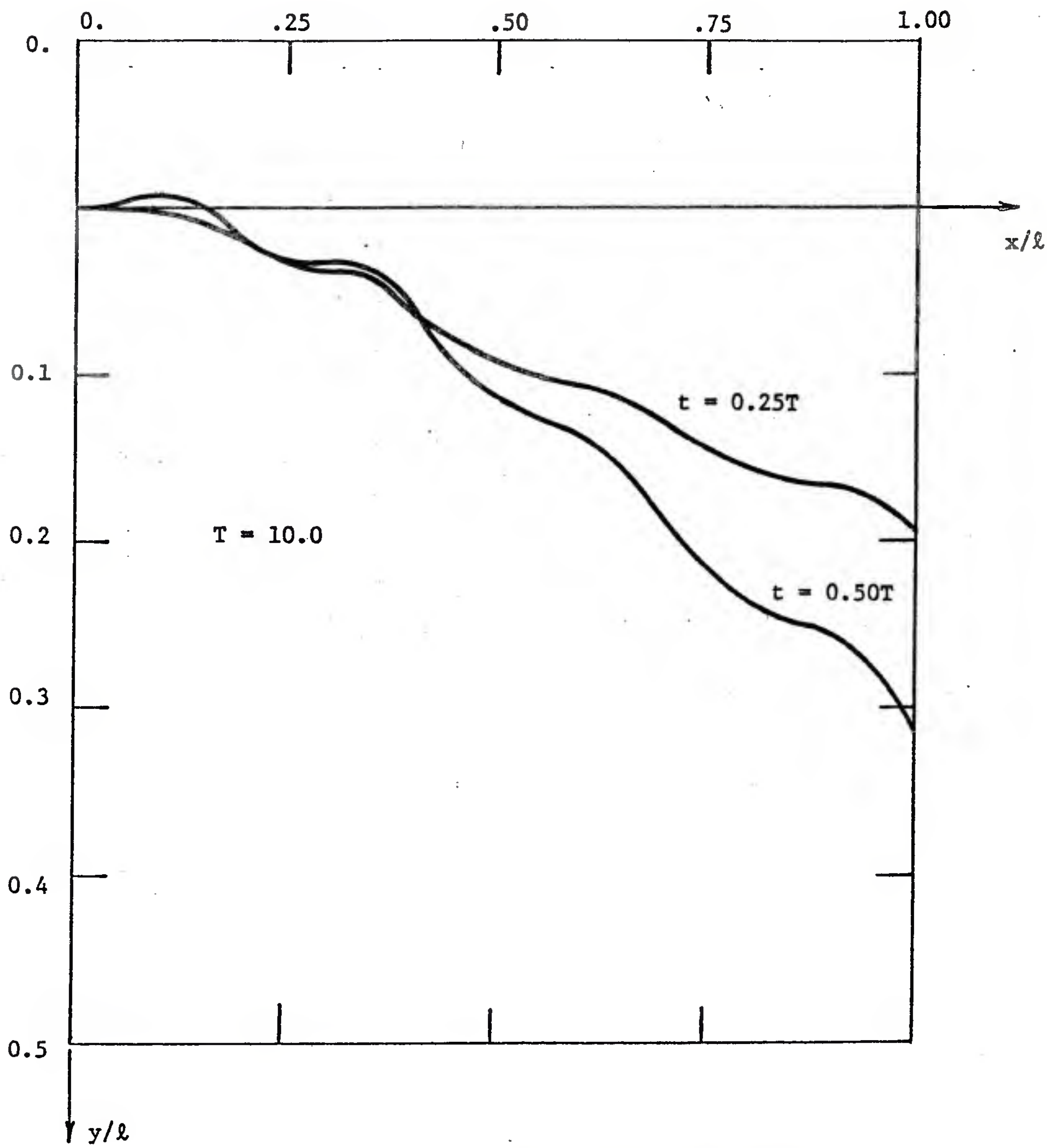

Figure 3. Deflection(s) of a Cantilevered Beam Under a Moving Couple.

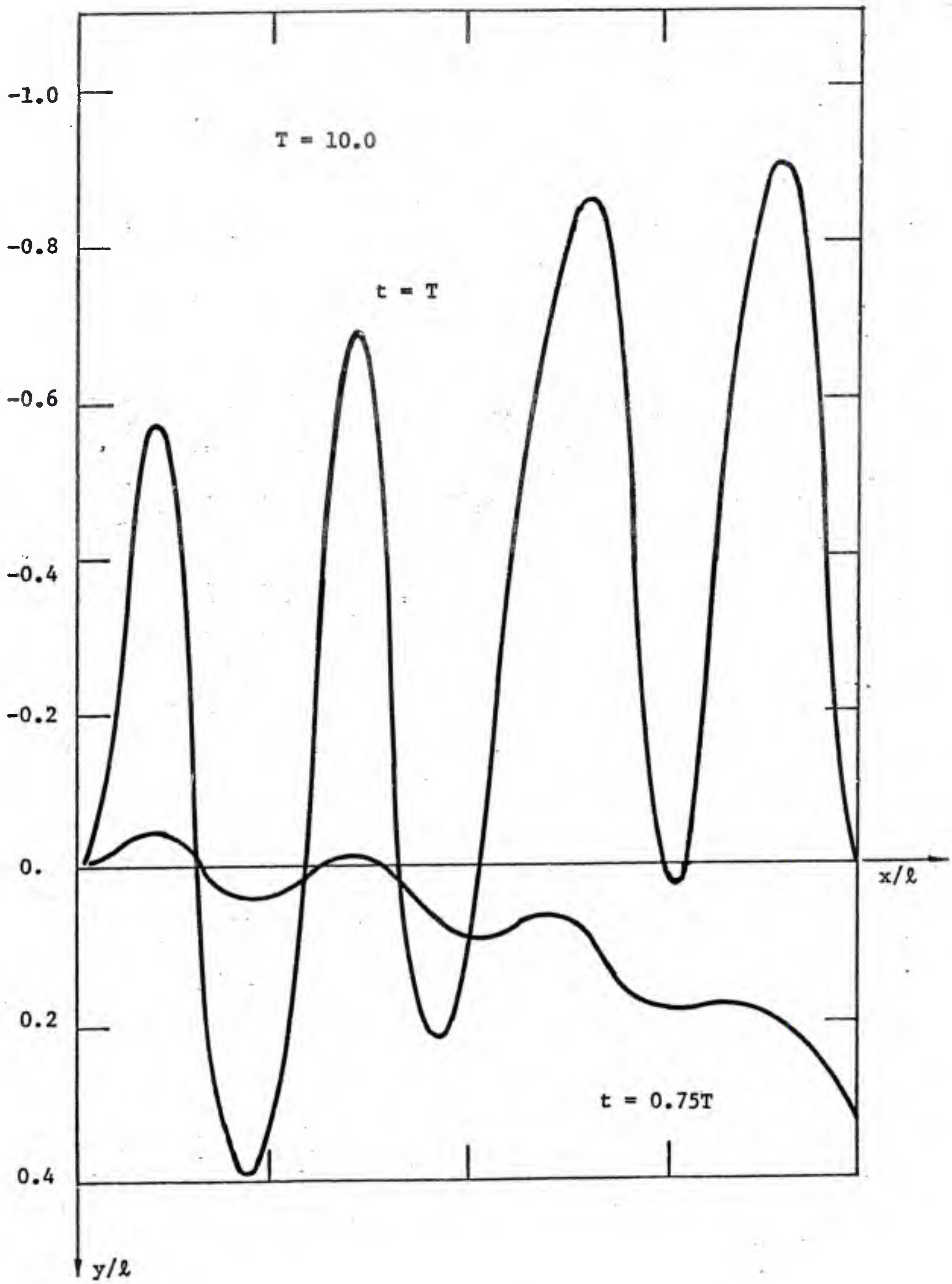

Figure 4. Deflection(s) of a Cantilevered Beam Under a Moving Couple.

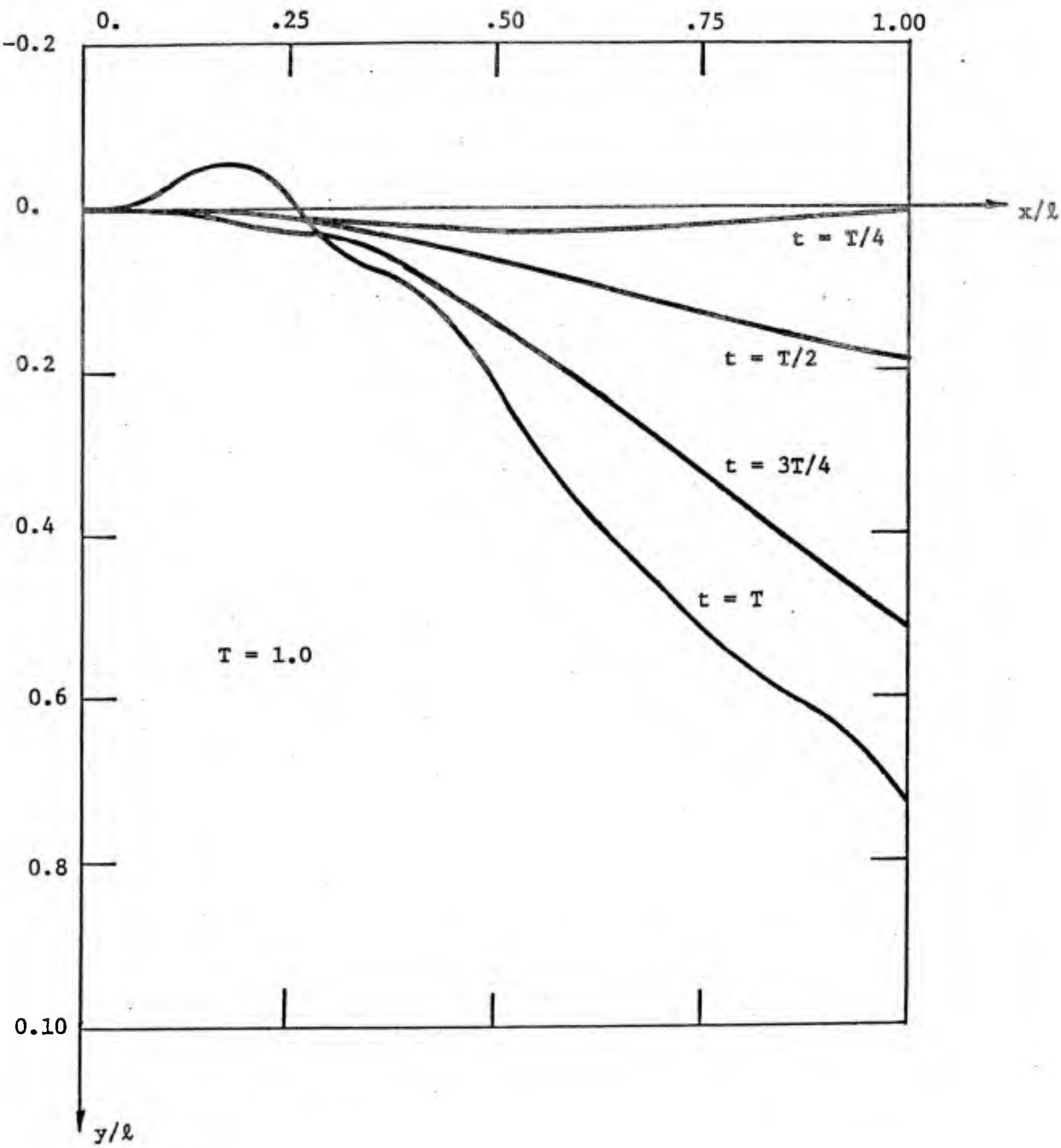

Figure 5. Deflection(s) of a Cantilevered Beam Under a Moving Couple.

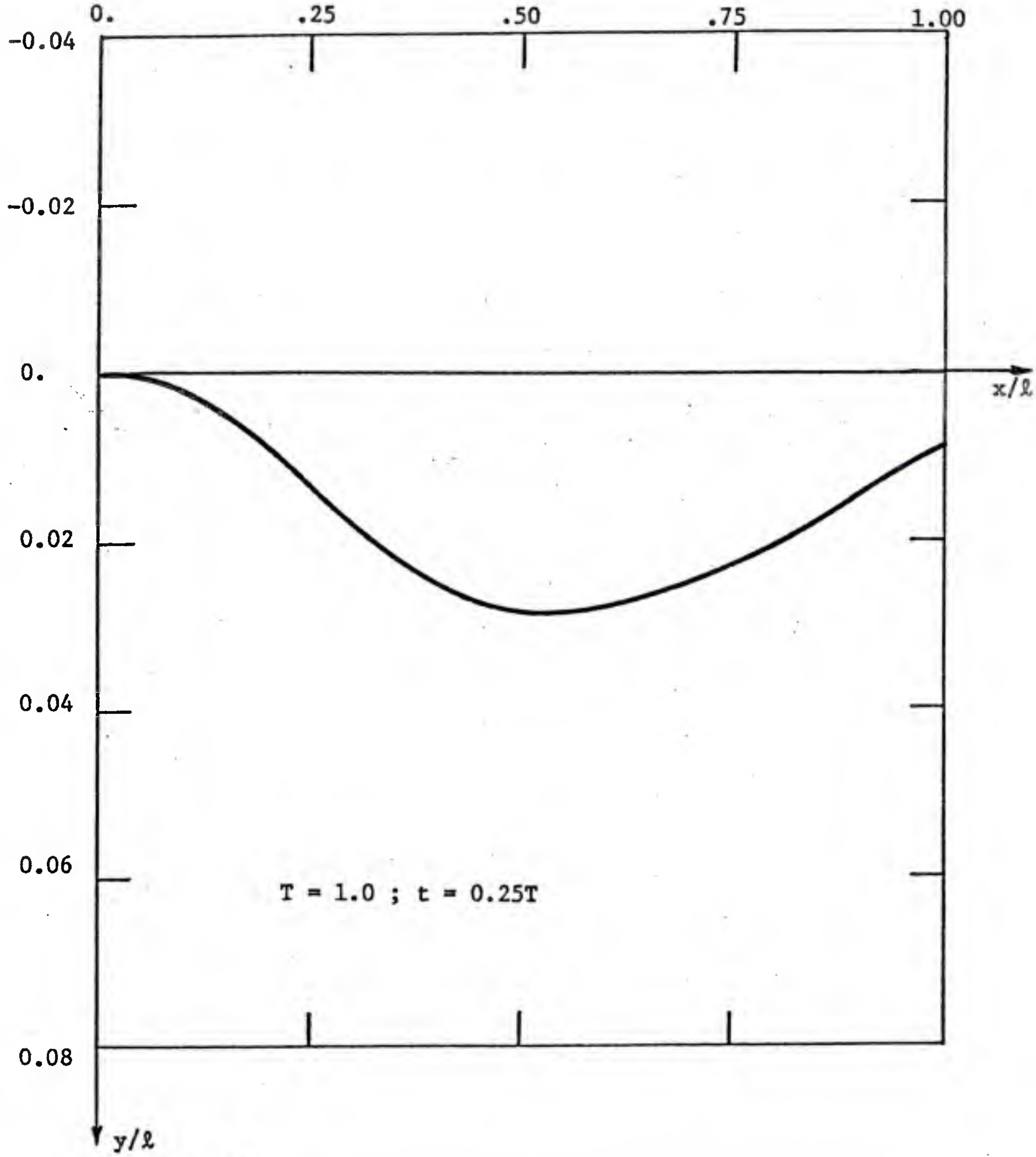

Figure 6. Deflection(s) of a Cantilevered Beam Under a Moving Couple.

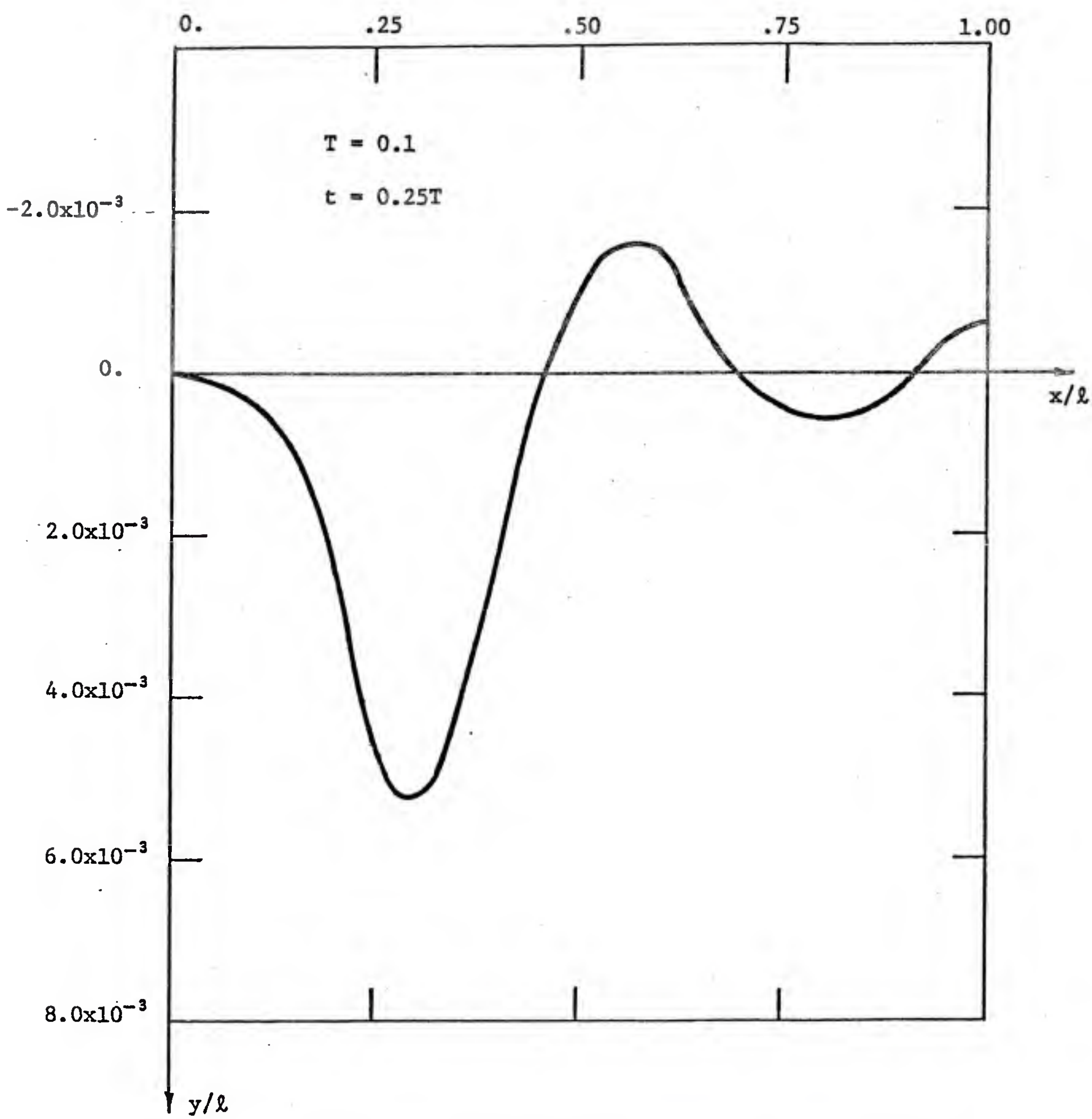

Figure 7. Deflection(s) of a Cantilevered Beam Under a Moving Couple.

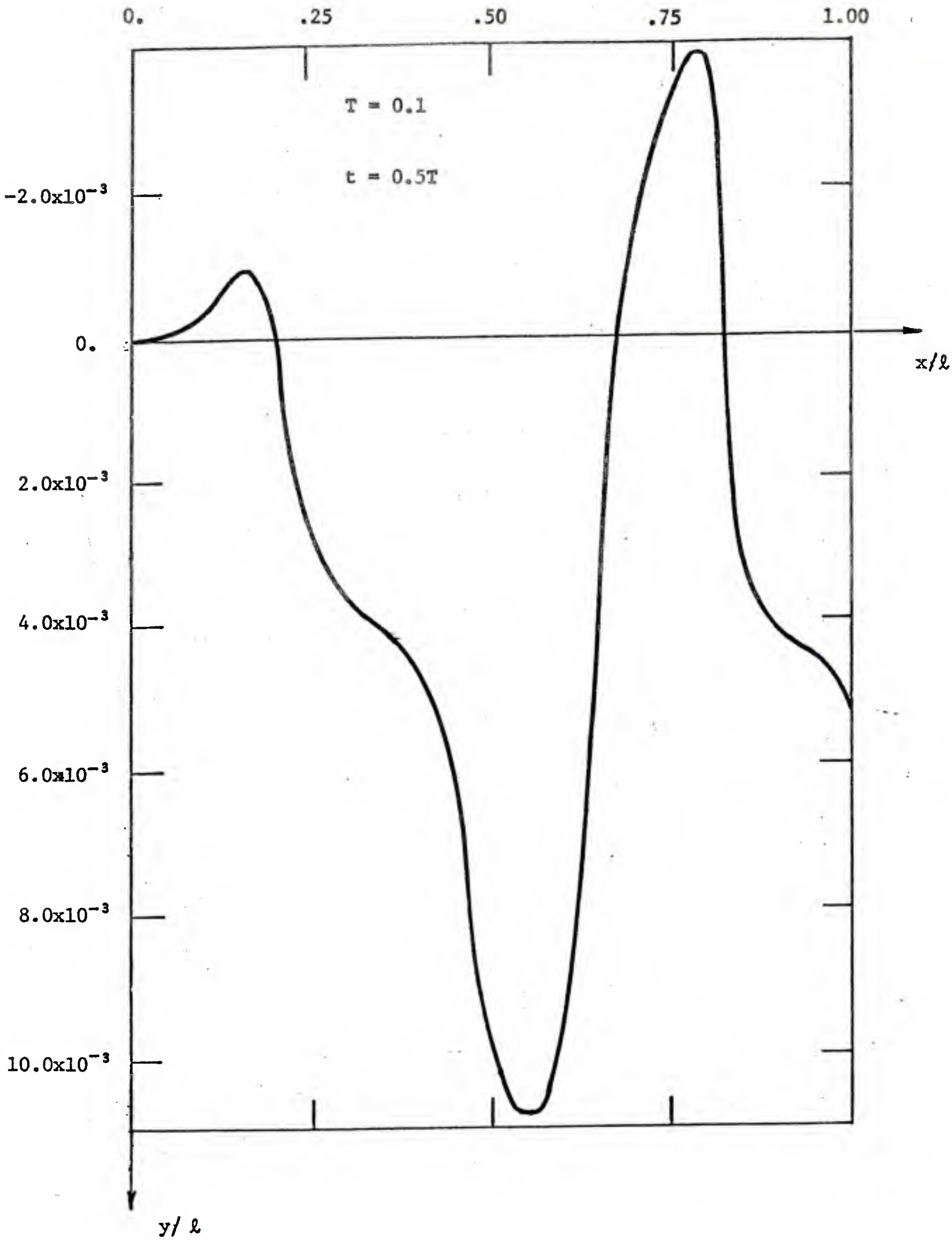

Figure 8. Deflection(s) of a Cantilevered Beam Under a Moving Couple.

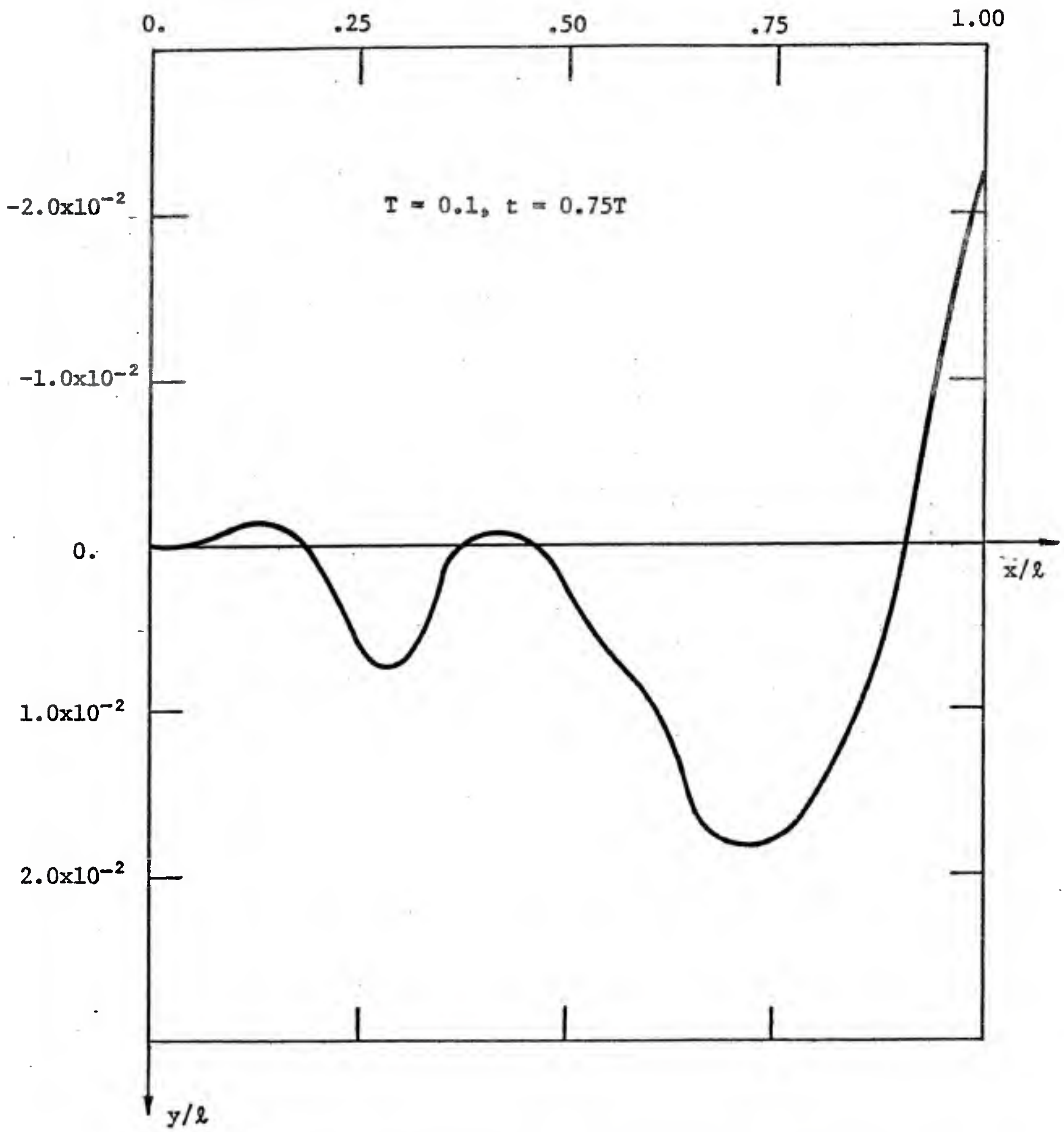

Figure 9. Deflection(s) of a Cantilevered Beam Under a Moving Couple.

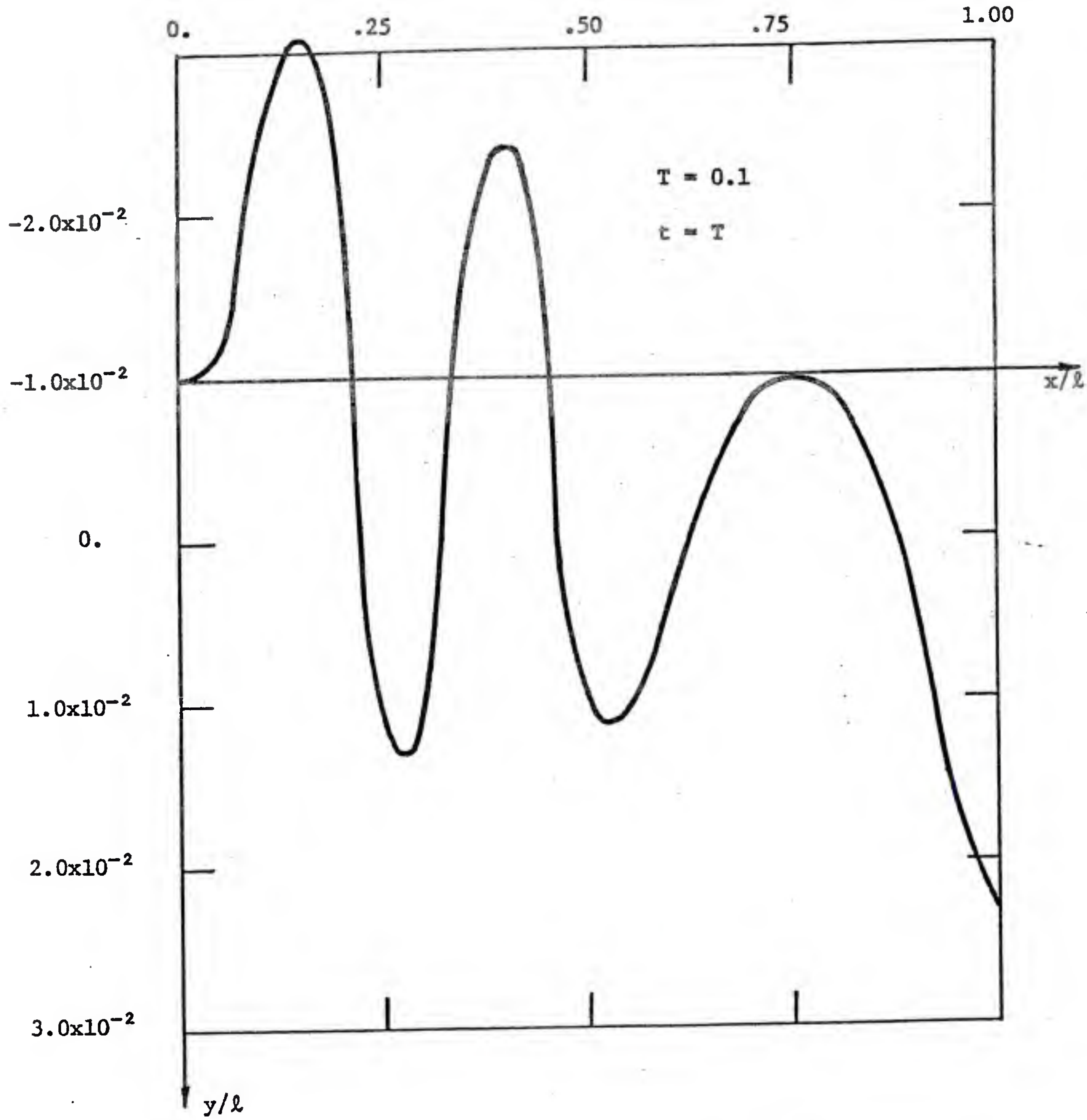

Figure 10. Deflection(s) of a Cantilevered Beam Under a Moving Couple.

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