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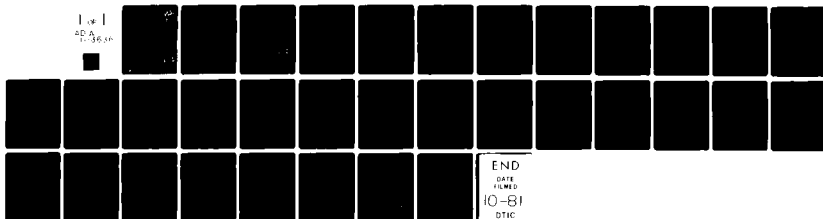
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PROPERTIES OF T^2 CHARTS IN
MONITORING PROCESS MEANS

RESEARCH REPORT NO. Q-10 ✓

D. R. JENSEN AND Y. V. HUI

JUNE 1981

U. S. ARMY RESEARCH OFFICE

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FIGURE 1. Geometric distributions of run lengths for the X^2 and T^2 charts monitoring at level $\alpha = 0.05$ for the case $p=2$, $n=6$, with $\delta=3.0$ and $\delta=6.75$.

PROPERTIES OF T^2 CHARTS IN
MONITORING PROCESS MEANS

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and State University

0. Abstract. Hotelling's T^2 and related charts are studied in monitoring the means of a multidimensional production process. The T^2 chart is shown to be optimal in a class of procedures in that it signals more quickly than other procedures in the class when the process is not in control. Nonstandard properties of the distributions of run lengths of these charts are studied when (i) certain parameters are estimated in a base period and modified procedures are followed using these estimates, (ii) the process is a drifting process, and (iii) the assumption of independent Gaussian vector observations is replaced by the assumption that observations are generated from a spherical process. For these cases stochastic bounds on the actual run-length distributions are given in terms of geometric distributions, and certain monotone properties of run lengths are established under drifting.

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Key words and phrases. Multicharacteristic quality control, Hotelling's T^2 charts, run-length distributions, drifting processes.

1. Introduction. Various control charts are available for monitoring the output of a production process. Shewhart's (1926) $\bar{X}$ chart is standard for monitoring the mean level of a single quality characteristic, and Hotelling's (1947) X^2 and T^2 charts are used for monitoring the means of several characteristics. In these charts successive values of a statistic are plotted against time, and a chart signals at level α that the process is not in control when the statistic exceeds a control limit c_α . Standard assumptions are that independent random samples of n observations each are taken in succession from a p -dimensional Gaussian process having the mean vector μ and the dispersion matrix Σ , and μ_0 is the value of μ when the process is in control. For the X^2 chart the statistics are

$$X_i^2 = n(\bar{Y}_i - \mu_0)' \Sigma^{-1} (\bar{Y}_i - \mu_0), \quad i = 1, 2, \dots \quad (1.1)$$

where $\bar{Y}_i$ is the vector of means from the i th sample, and the control limit is $c_\alpha = \chi_\alpha^2(p)$, the 100(1- α) percentile of the central chi-squared distribution having p degrees of freedom. For the T^2 chart the statistics are

$$T_i^2 = n(\bar{Y}_i - \mu_0)' S_i^{-1} (\bar{Y}_i - \mu_0), \quad i = 1, 2, \dots \quad (1.2)$$

where S_i is the sample dispersion matrix for use when Σ is unknown and $n > p$, and the control limit is $c_\alpha = T_\alpha^2(p, n-1)$, the 100(1- α) percentile of Hotelling's (1931) distribution having the parameters p and $n-1$. These procedures were developed further and applied by Jackson (1956, 1959), Jackson and Morris (1957), Jackson and Bradley (1961a, b), Ghare and Torgersen (1968), and Jackson and Mudholkar (1979), including diagnostic procedures for assigning probable causes when a chart signals that the monitored process is not in control.

Our purpose here is to study the $\bar{X}^2$ and T^2 charts and some modifications of these under standard and nonstandard assumptions. To be effective a chart should signal infrequently when the process is in control and more frequently as the process shifts from control. These properties are embodied in the distribution of the run length of a chart, i.e. the number of successive samples taken before the chart signals that the process is not in control. Run-length distributions are useful in comparing monitoring procedures, and their moments enter naturally into cost models for designing economic inspection policies. In Shewhart charts successive statistics typically are independent; if in addition the process is stationary, then the distribution of run lengths is geometric with parameter equal to the probability of exceeding the control limit on any sampling occasion. Under standard assumptions the run lengths of the $\bar{X}^2$ and T^2 charts are geometric, assuring that these charts eventually will signal with unit probability. Following the supporting materials of Section 2, we develop further properties of the standard T^2 charts in Section 3 including optimal properties.

Limitations are inherent in the standard $\bar{X}^2$ and T^2 charts, and these inhibit use of the charts. Process dispersion parameters frequently are unknown for use in the $\bar{X}^2$ charts, and the control value μ_0 may be unknown for use in either chart. Moreover, the T^2 chart requires that $n > p$ in each sample to ensure the invertibility of S_i , whereas in practice small but frequent samples routinely are taken. In the case of a single characteristic a common practice is to estimate such parameters in a base period of sufficient length to give reliable estimates, and to use these estimates in lieu of the unknown parameters. In the case of several

characteristics this leads to modified X^2 and T^2 charts as studied in Section 4. One consequence is that run lengths of these charts are not geometric owing to dependencies among the successive statistics.

Nonstandard properties of the X^2 and T^2 charts are examined further in Sections 5 and 6. Section 5 deals with drifting processes; their run lengths are not geometric because successive statistics are not distributed identically. Nonetheless, a basic monotonicity property of the run lengths is established and used to bound the actual run-length distributions in terms of geometric distributions. In Section 6 the Gaussian assumption is weakened to the case that observations are generated from a spherical process, and these charts are studied under the weakened assumptions.

2. Preliminaries. Let V be a finite-dimensional linear space; examples are the n -dimensional Euclidean space R^n and the space $F_{n \times m}$ of real $(n \times m)$ matrices. Of two random variables U and V , U is said to be stochastically larger than V if, for every t , $P(U > t) \geq P(V > t)$. This in turn implies that $E(U) \geq E(V)$ whenever the expected values are defined. Let $\{G(t; \alpha); \alpha \in (0, 1)\}$ be cumulative distribution functions (cdf's) of geometric distributions having argument t and parameter α ; then the family $\{G(t; \alpha); \alpha \in (0, 1)\}$ is stochastically decreasing in α . If $\mu(\cdot)$ and $\nu(\cdot)$ are two probability measures on R^N , $\mu(\cdot)$ is said to be more peaked about $\theta \in R^N$ than $\nu(\cdot)$ in the sense of Sherman (1955) if, for every compact convex set $E \subset R^N$ symmetric about θ under reflection, $\mu(E) \geq \nu(E)$.

A basic inequality is the following; a standard proof uses convexity and Jensen's inequality.

LEMMA 1. Let $\{X_0, X_1, \dots, X_t\}$ be mutually independent random elements

with values in $V_0 \times V^t$ such that $\{X_1, X_2, \dots, X_t\}$ are distributed identically. For any function $\phi: V_0 \times V \rightarrow R^1$ and any constant c ,

$$P[\phi(X_1, X_0) \leq c, \dots, \phi(X_t, X_0) \leq c] \geq \prod_{i=1}^t P[\phi(X_i, X_0) \leq c].$$

The available quality control procedures for variables are based almost exclusively on Gaussian assumptions. To study the effects of departures from normality we consider distributions having the weaker property of spherical symmetry, including such heavy-tailed distributions as the spherical Cauchy law and other spherical stable distributions. If $\underline{Y} \in R^N$ is a random element having the distribution $L(\underline{Y})$, then $L(\underline{Y})$ is said to be spherically symmetric if, for any $(N \times N)$ orthogonal matrix Q , $L(Q\underline{Y}) = L(\underline{Y})$. Properties of these and related distributions were considered by Kelker (1970). Let $S_N(\underline{\theta}, I_N)$ be the class of N -dimensional spherical distributions symmetric about $\underline{\theta} \in R^N$, i.e. if $L(\underline{Y}) \in S_N(\underline{\theta}, I_N)$, then $L(\underline{Y} - \underline{\theta})$ is spherical. If M is a linear subspace of R^N , a function $\phi: R^N \rightarrow R^k$ is said to be location-invariant with respect to M if, for every $\xi \in M$ and $\underline{y} \in R^N$, $\phi(\underline{y} + \xi) = \phi(\underline{y})$. An important invariance property of distributions derived from $S_N(\underline{\theta}, I_N)$ is the following; a proof is given in Jensen and Good (1981).

LEMMA 2. Let $L(\underline{Y})$ be spherically symmetric about $\underline{\theta} \in M$, and let $\phi: R^N \rightarrow R^k$ be location-invariant with respect to M and be homogeneous of degree zero. Then the distribution $L(\phi(\underline{Y}))$ on R^k is invariant for all distributions $L(\underline{Y})$ in the class $\{S_N(\underline{\theta}, I_N); \underline{\theta} \in M\}$.

Let N be countable and let $\{U(t); t \in N\}$ be a stochastic process with values in R^p such that, for each k and $\{t_1, t_2, \dots, t_k\} \in N$, the joint distribution of $\{U(t_1), \dots, U(t_k)\}$ considered as a distribution on R^{pk} is

spherical; then $\{U(t); t \in N\}$ is called a spherical process. A countable process is spherical if and only if it is a scale mixture of spherical Gaussian processes, i.e. if for each k and $\{t_1, t_2, \dots, t_k\} \in N$, the joint probability density function (pdf) of $\underline{U} = [U(t_1), \dots, U(t_k)]$ of order $(p \times k)$ is given by

$$f(\underline{U}) = (2\pi)^{-pk/2} \int_0^\infty r^{-pk} \exp(-\text{tr } \underline{U}' \underline{U} / 2r^2) dH(r) \quad (2.1)$$

with $H(\cdot)$ a mixing distribution on $(0, \infty)$; cf. Hartman and Wintner (1940). Various choices for $H(\cdot)$ yield different spherical processes, including the spherical Gaussian process.

In the context of T^2 charts, we assume that successive vector observations are generated throughout the monitoring period from a spherical process and thus are dependent except in the Gaussian case. In particular, partition N into disjoint subsets $\{N_0, N_1, \dots\}$ corresponding to successive sampling periods with N_0 a base period of length m and succeeding periods of length n . Let $\underline{u}(t)$ and $\underline{\Sigma}(t)$ be step functions on N , constant within samples with possible jumps between samples, such that $\underline{u}(t) \in R^p$ and $\underline{\Sigma}(t)$ $(p \times p)$ is positive definite. Our model for generating observations from a spherical process $\{U(t); t \in N\}$ with values in R^p is

$$\underline{Y}_{0j} = \underline{\Sigma}^{1/2}(t_j) \underline{U}(t_j) + \underline{u}(t_j); \quad t_j \in N_0, \quad j = 1, 2, \dots, m; \quad (2.2)$$

$$\underline{Y}_{1j} = \underline{\Sigma}^{1/2}(t_j) \underline{U}(t_j) + \underline{u}(t_j); \quad t_j \in N_1, \quad j = 1, 2, \dots, n; \quad (2.3)$$

where $i = 1, 2, \dots$. In monitoring location parameters the usual assumption is that scale parameters remain stationary, i.e. that $\underline{\Sigma}(t) \equiv \underline{\Sigma}$ for every $t \in N$. We return to this point later.

3. Properties of T^2 Charts. In this section we study Hotelling's (1947) T^2 charts under the standard validating assumptions. Specifically, we establish the propensity of these charts to signal when a process is not in control, using stochastic comparisons of run lengths.

Let $\underline{Y}_i = [Y_{i1}, Y_{i2}, \dots, Y_{in}]$ be the vector values observed during the i th sampling period, and let $T(\underline{Y}_i)$ be the statistic to be charted using a procedure for monitoring the means of the sampled process. In testing $H: \underline{\mu} = \underline{\mu}_0$ against $A: \underline{\mu} \neq \underline{\mu}_0$ using a single sample, it is known that the T^2 test (i) is invariant under nonsingular linear transformations of the data and thus is free of the arbitrary coordinates used to represent the observations, and (ii) depends on $\underline{Y}_i$ only through $\bar{\underline{Y}}_i$ and S_i , and its power depends on the parameters only through the noncentrality parameter $\lambda = n(\underline{\mu} - \underline{\mu}_0)' \Sigma^{-1} (\underline{\mu} - \underline{\mu}_0)$. Henceforth let τ_1 be the class of all procedures monitoring at level α using statistics $\{T(\underline{Y}_1), T(\underline{Y}_2), \dots\}$ having the property (i), and let τ_2 be the class of procedures monitoring at level α on each occasion and having property (ii). In the case of a single sample these classes provide α -level tests for $H: \underline{\mu} = \underline{\mu}_0$ against $A: \underline{\mu} \neq \underline{\mu}_0$, and it is widely known that the T^2 test is uniformly most powerful among tests in the classes τ_1 and τ_2 on a fixed sampling occasion (cf. Anderson (1958), Section 5.5). The notions of level and power do not carry over to the entire monitoring period, however, as control charts typically signal with unit probability even when the process is in control. Nonetheless, we next establish that the T^2 chart is optimal among procedures in the classes τ_1 and τ_2 in the strong sense that its run length is stochastically smallest at a fixed alternative $\underline{\mu} \neq \underline{\mu}_0$. The T^2 chart thus tends to signal more quickly than other procedures in these classes when the process is not in

control.

THEOREM 1. In monitoring the means of a p -dimensional Gaussian process, suppose the process is stationary with means $\underline{\mu} \neq \underline{\mu}_0$ and dispersion matrix $\underline{\Sigma}$. Of all invariant procedures monitoring at level α in the class τ_1 , and of all α -level monitoring procedures in the class τ_2 , the run length of the T^2 chart is stochastically smallest.

Proof. By stationarity, the mutual independence of $\{T(\underline{Y}_1), T(\underline{Y}_2), \dots\}$ and their identical distributions, it follows for all procedures in τ_1 and τ_2 that their run lengths are geometric with parameter equal to the probability of exceeding a control limit on any particular occasion. Because the T^2 test is optimal among α -level tests in τ_1 and τ_2 , the probability of exceeding the respective control limit at a fixed $\underline{\mu} \neq \underline{\mu}_0$ is at least as large for T^2 as for any other procedure in τ_1 or τ_2 . The proof is completed on recalling that the family $\{G(t; \beta); \beta \in (0, 1)\}$ of geometric distributions decreases stochastically in β , thus assuring that the run length of the T^2 chart is stochastically minimal.

We turn next to modifications of the standard charts which might be made in response to the practical difficulties of implementing those procedures.

4. Modified Charts. Both the standard X^2 and T^2 charts require that the control value $\underline{\mu}_0$ be known, and in addition the X^2 chart requires the value of $\underline{\Sigma}$. Often these parameters are unknown. In the one-dimensional case it is common practice to estimate such parameters in a base period when the process is in control, and to use these estimates in lieu of the unknown parameters. Carried over to the multidimensional case, this approach

yields modifications to the standard charts as follows. Let $\underline{Y}_0 = [\underline{Y}_{01}, \underline{Y}_{02}, \dots, \underline{Y}_{0m}] \in F_{p \times m}$ consist of sample observations in a base period of length m yielding one or both the estimates, $\bar{\underline{Y}}_0$ for $\underline{\mu}_0$ and $\underline{S}_0$ for $\underline{\Sigma}$. When $\underline{\Sigma}$ is unknown it is assumed that $m > p$ to ensure the invertibility of $\underline{S}_0$. The modifications of the standard charts to be considered here are set forth in Table 1, where the listed statistics are appropriate versions of (1.1) and (1.2) together with updated versions of their control limits. The latter are chosen to assure that the modified charts will continue to monitor at level α on each sampling occasion.

Note that the Type 2 and Type 3 modifications may be considered as substitutes for both X^2 and T^2 in (1.1) and (1.2). Practical advantages of the Type 2 chart over the standard T^2 chart are (i) only one matrix inversion is needed using the Type 2 modification, and (ii) sample sizes after the base period need not exceed p . This of course assumes stationarity of the process dispersion parameters, which may be checked periodically using the procedures of Hotelling (1947) or of Montgomery and Wadsworth (1972).

Run-length distributions of these modified charts generally are intractable; they are not geometric owing to dependencies among the successive statistics charted. However, the main result of this section is that all these distributions may be bounded below in terms of standard geometric distributions. That the modified charts eventually do signal is shown in the proof of the theorem.

THEOREM 2. Denote the typical modified statistic of Table 1 by $T(\underline{Y}_1, \underline{Y}_0)$ and its upper control limit by c_α ; suppose $\{\underline{Y}_1, \underline{Y}_2, \dots\}$ are observations from a stationary process; and let N be the run length of the modified chart. Then for every positive integer t ,

$$P(N > t) \geq 1 - G(t; \beta)$$

where β is the marginal probability $\beta = P[T(Y_1, Y_0) > c_\alpha]$.

Proof. Let M be geometric with distribution $G(t; \beta)$; fix t ; and for this value note that $P(M > t) = (1 - \beta)^t$. Similarly express $P(N > t)$ in terms of the finite-dimensional distribution of $\{T(Y_1, Y_0), \dots, T(Y_t, Y_0)\}$, and apply Lemma 1 to infer that

$$\begin{aligned} P(N > t) &= P[T(Y_1, Y_0) \leq c_\alpha, \dots, T(Y_t, Y_0) \leq c_\alpha] \\ &\geq \prod_{i=1}^t P[T(Y_i, Y_0) \leq c_\alpha] = (1 - \beta)^t = P(M > t). \end{aligned} \quad (4.1)$$

The proof using these finite-dimensional results will be complete if it can be shown that the modified charts eventually signal with unit probability. To see this, use the expression given for $P(N > t)$ and the conditional independence of $\{T(Y_1, Y_0), \dots, T(Y_t, Y_0)\}$ given Y_0 to write

$$P(N > t) = \int_{F_{p \times m}} \{P[T(Y_i, Y_0) \leq c_\alpha | Y_0]\}^t dF(Y_0) \quad (4.2)$$

where $F(\cdot)$ is the cdf of Y_0 and Y has the same distribution as Y_i for $i = 1, 2, \dots, t$. Clearly the function

$$H(t, Y_0) = \{P[T(Y, Y_0) \leq c_\alpha | Y_0]\}^t \quad (4.3)$$

satisfies $H(t, Y_0) < 1$ and $H(t, Y_0) \rightarrow 0$ as $t \rightarrow \infty$ for each fixed Y_0 . Moreover,

$\int_{F_{p \times m}} dF(Y_0) = 1$. From the dominated convergence theorem it follows that

$$\lim_{t \rightarrow \infty} P(N > t) = \int_{F_{p \times m}} \lim_{t \rightarrow \infty} H(t, Y_0) dF(Y_0) = 0. \quad (4.4)$$

This shows that the modified charts eventually signal with unit probability and thus that the proof holds for every $t > 0$.

We note that Theorem 2 holds under weakened assumptions. The role of Gaussian assumptions in that theorem is to supply the control limit c_α ; the theorem holds otherwise as long as $\{Y_0, Y_1, Y_2, \dots\}$ are mutually independent and $\{T(Y_1, Y_0), T(Y_2, Y_0), \dots\}$ are identically distributed. In particular, stationarity is not required for the Type 4 modification as long as the noncentrality parameters $\{\lambda_i = [nm/(n+m)](\mu_i - \mu_0)' \Sigma_i^{-1} (\mu_i - \mu_0); i = 1, 2, \dots\}$ are held constant.

5. Drifting Processes. Thus far we have supposed that the monitored process is stationary beyond a base period, whether or not it is in control. Combined with independence, stationarity leads to charts whose run lengths are geometric. This section develops properties of the X^2 and T^2 charts in monitoring the means of nonstationary multidimensional processes which we call drifting processes. The run lengths for drifting processes are not geometric even under independence because the successive statistics are not distributed identically. Such distributions are complicated further by their dependence on a countable sequence of parameters. Fortunately, intractable distributions of this type can be bounded by simpler distributions under suitable conditions on the sequence of parameters.

Let $\underline{\delta} = (\delta_1, \delta_2, \dots)$ be a sequence of parameters for a drifting process; let $F(t; \underline{\delta})$ be the distribution of run lengths for the corresponding chart; and let $\mathcal{D}$ be the set of all bounded sequences. Two sequences $\underline{\delta}$ and $\underline{\delta}^*$ in $\mathcal{D}$ are said to be ordered as $\underline{\delta} \leq \underline{\delta}^*$ whenever $\delta_i \leq \delta_i^*$ for all i . Under this partial ordering an element $\underline{\delta}_m \in \mathcal{D}_0$ is said to be minimal for $\mathcal{D}_0 \subset \mathcal{D}$, and $\underline{\delta}_M \in \mathcal{D}_0$ to be maximal for $\mathcal{D}_0$, if $\underline{\delta}_m \leq \underline{\delta} \leq \underline{\delta}_M$ for every $\underline{\delta} \in \mathcal{D}_0$. The family $\{F(t; \underline{\delta}); \underline{\delta} \in \mathcal{D}\}$ of distributions is said to be stochastically decreasing in $\underline{\delta}$ if, for any $\underline{\delta}$ and $\underline{\delta}^*$ in $\mathcal{D}$, the ordering $\underline{\delta} \leq \underline{\delta}^*$ implies the stochastic

ordering $F(t; \underline{\delta}^*) \geq F(t; \underline{\delta})$ for every t . For drifting processes this property captures the notion that a chart will signal more quickly as the process drifts further from control.

An important property of a stochastically decreasing family of run-length distributions is that an envelope of curves can be constructed for certain members of the family in terms of simpler distributions. Given a bounded sequence $\underline{\delta} = (\delta_1, \delta_2, \dots)$, let $\delta_m = \inf(\delta_1, \delta_2, \dots)$ and $\delta_M = \sup(\delta_1, \delta_2, \dots)$, and define $\underline{\delta}(m) = (\delta_m, \delta_m, \dots)$ and $\underline{\delta}(M) = (\delta_M, \delta_M, \dots)$. A basic result is the following.

LEMMA 3. Suppose the family $\{F(t; \underline{\delta}); \underline{\delta} \in \mathcal{D}\}$ of run-length distributions is stochastically decreasing in $\underline{\delta}$. Then for any $\underline{\delta} \in \mathcal{D}$, the inequalities

$$F(t; \underline{\delta}(M)) \geq F(t; \underline{\delta}) \geq F(t; \underline{\delta}(m))$$

hold for every $t > 0$. Moreover, these bounds hold for every $F(t; \underline{\delta}^*)$ such that $\underline{\delta}(m) \leq \underline{\delta}^* \leq \underline{\delta}(M)$.

Proof. The first conclusion is a consequence of the transparent ordering $\underline{\delta}(m) \leq \underline{\delta} \leq \underline{\delta}(M)$ and the fact that the family $\{F(t; \underline{\delta}); \underline{\delta} \in \mathcal{D}\}$ is stochastically decreasing in $\underline{\delta}$. The second conclusion follows on repeating these arguments for any $\underline{\delta}^*$ in $\underline{\delta}(m) \leq \underline{\delta}^* \leq \underline{\delta}(M)$.

It may be noted that $F(t; \underline{\delta})$ depends on the path $\underline{\delta} = (\delta_1, \delta_2, \dots)$ of the drifting process, whereas the upper and lower bounds do not. This fact simplifies a study of the bounding distributions, which are geometric in some important cases.

Lemma 3 supports bounds on distributions of run lengths for the standard X^2 and T^2 charts and their modifications under drifting. What is needed in each case is to identify the appropriate parameter sequence $\underline{\delta}$

and to show that $\{F(t; \underline{\delta}); \underline{\delta} \in \mathcal{D}\}$ is stochastically decreasing in $\underline{\delta}$.

To fix ideas, first consider the X^2 chart with run length N under a drifting process with parameters $\{(\mu_1, \Sigma_1), (\mu_2, \Sigma_2), \dots\}$. It follows that

$$P(N > t) = \prod_{i=1}^t P(X_{i-\alpha}^2 < c_\alpha) \quad (5.1)$$

where $L(X_1^2) = \chi^2(p, \delta_1)$, the noncentral chi-squared distribution with p degrees of freedom and the noncentrality parameter $\delta_1 = n(\mu_1 - \mu_0)' \Sigma_1^{-1} (\mu_1 - \mu_0)$. The parameter sequence for $F(t; \underline{\delta})$ accordingly is $\underline{\delta} = (\delta_1, \delta_2, \dots)$, and because $P(X_{i-\alpha}^2 < c_\alpha)$ is a decreasing function of δ_i , it follows that (i) the family $\{F(t; \underline{\delta}); \underline{\delta} \in \mathcal{D}\}$ is stochastically decreasing in $\underline{\delta}$, and (ii) $F(t; \underline{\delta}(m)) = G(t; \alpha(m))$ and $F(t; \underline{\delta}(M)) = G(t; \alpha(M))$, where $\alpha(m)$ and $\alpha(M)$ are the probabilities of exceeding the control limits on those occasions for which $\delta_i = \delta_m$ and $\delta_j = \delta_M$, respectively. Similar developments apply to the standard T^2 chart using the monotonicity of the noncentral T^2 distribution in its noncentrality parameter δ_i , leading to the following.

THEOREM 3. Let $F(t; \underline{\delta})$ with $\underline{\delta} \in \mathcal{D}$ be the run-length distribution of either the standard X^2 or T^2 chart under a drifting process with parameter sequence $\{\delta_i = n(\mu_i - \mu_0)' \Sigma_i^{-1} (\mu_i - \mu_0); i = 1, 2, \dots\}$. Then the stochastic bounds

$$G(t; \alpha(M)) \geq F(t; \underline{\delta}) \geq G(t; \alpha(m))$$

apply for every $t > 0$, where $\alpha(m)$ and $\alpha(M)$ are probabilities of exceeding the control limits when $\delta_i = \delta_m = \inf(\delta_1, \delta_2, \dots)$ and $\delta_j = \delta_M = \sup(\delta_1, \delta_2, \dots)$, respectively.

Further conclusions are evident. Because $\underline{0} = (0, 0, \dots) \in \mathcal{D}$ and is minimal for $\mathcal{D}$, it follows that the geometric distribution $G(t; \alpha)$ arising

when the process is in control stochastically dominates the run-length distribution for any drifting Gaussian process. Because the former signals with unit probability, it follows that the standard X^2 and T^2 charts eventually signal under drifting. The behavior of the standard charts may be compared under two drifting processes in terms of $\delta_i = n(\mu_i - \mu_0)' \Sigma_i^{-1} (\mu_i - \mu_0)$. When $\Sigma_i \equiv \Sigma$ for all i , the inequality $\delta_i \geq \delta_i^* = n(\mu_i^* - \mu_0)' \Sigma^{-1} (\mu_i^* - \mu_0)$ asserts that μ_i is more distant from μ_0 than μ_i^* in a non-Euclidean metric. The stochastic ordering $F(t; \delta) \geq F(t; \delta^*)$ assures that the chart tends to signal more quickly as the process drifts further from control. Even if a process is stationary in its means, i.e. $\mu_i \equiv \mu$ for all i , the charts benefit from successively tightened dispersion parameters through refinement of the process. Specifically, let $\{\Sigma_1, \Sigma_2, \dots\}$ and $\{\Sigma_1^*, \Sigma_2^*, \dots\}$ be the dispersion matrices of two processes such that $\Sigma_i^* - \Sigma_i$ is positive semidefinite for all i . It follows that $\delta_i \geq \delta_i^*$, i.e.

$$n(\mu - \mu_0)' \Sigma_i^{-1} (\mu - \mu_0) \geq n(\mu - \mu_0)' \Sigma_i^{*-1} (\mu - \mu_0), \quad (5.2)$$

hence that $F(t; \delta) \geq F(t; \delta^*)$ for every $t > 0$, and thus that the chart for the former process signals more frequently than the latter at a fixed $\mu \neq \mu_0$.

Corresponding results hold for modified charts as developed in Section 4. If N is the run length of a modified chart we have, in the notation of Section 4,

$$\begin{aligned} P(N > t) &= P(T(Y_1, Y_0) \leq c_\alpha, \dots, T(Y_t, Y_0) \leq c_\alpha) \\ &= \int_F \prod_{i=1}^t P[T(Y_i, Y_0) \leq c_\alpha | Y_0] dF(Y_0) \end{aligned} \quad (5.3)$$

for every positive integer t . For modified charts of Type 1 and Type 4 the family $\{F(t; \delta); \delta \in \mathcal{D}\}$ of run-length distributions is stochastically

decreasing in δ , conditionally using properties of X^2 and T^2 charts as in developments leading to Theorem 3, and unconditionally because the inequality holds point-wise at each fixed Y_0 . For Type 2 and Type 3 modifications the needed ordering is supplied in the following lemma under the assumption that the process dispersion parameters are stationary. Recall that these procedures otherwise would not be used.

LEMMA 4. Suppose the process dispersion parameters are stationary. Then the family $\{F(t; \delta); \delta \in \mathcal{D}\}$ of run-length distributions is stochastically decreasing in δ for both the modified T^2 charts of Type 2 and Type 3.

Proof. For Type 3 charts temporarily fix $\bar{Y}_0$ and note that these essentially are of Type 2 conditionally. For Type 2 charts we have

$$P(N > t) = P(T(\bar{Y}_1, \bar{Y}_0) \leq c_\alpha, \dots, T(\bar{Y}_t, \bar{Y}_0) \leq c_\alpha) \quad (5.4)$$

where $T(\bar{Y}_i, \bar{Y}_0)$, as defined in Table 1, may be rewritten as

$$T(\bar{Y}_i, \bar{Y}_0) = n(\bar{Y}_i - \bar{\mu}_0)' \Sigma^{-1} (\bar{Y}_i - \bar{\mu}_0) / w_i \quad (5.5)$$

with

$$w_i = (\bar{Y}_i - \bar{\mu}_0)' \Sigma^{-1} (\bar{Y}_i - \bar{\mu}_0) / (\bar{Y}_i - \bar{\mu}_0)' S_0^{-1} (\bar{Y}_i - \bar{\mu}_0) \quad (5.6)$$

for $i = 1, 2, \dots, t$. A standard result is that $L(w_i) = \chi^2_{(m-p+1)}$ independently of $\bar{Y}_i$. Write $\underline{w} = (w_1, w_2, \dots, w_t)$ and let $G(\underline{w})$ be their joint cdf. Now recalling the definition of X_i^2 , writing (5.5) as $T(\bar{Y}_i, \bar{Y}_0) = X_i^2 / w_i$, and noting that $\{T(\bar{Y}_1, \bar{Y}_0), \dots, T(\bar{Y}_t, \bar{Y}_0)\}$ are conditionally independent given $\underline{w}$, we evaluate (5.4) as

$$P(N > t) = \int_{R_+^t} \prod_{i=1}^t P(X_i^2 \leq w_i c_\alpha | \underline{w}) dG(\underline{w}) \quad (5.7)$$

where R_+^t is the positive orthant of R^t . Because X_1^2 conditionally has a non-central chi-squared distribution with noncentrality parameter $\delta_1 = n(\mu_1 - \mu_0)' \Sigma^{-1}(\mu_1 - \mu_0)$, the conditional probability $P(X_1^2 \leq w_1 c_\alpha | \underline{w})$ decreases as δ_1 increases, point-wise for each fixed w_1 , $i = 1, 2, \dots, t$. For Type 2 charts the desired property holds unconditionally on taking expectations using (5.7). For Type 3 charts these conditional results hold point-wise for each fixed $\bar{y}_0$ and thus unconditionally using a standard argument.

The foregoing results now may be combined with Lemma 3 to give the following.

THEOREM 4. Let $F(t; \underline{\delta})$ with $\underline{\delta} \in \mathcal{D}$ be the run-length distribution of a modified chart under a drifting process with parameters

(i) $\{\delta_i = n(\mu_i - \mu_0)' \Sigma_i^{-1}(\mu_i - \mu_0); i = 1, 2, \dots\}$ for Type 1 and Type 4 charts, and

(ii) $\{\delta_i = n(\mu_i - \mu_0)' \Sigma_i^{-1}(\mu_i - \mu_0); i = 1, 2, \dots\}$ for Type 2 and Type 3 charts. Then the bounds

$$F(t; \underline{\delta}(M)) \geq F(t; \underline{\delta}) \geq F(t; \underline{\delta}(m))$$

hold for every positive integer t , where $\underline{\delta}(m) = (\delta_m, \delta_m, \dots)$ and $\underline{\delta}(M) = (\delta_M, \delta_M, \dots)$ with $\delta_m = \inf(\delta_1, \delta_2, \dots)$ and $\delta_M = \sup(\delta_1, \delta_2, \dots)$.

It is of interest that $F(t; \underline{\delta}(m))$ and $F(t; \underline{\delta}(M))$ are precisely distributions of the types considered in Section 4. Theorem 2 accordingly yields the bounds

$$F(t; \underline{\delta}(m)) \leq G(t; \alpha(m)) \text{ and } F(t; \underline{\delta}(M)) \leq G(t; \alpha(M))$$

where $\alpha(m)$ and $\alpha(M)$ are the probabilities of exceeding the upper control

limit on those occasions for which $\delta_1 = \delta_m$ and $\delta_j = \delta_M$, respectively.

It was noted earlier that envelopes for the run-length distributions of certain drifting processes may be constructed using geometric distributions. To illustrate, envelopes are given in Figure 1 for the standard X^2 and T^2 charts, each monitoring at level $\alpha = 0.05$ with $p = 2$ and $n = 6$. The case $\delta = 0$ applies when the monitored process is in control. For each type of chart the curves labeled $\delta = 3.0$ and $\delta = 6.75$ represent stationary processes having the parameter sequences $\underline{\delta}_1 = (3.0, 3.0, \dots)$ and $\underline{\delta}_2 = (6.75, 6.75, \dots)$, respectively. These curves comprise an envelope containing the distributions of run lengths for all drifting processes satisfying

$$3.0 \leq \delta_m \leq \delta_1 \leq \delta_M \leq 6.75$$

using X^2 or T^2 charts as appropriate. Entries in Figure 1 were found on converting X^2 and T^2 into F-statistics and using P.C. Tang's tables as given in Graybill (1961) to determine the probability β of surpassing the α -level control limit on any particular occasion. Values of $G(t; \beta)$ then were computed directly for various values of t and were graphed.

It is informative to compare curves for the X^2 and T^2 charts at a fixed $\mu \neq \mu_0$. Then the run length of the X^2 chart is stochastically smaller than that of the T^2 chart. This property may be observed in Figure 1 for the cases treated there, and it can be shown analytically for any choice of p , n , and $\mu \neq \mu_0$. The greater efficiency of using X^2 when appropriate on any monitoring occasion thus translates into a greater propensity to signal that $\mu \neq \mu_0$ in comparison with the T^2 chart.

The optimality of the standard T^2 chart among procedures in τ_1 and τ_2 was established in Section 3 for stationary processes not in control. This

optimality carries over to drifting processes as well.

THEOREM 5. Consider a p -dimensional Gaussian process with drifting parameters $\{(\underline{\mu}_1, \underline{\Sigma}_1), (\underline{\mu}_2, \underline{\Sigma}_2), \dots\}$ on successive sampling occasions, to be monitored for means. Of all invariant procedures monitoring at level α in the class τ_1 , and of all α -level monitoring procedures in the class τ_2 , the run length of the T^2 chart is stochastically smallest.

Proof. Let $\{T(\underline{Y}_1), T(\underline{Y}_2), \dots\}$ be the statistics and c_α the control limit of any procedure in τ_1 or τ_2 , and let N be its run length. Because

$$P(N > t) = \prod_{i=1}^t P(T(\underline{Y}_i) \leq c_\alpha),$$

it is clear that (i) the run-length distribution $F(t; \underline{\beta})$ of the chart depends on the parameters $\underline{\beta} = (\beta_1, \beta_2, \dots)$ with $\{\beta_i = P(T(\underline{Y}_i) > c_\alpha); i = 1, 2, \dots\}$, and the family $\{F(t; \underline{\beta}); \underline{\beta} \in \mathcal{D}\}$ decreases stochastically in $\underline{\beta}$. Let $\underline{\beta}$ be the sequence of parameters for any procedure in τ_1 or τ_2 and let $\underline{\beta}^*$ be the corresponding sequence for the T^2 chart. Because on each sampling occasion T^2 is optimal among procedures in τ_1 and τ_2 , it follows that $\underline{\beta} \leq \underline{\beta}^*$. The conclusion of the theorem now follows from the stochastically decreasing character of $\{F(t; \underline{\beta}); \underline{\beta} \in \mathcal{D}\}$.

6. Monitoring Spherical Processes. Many findings of the foregoing sections carry over to observations generated from a spherical process $\{\underline{U}(t); t \in N\}$ as in Section 2. Consider the standard X^2 and T^2 charts using outcomes $\{\underline{Y}_{ij}(t)\}$ of the vector-valued process defined in (2.3) having the location parameters $\underline{\mu}(t)$ and the dispersion parameters $\underline{\Sigma}(t)$. For convenience we drop the arguments and index these parameters as $(\underline{\mu}_i, \underline{\Sigma}_i)$ for the i th monitoring period. The principal properties of these

charts under stationary and drifting processes induced from a spherical process are the following.

THEOREM 6. Let $c_\alpha(X)$ and $c_\alpha(T)$ be α -level control limits for the X^2 and T^2 charts for use with observations generated from a spherical process.

Let N_X and N_T be the run lengths of these charts having the distributions $F_X(t; \underline{\delta})$ and $F_T(t; \underline{\delta})$ with parameters $\{\delta_i = n(\mu_i - \mu_0)' \Sigma_i^{-1} (\mu_i - \mu_0); i = 1, 2, \dots\}$.

Then

- (i) $c_\alpha(T) = T_\alpha^2(p, n-1)$;
- (ii) $P(N_T \leq t) = G(t; \alpha)$ when $\mu(t) = \mu_0$ for all t ;
- (iii) $P(N_X > t) \geq 1 - G(t; \beta)$ when the process is stationary, where $\beta = P(X^2 > c_\alpha(X))$; and
- (iv) the families $\{F_X(t; \underline{\delta}); \underline{\delta} \in \mathcal{D}\}$ and $\{F_T(t; \underline{\delta}); \underline{\delta} \in \mathcal{D}\}$ are stochastically decreasing in $\underline{\delta} = (\delta_1, \delta_2, \dots)$.

Proof. (i) Starting with $\underline{Y}_1 = [Y_{11}, Y_{12}, \dots, Y_{1n}]$, make the location-scale changes $\underline{Z}_1 = [Z_{11}, Z_{12}, \dots, Z_{1n}]$ with $Z_{1j} = \Sigma_1^{-1/2} (Y_{1j} - \mu_0)$ and note that

$$T_1^2 = n(\bar{\underline{Y}}_1 - \mu_0)' \bar{S}_1^{-1} (\bar{\underline{Y}}_1 - \mu_0) = n \bar{\underline{Z}}_1' \bar{W}_1^{-1} \bar{\underline{Z}}_1 \quad (6.1)$$

where $\bar{\underline{Z}}_1$ and $\bar{W}_1$ respectively are the sample mean vector and the sample dispersion matrix computed from $\underline{Z}_1$. Moreover, $L(\underline{Z}_1)$ is spherical on $F_{p \times n}$.

Thus choosing M as $\{0\} \subset F_{p \times n}$ and noting that $\phi(\underline{Z}_1) = n \bar{\underline{Z}}_1' \bar{W}_1^{-1} \bar{\underline{Z}}_1$ is homogeneous of degree zero in its argument, we apply Lemma 2 with $\phi(\underline{Z}_1) = T_1^2$ to infer that $L(T_1^2) = T^2(p, n-1)$ for any underlying spherical process when $\mu(t) = \mu_0$ for all t . The conclusion $c_\alpha(T) = T_\alpha^2(p, n-1)$ follows immediately.

(ii) For each fixed t write

$$P(N_T > t) = P(T_1^2 \leq c_\alpha(T), \dots, T_t^2 \leq c_\alpha(T)); \quad (6.2)$$

define $\underline{Z} = [Z_1, Z_2, \dots, Z_t] \in (F_{p \times n})^t$ in terms of the standardized variables introduced in the proof of (i); and note that $L(\underline{Z})$ is spherical on $(F_{p \times n})^t$. Accordingly take M as $\{0\} \in (F_{p \times n})^t$; observe that $\phi(\underline{Z}) = (n\bar{Z}_1' W_1^{-1} \bar{Z}_1, \dots, n\bar{Z}_t' W_t^{-1} \bar{Z}_t)$ is homogeneous of degree zero in its argument; and apply Lemma 2 once more with $\phi(\underline{Z}) = (T_1^2, \dots, T_t^2)$ to conclude that $L(T_1^2, \dots, T_t^2)$ is identical to its normal-theory form when $\mu(t) \equiv \mu_0$. Conclusion (ii) now follows directly from (6.1) and (6.2) and results known for the Gaussian case.

(iii) From the representation (2.1) for the underlying spherical process, $P(N_X > t)$ may be written as

$$\begin{aligned} P(N_X > t) &= P(X_1^2 < c_\alpha(X), \dots, X_t^2 < c_\alpha(X)) \\ &= \int_0^\infty \prod_{i=1}^t P(X_i^2 < c_\alpha(X) | r) dH(r) \end{aligned} \quad (6.3)$$

because $\{X_1^2, \dots, X_t^2\}$ are conditionally independent given r . Under the hypothesis of stationarity (6.3) becomes

$$P(N_X > t) = \int_0^\infty [P(X^2 < c_\alpha(X) | r)]^t dH(r) \quad (6.4)$$

where X^2 has the typical distribution of $\{X_1^2, \dots, X_t^2\}$. Conclusion (iii) now follows from the arguments leading to Lemma 1.

(iv) Consider $\{F_X(t; \delta); \delta \in \mathcal{D}\}$. From (6.3) it is clear that the conditional distribution $L(r^{-2} X_1^2 | r)$ is $\chi^2(p, \delta_1(r))$ with the noncentrality parameter

$$\delta_1(r) = n(\mu_1 - \mu_0)' \Sigma_1^{-1} (\mu_1 - \mu_0) / r^2. \quad (6.5)$$

That the conditional run-length distributions $\{F_X(t; \delta | r); \delta \in \mathcal{D}\}$ are stochastically decreasing in δ follows as in the proof of Theorem 3 point-wise for each fixed r . The property holds unconditionally on applying a

standard argument. The case of $\{F_T(t; \delta); \delta \in \mathcal{D}\}$ is treated similarly.

Conclusion (i) assures that the standard T^2 chart with the normal-theory control limit c_α is appropriate for monitoring any spherical process after location and scale changes, at level α . However, the value $c_\alpha(X)$ required for monitoring at level α using the X^2 chart generally depends on the underlying spherical process. Conclusion (ii) of Theorem 6 establishes the familiar normal-theory run-length distribution of the T^2 chart for any spherical process when the process is in control.

Results for modified charts given in Sections 4 and 5 under Gaussian assumptions carry over to spherical processes. Proofs for these extensions follow the pattern of proof for conclusions (iii) and (iv) of Theorem 6. In view of the representation (2.1), conditional properties already have been established in the earlier sections point-wise for each r ; unconditional properties then yield to a standard argument. Further details are omitted in the interest of brevity.

Our final results deal with the relative performance of a given chart under alternative spherical processes. By a given chart is meant either the X^2 or T^2 chart with a fixed value for its control limit. Let $\mu(\cdot)$ and $\nu(\cdot)$ be finite-dimensional measures characterizing two zero-mean stochastic processes. If for every N , $\mu(\cdot)$ is more peaked about $0 \in \mathbb{R}^N$ than $\nu(\cdot)$ in the sense of Sherman (1955), then the μ -process is said to be more peaked about the zero function than the ν -process. In the case of spherical processes having the representation (2.1), it can be shown that of two such processes having the mixing distributions $H_\mu(r)$ and $H_\nu(r)$, the μ -process is more peaked about zero than the ν -process if and only if $H_\mu(r) \geq H_\nu(r)$ for each $r \geq 0$. Our principal result is the following.

THEOREM 7. Let $F_\mu(t; \delta)$ and $F_\nu(t; \delta)$ be the run-length distributions of a chart under drifting processes having the same parameters

$$\{\delta_i = n(\mu_i - \mu_0)' \Sigma_i^{-1} (\mu_i - \mu_0); i = 1, 2, \dots\}$$

and generated from spherical processes having the measures $\mu(\cdot)$ and $\nu(\cdot)$, respectively. If the μ -process is more peaked about zero than the ν -process, then

(i) $F_\nu(t; 0) \geq F_\mu(t; 0)$ for the X^2 chart when the processes are in control; and

(ii) $F_\mu(t; \delta) \geq F_\nu(t; \delta)$ for the T^2 chart.

Proof. (i) Let $c(X)$ be the control limit for the X^2 chart, and let

$$A_i = \{Z_i \in F_{p \times n} \mid X_i^2 = n \bar{Z}_i' \bar{Z}_i \leq c(X)\} \quad (6.6)$$

in the notation used in the proof of Theorem 6. On letting $A = A_1 \cap A_2 \cap \dots \cap A_t$, we infer from the first expression on the right of (6.3) that

$$[P(N_\mu > t) - P(N_\nu > t)] = [\mu(A) - \nu(A)]. \quad (6.7)$$

But A is a convex subset of $(F_{p \times n})^t$ symmetric about 0 under reflection.

Under the hypothesis that $\mu(\cdot)$ is more peaked about 0 than $\nu(\cdot)$, it follows that $[\mu(A) - \nu(A)] \geq 0$, which is equivalent to conclusion (i).

(ii) Consider the standard T^2 chart with control limit c_α . Let N be a typical run length and let N_μ and N_ν be the run lengths of the chart under the two processes. For any underlying spherical process note as in (6.3) that

$$\begin{aligned} P(N > t) &= P(T_1^2 \leq c_\alpha, \dots, T_t^2 \leq c_\alpha) \\ &= \int_0^\infty \prod_{i=1}^t P(T_i^2 \leq c_\alpha \mid r) dH(r) \end{aligned} \quad (6.8)$$

where $\{T_1^2, T_2^2, \dots, T_t^2\}$ are conditionally independent given r , and the conditional distribution $L(T_1^2 | r)$ is a noncentral Hotelling's (1931) distribution depending on r only through the noncentrality parameter $\delta_1(r)$ given by (6.5). Define

$$G(t; \underline{\delta}(r)) = \prod_{i=1}^t P(T_i^2 \leq c_\alpha | r) \quad (6.9)$$

with $\underline{\delta}(r) = (\delta_1, \delta_2, \dots, \delta_t)/r^2$. Applying the relation (6.8) twice and using (6.9), we evaluate the difference $D(t) = [P(N_v > t) - P(N_\mu > t)]$ as

$$D(t) = \int_0^\infty G(t; \underline{\delta}(r)) dH_v(r) - \int_0^\infty G(t; \underline{\delta}(r)) dH_\mu(r). \quad (6.10)$$

Under the hypothesis that $H_v(\cdot)$ dominates $H_\mu(\cdot)$ stochastically, there are increasing functions $\xi(v)$ and $\eta(v)$ with $\xi(v) \leq \eta(v)$ and a random variable V with distribution $H(\cdot)$ such that $L(\xi(V)) = H_\mu(\cdot)$ and $L(\eta(V)) = H_v(\cdot)$; cf. Lehmann (1959), page 73. It follows that

$$D(t) = \int_0^\infty [G(t; \underline{\delta}(\eta(v))) - G(t; \underline{\delta}(\xi(v)))] dH(v). \quad (6.11)$$

But $G(t; \underline{\delta}(r))$ is an increasing function of r ; because $\eta(v) \geq \xi(v)$ for each v , it follows that the integrand on the right of (6.11) is point-wise non-negative and thus the integral is nonnegative. This gives $P(N_v > t) \geq P(N_\mu > t)$, which is equivalent to conclusion (ii) of the theorem.

It was noted earlier that the control limit $c_\alpha(X)$ for monitoring at level α using the X^2 chart generally depends on the underlying spherical process. Suppose the control limit $c_\alpha^*(X)$ is chosen for some reference distribution such as the Gaussian distribution. Provided the processes are in control, the run-length for the reference distribution as a bound

stochastically dominates the run lengths for all spherical processes less peaked than the reference, and is dominated stochastically by the run lengths for all spherical processes more peaked than the reference distribution. These results are consequences of conclusion (i) of Theorem 7. Conclusion (ii) of that theorem assures that the T^2 chart will tend to signal more frequently under drifting, the more peaked the measure of the underlying process.

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TABLE 1. Statistics and their control limits in modified χ^2 and T^2 charts for monitoring the means of a multidimensional production process.

Modification Type	Statistic	Control Limit
1	$[nm/(n+m)] (\bar{Y}_i - \bar{Y}_0)' \Sigma^{-1} (\bar{Y}_i - \bar{Y}_0)$	$\chi^2_{\alpha}(p)$
2	$n(\bar{Y}_i - \mu_0)' S_0^{-1} (\bar{Y}_i - \mu_0)$	$T^2_{\alpha}(p, m-1)$
3	$[nm/(n+m)] (\bar{Y}_i - \bar{Y}_0)' S_0^{-1} (\bar{Y}_i - \bar{Y}_0)$	$T^2_{\alpha}(p, m-1)$
4	$[nm/(n+m)] (\bar{Y}_i - \bar{Y}_0)' S_1^{-1} (\bar{Y}_i - \bar{Y}_0)$	$T^2_{\alpha}(p, n-1)$

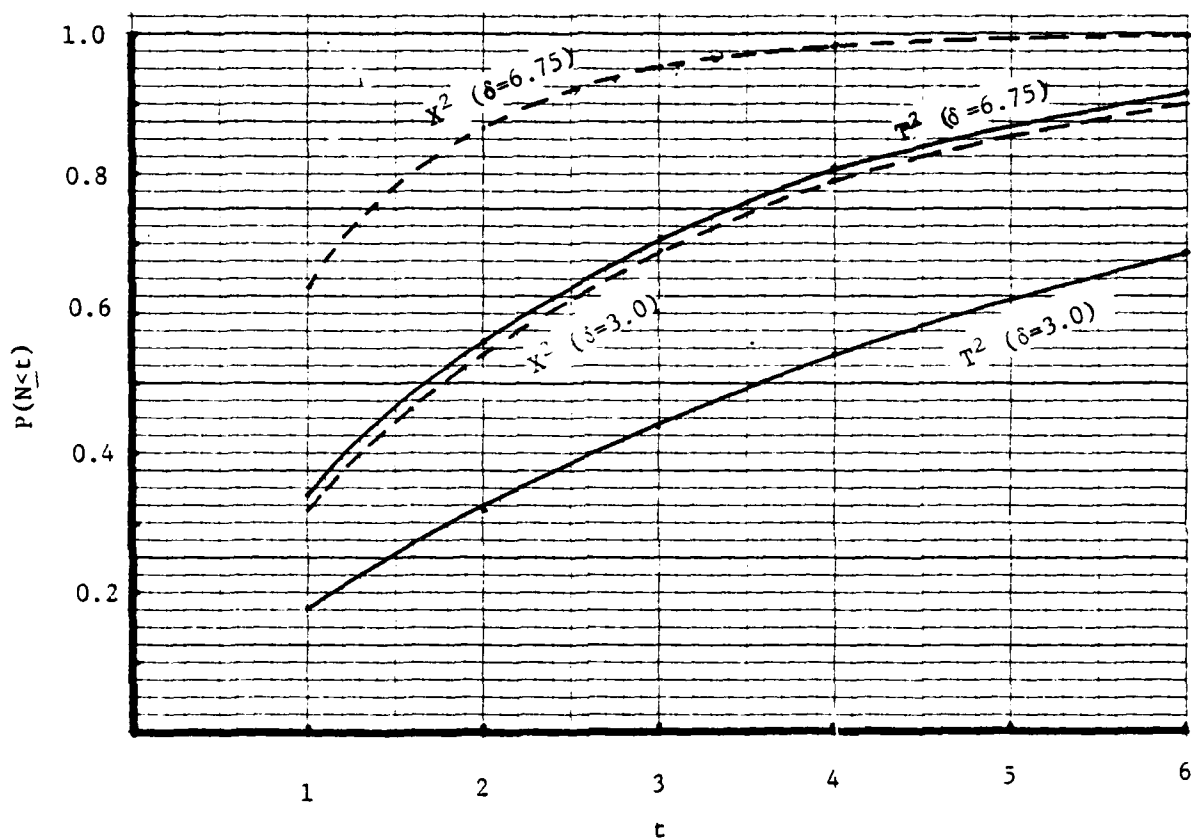


FIGURE 1. Geometric distributions of run lengths for the X^2 and T^2 charts monitoring at level $\alpha = 0.05$ for the cases $p=2$, $n=6$, with $\delta=3.0$ and $\delta=6.75$.

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20. (continued)

the distributions of run lengths of these charts are studied when (i) certain parameters are estimated in a base period and modified procedures are followed using these estimates, (ii) the process is a drifting process, and (iii) the assumption of independent Gaussian vector observations is replaced by the assumption that the observations are generated from a spherical process. For these cases stochastic bounds on the actual run-length distributions are given in terms of geometric distributions, and certain monotone properties of run lengths are established under drifting.

