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ON THE HAAR AND WALSH SYSTEMS ON A TRIANGLE.(U)
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ON THE HAAR AND WALSH SYSTEMS ON A TRIANGLE

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Y. Y. Feng* and D. X. Qi**

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ABSTRACT

In this paper we establish the Haar and Walsh systems on a triangle. These systems are complete in $L_2(\Delta)$. The uniform convergence of the Haar-Fourier series and the uniform convergence by group of the Walsh-Fourier series for any continuous function are proved.

AMS (MOS) Subject Classification: 41A15

Key Words: Haar function, Walsh function, Orthogonal system, Series expansion, Uniform convergence, Triangular domain

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SIGNIFICANCE AND EXPLANATION

A number of papers have been concerned with developing the theories of discontinuous orthonormal systems and their applications. In particular, the Haar and Walsh systems are presently the most important examples of nonsinusoidal functions, and have proved most useful in communication.

Some authors have studied the properties of approximation from the mathematical point of view. It seems interesting and helpful for both theory and practice to investigate the Haar and Walsh functions for a multivariate setting. In fact, many signals in communications and other functions are of several variables (for instance, TV signals have two space variables and the time variable). If the domain of definition of the system is tensor product, then the existing systems are readily extended to several variables.

The problem is how to construct an orthonormal system on a triangular domain in the plane, or more generally, on a simplex in n-dimensional space. This paper defines the Haar and Walsh system on a triangle domain, proves the orthogonality and completeness in L_2. Also the uniform convergence for the Haar-Fourier series, uniform convergence by group for the Walsh-Fourier series are studied. All of these results can be generalized easily to n dimensions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON THE HAAR AND WALSH SYSTEMS ON A TRIANGLE

Y. Y. Feng* and D. X. Qi**

1. Introduction

No doubt, it is interesting and useful to study multivariate Haar and Walsh functions either in theory or in practice. If we investigate on a domain which can be considered a Cartesian product, then the functions are readily extended to several variables from the one variable. Setting by the tensor product construct Harmuth has shown those kinds of multivariate systems in his book [5] and pointed out the applications in communication.

In this paper we attempt to focus on a triangle, or more generally on a simplex in n-dimensional space. We were unable to find any paper about it. Perhaps it puzzles some people temporarily.

The main contribution of this paper is to establish the Haar and Walsh system with two variables on a triangle. We prove their orthonormality and completeness in Hilbert space L_2. Moreover, the corresponding Haar-Fourier series and Walsh-Fourier series for any continuous function are uniformly convergent and convergent by group respectively.

It is easy to generalize these results to the n-dimensional simplex. For simplicity we will discuss only the two-dimensional triangle.

Now we should explain some preliminaries and notations.

The Haar functions on $[0,1]$ are defined as follows:

$$X_0(t) := 1 \quad \text{for} \quad 0 \leq t \leq 1 ,$$

and

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$$\chi_n^{(k)}(t) := \begin{cases} \sqrt{2^n}, & \text{for } \frac{2k-2}{2^{n+1}} < t < \frac{2k-1}{2^{n+1}}, \\ -\sqrt{2^n}, & \text{for } \frac{2k-1}{2^{n+1}} < t < \frac{2k}{2^{n+1}}, \\ 0, & \text{elsewhere in } [0,1] \end{cases} \tag{1.1}$$

$$k = 1,2,3,\ldots,2^n; \qquad n = 1,2,3,\ldots,\infty .$$

The Walsh functions on $[0,1]$ consist of the following ones:

$$W_0(t) := \quad 1 \quad \text{for} \quad 0 \leq t \leq 1 ,$$

$$W_1(t) := \begin{cases} 1 & \text{for } 0 \leq t < \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} < t \leq 1 , \end{cases} \tag{1.2}$$

$$W_{n+1}^{(2k-1)}(t) := \begin{cases} W_n^{(k)}(2t), & \text{for } 0 \leq t < \frac{1}{2}, \\ (-1)^{k+1} W_n^{(k)}(2t-1), & \text{for } \frac{1}{2} < t \leq 1 , \end{cases}$$

$$W_{n+1}^{(2k)}(t) := \begin{cases} W_n^{(k)}(2t), & \text{for } 0 \leq t < \frac{1}{2}, \\ (-1)^{k} W_n^{(k)}(2t-1), & \text{for } \frac{1}{2} < t \leq 1 , \end{cases}$$

$$k = 1,2,3,\ldots,2^n, \qquad n = 1,2,3,\ldots,\infty .$$

Some detailed investigation of the Haar and Walsh systems can be found in [1], [3], [5].

In order to generalize the Haar and Walsh systems to the two-dimensional case we should explain our representation in this paper. The Cartesian coordinates are not very convenient for triangular elements, and a special type of coordinate system called area coordinates should be used.

Referring to Figure 1 it is seen that the internal point P will divide the triangle ABC into three smaller triangles, and depending on the position of the point P, the area of each one of the triangles PAB, PBC, PCA can vary from zero to $|\Delta|$, which is the area of the triangle ABC. In other words, the ratios $\frac{a}{|\Delta|}$, $\frac{b}{|\Delta|}$ and $\frac{c}{|\Delta|}$ will take up any value between zero and unity. Here a, b, c are the area of triangles PBC, PCA, PAB respectively.

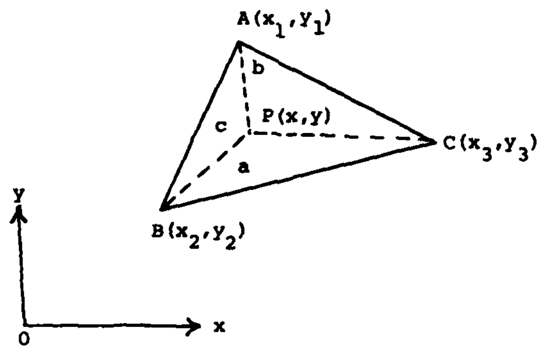


Figure 1

These ratios are called area coordinates, and they are defined by $\ell_1 := \frac{a}{|\Delta|}$, $\ell_2 := \frac{b}{|\Delta|}$, $\ell_3 := \frac{c}{|\Delta|}$.

It is easy to see that

$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} .$$

If two points P and Q are in two similar triangles respectively, and have the same area coordinates, then we denote them by $P \sim Q$.

2. An orthonormal sequence χ on a triangular domain

Suppose Δ (or ΔABC) is any triangle on a plane and $|\Delta| = 1$ is the area of the ΔABC. If D, E, F are midpoints of AB, BC, CA respectively, connecting DE, EF, FD, we divide the Δ into four similar small triangles ΔADF, ΔDBE, ΔFEC, ΔEFD. We call them $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ respectively.

We define the sequence χ as follows:

$$\chi_0(P) := 1 \quad \text{for} \quad P \in \Delta ,$$

$$\chi_1^{(1)}(P) := \begin{cases} 1 & \text{for} \quad P \in \Delta_1 \cup \Delta_2 , \\ -1 & \text{for} \quad P \in \Delta_3 \cup \Delta_4 , \end{cases}$$

$$\chi_1^{(2)}(P) := \begin{cases} \sqrt{2} & \text{for} \quad P \in \Delta_1 , \\ -\sqrt{2} & \text{for} \quad P \in \Delta_2 , \\ 0 & \text{for} \quad P \in \Delta_3 \cup \Delta_4 , \end{cases}$$

$$\chi_1^{(3)}(P) := \begin{cases} \sqrt{2} & \text{for} \quad P \in \Delta_3 , \\ -\sqrt{2} & \text{for} \quad P \in \Delta_4 , \\ 0 & \text{for} \quad P \in \Delta_1 \cup \Delta_2 , \end{cases}$$

$$\cdots \ \cdots \ \cdots \ \cdots$$

$$\chi_n^{(3j+i)}(P) := \begin{cases} 2\chi_{n-1}^{(i)}(Q) & \text{for} \quad P \in \Delta_{j+1} , \\ 0 & \text{for} \quad P \in \Delta \backslash \Delta_{j+1} , \end{cases} \tag{2.1}$$

where $Q \in \Delta$, $Q \sim P$, $j = 0,1,2,3$, $i = 1,2,\ldots,3\cdot 4^{n-2}$, $n = 2,3,\ldots$.

At a point of discontinuity, let the value of these functions be the average.

Now we consider the orthogonality of the sequence χ. We prove the following theorem.

<u>Theorem 1</u>. The sequence χ defined by (2.1) is orthonormal.

<u>Proof</u>. At first, it is easy to check that when $n \leq 2$ the sequence $\{\chi_n^{(j)}\}$ is orthonormal. We suppose that the theorem holds for $n \leq N$. For $2 \leq m \leq N+1$; $j_1, j_2 = 0,1,2,3$; $i_1 = 1,2,\ldots,3\cdot 4^{N-1}$; $i_2 = 1,2,\ldots,3\cdot 4^{m-2}$, by (2.1) and induction

hypothesis, we get

$$\int_{\Delta} \chi_{N+1}^{(3j_1+i_1)}(P)\chi_m^{(3j_2+i_2)}(P)dP = 4\delta_{j_1,j_2}\int_{\Delta_{j_1+1}} \chi_N^{(i_1)}(Q)\chi_{m-1}^{(i_2)}(Q)dP$$

$$= \delta_{j_1,j_2}\int_{\Delta} \chi_N^{(i_1)}(Q)\chi_{m-1}^{(i_2)}(Q)dP$$

$$= \delta_{j_1,j_2}\delta_{i_1,i_2}\delta_{N,m-1} .$$

It is easy to verify that

$$\int_{\Delta} \chi_{N+1}^{(3j+i_1)}(P)\chi_1^{(i_2)}(P)dP = \int_{\Delta} \chi_{N+1}^{(3j+i_1)}(P)\chi_0(P)dP = 0 .$$

Therefore the theorem holds for $n = N + 1$, and this finishes the induction. ∎

3. Convergence properties

The triangle Δ has been divided into four similar smaller triangles Δ_i $(i = 1,2,3,4)$. Now set

$$\Delta_{1,i} := \Delta_i \qquad (i = 1,2,3,4) .$$

For each $\Delta_{1,i}$, we divide it into four similar smaller triangles in the same way as we did before. We order them as $\Delta_{2,1},\Delta_{2,2},\ldots,\Delta_{2,16}$ such that

$$\Delta_{1,i} = \Delta_{2,4i} \cup \Delta_{2,4i-1} \cup \Delta_{2,4i-2} \cup \Delta_{2,4i-3}, \qquad i = 1,2,3,4 .$$

We continue this process. For any n we get a sequence $\Delta_{n,1},\Delta_{n,2},\ldots,\Delta_{n,4^n}$ such that

$$\Delta_{n-1,i} = \Delta_{n,4i} \cup \Delta_{n,4i-1} \cup \Delta_{n,4i-2} \cup \Delta_{n,4i-3},$$
$$i = 1,2,3,\ldots,4^{n-1}, \quad n = 1,2,3,\ldots,\infty .$$

$$\Delta_{0,1} := \Delta .$$

Define a function sequence $\{f_{n,i}\}$ on the Δ:

$$f_0(P) := 1 \quad \text{for } P \in \Delta ,$$

$$f_{1,i}(P) := \begin{cases} 1 & \text{for } P \in \Delta_{1,i} , \\ 0 & \text{for } P \in \Delta \backslash \Delta_{1,i}, \end{cases} \qquad i = 1,2,3,4 ,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$f_{n,i}(P) := \begin{cases} 1 & \text{for } P \in \Delta_{n,i} , \\ 0 & \text{for } P \in \Delta \backslash \Delta_{n,i} , \end{cases} \tag{3.1}$$

$$i = 1,2,3,\ldots,4^n, \quad n = 1,2,3,\ldots,\infty .$$

It is obvious that the sequence $\{f_{n,i}\}$ is orthogonal.

Let

$$M_n := \text{span}(f_{n,1},f_{n,2},\ldots,f_{n,4^n}) \quad (n > 0) . \tag{3.2}$$

Thus

$$\dim M_n = 4^n .$$

For convenience, sometimes we use notation

$$X_1 := X_0 ,$$
$$X_{4^{n-1}+i} := X_n^{(i)}, \quad n > 1, \quad i = 1,2,\ldots,3\cdot 4^{n-1} . \tag{3.3}$$

Set

$$H_n := \mathrm{span}(\chi_1, \chi_2, \ldots, \chi_n) .$$

It is clear that

$$H_{4^n} = M_n , \tag{3.4}$$

since $H_{4^n} \subset M_n$ and $\dim H_{4^n} = \dim M_n = 4^n$.

We define

$$L_2(\Delta) := \{f \mid \int_\Delta f^2 d\sigma < \infty \}$$

and

$$\|f\|_2^2 := \int_\Delta f^2 d\sigma .$$

Then the Fourier series of a given function $F \in L_2(\Delta)$ in terms of the function sequence $\{\chi_n\}$ is

$$F \sim \sum_{i=1}^{\infty} \alpha_i \chi_i \tag{3.5}$$

with

$$\alpha_i := \int_\Delta F(P)\chi_i(P)dP .$$

Let

$$P_n F := \sum_{i=1}^{n} \alpha_i \chi_i(P) \tag{3.6}$$

be the n-th partial sum of the series (3.5).

From the orthogonality of sequence $\{\chi_n\}$ we know that $P_n F$ is the best L_2-approximation to F from H_n. Hence it is convergent to F if F is in $L_2(\Delta)$, since H_n is dense in $L_2(\Delta)$. Thus we get the following theorem.

Theorem 2. If $F \in L_2(\Delta)$, then

$$\lim_{n\to\infty} \|F - P_n F\|_2 = 0 .$$

In order to study the uniform convergence we let

$$C(\Delta) := \{f \mid f \text{ is continuous on } \Delta\}$$

and

$$\|f\|_\infty := \max_{P\in\Delta} |f(P)| \ .$$

For $F \in C(\Delta)$ we define

$$P_n^{(j)}F := \int_\Delta F\chi_0 d\sigma \cdot \chi_0 + \int_\Delta F\chi_1^{(1)} d\sigma \cdot \chi_1^{(1)} + \cdots + \int_\Delta F\chi_n^{(j)} d\sigma \cdot \chi_n^{(j)} \ . \tag{3.7}$$

Set

$$K_0(P,Q) := \chi_0(P)\chi_0(Q), \quad (\text{for } P,Q \in \Delta)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$K_n^{(j)}(P,Q) := \chi_0(P)\chi_0(Q) + \chi_1^{(1)}(P)\chi_1^{(1)}(Q) + \cdots + \chi_n^{(j)}(P)\chi_n^{(j)}(Q) \ , \tag{3.8}$$

$$j = 1,2,\ldots,3\cdot 4^{n-1}, \qquad n = 1,2,\ldots \ .$$

Thus

$$P_n^{(j)}F(P) = \int_\Delta K_n^{(j)}(P,Q)F(Q)dQ \ . \tag{3.9}$$

Let $A := (a_{ij})$ $(i,j = 1,2,3,\ldots,4^n)$ be any $4^n \times 4^n$ $(n = 1,2,\ldots)$ matrix and $G(P,Q)$ be any function defined on $\Delta \times \Delta$.

The notation $G(P,Q) \leftrightarrow A$ means that the value of $G(P,Q)$ is a_{ij} when $P \in \Delta_{n,j}$, $Q \in \Delta_{n,i}$. It leads to the following relationship:

$$\chi_0(P)\chi_0(Q) \leftrightarrow \sigma_0 := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} ,$$

$$\chi_1^{(1)}(P)\chi_1^{(1)}(Q) \leftrightarrow \sigma_1 := \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} , \tag{3.10}$$

$$X_1^{(2)}(P)X_1^{(2)}(Q) \leftrightarrow \sigma_2 := \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_1^{(3)}(P)X_1^{(3)}(Q) \leftrightarrow \sigma_3 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}.$$

In order to write those more shortly, we use the notation

$$\text{diag block}(A_1, A_2, \ldots, A_m) := \begin{pmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_m \end{pmatrix}, \qquad (3.11)$$

where A_i is square submatrix.

Using (3.11) we get

$$K_1^{(1)}(P,Q) \leftrightarrow \sigma_0 + \sigma_1 = \text{diag block}\left(\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\right),$$

$$K_1^{(2)}(P,Q) \leftrightarrow \sigma_0 + \sigma_1 + \sigma_2 = \text{diag block}\left(4, 4, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\right),$$

$$K_1^{(3)}(P,Q) \leftrightarrow \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 = \text{diag block}(4,4,4,4),$$

where diag block$(4,4,4,4) = 4I_4$, I_n is $n \times n$ identity matrix. We denote the $m \times m$ zero-element matrix by O_m below.

Since

$$X_2^{(i)}(P)X_2^{(i)}(Q) \leftrightarrow \text{diag}(4\sigma_i, O_4, O_4, O_4), \qquad (i = 1,2,3)$$

we get

$$K_2^{(i)}(P,Q) \leftrightarrow \text{diag block}\Big(4\sum_{j=0}^{i}\sigma_j, 4I_4, 4I_4, 4I_4\Big)\ ,$$

$$K_2^{(3+i)}(P,Q) \leftrightarrow \text{diag block}\Big(4^2I_4, 4\sum_{j=0}^{i}\sigma_j, 4I_4, 4I_4\Big)\ ,$$

$$K_2^{(6+i)}(P,Q) \leftrightarrow \text{diag block}\Big(4^2I_4, 4^2I_4, 4\sum_{j=0}^{i}\sigma_j, 4I_4\Big)\ ,$$

and

$$K_2^{(9+i)}(P,Q) \leftrightarrow \text{diag block}\Big(4^2I_4, 4^2I_4, 4^2I_4, 4\sum_{j=0}^{i}\sigma_j\Big), \qquad (i = 1,2,3)$$

especially

$$K_2^{(12)}(P,Q) \leftrightarrow 4^2I_{16} = \text{diag block}(4^2I_4, 4^2I_4, 4^2I_4, 4^2I_4)\ .$$

Suppose in the general case that

$$K_n^{(3\cdot 4^{n-1})}(P,Q) \leftrightarrow 4^nI_{4^n}\ . \qquad (3.12)$$

By definition of (2.1) and (3.8),

$$K_{n+1}^{(\ell)}(P,Q) \leftrightarrow \text{diag block}(K_{1,1}, K_{2,2}, \ldots, K_{4^n,4^n})$$

where each $K_{i,i}$ is a 4×4 matrix. More precisely

$$K_{n+1}^{(i)}(P,Q) \leftrightarrow \text{diag block}\Big(4^n\sum_{t=0}^{i}\sigma_t, 4^nI_4, \ldots, 4^nI_4\Big), \qquad (i = 1,2,3)\ ,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \qquad (3.13)$$

$$K_{n+1}^{(3j+i)}(P,Q) \leftrightarrow \text{diag block}\Big(4^{n+1}I_4, \ldots, 4^n\sum_{t=0}^{i}\sigma_t, \ldots, 4^nI_4\Big)$$

where the term $4^n\sum_{t=0}^{i}\sigma_t$ is the $(j+1)$-th block, $(j = 1,2,\ldots,4^n - 1)$ in particular

$$K_{n+1}^{(3\cdot 4^n)}(P,Q) \leftrightarrow 4^{n+1} I_{4^{n+1}} ,$$

therefore for any $n = 1,2,\ldots$ (3.12) holds.

Suppose

$$a \in \Delta_{n,4(\ell-1)+i} \subset \Delta_{n-1,\ell}, \qquad i = 1,2,3,4 .$$

By (3.9) we know

$$P_n^{(j)}F(a) = \int_\Delta K_n^{(j)}(a,Q)F(Q)dQ . \tag{3.14}$$

By (3.13), (3.14) we obtain

$$P_n^{(j)}F(a) = 4^{n-1} \int_{\Delta_{n-1,\ell}} F(Q)dQ = \frac{1}{|\Delta_{n-1,\ell}|} \int_{\Delta_{n-1,\ell}} F(Q)dQ \tag{3.15}$$

for $j < 3(\ell - 1)$ and

$$P_n^{(j)}F(a) = 4^n \int_{\Delta_{n,\ell}} F(Q)dQ = \frac{1}{|\Delta_{n,\ell}|} \int_{\Delta_{n,\ell}} F(Q)dQ \tag{3.16}$$

for $j > 3\ell$.

For $j = 3\ell -2,\ 3\ell -1$ we have

$$P_n^{(3\ell-2)}F(a) = \frac{2}{|\Delta_{n-1,\ell}|} \Big(\int_{\Delta_{n,4(\ell-1)+i}} F(Q)dQ + \int_{\Delta_{n,4(\ell-1)+i+1}} F(Q)dQ\Big) \tag{3.17}$$

($i = 1$ or 3) and

$$P_n^{(3\ell-1)}F(a) = \begin{cases} \dfrac{1}{|\Delta_{n,\ell}|} \displaystyle\int_{\Delta_{n,4(\ell-1)}} F(Q)dQ \\ \qquad a \in \Delta_{n,4(\ell-1)+i}, \qquad i = 1,2 , \\ \dfrac{2}{|\Delta_{n-1,\ell}|} \Big(\displaystyle\int_{\Delta_{n,4(\ell-1)+3}} F(Q)dQ + \int_{\Delta_{n,4(\ell-1)+4}} F(Q)dQ\Big), \quad i = 3,4 . \end{cases} \tag{3.18}$$

In any case, from (3.15) to (3.18) we conclude

$$\lim_{n\to\infty} P_n^{(j)}F(a) = \frac{1}{|\Delta_a|} \int_{\Delta_a} F(Q)dQ = F(a) , \tag{3.19}$$

where $\Delta_a \in \{\Delta_{m,i}\}$ and $a \in \Delta_a$, $|\Delta_a| \to 0$ when $n \to \infty$.

It is easy to check that

$$\int_{\Delta} |K_n^{(j)}(p,Q)|\,dQ = 1 \ . \tag{3.20}$$

Now (3.19), (3.20) imply the following theorem.

<u>Theorem 3</u>. For $F \in C(\Delta)$ $\lim_{n\to\infty} \|P_n^{(j)}F - F\|_\infty = 0$ $(j = 1,2,\ldots,3\cdot 4^{n-1})$.

4. On the Walsh system

Naturally there exist some different forms of definition which are equivalent. We use area coordinates to define the two-variable Walsh function. Some notations follow the case of the Haar functions.

$$W_0(P) := 1 \qquad \text{for} \quad P \in \Delta\ ;$$
$$W_{n+1}^{(i)}(P) := W_n^{(i)}(Q) \quad \text{for} \quad P \in \Delta, \qquad i = 1,2,\ldots,4^n\ ,$$

$$W_{n+1}^{(4^n+i)}(P) := \begin{cases} \lambda & \text{for} \quad P \in \Delta_1 \cup \Delta_3\ , \\ -\lambda & \text{for} \quad P \in \Delta_2 \cup \Delta_4\ , \end{cases}$$

$$W_{n+1}^{(2\cdot 4^n+i)}(P) := \begin{cases} \lambda & \text{for} \quad P \in \Delta_1 \cup \Delta_2\ , \\ -\lambda & \text{for} \quad P \in \Delta_3 \cup \Delta_4\ , \end{cases}$$

$$W_{n+1}^{(3\cdot 4^n+i)}(P) := \begin{cases} \lambda & \text{for} \quad P \in \Delta_1 \cup \Delta_4\ , \\ -\lambda & \text{for} \quad P \in \Delta_2 \cup \Delta_3\ , \end{cases} \tag{4.1}$$

where

$$\lambda := W_n^{(i)}(Q), \quad Q \in \Delta, \quad Q \sim P, \quad i = 1,2,3,\ldots,4^n, \quad n = 0,1,2,\ldots\ .$$
$$W_0^{(1)}(P) := W_0(P) = 1 \quad \text{for} \quad P \in \Delta\ .$$

This finishes the definition of the sequence W.

Sometimes we prefer W_ℓ $(\ell = 1,2,3,\ldots,4^{m+1})$ to $W_{m+1}^{(j\cdot 4^m+i)}$ with $\ell = j\cdot 4^m + i$ $(i = 1,2,\ldots,4^m,\ j = 0,1,2,3)$.

At a point of discontinuity, the values of these functions are taken as the average.

Figures 2 and 3 show the Walsh sequence when $n = 0,1$.

Before the discussion of the orthogonality of the Walsh system we introduce the Hadamard matrix ([2],[4], p. 207).

The Hadamard matrix is a square array whose elements consist only of $+1$ and -1 and whose rows (and columns) are orthogonal to one another. Obviously the lowest order nontrivial Hadamard matrix is of the order two, viz.

$$H_2 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{4.2}$$

Higher order matrices whose orders are powers of two can be obtained from the recurrent relationship

$$H_n = H_{n/2} \otimes H_2 \tag{4.3}$$

where $\otimes$ denotes the direct or Kronecker product and n is a power of two. The direct product means replacing each element in the matrix by the H_2. With the help of the Hadamard matrix the one-dimensional Walsh function can be defined [1]. In the two-dimensional case we should use the 4×4 matrix

$$H_4 = H_2 \otimes H_2$$

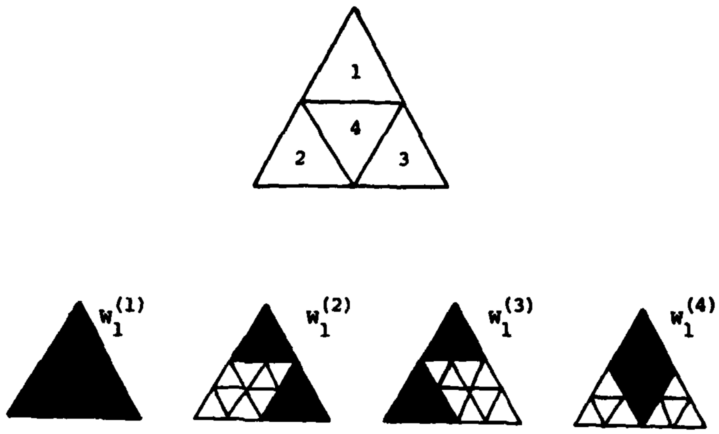
and get the recurrent relationship

$$H_N = H_{N/4} \otimes H_4 . \tag{4.4}$$

The Hadamard matrix (4.4) corresponds to the Walsh sequence $\{W_i\}$ $(i = 1,2,\ldots,4^N)$ for a given N.

Figures 2 and 3 show the Walsh sequence associated with the Hadamard matrix.

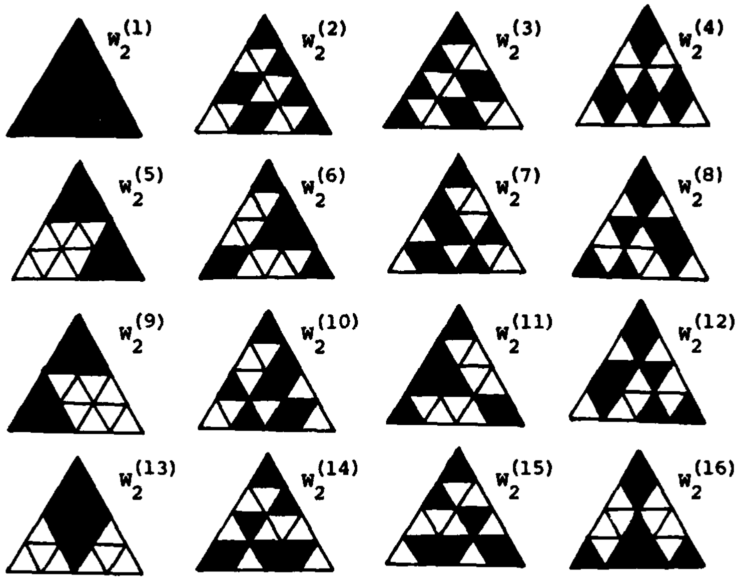
In Figures 2 and 3 black areas represent $+1$, white areas -1. The following triangle shows a certain order.



$$n = 0 \qquad \begin{array}{c} W_1^{(1)} \leftrightarrow \\ W_1^{(2)} \leftrightarrow \\ W_1^{(3)} \leftrightarrow \\ W_1^{(4)} \leftrightarrow \end{array} \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} =: H_4$$

(where we omit 1 in these elements of the Hadamard matrix)

Figure 2



$W_2^{(1)}$	↔	+ + + +	+ + + +	+ + + +	+ + + +	
$W_2^{(2)}$	↔	+ − + −	+ − + −	+ − + −	+ − + −	
$W_2^{(3)}$	↔	+ + − −	+ + − −	+ + − −	+ + − −	
$W_2^{(4)}$	•	+ − − +	+ − − +	+ − − +	+ − − +	
$W_2^{(5)}$	•	+ + + +	− − − −	+ + + +	− − − −	
$W_2^{(6)}$	•	+ − + −	− + − +	+ − + −	− + − +	
$W_2^{(7)}$		+ + − −	− − + +	+ + − −	− − + +	
$W_2^{(8)}$		+ − − +	− + + −	+ − − +	− + + −	$= H_4 \otimes H_4$
$W_2^{(9)}$		+ + + +	+ + + +	− − − −	− − − −	
$W_2^{(10)}$		+ − + −	+ − + −	− + − +	− + − +	
$W_2^{(11)}$		+ + − −	+ + − −	− − + +	− − + +	
$W_2^{(12)}$		+ − − +	+ − − +	− + + −	− + + −	
$W_2^{(13)}$		+ + + +	− − − −	− − − −	+ + + +	
$W_2^{(14)}$	•	+ − + −	− + − +	− + − +	+ − + −	
$W_2^{(15)}$	•	+ + − −	− − + +	− − + +	+ + − −	
$W_2^{(16)}$	•	+ − − +	− + + −	− + + −	+ − − +	

Figure 3

Now we notice that

$$P_n F := \sum_{i=1}^{n} \alpha_i W_i$$

is the best L_2-approximation to a given function F from

$$\hat{M}_n := \operatorname{span}(W_i)_1^n$$

where

$$\alpha_i := \int_\Delta F(P) W_i(P) dP .$$

Hence it is convergent to F if F is in L_2, since $\hat{M}_n$ is dense in L_2. I.e.

__Theorem 4__. If $F \in L_2(\Delta)$, then $\lim_{n\to\infty} \|F - P_n F\|_2 = 0$.

Since $\hat{M}_{4^n} = M_n$ and from Theorem 3 we get the following theorem.

__Theorem 5__. Let $F \in C(\Delta)$, P_{4^n} be L_2-projector onto M_{4^n} on Δ, then

$$\lim_{n\to\infty} \|F - P_{4^n} F\|_\infty = 0 .$$

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In this paper we establish the Haar and Walsh systems on a triangle. These systems are complete in $L_2(\Delta)$. The uniform convergence of the Haar-Fourier series and the uniform convergence by group of the Walsh-Fourier series for any continuous function are proved.