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OPTIMAL PRICING AND ADVERTISING POLICIES FOR NEW PRODUCT OL160P--ETC(U)
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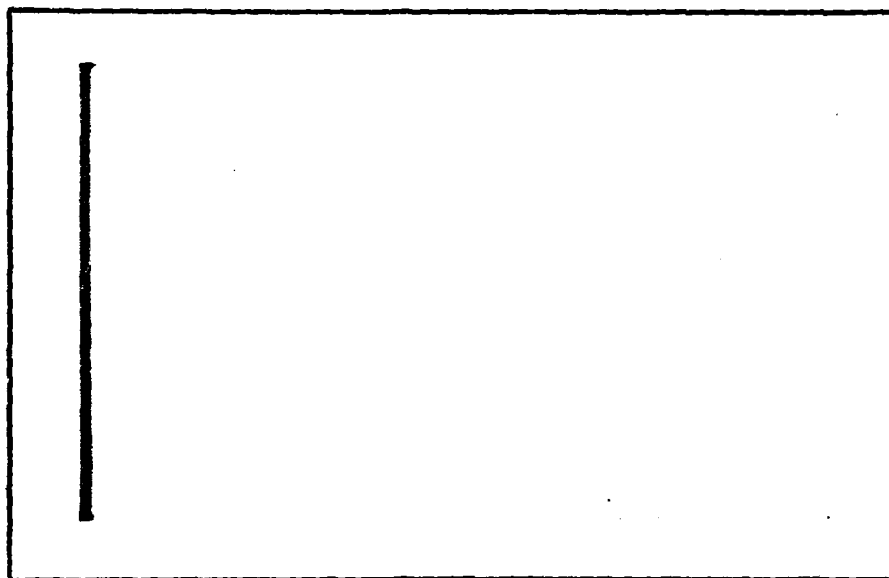
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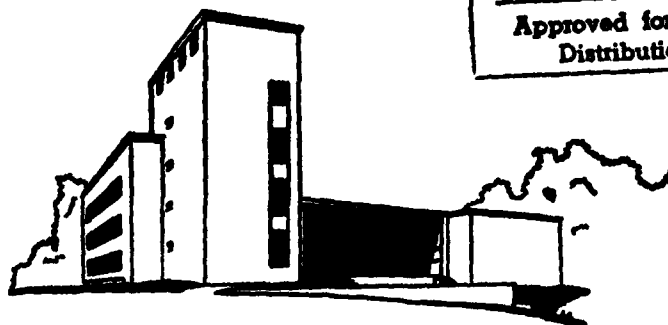
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Management Science Research Report No. 461

(6) OPTIMAL PRICING AND ADVERTISING POLICIES
FOR NEW PRODUCT OLIGOPOLY MODELS . Revision 1

by

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July, 1980
Revised August, 1981

(12) 43

(1) / AUG 81

(14) MSRR 461-REV
WP-5-11-1-REV

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Optimal Pricing and Advertising Policies
for New Product Oligopoly Models

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Gerald L. Thompson and Jinn-Tsair Teng

ABSTRACT

↓
In this paper our previous work on monopoly and oligopoly new product models is extended by the addition of pricing as well as advertising control variables. These models contain Bass's demand growth model, and the Vidale-Wolfe and Ozga advertising models, as well as the production learning curve model and an exponential demand function.

The problem of characterizing an optimal pricing and advertising policy over time is an important question in the field of marketing as well as in the areas of business policy and competitive economics. These questions are particularly important during the introductory period of a new product, when the effects of the learning curve phenomenon and market saturation are most pronounced.

We consider first the monopoly case with linear advertising cost, exponential demand, and three different pricing rules: the optimal variable pricing, the instantaneous marginal pricing, and the optimal constant pricing rules. Several theoretical results are established for these rules including the facts that the instantaneous marginal pricing rule is a myopic version of the optimal pricing rule and the optimal constant pricing rule is a weighted average over time of the instantaneous marginal pricing rule. Another surprising result is that, after the market is at least half saturated, a pulse of

advertising must be preceded by a significant drop in price. Numerical solutions of a number of examples are discussed.

Oligopolistic models are analyzed as non-zero-sum differential games in the rest of the paper. The state and adjoint equations are easy to write down, but impossible to solve in closed form. Hence we describe how to reformulate these models as discrete differential games, and give a numerical algorithm for finding open loop Nash solutions. The latter was used to solve three triopoly models. In each case it was found that optimal prices and advertising rates start high and steadily decline.

KEY WORDS

Oligopoly

Control Theory

Advertising models

Production learning curve

Differential games

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Optimal Pricing and Advertising Policies

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1. INTRODUCTION

In a previous paper [20] we considered monopoly and oligopoly advertising models for a new product during its introductory period when the production learning curve phenomena is most pronounced. The models given in that paper were based on Bass's classic demand growth model [1, 3], the Vidale-Wolfe [22] and Ozga [15] advertising models, and the production learning curve model. [9, 21]. Optimal advertising policies were found for the monopoly models by applying Green's theorem. However, we found that the oligopoly (specifically triopoly) non-zero-sum differential game models we considered did not have closed form solutions so that we were only able to obtain computer generated and graphed solutions for specific numerical instances of the models. Nevertheless these solutions were intuitively reasonable and informative.

In the present paper we extend the results of our previous paper to models in which price as well as advertising is a control variable. In Section 2 we discuss the monopoly case in which we add an exponential demand function to the state equation in a manner similar to that proposed by Robinson and Lakhani in [15]. With that model we discuss three different pricing rules, the optimal variable pricing rule, the instantaneous marginal pricing rule, and the optimal constant pricing rule. The reason for discussing these three pricing rules is to determine which pricing rule is the best for different objective functions. For instance, the optimal variable pricing rule is

the best policy for maximizing net profit without consideration of the transaction costs of changing prices; the instantaneous marginal pricing rule is the best for maximizing the ending cumulative market share; and the optimal constant pricing rule gives the largest profit if we consider the transaction costs of changing prices, provided the price elasticity of demand is sufficient large.

In Theorem 1 of Section 2.1 we prove several results that hold for all three pricing rules. First, that optimal advertising is zero when the market is almost saturated. Second, a higher value of the production learning coefficient or a lower value of the demand coefficient causes the optimal advertising rate to be higher. Simple economic interpretations are as follows: (1) A higher value of the production learning coefficient implies a lower value of production cost. Hence, in this case we can afford to do more advertising and increase the profit. (2) A lower value of the demand coefficient implies that demand is sensitive to the advertising rate. In this case, it pays to do more advertising. Third, that high values of the production learning coefficient or of the demand coefficient cause the optimal price to be low. Simple economic interpretations are as follows: (1) A higher value of the production learning coefficient implies a lower value of production cost. In this case, we can lower the price to increase the sales volume and profit. (2) A higher value of the demand coefficient implies a higher value of the price elasticity of demand. In this case, we have to lower the price in order to increase the sales volume and profit. Finally, it is shown that after the market is more than half saturated, no advertising will be done unless preceded (and usually accompanied) by a significant price drop.

In Section 2.2 we show that the instantaneous marginal pricing rule is "myopic" in that it differs from the optimal variable pricing rule by the factor $-\lambda$, where λ is the adjoint variable. The adjoint variable satisfies

a differential equation, and its value depends on future demands and advertising expenditures. Theorem 2 characterizes the optimal advertising policy for the instantaneous marginal pricing case.

In Section 2.3 we obtain the optimal constant price rule and observe that it can be interpreted as the weighted mean value of the instantaneous marginal pricing rule. We also characterize in Theorem 3 the optimal advertising policy in a way that is similar to the marginal pricing case. Theorem 4 asserts that for the optimal constant pricing rule, the optimal advertising trajectory will have at most one interval, or pulse, of advertising. This same result still holds under slightly modified assumptions for the marginal pricing rule. However, it definitely is false for the optimal variable pricing rule. A counter-example is presented in the next section.

In Section 2.4 we discuss numerical solutions of 10 different examples with randomly generated parameters. We present graphical solutions for Example 2, which has the "two pulse" advertising solution alluded to earlier. In all of these 10 examples, the optimal selling price started high and steadily decreased with time. Such behavior is in accordance with observed experience with new product introductions [8]. However, we also were able to construct an example for which the optimal selling price increased to a peak and then decreased.

In the rest of the paper we discuss oligopolistic versions of price advertising models in which the advertising costs are quadratic and the price is assumed to be determined by the dominant firm. The state equations are straight forward extensions of the monopoly case. However, the adjoint equations and optimal advertising and pricing policies are so complicated that it is impossible to find closed form solutions for the solutions. Hence we were

forced to reformulate the problem as a discrete non-zero-sum differential game model, solve it and plot the solution with the aid of a computer. In Section 3.1 we describe the numerical algorithm for finding open loop Nash solutions to the model, and in Section 3.2 we discuss the numerical solutions of three triopoly examples. In all of these examples the optimal price steadily declined. It was also true for these examples that the optimal advertising starts high and steadily decreases with time.

2. MONOPOLY MODELS

For many marketing situations, advertising is the most important strategic element in determining the market share achieved by a firm. For most consumer products during their introductory phase there frequently is little change over time in the form of the product, the channels of its distribution, etc. Price is another important element in a marketing strategy which influences both the demand for and the profit obtained from a product. Here we shall consider demand growth models in which both advertising and pricing are control variables which can determine profits and market share.

For simplicity, economists generally assume that the demand of a product is a linear, a power, or an exponential function of the price of the product. In this section, we assume the demand is an exponential function of the price since there is little significant difference in results obtained from using any of these assumptions. We shall explore the optimal advertising monopoly models under optimal variable, instantaneous marginal, and optimal constant pricing policies and compare their performance with several different parameter settings.

To define the monopoly models we introduce the following notation:

T = terminal time

ρ = discount rate

$x(t)$ = cumulative market share at time t ; we assume

$$x(t) \in (0,1] \text{ and } x(0) = x_0$$

$$c(t) = c_0(x_0/x(t))^f$$

= learning curve production cost at time t ; c_0

x_0 and f (the learning coefficient) are constants;

we assume $c(t) > 0$, $x_0 > 0$, $c(0) = c_0 > 0$, and $0 < f \leq 1$.

$u(t)$ = advertising rate at time t ; we assume $u(t) \in [0,U]$,

where U is a positive constant which is the maximum permissible advertising rate. (U is determined by the advertising budget, media limitations. etc.)

$p(t)$ = prices at time t

$\alpha u(t) + \beta$ = linear advertising cost; we assume α and β are constants,

and $\alpha > 0$, $\beta \geq 0$. The number β represents the fixed costs such as, listing the product in the catalog, registering a new brand name, etc.

$\lambda(t)$ = the adjoint variable.

As in [20], we assume that the marginal sales volume, $\dot{x} = dx/dt$, is proportional to the advertising rate as follows:

$$\dot{x} \sim \underbrace{(\gamma_1 + \gamma_2 u)(1-x)}_{\substack{\text{Advertising} \\ \text{affects} \\ \text{innovators}}} + \underbrace{(\gamma_3 + \gamma_4 u)(1-x)x}_{\substack{\text{Advertising} \\ \text{affects} \\ \text{imitators}}}, \quad (1)$$

rewriting, we can give another interpretation:

$$\dot{x} \sim \underbrace{(1-x)}_{\text{Saturation effect}} \underbrace{(\gamma_1 + \gamma_2 u + \gamma_3 x)}_{\text{Main effects}} \underbrace{+ \gamma_4 ux}_{\text{Interaction effect}} \quad (1')$$

where $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are nonnegative constants. As noted in [20], the right hand side of (1) is a combination of Bass's demand growth model (in which $\gamma_2 = \gamma_4 = 0$), and the Ozga [15] (in which $\gamma_1 = \gamma_2 = \gamma_3 = 0$) and Vidale-Wolfe [22] (in which $\gamma_1 = \gamma_2 = \gamma_4 = 0$) advertising models. Horsky and Simon have formulated a similar model using the natural logarithm function for advertising effect [10]. Following Robinson and Lakhani [16] and Dolan and Jenland [7], We assume that the demand is proportional to an exponential function of the price, i.e.,

$$x \sim e^{-gp} \quad (2)$$

where the demand coefficient g is a given positive constant. Combining these equations (1) and (2), we obtain the state equation of a monopolistic advertising and pricing model as follows:

$$\dot{x} = [(\gamma_1 + \gamma_2 u)(1-x) + (\gamma_3 + \gamma_4 u)(1-x)x]e^{-gp}, \quad x(0) = x_0. \quad (3)$$

Subject to (3), the monopolist wants to maximize his profit. Mathematically his problem is equivalent to the following optimal control problem:

$$\begin{aligned} & \text{maximize } \{J = \int_0^T e^{-\rho t} [(p-c)\dot{x} - (\alpha u + \beta)] dt\} \\ & 0 \leq u \leq U; p \geq 0 \end{aligned} \quad (4)$$

subject to (3), and the definition $c(t) = c_0(x_0/x(t))^f$.

Note that: (1) Horsky and Simon [10], Ozga [15], and Vidale and Wolfe [22] treated advertising as the only control variable. Bass and Bultez [3], Dolan and Jeuland [7], and Robinson and Lakhani [16] considered price as the only control variable. In this paper, we discuss a model in which both pricing and advertising are control variables. Kotowitz and Mathewson [12] have a slightly different monopoly model which does not consider price and advertising simultaneously and does not have a learning curve production function. (2) Since $p \rightarrow +\infty$ implies $\dot{x} \rightarrow 0$ in (2) and $J \rightarrow 0$ in (4) and $p \leq 0$ implies $J < 0$, we know that the constraint $p \geq 0$ in (4) is redundant and $p = 0$ cannot occur in any optimal solution of the model.

2.1 Optimal Variable Pricing Rule

To apply the maximum principle, we formulate the current-value Hamiltonian [13, 19] using the adjoint variable as follows:

$$\begin{aligned} H &= (p-c)\dot{x} - (\alpha u + \beta) + \lambda \dot{x} \\ &= [(p-c+\lambda)(\gamma_2+\gamma_4x)(1-x)e^{-\beta p} - \alpha]u + (p-c+\lambda)(\gamma_1+\gamma_3x)e^{-\beta p} - \beta. \end{aligned} \quad (5)$$

The current-value Hamiltonian is linear in u so that the optimal control rule for u is bang-bang, that is,

$$u^* = \begin{cases} 0 & \text{if } D < 0 \\ \text{undefined} & \text{if } D = 0 \\ U & \text{if } D > 0 \end{cases} \quad (6)$$

where

$$D = (p-c+\lambda)(\gamma_2+\gamma_4x)(1-x) - ae^{gp} \quad (7)$$

is the advertising switching function.

To find the optimal control rule for the price p we maximize H by differentiating it with respect to p and set the result to zero which gives the optimal variable price rule:

$$p^* = c + \frac{1}{g} - \lambda. \quad (8)$$

The current-value adjoint variable λ satisfies the following differential equation:

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial x} = \rho\lambda - [(\gamma_3+\gamma_4u)(1-2x) - (\gamma_1+\gamma_2u)]\left(\frac{1}{g}\right)e^{-gp^*} \quad (9)$$

together with its transversality condition $\lambda(T) = 0$. An economic interpretation of λ can be found in [19]. Briefly, the value of $\lambda(t)$ at time t evaluates the future effects on profits of making a small change of $x(t)$ at time t . Its role in optimal control theory is similar to the role of dual variables in linear programming. The sign of and values of λ as well as the values of the optimal variable price rule p^* are impossible to predict. However, from our computational experience, we found that λ was always negative when g was sufficiently small. In the derivation of (9) we used (8). Equation (8) can also be used to rewrite (7) as

$$D = \frac{1}{g}(\gamma_2+\gamma_4x)(1-x) - ae^{gp^*} \quad (10)$$

which is the simplified form of the switching function.

Equation (6), (8) and (10) imply the following basic results (which will also hold true for the marginal or optimal constant pricing rules to be discussed later.)

Theorem 1. (a) When the market is almost saturated the optimal advertising rate must be zero, i.e.,

$$u^* = 0 \quad \text{as } x \rightarrow 1 \quad . \quad (11)$$

(b) A higher value of the learning coefficient f or a lower value of the demand coefficient g causes the optimal advertising to be higher; i.e.

$$u^* \uparrow \quad \text{when } f \uparrow \quad \text{or} \quad g \downarrow \quad . \quad (12)$$

(c) A higher value of the learning coefficient f or of the demand coefficient g causes the optimal price to be lower; i.e.,

$$p^* \downarrow \quad \text{when } f \uparrow \quad \text{or} \quad g \downarrow \quad . \quad (13)$$

(d) If $x > x^D = \frac{1}{2} - \frac{\gamma_2}{2\gamma_4}$ then a necessary condition that the optimal policy should change from no advertising to positive advertising is that there should first be a price drop.

Proof. The proof of (a) and (b) is trivial by looking at (8) and (10); and the proof of (c) follows quickly from (8).

To prove (d) we note from (10) that the switching function D is the difference between the function $h(x) = \frac{1}{g}(\gamma_2 + \gamma_4 x)(1-x)$ and ae^{gp^*} . It is easy to show that h is a concave function with its maximum value at the argument $x^D = \frac{1}{2} - \frac{\gamma_2}{2\gamma_4} < \frac{1}{2}$. Hence h is strictly decreasing for $x \geq x^D$. Therefore if $x \geq x^D$ and $h(x) < ae^{gp^*}$ so that from (6) advertising is not

optimal, the only way the reverse inequality can hold which would make advertising optimal, is for p^* to decrease faster than $h(x)$ decreases over time. That is, a price drop must precede an advertising pulse.

Note that x^D is non positive when $\gamma_2 \geq \gamma_4$, i.e., when the advertising coefficient for the innovators is greater than the advertising coefficient for the imitators. In this case price drops are always needed in order to change advertising rates. In the other case, when $\gamma_2 < \gamma_4$, that is when advertising has more effect on imitators than it has on innovators, price drops are not needed as frequently.

In the later numerical examples we will see examples of the two results stated in Theorem 1(a) and (d).

Because equations (3), (6), (8), (9), and (10) form a complicated two point boundary value problem, it is impossible to find a closed form solution for x when the optimal variable pricing rule is used. However, we can reformulate the problem as a discrete control problem, let a computer solve it and plot the resulting solutions. A simple algorithm can easily be devised to solve this discrete problem, but, because it is similar to the algorithm used to solve the triopoly problem presented in Section 3, we do not give details here.

2.2 Instantaneous Marginal Pricing Rule

Suppose we keep the optimal advertising rule (6), but change the price rule from (8) to the instantaneous marginal pricing rule,

Instantaneous Marginal Revenue = Instantaneous Marginal Cost

or

$$\frac{\partial}{\partial x} [p\dot{x}] = \frac{\partial}{\partial x} [c\dot{x}] \quad ,$$

then after differentiation and making some algebraic manipulations we have

$$p^+ = c + \frac{1}{g} \quad (14)$$

which is similar to the optimal variable pricing rule, p^* given in (8) except for the term $-\lambda$.

Note that in this case, the prices always decline as the production cost decreases since g is a constant. This is sometimes called "pricing along the learning curve," see [2,8].

It is extremely difficult to find a closed form solution for the optimal advertising rule when the marginal price rule (14) is used in (3) and (4) by applying the maximum principle. However, since u appears linearly in both the state and the objective functions we can apply Green's theorem as we did in [14]. From the state equation (3) we obtain the formal relation

$$u dt = \frac{e^{gP} dx - (\gamma_1 + \gamma_3 x)(1-x) dt}{(\gamma_2 + \gamma_4 x)(1-x)} \quad (15)$$

Substituting this into (4) we can rewrite the objective function as a line integral along any smooth curve Γ_1 in the (t, x) space as follows

$$J_{\Gamma_1} = \int_{\Gamma_1} (P dx + Q dt) \quad (16)$$

where the functions P and Q are given by

$$P = \left[\frac{1}{g} - \frac{\alpha e^{gP}}{(\gamma_2 + \gamma_4 x)(1-x)} \right] e^{-\rho t} \quad (17)$$

$$Q = \left[\frac{(\gamma_1 + \gamma_3 x)\alpha}{\gamma_2 + \gamma_4 x} - \beta \right] e^{-\rho t} \quad (18)$$

Let Γ be a simple closed curve which bounds a region R . Applying Green's theorem we obtain

$$\begin{aligned} J_{\Gamma} &= \iint_R \left\{ \frac{\partial P}{\partial t} dx - \frac{\partial Q}{\partial x} dt \right\} \\ &= \iint_R e^{-\rho t} I(x) dt dx \end{aligned} \quad (19)$$

where

$$I(x) = \rho \left[-\frac{1}{g} + \frac{\alpha e^{gp}}{(\gamma_2 + \gamma_4 x)(1-x)} \right] - \frac{\alpha(\gamma_2 \gamma_3 - \gamma_1 \gamma_4)}{(\gamma_2 + \gamma_4 x)^2} \quad (20)$$

and p is given by (14).

Lemma 1. Let Γ_1 and Γ_2 be the lower and upper feasible arcs of a simple closed curve $\Gamma = \Gamma_1 - \Gamma_2$, and let R be the region enclosed by Γ . If $I(x) \geq 0$ in R then the lower arc Γ_1 is more profitable than the upper arc Γ_2 . Similarly if $I(x) \leq 0$ in R then Γ_2 is more profitable than Γ_1 .

Proof. Since $J_{\Gamma} = J_{\Gamma_1} - J_{\Gamma_2}$ the result follows easily from (19). For analogous arguments see [14, 19].

Lemma 1 and (14) now imply the following closed form solution for the optimal advertising.

Theorem 2. If the firm adopts the marginal pricing rule (14), then the optimal advertising control is

$$u^* = \begin{cases} 0 & \text{if } I(x) > 0 \\ \text{undefined} & \text{if } I(x) = 0 \\ U & \text{if } I(x) < 0 \end{cases} \quad (21)$$

$$\text{where } I(x) = \rho \left[-\frac{1}{g} + \frac{\alpha e^{1+gc_0(x_0/x)^f}}{(\gamma_2 + \gamma_4 x)(1-x)} \right] - \frac{\alpha(\gamma_2 \gamma_3 - \gamma_1 \gamma_4)}{(\gamma_2 + \gamma_4 x)^2}.$$

Corollary 1. (a) $u^* = 0$ as $x \rightarrow 1$.

(b) $u^* \uparrow$ when $f \uparrow$ or $g \uparrow$.

Proof. Since $I(x) > 0$ as $x \rightarrow 1$, and $I(x) \downarrow$ when $f \uparrow$ or $g \uparrow$, the results follow immediately from (21).

Note that Corollary 1(b) means that products which can be made more efficiently or which have higher consumer appeal can support a higher level of advertising; this is an intuitively appealing result.

2.3 Optimal Constant Pricing Rule

If the firm wants to maintain its price constant over a given time-horizon, it would choose an optimal constant price, $\bar{p}$, such that the following conditions are satisfied:

$$\frac{\partial J}{\partial p} = 0 \quad \text{and} \quad \frac{\partial^2 J}{\partial p^2} < 0, \quad (22)$$

where J is given as in (4). After carrying out the differentiations and making some algebraic manipulations, we have the unique optimal constant price

$$\bar{p} = \frac{\int_0^T e^{-\rho t} (c + \frac{1}{g}) \dot{x} dt}{\int_0^T e^{-\rho t} \dot{x} dt} \quad (23)$$

which can be interpreted as the weighted mean value of the marginal pricing rule (14).

Similarly, we can apply Green's theorem as was done in 2.2 and get the following closed form solution for the optimal advertising given any constant price rule.

Theorem 3. If the firm maintains its price constant over T horizon time, then its optimal advertising control is

$$u^* = \begin{cases} 0 & \text{if } K(x) > 0 \\ \text{undefined} & \text{if } K(x) = 0 \\ U & \text{if } K(x) < 0 \end{cases} \quad (24)$$

where

$$K(x) = \rho \left[-\bar{p} + c_0(x_0/x)^f + \frac{ae^{g\bar{p}}}{(\gamma_2 + \gamma_4 x)(1-x)} \right] - \frac{\alpha(\gamma_2\gamma_3 - \gamma_1\gamma_4)}{(\gamma_2 + \gamma_4 x)^2} \quad (25)$$

Corollary 2. (a) $u^* = 0$ as $x \rightarrow 1$.

(b) $u^* \uparrow$ when $f \uparrow$ or $g \uparrow$.

By testing the limits of $K(x)$ and by calculating its second derivative we can easily establish the following results.

Lemma 2. (a) if c_0, x_0, f , and $\gamma_2 + \gamma_4$ are positive then $K(x) \rightarrow \infty$ as $x \rightarrow 0$ or 1 .

(b) If $\gamma_2\gamma_3\gamma_4 = 0$ or $\gamma_2\gamma_3 \leq \gamma_1\gamma_4$ then $K''(x) \geq 0$ so that $K(x)$ is a convex function of x .

The proof of Lemma 2(a) is obtained by taking the limit of the expression in (25). The proof of Lemma 2(b) is obtained by considering the various cases in which the hypotheses are true. Details are omitted.

Note that the hypotheses of Lemma 2(b) are true for both the Vidale-Wolfe [22] and the Ozga [15] advertising models. In fact, if ρ is sufficiently large then $K(x)$ is convex even though the hypotheses of Lemma 2(b) do not hold.

Assuming the truth of the hypotheses of Lemma 2(b), it follows that there is exactly one argument $\bar{x}$ in the open interval $(0,1)$ which gives the global minimum of $K(x)$. If $K(\bar{x}) < 0$ then there exist exactly two arguments x_1 and x_2 such that

$$K(x_1) = K(x_2) = 0 \quad \text{and} \quad 0 < x_1 < \bar{x} < x_2 < 1. \quad (26)$$

Because K is a convex function it is easy to find x_1 and x_2 by using Newton's or some other search method.

Lemma 2 and Theorem 3 now imply the following theorem.

Theorem 4. If the assumptions of Lemma 2 hold then there are only two different optimal advertising strategies:

- (a) If $K(\bar{x}) \geq 0$ then $u^* \equiv 0$ is the optimal control. (27)
- (b) If $K(\bar{x}) < 0$ then the optimal advertising control is

$$u^* = \begin{cases} 0 & \text{if } 0 \leq x < x_1 \quad \text{or} \quad x_2 < x \leq 1 \\ \text{undefined} & \text{if } x = x_1 \quad \text{or} \quad x_2 \\ U & \text{if } x_1 < x < x_2 \end{cases} \quad (28)$$

This Theorem tells us that there are only three possible different kinds of optimal advertising policies as follows: (A) No advertising at all over the whole time horizon; (B) Advertising during an initial time, then no advertising when the market becomes saturated; and (C) No advertising, then advertising finally no advertising again.

Theorem 4 is still true for the marginal pricing rule if we make some slight modifications of the assumptions of Lemma 2. But Theorem 4 is not true for the optimal variable pricing rule in (8) in general. For a counter example see Figure 2 in Section 2.4.

2.4 Comparisons among Optimal Variable, Optimal Constant Price, and Instantaneous Marginal Price Rules

The definition of a monopoly model requires the setting of 13 parameters $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, g, f, \alpha, \beta, x_0, c_0, \rho, T, U)$ so that it is very difficult to predict

comparative behavior or to prove general results comparing the performance of the optimal variable, optimal constant, and instantaneous marginal price strategies. It is also clearly impractical to explore numerically very much of the parameter space. Therefore, we have chosen 10 random examples whose parameters are given in Table 1, and computed the discounted profits and the ending cumulative market shares for each of the three different pricing strategies; these results are shown in Table 2.

Table 2 indicates that use of the optimal variable pricing rule gives the largest objective value, use of the optimal constant pricing rule is next, and use of the instantaneous marginal pricing rule gives the smallest value. But the order is reversed when the ending cumulative market shares for each rule are compared. In many cases the size of the ending cumulative market share is extremely important for the future of a firm (e.g., to maintain leadership in other markets), see Fig. 2-4.

In Table 2, examples 2, 4, 6, and 7 all have $g = .1$ and all indicate that the optimal variable pricing rule gives larger profits than the other two rules (provided we ignore the costs of changing prices). However, for examples 1, 3, 5, 8, 9, and 10 in Table 2, g is always greater than .1, and the three pricing rules yield the same total profit and ending market share. The reason for this difference is that g is proportional to the elasticity of demand, so that a small value of g means a small elasticity of demand, i.e., demand is not sensitive to price. (For instance, if $g = 0$, then demand is completely independent of the price.) Therefore to maximize profit a high price strategy is best. On the other hand, if g is large then the elasticity of demand is large too, so that demand is sensitive to price, and the marginal price is almost equal to the marginal production cost, see (14). So we cannot adopt a significantly higher or lower price than (14). So there is little significant difference among the three pricing rules if g is large.

We were unable to prove anything about the properties of the pricing trajectory $p(t)$ when the model employed the optimal variable pricing rule given in (8). However, from our computational experience we know that the optimal variable price, as well as the marginal price, decreases in time in each of the 10 cases given in Table 1. We were also able to find a case (shown in Fig. 1) in which the optimal variable price begins with a low initial price, then increases in time, and finally decreases, a result which was similar but less extreme than that previously by Robinson and Lakhani [16].

In general, the advertising trajectory $u(t)$ when using either the constant price or marginal price rule obeys Theorem 4, in which there are only three quantitatively different kinds of optimal advertising policies as follows: (A) No advertising at all over the whole time horizon; (B) Advertising during an initial time, then no advertising when the market becomes saturated; (C) No advertising, then advertising finally no advertising again. For examples, see Fig. 3 and 4.

However, the advertising trajectory for the model which uses the optimal variable price rule (8) is very complicated. As an example, we present graphically in Figure 2 the computer generated solutions of Example 2 of Table 1 and explain why they are so complicated. From (10), we know the switching function of u is $D = (\gamma_2 + \gamma_4 x)(1-x)/g - ae^{sp^*}$. At the initial time, ae^{sp^*} is greater than $(\gamma_2 + \gamma_4 x)(1-x)/g$ so that no advertising is done in this period as shown in Figure 2 and 5. When x reaches about 0.23, the optimal variable price dips rapidly and ae^{sp^*} becomes smaller than $(\gamma_2 + \gamma_4 x)(1-x)/g$ so that using the optimal control rule (6), we set $u^* = 10$ and the cumulative market share x increases rapidly as shown in Figures 2 and 5. Such an impact makes price p^* go down sharply too such that ae^{sp^*} is lower than $(\gamma_2 + \gamma_4 x)(1-x)/g$ again. When x is close to T , x is approaching to 1, and ae^{sp^*} is always greater

than $(\gamma_2 + \gamma_4 x)(1-x)/g$ so that there is no advertising at the terminal time. This example also illustrates Theorem 1(e) since a rapid decrease of the optimal variable price on two occasions causes pulses of advertising, see Figure 2 for illustration.

Comparison of the solutions of the two examples in Figures 1 and 2 are interesting. Note that in Figure 1 the inequalities $\gamma_1 < \gamma_3$ and $\gamma_2 < \gamma_4$ hold while in Figure 2 the reverse inequalities hold. In other words in Figure 1 innovation is much less important than imitation while in Figure 2 the reverse holds. In Figure 1 advertising is always 0, and price starts relatively high, rises slightly, and drops off somewhat at the end, which means the monopolist is passive and merely offers his product to the marketplace and waits for knowledge about it to diffuse through imitation only. In Figure 2, the monopolist is very active doing both "skimming pricing" and "pulse advertising." He begins with a high price, skimming off profits from the early innovators; then later he makes sudden drops in price followed by intense advertising campaigns to encourage new buyers and increase sales volume. In between these sales campaigns, the imitation effects continue to increase sales.

Samples	γ_1	γ_2	γ_3	γ_4	α	β	x_0	c_0	f	U	g	T	ρ
Range	.1-.4	.5-.9	.1-.4	.5-.9	.05-.10	0	.1-.5	10-30	.1-.5	5-10	.1-.5	20-150	.001-.005
1	.3	.6	.2	.9	.05	0	.5	14	.2	8	.3	23	.001
2	.4	.7	.1	.5	.09	0	.2	14	.1	10	.1	56	.005
3	.4	.9	.2	.8	.08	0	.4	30	.1	7	.2	42	.002
4	.2	.8	.4	.6	.10	0	.1	20	.4	6	.1	81	.001
5	.4	.8	.3	.7	.06	0	.2	21	.3	7	.4	119	.002
6	.1	.5	.3	.6	.07	0	.4	30	.2	8	.1	149	.002
7	.3	.6	.1	.5	.09	0	.3	24	.5	10	.1	109	.003
8	.1	.8	.4	.5	.05	0	.5	21	.2	9	.5	90	.002
9	.2	.6	.1	.8	.06	0	.4	21	.2	9	.2	77	.003
10	.3	.6	.1	.8	.05	0	.1	11	.2	7	.3	135	.003

Table 1. Randomly chosen parameters for 10 examples solved by a discrete control monopoly model for all three pricing rules.

Random Samples	J_c/J_0	J_m/J_0	$x_0(T)/x_m(T)$	$x_c(T)/x_m(T)$
1	1.00	1.00	1.00	1.00
2	0.62	0.55	0.94	1.00
3	1.00	1.00	1.00	1.00
4	0.53	0.49	1.00	1.00
5	1.00	1.00	1.00	1.00
6	0.62	0.49	0.73	0.98
7	0.80	0.68	0.97	1.00
8	1.00	1.00	1.00	1.00
9	1.00	1.00	1.00	1.00
10	1.00	1.00	1.00	1.00
Average	0.86	0.82	0.96	1.00
Standard Deviation	0.20	0.24	0.08	0.01

Table 2. Ratio Values of the Objective Values J , and the Ending Cumulative Market Shares $x(T)$. Note that the subscripts 0, c, and m indicate the optimal, constant and marginal pricing rules, respectively.

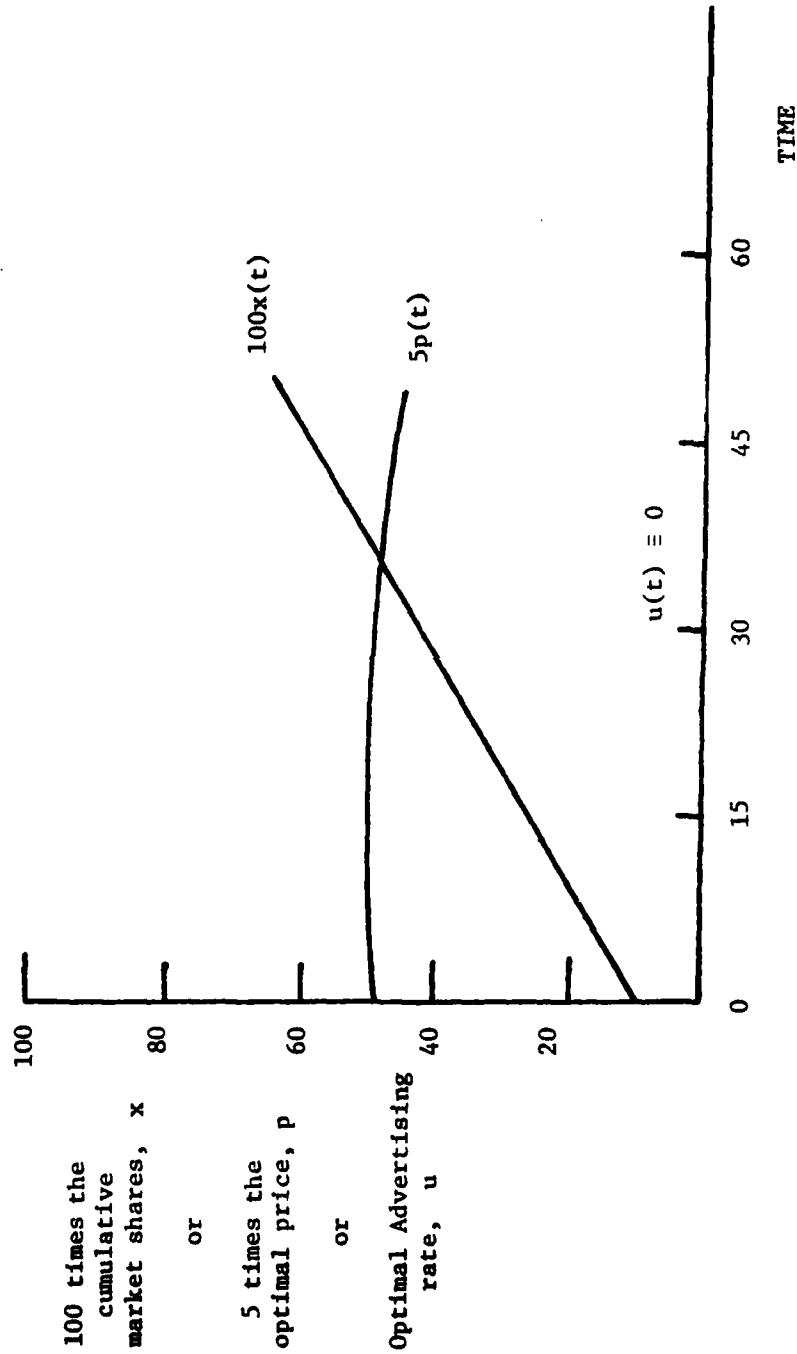


Figure 1. Computer plots using the optimal price rule for the monopoly model;

here $\gamma_1 = 0.2$, $\gamma_2 = 0.3$, $\gamma_3 = 0.4$, $\gamma_4 = 0.6$, $\alpha = 0.1$, $\beta = 0$,
 $x_0 = 0.1$, $c_0 = 10$, $f = 0.3$, $U = 10$, $g = 0.3$, $T = 50$, $p = 0.005$,
 $J = 1.33$, and $x_{50} = 0.65$. In this example the price starts low,
increases somewhat, and then decreases.

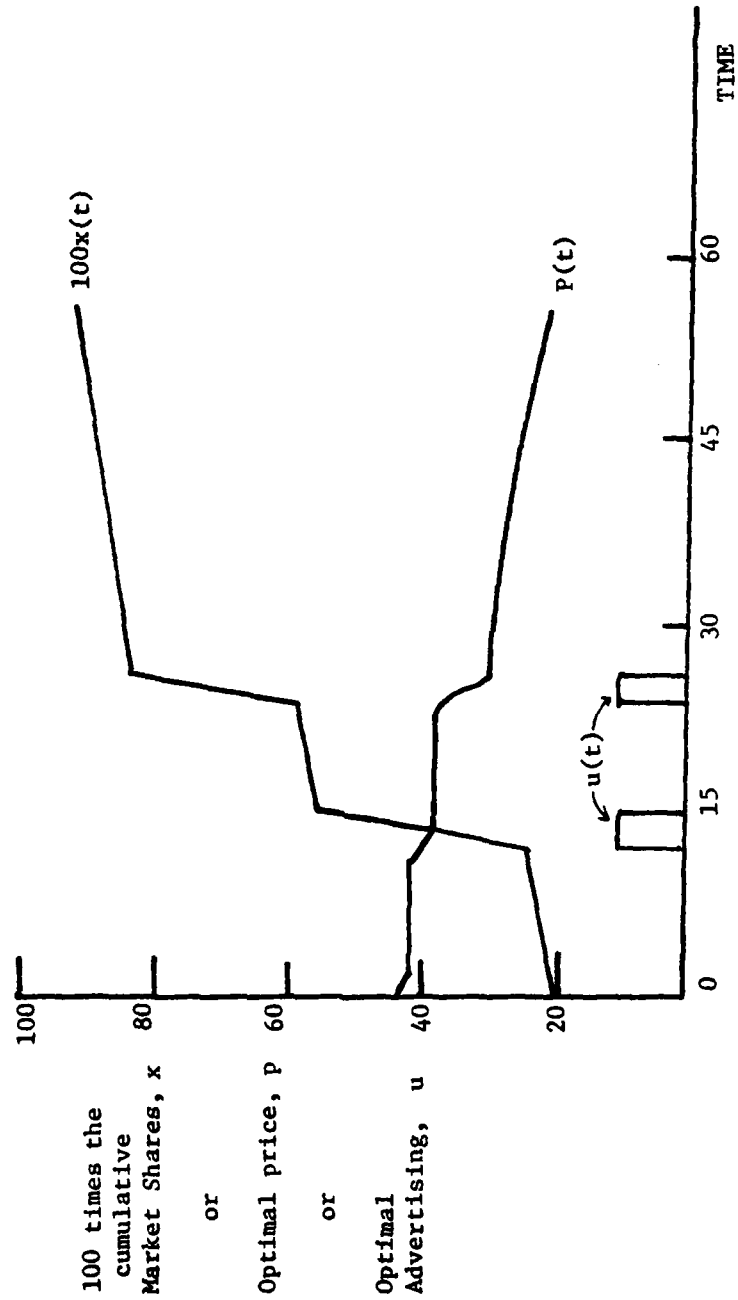


Figure 2. Computer plots using the optimal price rule for the monopoly model; data are given as random sample 2 in Table 1.

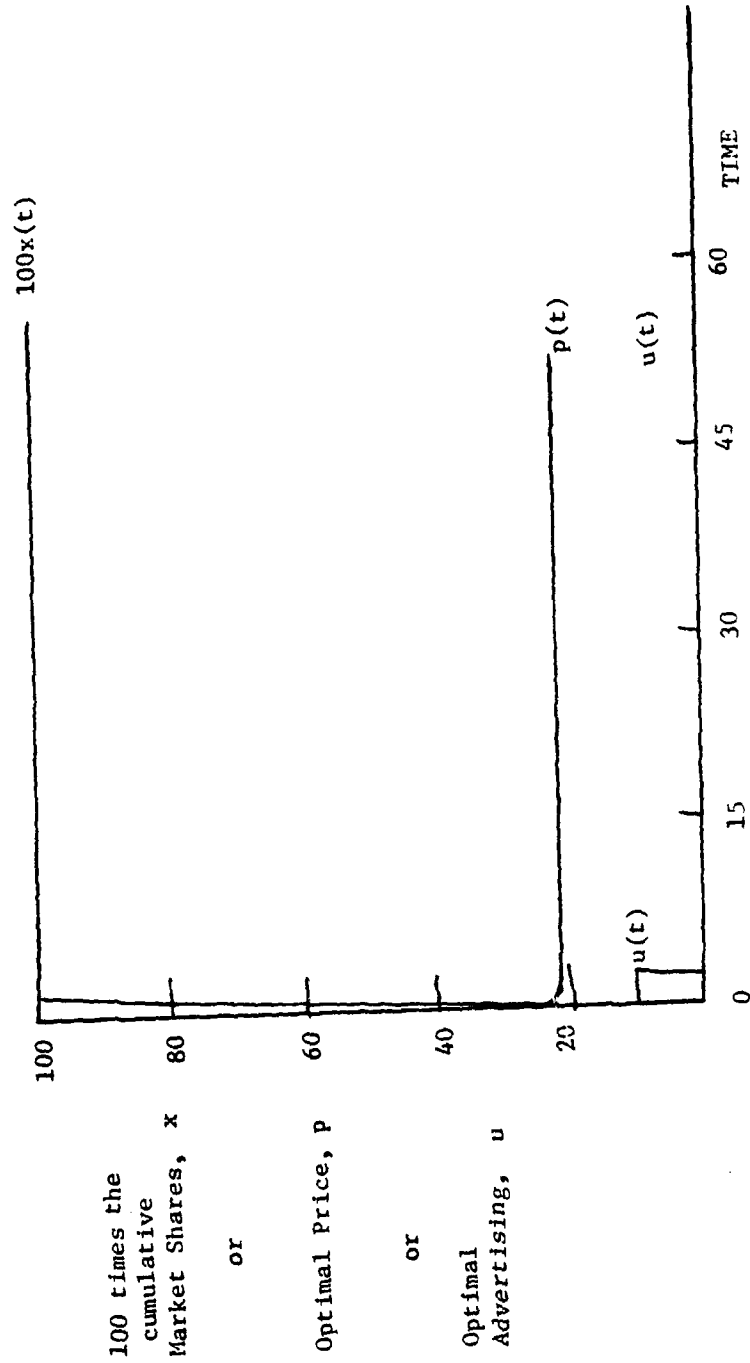


Figure 3. Computer plots using the marginal price rule for the monopoly model; data are given as random sample 2 in Table 1.

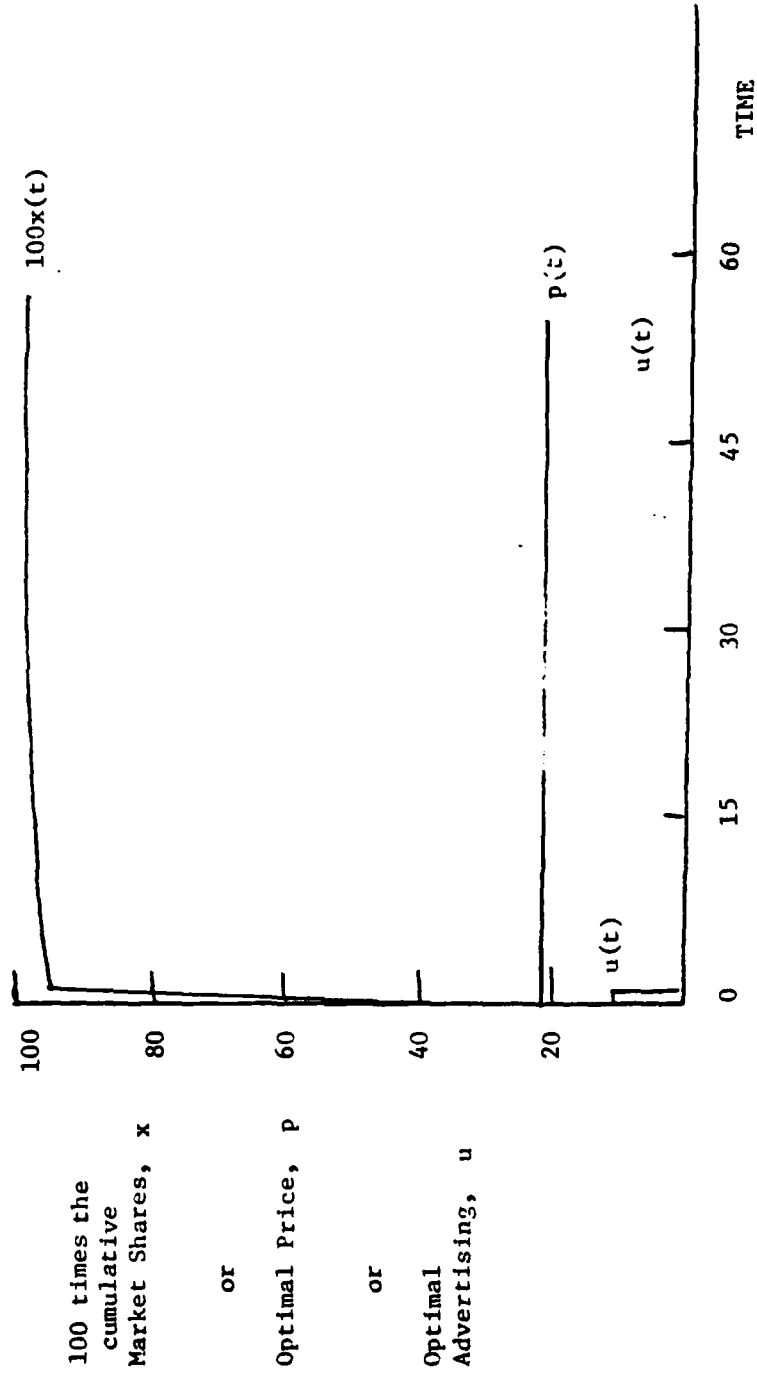


Figure 4. Computer plots using the constant price rule for the model; data are given as random sample 2 in Table 1.

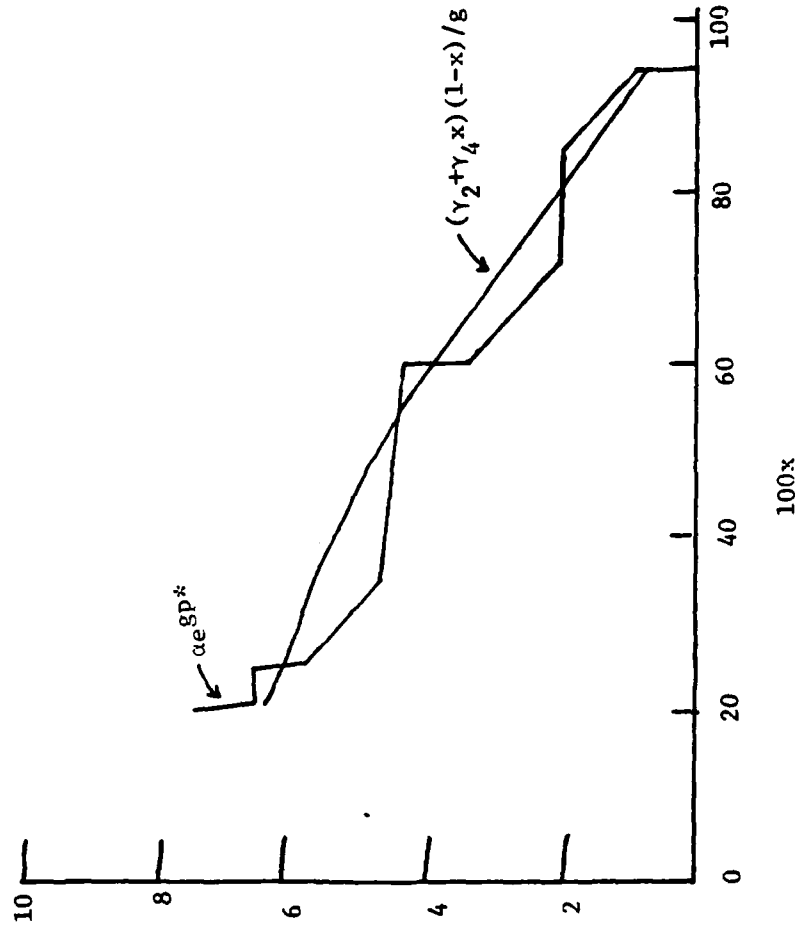


Figure 5. Computer plots of two parts of switching function D for the illustration in Figure 2: here $D = (\gamma_2 + \gamma_4, x) (1-x) / g - ae\ gp^*$.

3. THE OLIGOPOLISTIC PRICE LEADER ADVERTISING MODELS

Here we shall extend the model from the one player case to the n player price advertising non zero sum differential game. The procedure is similar to that we employed previously in [20].

To define the oligopoly models we first state the following notation:

n = number of players

T = terminal time

ρ_i = discount rate for player i

$x_i(t)$ = cumulative market share for player i at time t ; We

assume $x_i(t) \geq 0$ and $x_i(0) = x_{i0} > 0$.

$x(t) = \sum_{i=1}^n x_i(t)$ = cumulative total production at time t ;

We assume $0 \leq x(t) \leq 1$.

W_i = salvage value constant for player i .

$c_i = c_{i0}(x_{i0}/x_i)^{f_i}$ = production cost for player i at time t ;

c_{i0} and f_i are constants; We assume $c_i(0) = c_{i0} > 0$, and $f_i > 0$.

$u_i(t)$ = advertising rate for player i at time t : We assume

$u_i(t) \geq 0$.

$p_i(t)$ = price for player i at time t

$A_i = \alpha_i u_i^2 + \beta_i u_i + \delta_i$ = quadratic advertising cost for player i ;

We assume α_i, β_i and δ_i are constants, and $\alpha_i > 0$.

Note that in this section we use a quadratic advertising cost instead of a linear advertising cost as in Section 2. (We couldn't obtain closed form solutions if we use a quadratic advertising cost in Section 2.) We also consider salvage values for each player for his cumulative market shares at terminal time T .

We consider in this section a market in which there is a single selling price, decided by the largest competitor, say Firm 1, and assume first that the demand for each player's product is linear in the price. The state equation for player 1 then becomes

$$\dot{x}_1 = [(\gamma_{11} + \gamma_{12}u_1)(1-x) + (\gamma_{13} + \gamma_{14}u_1)(1-x)x_1] \frac{p_{U1} - p_1}{p_{U1} - p_{L1}}, x_1(0) = x_{10} \quad (29)$$

where γ_{ik} for $k = 1, \dots, 4$ are nonnegative constants and p_{U1}, p_{L1} are the upper and lower limits of p_1 , respectively.

If, as in Section 2, the demand is an exponential function of price, then the state equation for player 1 is

$$\dot{x}_1 = [(\gamma_{11} + \gamma_{12}u_1)(1-x) + (\gamma_{13} + \gamma_{14}u_1)(1-x)x_1] e^{-\beta_1 p_1}, x_1(0) = x_{10} \quad (30)$$

where β_1 is a positive constant for $i = 1, \dots, n$.

Subject to one of the two Equations (29) or (30), Firm 1 wants to maximize its profit. Mathematically this is equivalent to maximizing

$$J_1 = W_1 e^{-p_1 T} x_1(T) + \int_0^T e^{-p_1 t} \{ [p_1 - c_{10}(x_{10}/x_1)^{f_1}] \dot{x}_1 - (\alpha_1 u_1^2 + \beta_1 u_1 + \delta_1) \} dt. \quad (31)$$

The simultaneous maximization of all n functions J_i is impossible in general. We shall apply the differential game maximum principle [4, 11] find open loop Nash solutions as we did in [20]. Formal definitions of open loop Nash solutions are given in [19]. We shall not give a complete formal definition here. But simply note that an open loop Nash solution is a control trajectory $u_i(t)$ for $i = 1, 2, \dots, n$, such that $u_i(t)$ maximizes J_i given the assumption that all other controls $u_j(t)$ for $j = 1, \dots, i-1, i+1, \dots, n$ are held fixed. We formulate the current-value Hamiltonian for Firm 1 as follows:

$$H_i = [p_1 - c_{i0}(x_{i0}/x_i)^{f_i}] \dot{x}_i - (\alpha_i u_i^2 + \beta_i u_i + \delta_i) + \sum_{j=1}^n \lambda_{ij} \dot{x}_j \quad (32)$$

where the current-value adjoint variables λ_{ij} , $i, j = 1, 2, \dots, n$ satisfy the following differential equations:

$$\dot{\lambda}_{ij} = \rho_i \lambda_{ij} - \frac{\partial H_i}{\partial x_j}, \quad i, j = 1, 2, \dots, n. \quad (33)$$

Also, these adjoint variables must satisfy the transversality conditions:

$$\lambda_{ij}(T) = \begin{cases} W_i & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \quad i, j = 1, 2, \dots, n. \quad (34)$$

By differentiating H_i with respect to u_i and p_1 , and setting the results to zero, we find the optimal controls as follows:

$$u_i = \begin{cases} \left[\frac{[(p_1 - c_i + \lambda_{ii})(\gamma_{i2} + \gamma_{i4} x_i)(1-x) \frac{p_{U1} - p_1}{p_{U1} - p_{L1}} - \beta_i]}{(2\alpha_i)} \right] & \text{if state equations are (29)} \\ \left[\frac{[(p_1 - c_i + \lambda_{ii})(\gamma_{i2} + \gamma_{i4} x_i)(1-x)e^{-g_i p_1} - \beta_i]}{(2\alpha_i)} \right] & \text{if state equations are (30)} \\ 0 & \text{if the above value is less than 0.} \end{cases} \quad (35)$$

$$p_1 = \begin{cases} \text{Max}\{p_{L1}; [(c_1 + p_{U1}) \dot{x}_1 - \sum_{j=1}^n \lambda_{1j} \dot{x}_j] / (2\dot{x}_1)\} & \text{if state equations are (29)} \\ \text{Max}\{0; c_1 + (1/g_1) - \sum_{j=1}^n \lambda_{1j} [g_j \dot{x}_j / (g_1 \dot{x}_1)]\} & \text{if state equations are (30)} \end{cases} \quad (36)$$

It is impossible to find a closed form for the optimal advertising and pricing policies for this oligopolistic quadratic advertising cost model. Thus, we formulate the problem as discrete differential game problem and let the computer solve it in a stepwise manner.

3.1 A Numerical Algorithm for Finding Open Loop Nash Solutions

After specifying the necessary conditions for optimality of the differential game, there are n state equations (29) or (30) with n given initial conditions and n^2 transversality conditions (34) and the n equations for the advertising control variables, and one other equation for setting the pricing control variable.

Algorithm for the oligopolistic advertising and pricing models with quadratic advertising cost.

Step 1. Read the values of parameters.

Step 2. Calculate the starting values of $u_i(k)$, $x_i(k)$, and $p_i(k)$ from $k = 0$ to $k = T$ step by step as follows:

$u_i(k)$ = as in (35) and setting λ_{ii} to zero.

$x_i(k+1) = x_i(k) + \dot{x}_i(k)$

$p_i(k+1)$ = as in (36) and setting λ_{ii} to zero.

Store the values of u_i , x_i and p_i for all $i = 1, \dots, n$, and calculate J_i as shown in (31) then go to Step 3.

Step 3. Find the values of the adjoint variables backward in time by using the values of u_i , x_i and p_i , and the terminal conditions on the adjoint variables, and substituting them in (33). Go to Step 4.

Step 4. Update the new values of u_i , x_i and p_i forward in time step by step as follows:

$$u_i(k) = \text{as in (35)}$$

$$x_i(k+1) = x_i(k) + \dot{x}_i(k)$$

$$p_i(k+1) = \text{as in (36)}.$$

When $k+1 = T$, calculate J_i as shown in (31) and go to Step 5.

Step 5. Check the difference between the new values of J_i (or u_i or p_i) and the previously found values. If there is no significant difference then the algorithm is terminated. Otherwise, update the new J_i and go to Step 3.

Note that the above algorithm can solve other kinds of oligopolistic advertising and pricing models with quadratic advertising cost. All that is needed is to change the definitions of the various state and control rules. In fact, it is a straightforward extension of our algorithm in [20]. It also can solve Deal's duopolistic advertising problem [5], and the bilinear quadratic differential game of Deal, Sethi and Thompson [6]. Intuitively, our numerical algorithm seems to be easier to use than Deal's algorithm, because that method requires guessing all values for each competitor's controls for each discrete instant of time during the planning horizon. We also were able to solve the problems over much longer time periods than was done in [5, 6].

3.2 Numerical Results

We have applied the algorithm of the preceding section to find open loop Nash solutions for many different triopoly encounters. In each case, the algorithm exhibited stability and fast convergence in about 10 iterations.

To state a triopoly problem, there are 48 parameters to choose so that it is difficult to predict behavior or to prove general results. However, if we suppose that all 3 firms have the same values of parameters except one, then we observed the numerical results shown in Table 3.

A Larger Value of the Following Parameter in the Triopoly Models Causes Larger Values of u_i, J_i, and $x_i(T)$: $W_i, \gamma_{i1}, \gamma_{i2}, \gamma_{i4}, f_i$
A Smaller value of the Following Parameter in the Triopoly Models Causes Larger Values of u_i, J_i, and $x_i(T)$: A_i, c_{i0}

Table 3. Effect of Larger Values of the Parameters on the Controls, Objective Value and Ending Cumulative Market Shares in the Triopoly Models.

We have run many triopolistic price advertising cases by using one or the other of the two different state equations defined in (29) and (30). The computational results obtained from each of those two different state equations are quite similar. It is sufficient to understand the outline of the optimal controls in our problems by assuming the state equations are as in (29). Here we present two examples whose state equations are given in (29), and whose parameters are shown in Table 4. The values of the objective function and the ending cumulative market shares for each competitor are shown in Table 5. We also show in Figures 6 and 7, computer generated graphs for the optimal controls u_i and p_i , for the state trajectories x_i and x .

Example Player		i	γ_{i1}	γ_{i2}	γ_{i3}	γ_{i4}	α_i	β_i	δ_i	W_i	x_{i0}	c_{i0}	f_i	T	ρ_i	p_{i0}	p_{i1}	p_{L1}
1	1	1	.004	.006	.004	.008	.01	.006	0	0	.20	22	.4	150	.005	29	35	20
	2	"	"	"	"	"	"	"	"	"	.15	25	.3	"	"	"	"	"
	3	"	"	"	"	"	"	"	"	"	.10	28	.2	"	"	"	"	"
2	1	1	.004	.006	.004	.008	.01	.006	0	60	.20	22	.4	150	.005	29.5	35	20
	2	"	"	"	"	"	"	"	"	"	.15	25	.3	"	"	"	"	"
	3	"	"	"	"	"	"	"	"	"	.10	28	.2	"	"	"	"	"

Table 4. Parameter Values of Two Examples of the One-Price Triopoly Problems.
Dittos indicate same values.

Example No.	Objective Values J_i			Ending Cumulative Market Shares $x_i(T)$		
	1	2	3	1	2	3
1	0.807	0.422	0.157	0.366	0.272	0.202
2	10.563	8.212	5.972	0.402	0.328	0.254

Table 5. Objective Values J_i , and ending Cumulative Market Shares $x_i(T)$, for the two Triopoly examples in The Advertising and Pricing Model.

In Example 1, Firm 3 has the smallest beginning market share and learning coefficient, and the largest initial production cost. This makes advertising not worthwhile for Firm 3, as shown in Figure 6.

The ending market share is frequently important to the future of a firm (to maintain market leadership). Of course, the way to increase the ending market share for firm 1 is to make the value of W_1 , the salvage value for player 1, larger. In Example 2, we assume that all 3 firms have higher salvage values $W_1 = 60$ than in Example 1. In this case, the advertising rate of each firm is

higher and the price is lower than in Example 1. As shown in Figure 7, the price goes down to the lower bound, 20, at time 100.

Note that we also compare the profits and the ending cumulative market shares for the optimal variable and the instantaneous marginal pricing rules. The results are similar to the monopoly case, i.e., if g is small enough then the optimal variable pricing rule has a significantly larger profit than the others. Otherwise, there is no significant difference among them. Here we present Example 3 whose parameters are shown in Table 6, and solve it by using the optimal variable and the instantaneous marginal pricing rules. Those computer plots are shown in Figure 8 and 9, respectively. In this example, we also used state equation (30) instead of (29).

Example 3	i	γ_{i1}	γ_{i2}	γ_{i3}	γ_{i4}	α_i	β_i	δ_i	W_i	x_{i0}	c_{i0}	f_i	g_i	T	ρ
	1	.3	.6	.1	.5	.09	0	0	0	.20	24	.4	.1	109	.003
	2	-	-	-	-	-	-	-	-	.15	-	.3	-	-	-
	3	-	-	-	-	-	-	-	-	.10	-	.2	-	-	-
		Objective Values			J_i		Ending Cumulative Market Shares $x_i(T)$								
Player		1	2	3	1	2	1	2	3						
Optimal price		2.93	2.80	2.67			0.38	0.32	0.26						
Marginal price		1.43	1.32	1.21			0.39	0.33	0.27						

Table 6. Parameters, Objective Values, and Ending Cumulative Market Shares of Example 3. Dashes indicate same values.

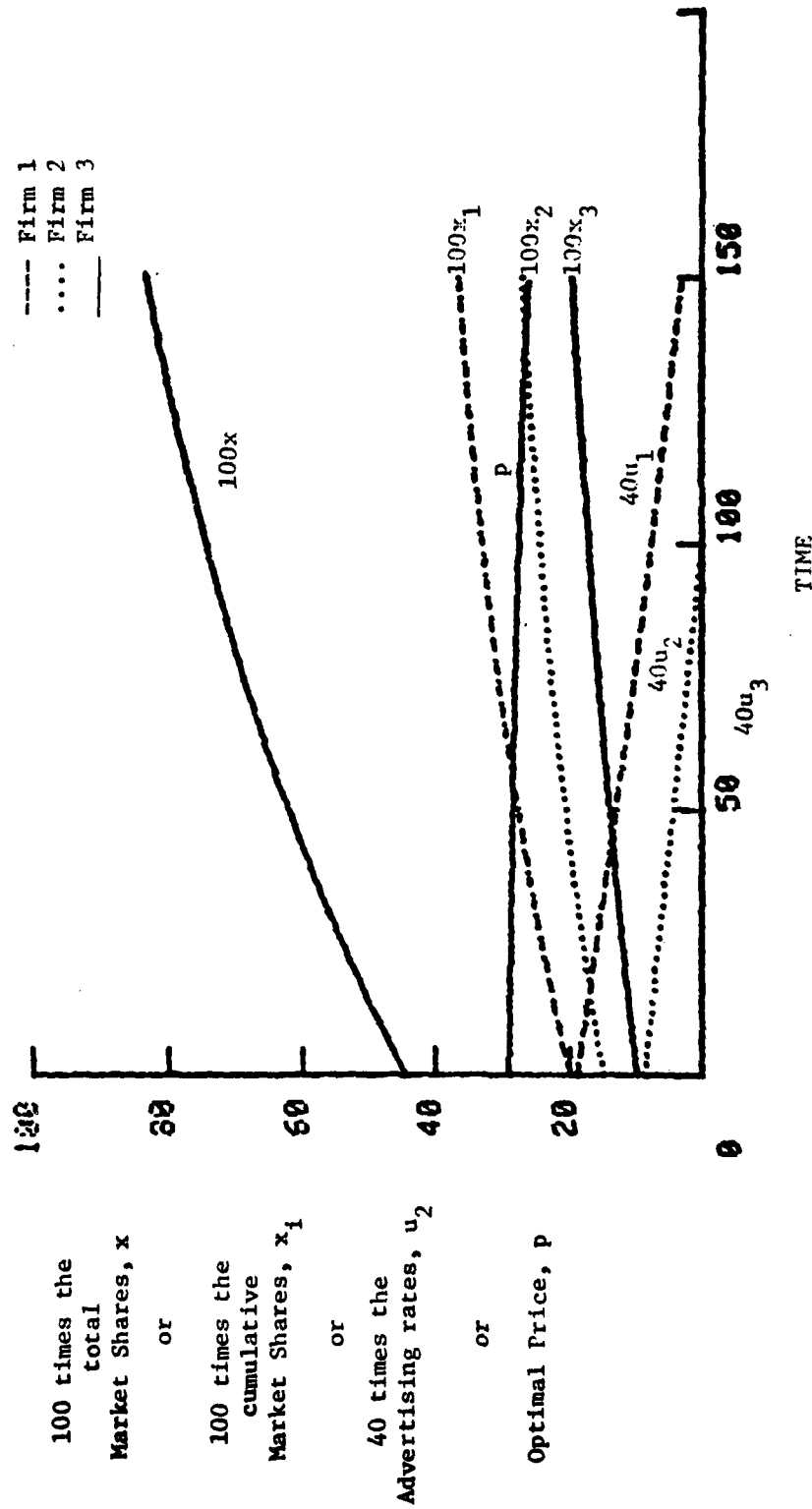


Figure 6. Computer plots of open loop Nash solutions for the first encounter of triopoly model; data are given as Example 1 in Table 5. Note that Firm 3 never advertises.

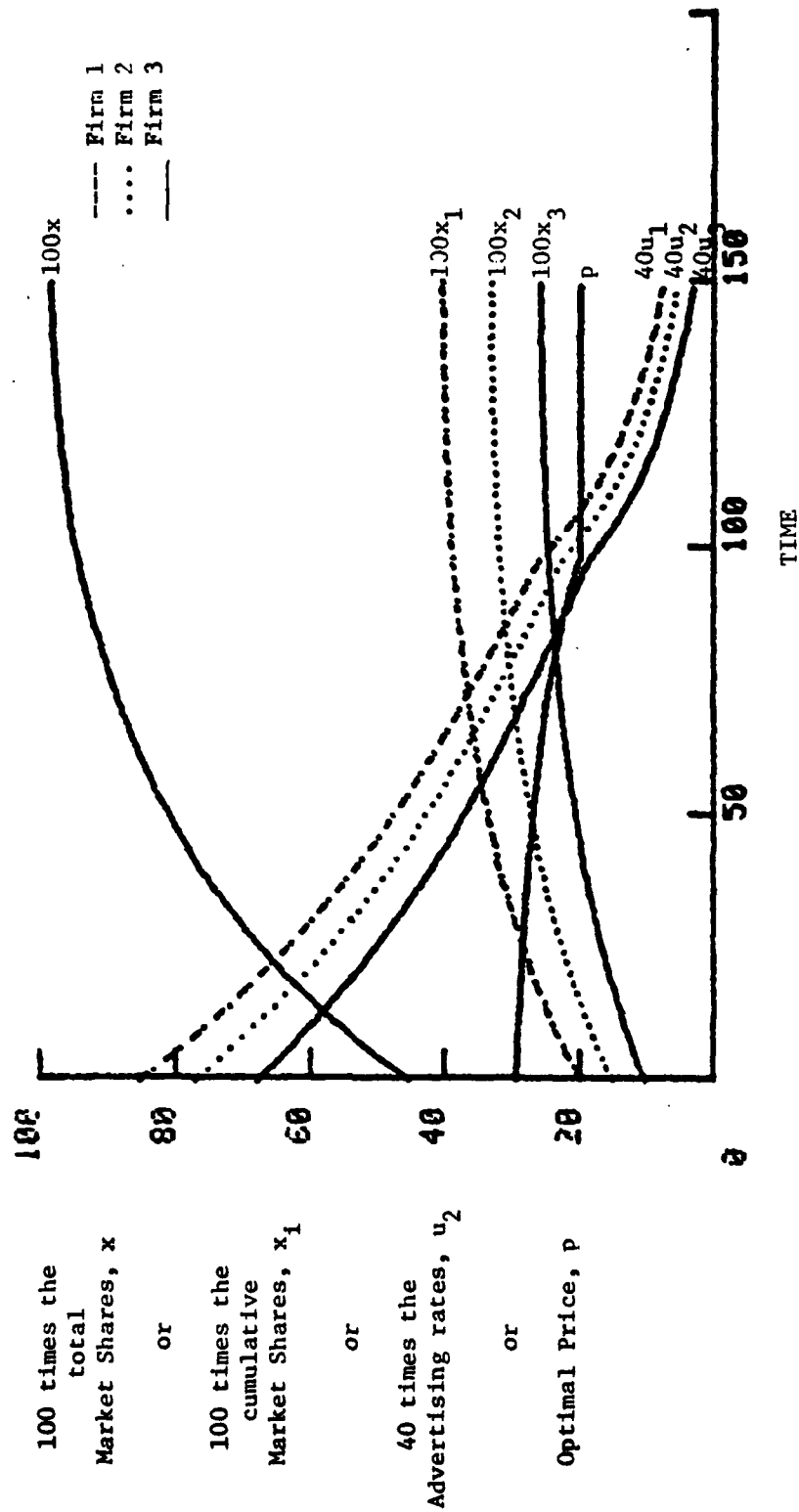


Figure 7. Computer plots of open loop Nash solutions for the second encounter of triopoly model; data are given as Example 2 in Table 5.

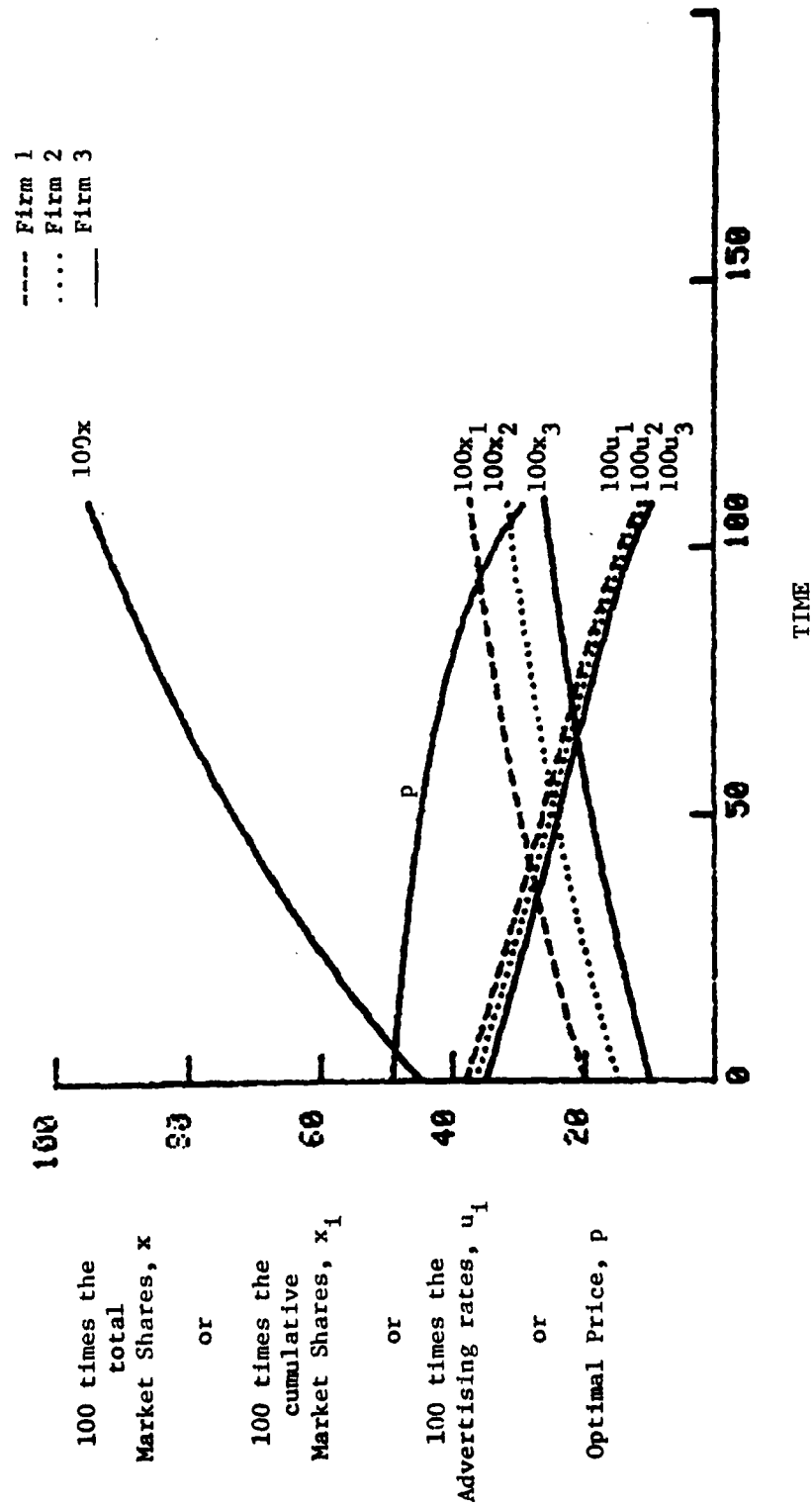


Figure 8. Computer plots of the optimal variable price and advertising for the third encounter of triopoly model; data are given as Example 3 in Table 6.

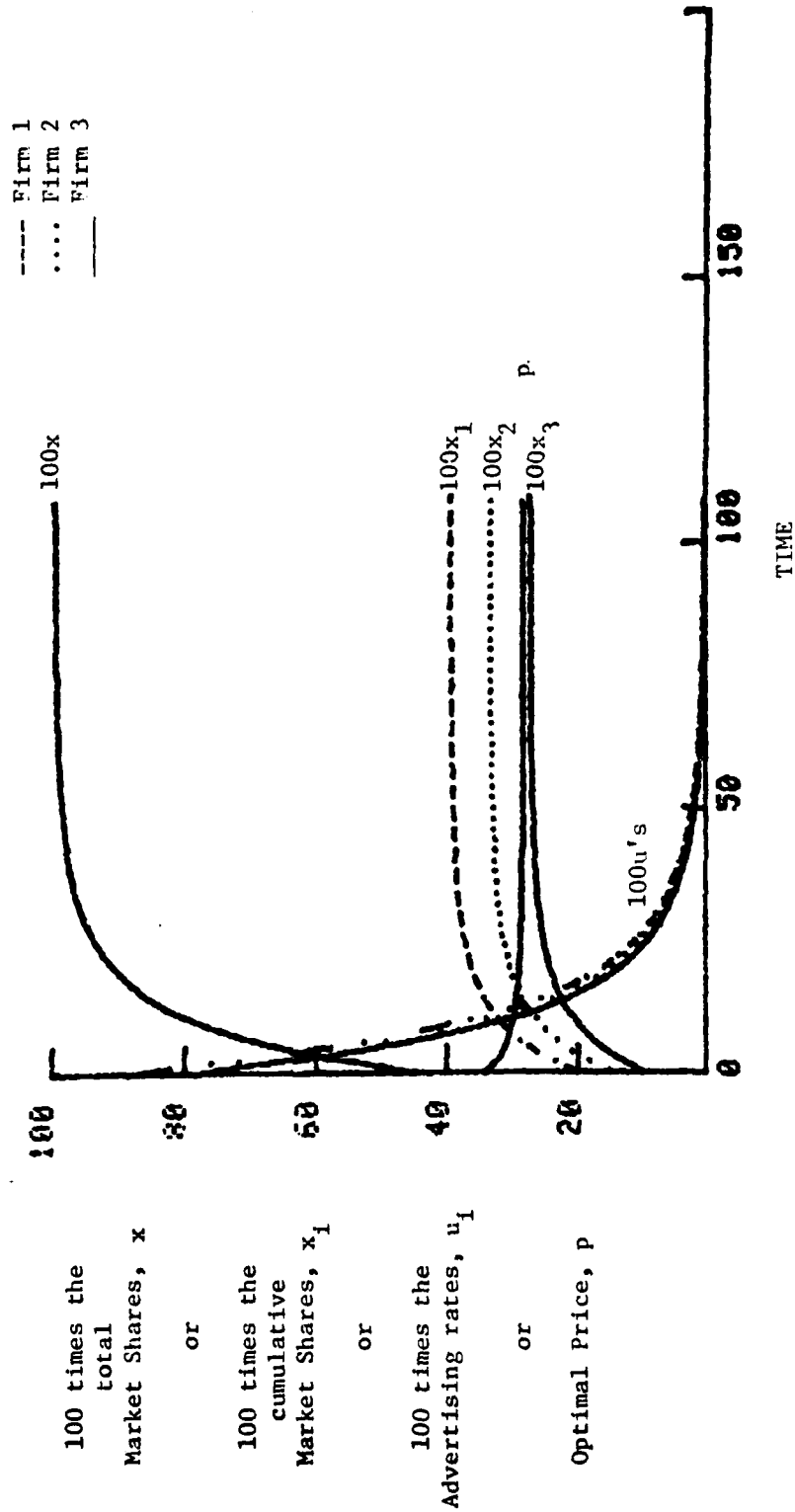


Figure 9. Computer plots of the instantaneous marginal price and advertising for the third encounter of triopoly model; data are given as Example 3 in Table 6.

4. Conclusions

The problem of characterizing an pricing and advertising policy over time is an important question in the field of marketing as well as in the areas of business policy and competitive economics. These questions are particularly important during the introductory period of a new product, when the effects of the learning curve phenomenon and market saturation are most pronounced.

In this paper, we have established a generalized pricing and advertising model for a new product, which contains as special cases a number of other authors' pricing or advertising models [1, 3, 7, 15, 16, 20, 22]. Theoretical results for the monopoly case were obtained. We also extended the monopoly model to an n competitor differential game oligopoly model, and gave a numerical algorithm for finding open loop Nash solutions.

Our oligopoly models can be extended in several different ways. For instance, we may assume that the number of potential customers is dependent on both advertising and price instead of being constant. Also, we could consider a model in which each competitor sets his own price instead of single industry price being decided by the dominant firm. Finally we could consider the problem of finding closed loop Nash solutions instead of open loop Nash solutions to the model.

References

- [1] Bass, F. M., "A New Product Growth Model for Consumer Durables," Management Science, Jan.(1969) pp. 215-227.
- [2] Bass, F. M., "The Relationship between Diffusion Rates, Experience Curves, and Demand Elasticities for Consumer Durable Technological Innovations," Krannert School of Management, Purdue Univ., Paper No. 660, March, 1978.
- [3] Bass, F. M., and A. V. Bultez, "Optimal Strategic Pricing Policies with Learning," Krannert Graduate School of Management, Purdue Univ., Working Paper, Sept., 1979.
- [4] Case, J. H., Economics and the Competitive Process, New York University Press, New York, 1979.
- [5] Deal, K. R., "Optimizing Advertising Expenditures in a Dynamic Duopoly," Operations Research, 27, (1979) pp. 682-692.
- [6] Deal, K. R., S. P. Sethi, and G. L. Thompson, "A Bilinear-Quadratic Differential Game in Advertising," in Control Theory in Mathematical Economics, Liu and Satinen, Eds. Marcel Dekker, Inc., New York, 1979, pp. 91-109.
- [7] Dolan, R. J., and A. P. Jeuland, "Experience Curves and Dynamic Demand Models: Implications for Optimal Pricing Strategies," Journal of Marketing, 45 (1981) pp. 52-62.
- [8] Fitzroy, P. T., Analytical Methods for Marketing Management, McGraw-Hill, Inc., New York, 1976.
- [9] Hirschmann, W. P., "Profit from the learning Curve," Harvard Business Review, Jan.-Feb. (1964), pp. 125-139.
- [10] Horsky, D., and L. S. Simon, "Advertising in a Model of New Product Diffusion," University of Rochester, Working paper, 1978.
- [11] Isaacs, R., Differential Games, Robert E. Krieger Publishing Co., Huntington, New York, 1975.
- [12] Kotowitz, Y., and F. Mathewson, "Informative Advertising and Welfare," American Economic Review, 69 (1979) pp. 284-294.
- [13] Lee, E. B., and L. Markus, Foundations of Optimal Control Theory, Wiley New York, 1968.
- [14] Meile, A., "Extremization of Linear Integral Equations by Greer's Theorem," in G. Leitman (ed.), Optimization Techniques, Academic Press, New York, 1962.

- [15] Ozga, S., "Imperfect Markets Through Lack of Knowledge," Quarterly Journal of Economics, (1960) pp. 29-52.
- [16] Robinson, B. and C. Lakhani, "Dynamic Price Models for New-Product Planning," Management Science, 21 (1975) pp. 1113-1122
- [17] Sethi, S. P., "Optimal Control of Vidale-Wolfe Advertising Model," Operations Research, 21 (1973) pp. 998-1013.
- [18] Sethi, S. P., "Dynamic Optimal Control Models in Advertising: A Survey," SIAM Review, 19 (1977) pp. 685-725.
- [19] Sethi, S. P. and G. L. Thompson, Optimal Control Theory: Applications to Management Science, Martinus Nijhoff Publishing Co., Boston, 1981.
- [20] Teng, J. T., and G. L. Thompson, "Oligopoly Models for Optimal Advertising when Production Costs Obey a Learning Curve," under revision for publication in Management Science.
- [21] The Boston Consulting Group, Perspectives on Experience, Boston: The Boston Consulting Group, 1972.
- [22] Vidale, M. L. and H. B. Wolfe, "An Operations Research Study of Sales Response to Advertising," Operations Research, 6 (1975), pp. 370-381.