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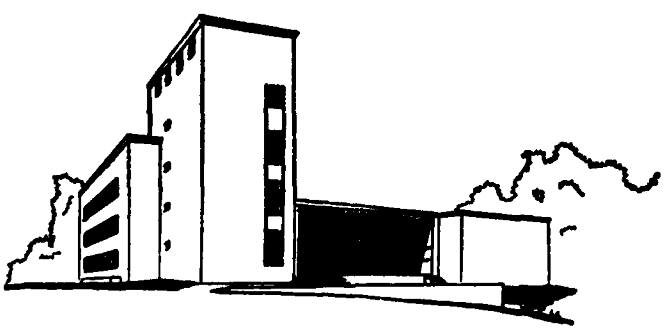

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STRONG PLANNING AND FORECAST HORIZONS IN A CONVEX PRODUCTION PLANNING PROBLEM

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Abstract

Strong Planning and Forecast Horizons
in a Convex Production Planning Problem

by

J. T. Teng, G. L. Thompson, and S. P. Sethi

We consider a production-inventory planning problem with time varying demands, convex production costs, and a warehouse capacity constraint. It is solved by use of the Lagrangian form of the maximum principle. The possible existence of strong planning and forecast horizons is demonstrated. When they exist, they permit the breaking up of the whole problem into a set of smaller problems which can be solved independently, because optimal decisions up to a strong planning horizon are completely independent of demand data beyond the next forecast horizon. A forward branch and bound algorithm is developed to determine such horizons and to solve the whole problem.

1. Introduction

This paper deals with a production planning problem which has a time-varying demand. In particular we consider the problem of finding a production schedule over a finite period [0,T] for a product having: (1) deterministic exogenous demand, (2) strictly convex production costs, (3) strictly increasing inventory costs and (4) and warehouse capacity constraint. This is an extension of a problem solved by Modigliani and Hohn [5]. Kleindorfer and Lieber [1] have treated a similar problem using the extrapolation approach in an optimal control framework; see also Thompson and Sethi [11]. We formulate the problem as an optimal control problem and solve it by using the Lagrangian form of the maximum principle. First, we characterize the form of the optimal production rate rule and inventory level rule. We then find strong planning and strong forecast horizons which permit the decomposition of the whole problem into a set of smaller problems, because the optimal decisions during the period up to the strong planning horizon are completely independent of the data beyond the corresponding forecast horizon. Finally, we derive a forward branch and bound algorithm to solve the whole problem and give a numerical example to illustrate the algorithm.

Earlier planning horizon results for production-inventory problems, obtained by using dynamic programming and variational arguments, have been given by Kunreuther and Morton [2, 3], Modigliani and Hohn [5], and Wagner and Whitin [12]. Other planning horizon results, derived by applying optimal control theory, were given by Kleindorfer and Lieber [1], Lieber [4], Morton [6], Pekelman [7, 8], and Thompson, Sethi, and Teng [10].

2. The Model and Its Solution

Consider a firm producing a homogeneous good and having an inventory warehouse. Define the following quantities:

- $I(t)$: inventory level at time t (state variable),
- $u(t)$: production rate at time t (control variable),
- $d(t)$: demand rate at time t (exogeneous variable); bounded and differentiable for all $t \geq 0$,
- T: length of planning period,
- $f(u)$: strictly convex nonnegative increasing production cost; twice differentiable for all $u \geq 0$,
- $h(I)$: strictly increasing nonnegative inventory holding cost; differentiable for all $t \geq 0$,
- W: upper bound of its warehouse capacity; $W>0$.

Suppose the firm wants to minimize its production and inventory costs to meet the given exogeneous demand rate. The resulting problem is mathematically equivalent to minimizing the following expression:

$$J=\int_0^T [f(u(t))+h(I(t))]dt \tag{1}$$

subject to

$$[\lambda] \qquad \dot{I}(t)=u(t)-d(t); \; I(0)=I_0 \leq W, \tag{2}$$

$$[\rho_1] \quad I \geq 0, \; [\rho_2] \quad I \leq W, \; [\zeta] \; u \geq 0, \tag{3}$$

where a dot denotes the first derivative with respect to time, $\lambda(t)$ is the adjoint variable of (2), and $\rho_1(t)$, $\rho_2(t)$ and $\zeta(t)$ are the Lagrange variables of the corresponding constraints as given in (3).

2.1 The Necessary Conditions for an Optimal Solution

To apply the maximum principle, see [9], we write the Hamiltonian function of (1) as

$$H=-f(u)-h(I)+\lambda(u-d). \tag{4}$$

The Lagrangian then will be

$$L=H-\rho_1(u-d)+\rho_2(d-u)+\zeta u. \tag{5}$$

The following necessary conditions hold (see [9]) for an optimal solution:

$$(\partial L/\partial u)=0=-f'(u)+\lambda+\rho_1-\rho_2, \quad \text{or}$$

$$u^*=\begin{cases} 0 & \text{if } \lambda+\rho_1-\rho_2<f'(0) \\ g(\lambda+\rho_1-\rho_2) & \text{if } \lambda+\rho_1-\rho_2>f'(0) \end{cases} \tag{6}$$

where

$$g=(f')^{-1}, \tag{7}$$

which it may be noted from the assumption on f, is a strictly increasing nonnegative function; the adjoint equation satisfies,

$$\dot{\lambda}=-\partial L/\partial I=h'(I) \tag{8}$$

and the transversality conditions are

$$\lambda(T)\geq 0 \text{ and } \lambda(T)I(T)=0; \tag{9}$$

the complementarity and nonnegativity conditions are

$$\rho_1 I,\ \rho_1\dot{I},\ \rho_2(W-I),\ \rho_2\dot{I},\ \zeta u=0, \text{ and } \rho_1,\ \rho_2,\ \zeta\geq 0; \tag{10}$$

$$\begin{cases} \text{the adjoint variable } \lambda \text{ is continuous except} \\ \text{possibly at an entry or an exit to the boundary} \\ \text{conditions } I(t)=0 \text{ or } I(t)=W\text{, the variables } I \text{ and} \\ \lambda+\rho_1-\rho_2 \text{ are continuous everywhere, and moreover} \\ \dot{\rho}_1\leq 0 \text{ and } \dot{\rho}_2\leq 0. \end{cases} \tag{11}$$

Note that t_{iw} is an initial or entry time to $I(t)=W$ if $I(t_{iw}^-)<W$ and $I(t_{iw})=I(t_{iw}^+)=W$, t_{fw} is a final or exit time to $I(t)=W$ if $I(t_{fw}^-)=I(t_{fw})=W$ and $I(t_{fw}^+)<W$. The definitions of t_{i0} and t_{f0} are similar and are obtained by changing W to 0.

2.2 Optimal Policies for Three Possible Cases

There are only three different possible cases for the values of $I(t)$ for some time interval S. We shall discuss the optimal policies in each of these cases.

Case 1 $0<I(t)<W$ for all $t\in S$

This implies $\rho_1=\rho_2=0$ and

$$\begin{cases} u^*=0 \quad \text{and } \dot{I}=-d \quad \text{if } \lambda<f'(0) \\ \\ u^*=g(\lambda) \text{ and } \dot{I}=g(\lambda)-d \text{ if } \lambda>f'(0) \end{cases} \tag{12}$$

Case 2 $I(t)=0$ and $\dot{I}(t)=0$ for all $t\in S$.

This implies

$$\rho_2=0,\ u^*=d,\ \text{and } f'(d)=\lambda+\rho_1. \tag{13}$$

Since $\dot{\rho}_1 \leq 0$, we have $\dot{\lambda}+\dot{\rho}_1 \leq h'(I)$. Using (13), this case can occur only if

$$f''(d)\dot{d} \leq h'(I) \tag{14}$$

Case 3 $I(t)=W$ and $\dot{I}(t)=0$ for all $t \in S$

This implies that

$$\rho_1=0,\ u^*=d, \text{ and } f'(d)=\lambda-\rho_2. \tag{15}$$

From the fact that $\dot{\rho}_2 \leq 0$, we have the following conclusion:

Case 3 can happen only if

$$f''(d)\dot{d} \geq h'(I).$$

Simple economic interpretations of Cases 2 and 3 are as follows: In Case 2, $f''(d)\dot{d} \leq h'(I)$ implies $f''(u)\dot{u} \leq h'(0)$ because $u=d$ and $I=0$. Taking integral of both sides over any interval $(t_1, t_2) \subset S$, we have

$$f'(u(t_2))-f'(u(t_1))=\int_{t_1}^{t_2} f''(u)\dot{u}dt \leq \int_{t_1}^{t_2} h'(0)dt=h'(0)(t_2-t_1)$$

or (17)

$$f'(u(t_2)) \leq f'(u(t_1))+h'(0)(t_2-t_1).$$

In (17), $f'(u(t_2))$ is the cost of making an additional small quantity at time t_2 by producing it at time t_2. On the other hand $f'(u(t_1))+h'(0)(t_2-t_1)$ is the cost of making an additional small quantity available at time t_2 by producing it at time t_1 and storing it until time t_2, hence including an inventory cost $h'(0)(t_2-t_1)$. Therefore, if inequality (17) holds, then it is not worthwhile to build up inventory for the future, i.e., keeping the warehouse empty is the optimal strategy. by a similar argument, if $f''(d)\dot{d}\geq h'(I)$ holds then it is desirable to build up inventory.

2.3 Theoretical Results

We shall characterize the optimal trajectories of the problem. First, let us define

$$\bar{I}_0=\int_0^T d(t)dt$$

which is the total demand during the planning period.

Theorem 1. The following three statements are equivalent:

(a) $I_0\geq\bar{I}_0$,

(b) $I(T)\geq 0$ and $\lambda(T)\leq f'(0)$,

(c) $u^*(t)=0$ for all t.

Proof. We prove that (a) implies (b) only. Other implications are trivial. If $I(T)>0$ then by (9) we have $\lambda(T)=0\leq f'(0)$. If $I(T)=0$ then $I_0=\bar{I}_0$ and $u^*=0$ for all t. Hence, $\lambda(T)\leq f'(0)$.

Corollary 1 $I_0<\bar{I}_0$ implies $I(T)=0$

Proof. If $I(T)>0$ then $\lambda(T)=0<f'(0)$. From Theorem 1, we have $I_0\geq\bar{I}_0$, which leads to a contradiction.

Theorem 2 We can conclude that $\dot{I}\leq 0$ for all $t\epsilon[0, T]$ under each of the following cases:

(i) $I_0\geq\bar{I}_0$

(ii) $I_0 < \bar{I}_0$ and $f''(d)\dot{d} \leq h'(I)$ for all $t \in [0, T]$

<u>Proof</u>. Case (i) immediately follows form Theorem 1. We now prove case (ii). Suppose not, i.e., $\dot{I}(t) > 0$ in some interval $(j, j+\varepsilon) \subset [0,T]$. This can happen only in Case 1 of Section 2.2. Hence, $\dot{I} = g(\lambda) - d > 0$ in $(j, j+\varepsilon)$. Since $f'' > 0$ from the strictly convexity of f we have that f' is a strictly increasing function. Thus, from (7), $g(\lambda) - d > 0$ if and only if $\lambda > f'(d)$. Because

$\dot{\lambda} = h'(I) \geq d[f'(d)]/dt = f''(d)\dot{d}$ for all t and $\lambda > f'(d)$ in $(j, j+\varepsilon)$ we get $\lambda > f'(d)$ for all $t > j$. Therefore, $\dot{I}(t) > 0$ for all $t > j$ which implies $I(T) > 0$ and leads to a contradiction to Corollary 1.

<u>Corollary 2</u> Under the cases specified in Theorem 2, the optimal path for I(t) is characterized by one of the following three cases:

(α) $I(t) > 0$ for all t (This case can happen if and only if $I_0 > \bar{I}_0$),

(β) $I(t) = 0$ for all t (This case can occur if and only if $I_0 = 0$),

(c) $I(t) > 0$ on $[0, t_1)$ and $I(t) = 0$ on $[t_1, T]$ (This case can happen if and ony if $0 < I_0 \leq \bar{I}_0$)

Next, we shall explore the case in which $f''(d)\dot{d} > h'(I)$, i.e., the case in which it pays to store inventory. For convenience, we define the function

$$\psi(t) = (\lambda + \rho_1 - \rho_2)(t) \tag{18}$$

Then we have the following results.

<u>Theorem 3</u>. If d is continuous and nondecreasing, then we have:

(1) ψ is continuous and nondecreasing,

(2) u^* is continuous and nondecreasing.

Proof. If $0<I<W$ for all $t \in S$ then by (12) we know that $\psi=\lambda$ is an increasing function because $\dot{\lambda}=h'(I)>0$. Otherwise, $I=0$ or $I=W$ implies that $\psi=f'(d)$ is also a nondecreasing function by (13) and (15). Since ψ must be continuous everywhere, see (11), we have that ψ is continuous and nondecreasing.

By (6) and (7), we know that a continuous and nondecreasing ψ will imply a continuous and nondecreasing u^* because g is a strictly function.

Corollary 3 If d is continuous and nondecreasing, then the optimal control path for $u^*(t)$ satisfies one of the following three cases:

(I) $u^*=0$ for all t

(II) $u^*>0$ for all t

(III) $u^*=0$ for $t \leq t_0$ and $u^*>0$ for $t>t_0$

Proof. It immdeiately follows from Theorem 3.

Theorem 4 If $f''(d)\dot{d}>h'(I)$ for all $t \in [0,T]$ then we have the following results: (1) $I(t)>0$ almost everywhere on $[0,T]$. (2) $u^*(t)$ will be one of the three cases as shown in Corollary 3.

Proof. $f''(d)\dot{d}>h'(I)$ implies Case 2 in Section 2.2 cannot happen. Therefore, $I(t)>0$ almost everywhere on $[0,T]$. Again, $f''(d)\dot{d}>h'(I)$ implies $\dot{d}>0$. Hence $u^*(t)$ satisfies one of the three cases shown in Corollary 3.

From Corollary 2 and Theorem 4, we can easily get the following results:

Theorem 5 Assume $I_0<\bar{I}_0$ and t' is choosen so that $f''(d)\dot{d}>h'(I)$ for $t<t'$ and $f''(d)\dot{d}\leq h'(I)$ for $t \geq t'$, i.e., it pays to

store production before but not after t'. Then the optimal path for I(t) satisfies one of the following two cases:

(A) $I(t)>0$ for all $t\epsilon[0,T]$

(B) There exists $t_2 \geq t'$ such that $I(t)>0$ on $(0,t_2)$ and $I(t)=0$ on $[t_2,T]$.

Theorem 6 Assume $I_0<\bar{I}_0$ and t' is choosen so that $f''(d)\dot{d} \leq h'(I)$ for $t \leq t'$ and $f''(d)\dot{d}>h'(I)$ for $t>t'$, i.e. it pays to store production after but not before t'. Then the optimal path for I(t) satisfies one of the following three cases:

(a) $I(t)>0$ for all $t\epsilon[0,T]$,

(B) There exists $t_3 \leq t'$ such that $I(t)=0$ on $[0,t_3]$ and $I(t)>0$ on (t_3, T),

(c) There exist $t_4<t_5 \leq t'$ such that $I(t)>0$ on $[0,t_4)$, $I(t)=0$ on $[t_4, t_5]$, and $I(t)>0$ on (t_5,T).

In general, the interval [0,T] will contain many subintervals in which $f''(d)\dot{d}<h'(I)$ or $f''(d)\dot{d}>h'(I)$. By repeatedly applying the results of Theorems 5 and 6 we can construct the solution by piecing together different solutions obtained from applications of these theorems.

3. Strong Planning Horizon Theorem

If optimal production decisions during $[0,t^*]$ can be shown to be completely independent of the data beyond $t^{**} \geq t^*$, then we call t^* a <u>strong planning horizon</u> and t^{**} <u>a strong forecast horizon</u>.

In this section, we will obtain the strong planning and strong forecast horizons for the problem.

<u>Lemma 1</u>. Let t^* and t^{**} be the two consecutive entries at which inventory hits the boundary constraint. Furthermore, let $I(t^*)=0$ and $I(t^{**})=W$, or $I(t^*)=W$ and $I(t^{**})=0$. Suppose $t^*<t^{**}$. Then t^* is a strong planning horizon, and t^{**} is a strong forecast horizon.

<u>Proof</u>. We will prove the case in which $I(t^*)=W$ and $I(t^{**})=0$. By using an analogous argument we could prove the other case in which $I(t^*)=0$ and $I(t^{**})=W$. We may assume, without loss of generality, that t^* is the first time at which the process enters, i.e., satisfies, the boundary condition $I(t)=W$. Let the optimal value of $\lambda(0)$ for the original problem be denoted by λ_0. If we can prove that $\lambda(t)=\lambda_0+\int_0^t h'(I)dt$ on $(0,t^*)$ is still an optimal solution to any problem having the same demand rate information as the original problem on $[0,t^{**}]$ and regardless of its value after t^{**}, then we are done. If the new optimal value of $\lambda(0)$ were $\tilde{\lambda}_0$ with $\tilde{\lambda}_0<\lambda_0$, then the new inventory level $\tilde{I}<I$ on $[0,t^*]$ because

$$\psi > \tilde{\psi} \Rightarrow u^* \geq \tilde{u}^* \text{ by using (6) and (7)}$$
$$\Rightarrow \dot{I}=u^*-d \geq \tilde{u}^*-d=\dot{\tilde{I}}.$$

Hence, $\tilde{I}(t^*)<W$. For an illustration see Figure 1. Since $\psi=\lambda-\rho_2=f'(d)$ on (t^*,t_0) where t_0 is the time of exit from the inventory constraint $I(t)=W$, we know

$$\dot{\psi}=f''(d)\dot{d} \geq h'(I) > h'(\tilde{I})=\dot{\tilde{\psi}}.$$

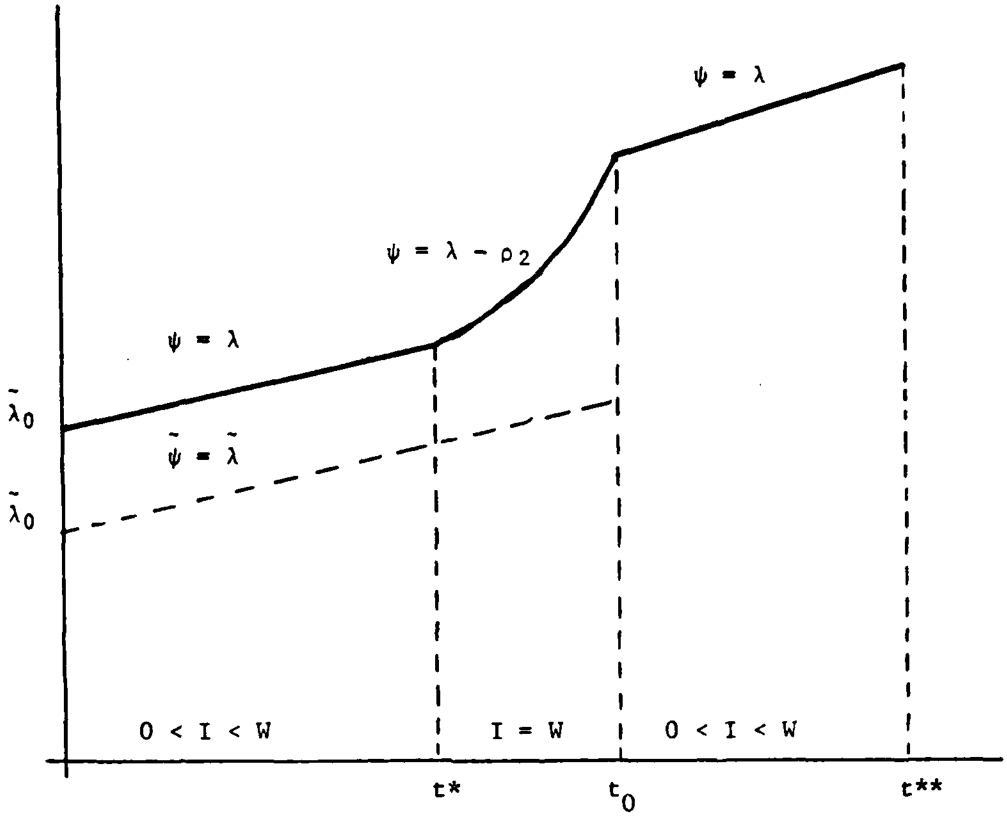

Figure 1

This implies that $\psi > \tilde{\psi}$, $\dot{I} \geq \dot{\tilde{I}}$, and $I > \tilde{I}$ on (t^*, t_0). Similarly, we can get

$$\tilde{\psi} < \psi,\ \dot{\tilde{I}} \leq \dot{I},\ \tilde{I} < I, \text{ and } \tilde{I} < W \text{ on } [t_0, t^{**}].$$

This implies

$$\tilde{I}(t^{**}) - \tilde{I}(t_0) = \int_{t_0}^{t^{**}} \dot{\tilde{I}}\,dt \leq \int_{t_0}^{t^{**}} \dot{I}\,dt = I(t^{**}) - I(t_0) = -W$$

or $\tilde{I}(t^{**}) < 0$ and leads to a contradiction. By using an analogous argument we can prove that if the new optimal value of $\lambda(0)$ were

$\tilde{\lambda}_0$ with $\tilde{\lambda}>\lambda_0$ then it leads to a contradiction again.

Theorem 7 (Strong Planning Horizon Theorem)

Let t_{iw} and t_{fw} be the initial and final times to $I(t)=W$ respectively, i.e. $I(t)=W$ for $t\epsilon[t_{iw},t_{fw}]$. Suppose t_{i0} is the next time at which the inventory reaches $I(t)=0$ boundary. Then all t before t_{fw} are strong planning horizons and all t after t_{i0} are strong forecast horizons, i.e. t_{fw} is a maximal strong planning horizon and t_{i0} is a minimal forecast horizon. The Theorem also holds when 0 and W are interchanged.

Proof. If t_{fw} is not a strong planning horizon then there exists a new optimal solution such that $t_0\epsilon(t_{iw},t_{fw})$ is the new exit form $I(t)=W$ boundary. The reasons are as follows: $t_0<t_{iw}$ will contradict to Lemma 1 and $t_0>t_{fw}$ implies that t_{fw} is a strong planning horizon. Similarly, using the same arguments as in Lemma 1 we can prove that the new exit t_0 will lead to a contradiction.

4. A Forward Branch and Bound Algorithm

The solution is trivial if $I_0\geq\bar{I}_0$. Therefore, we may assume, without loss of generality, that $I_0<\bar{I}_0$. Suppose j is a feasible solution to problem (1) during $[0,t_i]$. Then it is easy to show (proof omitted) that a lower bound for this policy j during $[0,T]$ is

$$\underline{J}^{(j)} = \int_0^{t_i}[f(u^{(j)}) + h(I^{(j)})]dt + f(\bar{u})[T-t_i], \tag{19}$$

where

$$\bar{u}=[\int_{t_i}^{T} d\,dt - I(t_i)]/(T-t_i), \tag{20}$$

$u^{(j)}$ and $I^{(j)}$ are the production rate and inventory level corresponding to the policy j, respectively. From (19) and (20), we also have that $\underline{J}_0=f(u_0)T$ is a lower bound to the value of J where $U_0=(\bar{I}_0-I_0)/T$. On the other hand, $u^*=d$ for all t is a feasible solution to problem (1). Thus, $J_0=h(I_0)t_0+\int_{t_0}^{T} f(d)dt$ where t_0 is the solution of $\int_0^{t_0} d\,dt=I_0$, is an upper bound of J.

We are now in a position to present the algorithm. For convenience, we shall say that the vertex v_j is <u>fathomed</u> if and only if no further exploration from this vertex can be profitable. Otherwise, we shall say that v_j is <u>unfathomed</u> or <u>alive</u>.

<u>Forward Branch and Bound Algorithm</u>

<u>Step 0</u> (Initialization) Begin at the live vertex v_0, where

$\underline{J}=\underline{J}_0$ and $J=J_0$. Go to Step 1.

<u>Step 1</u> (Branching) Assume that t_i is the first entry time to the constraint $I(t)=W$ or $I(t)=0$.

<u>Case 1.1</u> Suppose that t_i is the entry time of I=W. Then t_i should satisfy the following equations and contraints:

$$\lambda(t_i^-)=f'(d(t_i)) \tag{21}$$

$$\lambda(t)=\lambda(t_i^-)-\int_t^{t_i} h'(I)dt \quad \text{for all } t\le t_i \tag{22}$$

$$W-I_0>\int_0^t [g(\lambda)-d]dt>-I_0 \quad \text{for all } t\le t_i \tag{23}$$

$$W-I_0=\int_0^{t_i}[g(\lambda)-d]dt \tag{24}$$

$$f''(d(t_i))\dot{d}\geq h'(W) \qquad \text{at } t-t_i \tag{25}$$

If there exists one or more solutions, then we keep each such solution as successor vertices of v_0 and go to Case 1.2. Otherwise, there are no solutions for this branch, terminate this branch and go to Case 1.2.

<u>Case 1.2</u> Suppose that t_i is the entry of $I=0$. Then t_i satisfies the same constraints as in Case 1.1, except that (24) and (25) must be replaced by (26) and (27), respectively.

$$I_0=\int_0^{t_i}[g(\lambda)-d]dt \tag{26}$$

$$f''(d)\dot{d}\leq h'(0) \qquad \text{at } t=t_i\neq T \tag{27}$$

Again, we keep all solutions as successor vertices of v_0, if any, and go to Step 3.

<u>Step 2</u> (Fathoming by Bound) Check every new live vertex v_j. If $\underline{J}^{(j)}$ as in (19) is larger than or equal to J, then the vertex v_j is <u>fathomed</u>, i.e., no further exploration from this vertex can be profitable. Go to Step 4.

<u>Step 3</u> (Update Bound) Check each new live vertex v_j. If $t_i^{(j)}=T$ and $J^{(j)}\geq J$ then the vertex v_j is fathomed. If $t_i^{(j)}=T$ and $\underline{J}^{(j)}<J$ then let $J=\underline{J}^{(j)}$. Go to Step 2.

<u>Step 4</u> (Branching) If no live vertices exist, go to Step 7;

otherwise select a live vertex v_j. Let $t_i^{(j)}$ be the last time at which I reaches to W or 0. If $I(t_i^{(j)})=W$ then go to Step 5; otherwise go to Step 6.

<u>Step 5</u> Assume that t_{fw} is the exit time from $I(t)=W$, and t_i is the next entry time of $I=0$ or $I=W$. As in Step 1, we have two cases.

<u>Case 5.1</u> Suppose t_i is the entry time to $I=0$, and $t_{fw}<t_i$. Solving the following constraints, we can find the values of t_{fw} and t_i.

$$\lambda(t_{fw}^{+})=f'(d(t_{fw})) \tag{28}$$

$$\lambda(t_i^{-})=f'(d(t_i)) \tag{29}$$

$$\lambda(t)-\lambda(t_{fw}^{+})=\int_{t_{f_w}}^{t} h'(I)dt \quad \text{for all } t\in(t_{fw}, t_i) \tag{30}$$

$$-W<\int_{t_{f_w}}^{t} [g(\lambda)-d]dt<0 \quad \text{for all } t\in(t_{fw}, t_i) \tag{31}$$

$$f''(d)\dot{d}\geq h'(W) \quad \text{for all } t\in(t_i^{(j)}, t_{fw}) \tag{32}$$

$$-W=\int_{t_{f_w}}^{t_i}[g(\lambda)-d]dt \tag{33}$$

$$f''(d)\dot{d}\leq h'(0) \quad \text{at } t=t_i\neq T \tag{34}$$

If above constraints have one or more solutions, then we save them as successor vertices of v_j and go to Case 5.2. Otherwise, terminate this branch and go to Case 5.2.

Case 5.2 Let t_i be the entry time to $I(t)=W$. This case is similar to Case 5.1, except that (33) and (34) are replaced by (35) and (36), respectively.

$$0=\int_{t_{f_W}}^{t_i}[g(\lambda)-d]dt \tag{35}$$

$$f''(d)\dot{d}\geq h'(W) \qquad \text{at } t=t_i \tag{36}$$

If there exist some solutions to (t_{fW}, t_i) in Case 5.1 or Case 5.2, then we keep them as successor vertices of v_j and to to Step 3. Otherwise, v_j is fathomed and go to Step 4.

Step 6 Assume that t_{f0} is the exit time from $I(t)=0$, and t_i is the next entry time to $I=0$ or $I=W$.

Case 6.1 Suppose that t_i is the entry time to $I=0$. Solving the following constraints, we may get the values of t_{f0} and t_i.

$$\lambda(t_{f0}^{+})=f'(d(t_{f0})) \tag{37}$$

$$\lambda(t_i^{-})=f'(d(t_i)) \tag{38}$$

$$\lambda(t)-\lambda(t_{f0}^{+})=\int_{t_{f_0}}^{t}h'(I)dt \qquad \text{for all } t\epsilon(t_{f0}, t_i) \tag{39}$$

$$0<\int_{t_{f_0}}^{t}[g(\lambda)-d]dt<W \qquad \text{for all } t\epsilon(t_{f0}, t_i) \qquad (40)$$

$$f''(d)\dot{d}\leq h'(0) \qquad \text{for all } t\epsilon(t_i^{(j)}, t_{f0}) \qquad (41)$$

$$0=\int_{t_{f_0}}^{t_i}[g(\lambda)-d]dt \qquad (42)$$

$$f''(d)\dot{d}\leq h'(0) \qquad \text{at } t=t_i\neq T \qquad (43)$$

We keep all solutions as successors of v_j, if any, and go to Case 6.2.

<u>Case 6.2</u> Suppose that t_i is the entry time to I(t)=W. Again, this case is similar to Case 6.1, except that we replace (42) and (43) by (44) and (45), respectively.

$$W=\int_{t_{f_0}}^{t_i}[g(\lambda)-d]dt \qquad (44)$$

$$f''(d)\dot{d}\geq h'(W) \qquad \text{at } t=t_i \qquad (45)$$

If there exist any solutions to (t_{f0}, t_i) in Case 6.1 or Case 6.2 then save them and go to Step 3. Otherwise, check whether (41) is satisfied by setting t_{f0}=T or not. If yes, let I(t)=0 on $(t_i^{(j)}, T)$ be a feasible solution and go to Step 3. If not, v_j is fathomed and go to Step 4.

<u>Step 7</u> (Termination) $J=\underline{J}$ which is optimal.

To illustrate that the algorithm can find an optimal solution, we will solve a simple numerical example.

An Example

Suppose that $d(t)=60+10t-t^2$, $T=10$, $f(u)=u^2/2$, $h(I)=I$, $W=125/6$, and $I_0=17$. We then have that $f'(u)=u$, $f''(u)=1$, $g(u)=u$, $\lambda=1$, $\dot{d}=10-2t$, $f''(d)\dot{d}>h'(I)$ for $t<4.5$, and $f''(d)\dot{d}<h'(I)$ for $t>4.5$.

Step 0 $\underline{J}_0=f(15)\cdot 10=1,125$ and $J_0=29,027$

Step 1 Solving Case 1.1, we have

$$\lambda(t_i^-)=f'(d(t_i))=60+10t_i-t_i^2.$$

Substituting the result into (22) and (24), we obtain

$$W-I_0=\int_0^{t_i}[(9t_i-t_i)^2-9t+t^2]dt,$$

and $t_i=1$. Let the vertex corresponding this case is v_1. Next, we find that there is not feasible solution to Case 1.2. and go to Step 3.

Step 3 $t_i^{(1)}=1\neq T$ and go to Step 2.

Step 2 $\underline{J}^{(1)}<J$ and go to Step 4.

Step 4 There is only one live vertex v_1, and $I(t_i^{(1)})=W$. Go to Step 5.

Step 5 Solving (28)-(34) simultaneously, yields $t_{fw}=2$ and $t_i=7$. Let v_2 be the vertex corresponding to this case. Again, we find that there are no feasible solutions to Case 5.2. Go to Step 3.

Step 3 $t_i^{(2)}=7\neq T$ and go to Step 2.

Step 2 $\underline{J}^{(2)}<J$ and go to Step 4 again.

Step 4 There exists only one live vertex v_2, and $I(t_i^{(2)})=0$.

Go to Step 6.

Step 6 There are no feasible solutions to both Case 6.1 and 6.2. Since $f''(d)\dot{d} \le h'(0)$ for all $t \in [7, 10]$. We know that $I(t)=0$ on [7,10] is a feasible solution, let v_3 be the corresponding vertex to this, and go to Step 3.

Step 3 $J = \int_0^{10} [f(u^*)+h(I^*)]dt,$ (46)

$$\text{where} \quad u^* = \begin{cases} 68+t & \text{if } t<1 \\ 74+t & \text{if } 2<t<7 \\ 60+10t-t^2 & \text{otherwise,} \end{cases} \tag{47}$$

and

$$I^* = \begin{cases} 17+8t-9t^2/2+t^3/3 & \text{if } t<1 \\ 125/6 & \text{if } 1 \le t \le 2 \\ 49/6+14t-9t^2/2+t^3/3 & \text{if } 2<t<7 \\ 0 & \text{if } 7 \le t \le 10. \end{cases} \tag{48}$$

Go to Step 2.

Step 2 $\underline{J}^{(3)}=J$ so that v_3 is fathomed. Go to Step 4.

Step 4 There is no live vertex. Go to Step 7.

Step 7 J as in (46) is the optimal solution. This shows that t=2 is a maximal strong planning horizon and t=7 is a minimal strong forecast horizon, see Figure 2.

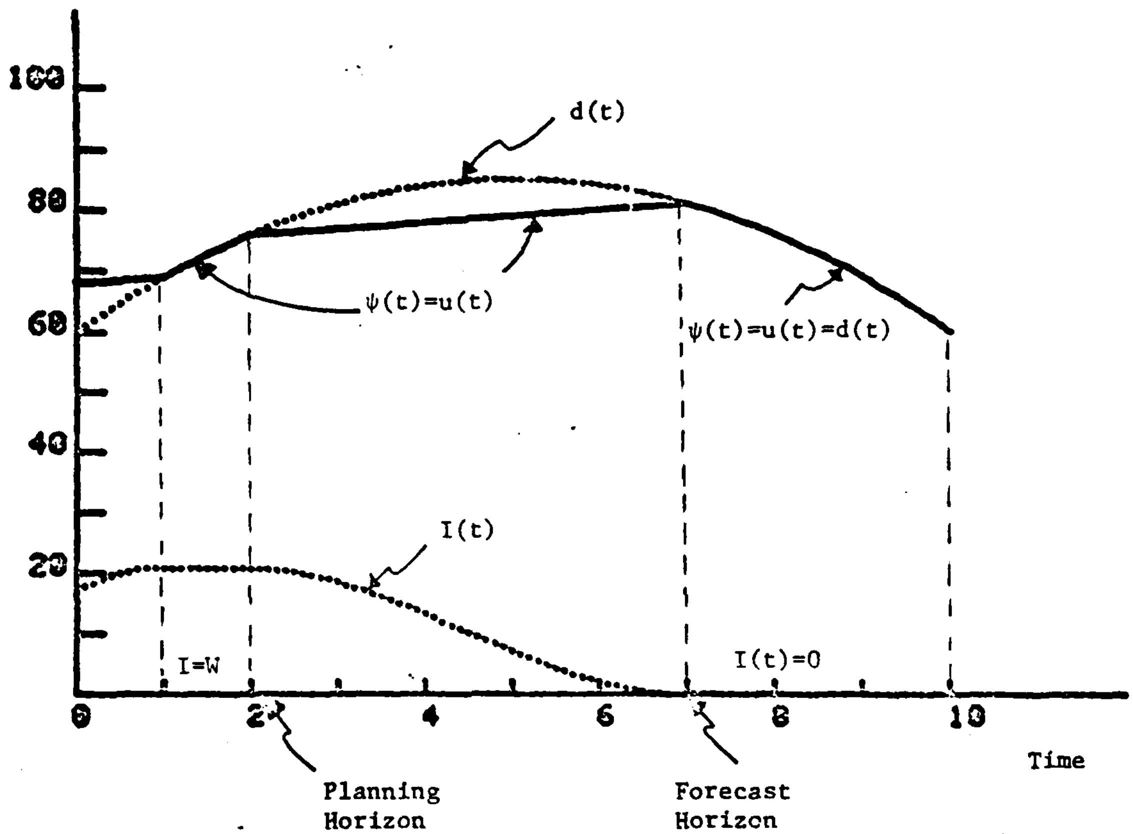

Figure 2. Trajectory for Example.

5. Conclusions

In this paper we have studied a general production inventory model with convex costs and an inventory upper bound. We characterized the optimal trajectories and showed that there could be strong planning and forecast horizons. When such horizons exist, we showed that the problem can be decomposed into a set of smaller problems. In Section 5 we presented a forward branch and bound algorithm which carries out this decomposition. The algorithm is illustrated with a simple example.

It would be possible to extend the results of this paper in several different ways. For instance more general demand functions could be considered. Also, if the state equation (2) is replaced by $\dot{I}=u-d-kI$, where $k>0$ is the rate of spoilage of the inventory, our strong planning horizon theorem is still true. The other theorems in this paper, with suitable modification, are also true.

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