

FEASIBLE OBJECTIVE CONFIDENCE BOUNDS  
FOR SYSTEM RELIABILITY

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M. VERNON JOHNS, JR.

TECHNICAL REPORT NO. 6

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ABSTRACT

The basic problem of determining objective (frequentistic) confidence bounds for the reliability of a series system based on failure data from tests of the independent components is addressed. The notion of confidence bounds based on orderings imposed on the sample space is exploited, and certain optimality considerations are incorporated. Advantage is taken of the simplifications resulting from the use of the Poisson approximation for data from highly reliable components. Tables of exact confidence bounds are produced for the case of two-component systems. These bounds are computed using sample orderings generated sequentially by a two-stage, prospective optimization procedure. A generalization of the Lindstrom-Madden technique is proposed for using the tables to find confidence bounds for systems consisting of more than two components with differing sample sizes.

Key Words: Reliability, series-system, confidence bounds, Poisson approximation, Lindstrom-Madden, sample orderings.

# FEASIBLE OBJECTIVE CONFIDENCE BOUNDS FOR SYSTEM RELIABILITY

By

M. Vernon Johns, Jr.

## 1. Introduction

Certainly one of the most basic statistical problems in the assessment of system reliability is that of determining a confidence bound on the reliability of a series system based on component failure data. The continuing appearance of papers concerned with this subject (e.g., Harris and Soms 1981, Butcher et al. 1978, and Winterbottom 1980) testifies to the elusiveness of solutions which are both practically feasible and acceptably precise. The present paper deals with the case of systems characterized by high intrinsic reliability ( $>90\%$ ) where the use of the Poisson approximation for the binomial distributions of component failure data does not introduce appreciable error. The emphasis is on objective, frequentistic confidence bounds which avoid the uncertainties of interpretation associated with posterior bounds obtained by Bayesian methods. The use of the Poisson distribution provides several advantages. It easily leads to valid results for cases involving zero observed failures for some components where maximum likelihood methods and other proposed approximations tend to break down (see, e.g., Madansky 1965 and Mann et al. 1974). It also permits the pooling of failure data for different components having the same test sample sizes. This potential reduction in the effective number of system components enhances the utility of tabulated bounds such as those presented here.

Because of the structure of the problem, universally optimal confidence bounds for system reliability (i.e., "uniformly most accurate" bounds in the sense of Lehmann 1959) do not generally exist. On the other hand, the ideas of Buehler (1957) may be exploited to produce a variety of valid confidence bounds based on total orderings of the sample points. Such bounds are exact in the sense that the desired coverage probability is guaranteed. The construction of good confidence bounds is thus reduced to the selection of suitable orderings imposed on the sample space. This is the approach adopted in the present study. Previous applications of these methods to reliability may be found, for example, in Harris and Soms (1980), Johnson (1969), and Lipow and Riley (1959).

Once the sample ordering approach has been chosen, there remains the problem of determining orderings which lead to confidence bounds which are "good" according to some criterion measuring the size (length) of the confidence region. An early proposal of the present author (Johns 1975) was to generate the ordering by means of a simple function of the observations which was asymptotically equivalent to the maximum likelihood confidence bound. This method guarantees asymptotic optimality when the numbers of component failures observed under testing is large. It was found, however, that the resulting bounds could be noticeably improved for small to moderate numbers of observed failures by more sophisticated methods. Another procedure investigated (Johns 1977) was a sequential method for generating the sample ordering starting at the origin (zero component failures observed) and selecting at each stage as the next sample

point the "adjacent" point producing the largest value for the lower bound on reliability. This method, while intuitively appealing, does not generally produce a "best" ordering as has been suggested by some investigators. In particular, it is improved upon by the method adopted in the present paper.

The non-existence of a unique optimal ordering leaves open the possibility of obtaining a confidence bound which is at least admissible by choosing the sample ordering to minimize the expected length of the confidence interval under some suitable prior distribution. Such a semi-Bayesian approach does not in any way impair the frequentistic interpretation of the confidence bounds obtained from the minimizing ordering. The implementation of such a minimization, while theoretically perfectly possible, turns out to be totally unfeasible computationally except for the earliest part of the ordering generated. Nevertheless, for a class of priors chosen to emulate certain properties of maximum likelihood, fragmentary orderings computed by this method provide a considerable justification for the two-stage "look-ahead" sequential method actually used to generate the tables which are a principal concern of this paper.

Suppose that the series system under consideration consists of  $k$  independent components and that the respective probabilities of component failure are  $p_i = 1 - q_i$ ,  $i = 1, 2, \dots, k$ . The system reliability  $R$  is given by

$$R = \prod_{i=1}^k q_i = \prod_{i=1}^k (1 - p_i) \quad . \quad (1.1)$$

If the observed numbers of failures for the  $k$  components are  $X_1, X_2, \dots, X_k$  based on independent tests with corresponding sample sizes  $n_1, n_2, \dots, n_k$ , we let  $\lambda_i = n_i p_i$ ,  $i = 1, 2, \dots, k$ , so that

$$R = \prod_{i=1}^k (1 - \lambda_i/n_i) \approx 1 - \sum_{i=1}^k \lambda_i/n_i . \quad (1.2)$$

The approximation on the right will be best when the  $p_i$ 's ( $= \lambda_i/n_i$ ) are all small which is just the case where  $R$  is close to one and the Poisson approximation for the distributions of the  $X_i$ 's is valid.

It will be convenient to express the problem in a canonical form by introducing some further notation. Let  $c = \sum_{i=1}^k 1/n_i$  and  $a_i = 1/cn_i$ ,  $i = 1, 2, \dots, k$ . Then letting  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  and

$$\theta(\lambda) = \sum_{i=1}^k a_i \lambda_i , \quad (1.3)$$

we have

$$R \approx 1 - c \sum_{i=1}^k a_i \lambda_i = 1 - c\theta(\lambda) . \quad (1.4)$$

We shall assume henceforth that the components are indexed so that  $n_1 \geq n_2 \geq \dots \geq n_k$  which implies that  $a_1 \leq a_2 \leq \dots \leq a_k$ . The problem of finding a lower confidence bound for  $R$  is thus reduced to that of finding an upper confidence bound for  $\theta(\lambda)$ . The fact that  $\theta(\lambda)$  is a convex combination of the  $\lambda_i$ 's facilitates tabulation of the bounds by reducing the number of classification variables by one. In principle, confidence bounds for  $R$  could be constructed directly without introducing the approximation (1.2). Such an approach would,

however, eliminate the possibility of constructing useful tables of bounds, since separate entries would be required for every configuration of  $(n_1, n_2, \dots, n_k)$ .

The vector of observations is  $X = (X_1, X_2, \dots, X_k)$  so that the sample space on which a total ordering must be imposed consists of all vectors  $x = (x_1, x_2, \dots, x_k)$  where the  $x_i$ 's are non-negative integers. It seems reasonable to confine attention to orderings which are consistent with the natural partial ordering induced by dominance (see Section 2), and we shall do so. In Section 2 it is shown that the best upper confidence bound for  $\theta(\lambda)$  with confidence coefficient  $1 - \alpha$  which is monotone in a prescribed total ordering (designated by the relation  $\leq$ ) is given by

$$t(x) = \sup_{\lambda \in S_\alpha^*(x)} \theta(\lambda) \quad , \quad (1.5)$$

where  $S_\alpha^*(x) = \{\lambda : P_\lambda \{X \leq x\} = \alpha\}$ . From (1.4) the lower confidence bound for system reliability is then given by

$$r(x) = 1 - ct(x) \quad . \quad (1.6)$$

Because of the additive property of independent Poisson observations the effective number of components in the system may be reduced if some sample sizes are equal (or nearly equal) as follows: Suppose  $n_1 = n_2 = \dots = n_r = n$  for some  $r$ ,  $2 \leq r < k$ . Then  $a_1 = a_2 = \dots = a_r = a^*$  (say) and letting  $\lambda^* = \sum_{i=1}^r \lambda_i$  we have

$$\theta(\lambda) = a^* \lambda^* + \sum_{i=r+1}^k a_i \lambda_i \quad (1.7)$$

Now  $X^* = \sum_{i=1}^r X_i$  has a Poisson distribution with parameter  $\lambda^*$  so that the right hand side of (1.7) has the correct form for the case of dimension  $k^* = k - r + 1$  except that the coefficients must each be divided by  $c_0 = a^* + a_{r+1} + \dots + a_k$  to preserve convexity. The confidence bound for the  $k^*$  dimensional case based on  $X^*, X_{r+1}, \dots, X_k$  and the normalized coefficients may then be multiplied by  $c_0$  to obtain the bound for the original  $\theta(\lambda)$  given by (1.7). Further reductions may be made in the same way if several groups of components have common sample sizes. If  $n_1, n_2, \dots, n_r$  are only approximately equal, their average may be used for  $n$  in the above calculations to obtain an approximate bound.

If all components are subjected to the same number of trials, we may take  $k$  equal to one and the problem reduces to the familiar one of finding an upper confidence bound for a single Poisson parameter.

The component failure data may be developed through independent testing of the components, or through testing of the complete system with the assignment of failures to the appropriate components.

Even in the latter case component sample sizes may differ if components are redesigned during the course of testing so that the trials and failures observed prior to redesign are not relevant to the reliability of the final version of the system.

It is worth noting that the same formal confidence bound problem arises from a somewhat different and more restrictive model involving continuous time sampling and exponentially distributed failure times. Thus, if we assume that the time to failure of the  $i$ -th component has the exponential probability density function  $\mu_i e^{-\mu_i t}$ ,  $\mu_i > 0$ ,  $t > 0$ , and if the fixed aggregate testing time for the  $i$ -th component is given by  $\tau_i$  and the total number of failures observed is  $X_i$ ,  $i=1,2,\dots,k$ , then the  $X_i$ 's which are sufficient statistics have independent Poisson distributions with corresponding parameters  $\lambda_i = \mu_i \tau_i$ . Such test data would be generated, for example, if a single unit of each component were placed on test and repaired whenever a failure occurred until the total test time for each component reached its corresponding limit,  $\tau_i$ . The system reliability for a single duty cycle (consisting of operation for a unit interval of time) is given by

$$R = \prod_{i=1}^k e^{-\mu_i \tau_i} = \exp \left\{ - \sum_{i=1}^k \mu_i \tau_i \right\} = \exp \left\{ - \sum_{i=1}^k \lambda_i / \tau_i \right\}, \quad (1.8)$$

and the confidence bound problem is essentially the same as the one previously introduced except that no approximations are required.

The general theory of confidence bounds based on sample orderings is discussed in Section 2. In Section 3 the case of systems having two components is considered in detail, and the rationale for, and use of, the tables for this case are explained. Section 4 includes suggestions for constructing approximate confidence bounds for cases of systems with  $k \geq 3$  by finding approximately equivalent cases with  $k = 2$ . Use of the maximum likelihood ratio bounds for cases where the data are beyond the limits of the available tables is also discussed.

## 2. Bounds and Orderings: Generalities

The idea of using sample orderings to generate confidence bounds was first introduced by Buehler (1957) who discussed the validity of the proposed method in the context of a specific reliability problem. Bol'shev and Loginov (1969) discuss the construction of confidence bounds monotone in the sample orderings generated by certain functions of the observations. In the following, which is a revision and extension of Johns (1975), we develop the theory with emphasis on the sample orderings themselves rather than possible generators of the orderings.

To develop the general ideas relating exact confidence bounds to sample orderings it is convenient to introduce a fairly abstract statistical model. Let the sample space  $\mathcal{X}$  be endowed with a measurable total ordering relation " $\preceq$ " and let  $X$  represent the random outcome of the experiment where the space of possible outcomes is  $\mathcal{X}$ . Suppose that the possible distributions of  $X$  are determined by the family of probability measures  $P_\lambda$ , indexed by  $\lambda$ , an element of the parameter space  $\Lambda$ . Our objective is to find a  $1 - \alpha$  level upper confidence bound for a specified real-valued function  $\theta(\lambda)$  defined on  $\Lambda$  where the range  $T$  of  $\theta(\lambda)$  is assumed to be closed and bounded below. The quantity  $\alpha \in (0,1)$  is regarded as fixed throughout. We make the following definitions and assumptions:

Definition D1. For each  $x \in \mathcal{X}$  let

$$S_\alpha(x) = \{\lambda : P_\lambda\{X \preceq x\} > \alpha\} . \quad (2.1)$$

Definition D2. For each  $x \in \mathcal{X}$  let

$$t(x) = \begin{cases} \sup\{\theta(\lambda) : \lambda \in S_\alpha(x)\} , & \text{if } S_\alpha(x) \text{ is non-empty} , \\ \inf T, & \text{otherwise} . \end{cases} \quad (2.2)$$

Remark 1. By D2 if  $\theta(\lambda) > t(x)$ , then necessarily  $P_\lambda\{X \preceq x\} \leq \alpha$ .

Assumption A1. For every subset  $C$  of  $\mathcal{X}$  having the property that if  $x \in C$  and  $y \preceq x$  then  $y \in C$ , there exists an ordered sequence

$x_1 \preceq x_2 \preceq \dots$  of elements of  $C$  such that  $C = \bigcup_{n=1}^{\infty} \{x : x \preceq x_n\}$ .

Assumption A2. For each  $x \in \mathcal{X}$ , if  $\theta(\lambda) = t(x)$ , then  $P_\lambda\{X \preceq x\} \leq \alpha$ .

Remark 2. By D1, D2, and A2 if  $\theta(\lambda) = t(x)$ , then  $\lambda \notin S_\alpha(x)$ , i.e., the supremum in D2 is never attained.

We now establish the following propositions.

Proposition P1. The function  $t(x)$  is monotone in the ordering on  $\mathcal{X}$ .

Proof: If  $x, y \in \mathcal{X}$  and  $x \preceq y$ , then  $S_\alpha(x) \subset S_\alpha(y)$  (D1) which in turn implies  $t(x) \leq t(y)$  (D2).  $\square$

Proposition P2. Under assumptions A1 and A2 the function  $t(x)$  is an upper confidence bound for  $\theta(\lambda)$  at level  $1 - \alpha$ . In particular,

$$P_\lambda \{\theta(\lambda) < t(X)\} \leq \alpha \quad \text{for all } \lambda \in \Lambda. \quad (2.3)$$

Proof: For arbitrary  $\lambda_0 \in \Lambda$ , let  $\theta_0 = \theta(\lambda_0)$  and  $C_0 = \{x: t(x) \leq \theta_0\}$ .

The result follows immediately for all  $\lambda_0$  for which  $C_0$  is empty.

Assume that  $C_0$  is non-empty. Then by P1 the set  $C_0$  possesses the

property required in A1 for the existence of a sequence  $\{x_n\} \subset C_0$

such that  $x_n \preceq x_{n+1}$  for all  $n$ , and  $C_0 = \bigcup_{n=1}^{\infty} \{x: x \preceq x_n\}$ . This

implies that, as  $n \rightarrow \infty$ ,

$$P_{\lambda_0} \{X \preceq x_n\} \uparrow P_{\lambda_0} \{X \in C_0\}. \quad (2.4)$$

But by Remark 1 and A2, for all  $n$ ,  $P_{\lambda_0} \{X \preceq x_n\} < \alpha$ . Hence

$P_{\lambda_0} \{X \in C_0\} \leq \alpha$  and the desired result follows.  $\square$

Proposition P3. Under assumptions A1 and A2, if  $\tilde{t}(x)$  is any T-valued confidence bound such that  $P_\lambda\{\theta(\lambda) < \tilde{t}(X)\} \geq 1 - \alpha$  for all  $\lambda \in \Lambda$ , then

(i)  $\sup_{y \preceq x} \tilde{t}(y) \geq t(x)$  for all  $x \in \mathcal{X}$ , and

(ii) if  $t(x)$  is monotone in the ordering on  $\mathcal{X}$ , then

$\tilde{t}(x) \geq t(x)$  for all  $x \in \mathcal{X}$ .

Proof: First we assume that  $\tilde{t}(x)$  is monotone and establish (ii).

Suppose there exists an  $x' \in \mathcal{X}$  such that  $\tilde{t}(x') < t(x')$ . Then  $S_\alpha(x')$  must be non-empty and by Remark 2 following A2, the sup defining  $t(x')$  is not attained. Hence there exists a  $\lambda' \in S_\alpha(x')$  such that  $\tilde{t}(x') < \theta(\lambda') < t(x')$ , and  $P_\lambda\{X \preceq x'\} > \alpha$ . Thus by the monotonicity of  $\tilde{t}(x)$ ,

$$P_\lambda\{\tilde{t}(X) \leq \theta(\lambda')\} \geq P_\lambda\{\tilde{t}(X) \leq \tilde{t}(x')\} = P_\lambda\{X \preceq x'\} > \alpha. \quad (2.5)$$

This contradicts the hypothesis that  $P_\lambda\{\theta(\lambda) < \tilde{t}(X)\} \geq 1 - \alpha$  for all  $\lambda$  and establishes (ii). To show (i) we let  $t^*(x) = \sup_{y \preceq x} \tilde{t}(y)$  and observe that  $t^*(x)$  is monotone in the ordering on  $\mathcal{X}$  and  $t^*(x) \geq \tilde{t}(x)$  for all  $x \in \mathcal{X}$ . Hence if  $\tilde{t}(x)$  is a  $1 - \alpha$  confidence bound for  $\theta(\lambda)$ , so is  $t^*(x)$  and applying (ii) to  $t^*(x)$  yields (i).  $\square$ .

In order to specialize these results in the direction of applications we henceforth assume that the parameter  $\lambda$  and the

observation  $X$  are both of dimension  $k$ , i.e.,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  and  $X = (X_1, X_2, \dots, X_k)$  where the  $\lambda_i$ 's are real and the  $X_i$ 's are random variables. Without essential loss of generality we assume that  $\Lambda$  contains the positive orthant. Within this framework we make the following additional assumptions:

Assumption A3. The function  $\theta(\lambda)$  is continuous and strictly increasing in each of the  $\lambda_i$ 's.

Assumption A4. For any  $x \in \mathcal{X}$ ,  $P_\lambda\{X \leq x\}$  is continuous in each of the  $\lambda_i$ 's.

Proposition P4. Assumptions A3 and A4 imply that Assumption A2 is satisfied.

Proof: Suppose that for some  $x' \in \mathcal{X}$  there exists a  $\lambda' \in \Lambda$  such that  $\theta(\lambda') = t(x')$  and  $P_{\lambda'}\{X \leq x'\} > \alpha$ . Then by A3 and A4 we can find a  $\lambda'' \in \Lambda$  with  $\lambda_i'' \geq \lambda_i'$  for all  $i$  and  $\lambda_{i_0}'' > \lambda_{i_0}'$  for some  $i_0$  such that  $\theta(\lambda'') > t(x')$  and  $P_{\lambda''}\{X \leq x'\} > \alpha$ . This contradicts D2 and the result follows.  $\square$

Corollary C1. If (i) the observations  $X_1, X_2, \dots, X_k$  are independent Poisson random variables with parameters  $\lambda_1, \lambda_2, \dots, \lambda_k$  respectively, and (ii)  $\theta(\lambda) = a_1\lambda_1 + a_2\lambda_2 + \dots + a_k\lambda_k$  where the  $a_i$ 's are positive, then  $t(x)$  given by D2 is an upper confidence bound for  $\theta(\lambda)$  at level  $1 - \alpha$ .

Proof: A1 is satisfied for any ordering since  $\mathcal{X}$  is discrete. A3 is clearly satisfied for  $\theta(\lambda)$  of the form given by (ii), and A4 is satisfied for Poisson random variables. The desired result follows from P4 and P2.  $\square$

The actual computation of the bound  $t(x)$  given by D2 is greatly facilitated if  $P_\lambda\{X \leq x\}$  is monotone in the components of  $\lambda$ . The following proposition gives conditions guaranteeing this property:

Assumption A5. The components of  $X$  are independent and for  $i=1,2,\dots,k$  the distribution of each  $X_i$  depends only on the corresponding  $\lambda_i$  and is stochastically increasing in  $\lambda_i$ .

Definition D3. We denote by  $(\preceq^*)$  the natural partial ordering of  $\mathcal{X}$  generated by componentwise dominance. That is,  $x \preceq^* y$  if and only if  $x_i \leq y_i$  for  $i=1,2,\dots,k$ , with  $x \prec^* y$  if at least one of these inequalities is strict. An arbitrary total ordering  $(\preceq)$  on  $\mathcal{X}$  is consistent with  $(\preceq^*)$  if  $x \preceq^* y$  implies  $x \preceq y$ .

Proposition P5. If the ordering on  $\mathcal{X}$  is consistent with the natural partial ordering and Assumption A5 is satisfied, then for each  $x \in \mathcal{X}$ ,  $P_\lambda\{X \leq x\}$  is non-increasing in each component of  $\lambda$ .

Proof: For  $y \in \mathcal{X}$  we introduce the representation  $y = (y_1, y^{(2)})$  where  $y^{(2)} = (y_2, y_3, \dots, y_k)$ . For real  $z$ , fixed  $x \in \mathcal{X}$ , and all  $y \in \mathcal{X}$ , let  $I_x(z, y^{(2)})$  be the indicator function of the set  $\{y^{(2)} : (z, y^{(2)}) \leq x\}$ .

Then

$$P_{\lambda}\{X \preceq x\} = E_{\lambda} I_x(X_1, X^{(2)}) \quad (2.6)$$

For any  $y \in \mathcal{X}$ , if  $z' \leq z''$ , then  $(z', y^{(2)}) \preceq^* (z'', y^{(2)})$ , by the consistency hypothesis, and  $I_x(z', y^{(2)}) \geq I_x(z'', y^{(2)})$ . Hence letting  $G_x(z) = E_{\lambda}\{I_x(X_1, X^{(2)}) | X_1 = z\}$  we see that  $G_x(z)$  is non-increasing in  $z$ . Thus, since the distribution of  $X_1$  is stochastically increasing in  $\lambda_1$  (A5), we conclude that  $E_{\lambda} I_x(X_1, X^{(2)}) = E_{\lambda} G_x(X_1)$  is non-increasing in  $\lambda_1$ . The same argument applies to the other components of  $\lambda$  establishing the desired result.  $\square$

Suppose that  $\Lambda$  is the non-negative orthant of  $R^{(k)}$  and let  $\mathcal{S}$  be the simplex  $\mathcal{S} = \{\lambda : \sum_{i=1}^k \lambda_i = 1\}$ . If A4 and A5 are satisfied, then by P5 we observe that for any  $\lambda \in \mathcal{S}$  and real  $c$ ,  $P_{c\lambda}\{X \preceq x\}$  is continuous and non-increasing in  $c$ . If the lower bound of  $P_{c\lambda}\{X \preceq x\}$  as  $c \rightarrow \infty$  is less than  $\alpha$  for all  $x \in \mathcal{X}$  and all  $\lambda \in \mathcal{S}$ , then for each  $x \in \mathcal{X}$  and  $\lambda \in \mathcal{S}$  there exists a smallest number  $b = b(x, \lambda)$  such that  $P_{b(x, \lambda)\lambda}\{X \preceq x\} = \alpha$ . The confidence bound  $t(x)$  defined by D2 is then given by

$$t(x) = \sup_{\lambda \in \mathcal{S}} \theta(b(x, \lambda)\lambda) \quad (2.7)$$

Now  $b(x, \lambda)$  is easily computed using root-finding techniques so that the computation of  $t(x)$  reduces to searching over  $\mathcal{S}$  for the maximum of  $\theta(b(x, \lambda)\lambda)$ . Many routines are available for implementing such searches. For the situation described in Corollary C1, the value of

b such that  $P_{b\lambda}\{X \leq x\} = \alpha$  is unique and (2.7) is a computationally feasible version of (1.5).

All of the above results apply mutatis mutandis to the construction of lower confidence bounds and hence confidence intervals.

Applications to the reliability of coherent systems involving the binomial or other distributions are possible. In particular, the above discussion applies directly to the binomial case under the transformation  $\lambda_i = -\log(1-p_i)$ ,  $i = 1, 2, \dots, k$ ;

$$\theta(\lambda) = \sum_{i=1}^k \lambda_i = -\log \prod_{i=1}^k (1-p_i).$$

### 3. Systems With $k = 2$

As was noted in Section 1, if the system has effectively only one component (e.g., when all sample sizes are equal), then the problem reduces to the well-known case of finding an upper confidence bound for a single Poisson parameter. Then in the notation of Section 1,  $\theta(\lambda) = \lambda$  and if  $t(x)$  is the confidence bound for  $\lambda$ , the lower confidence bound (1.6) for reliability  $R$  becomes

$$r(x) = 1 - t(x)/n, \quad (3.1)$$

where  $n$  is the (common) sample size.

The two component case ( $k = 2$ ) presents all of the difficulties of the general case. The principal problem is to generate an ordering of the sample points  $x = (x_1, x_2)$  which will lead to a "good" confidence bound  $t(x)$  computed using (1.5) or (2.7). Several different methods have been considered and implemented to varying extents

in the course of this investigation. These methods may be described briefly as follows:

- (i) The  $x$ 's are ordered according to the values of the function  $\tilde{t}(x) = a_1x_1 + a_2x_2 + z_\alpha \sqrt{a_1^2x_1^2 + a_2^2x_2^2}$ , where  $z_\alpha$  is the upper  $\alpha$ -th quantile of the standard normal distribution.
- (ii) The  $x$ 's are ordered according to the values of the approximate confidence bound obtained from the maximum likelihood ratio statistic (see Section 4).
- (iii) The ordering is generated sequentially by considering at each stage the group of points which are not yet ordered but could be adjoined without violating the natural partial ordering (see D3 of Section 2). The next point in the ordering is then chosen to be the "best" member of the candidate group, i.e., the point producing the smallest value of  $t(x)$  given by (1.5).
- (iv) The ordering is chosen so as to minimize  $E_G\{t(X)\}$  for some suitable prior distribution  $G$  over the values of  $\lambda$ .
- (v) The ordering is generated sequentially in the manner of (iii) above except that at each stage the candidate points for the next two steps are considered as pairs and the next point selected is the first step component which, together with the best available point for the second step, produces the smallest sum for the two values of  $t(x)$ . Note that the point that appears to be "best" two steps ahead may not actually be chosen when that stage is reached.

It is clear that none of these methods is special to the case  $k = 2$ . Method (i), based on the function  $\tilde{t}(x)$ , which is really a maximum likelihood estimate of an asymptotically valid confidence bound, was used to generate tables of bounds for the case  $k = 2$  in Johns (1975). Method (ii) does not improve substantially on Method (i) for moderate values of the  $X_i$ 's. Methods (i) and (ii) are asymptotically equivalent when at least one  $X_i$  becomes large (see Johns 1975) and indeed standard maximum likelihood results guarantee that both are asymptotically optimal. Method (iii) discussed in Johns (1977) was found to be a substantial improvement on (i) in the strong sense that when the Method (iii) ordering is used the values of  $t(x)$  are often smaller and only rarely slightly larger than the values for corresponding  $x$ 's produced by Method (i).

The semi-Bayesian approach of Method (iv), which minimizes the expected length of the confidence interval, is the only one of the five that is directly motivated by optimality considerations. The bound resulting from any reasonable prior must at least be admissible. In pursuing this approach it was decided in the spirit of objectivity and in the hope of rapid convergence to asymptotic optimality to choose a prior distribution leading to an unconditional probability mass function for  $x_1$  and  $x_2$  constant for constant values of the maximum likelihood estimator  $a_1 x_1 + a_2 x_2$  for  $\theta(\lambda)$ . In particular, the prior density for  $\lambda_1$  and  $\lambda_2$  was taken to be

$$g(\lambda_1, \lambda_2) = b_1 b_2 e^{-b_1 \lambda_1 - b_2 \lambda_2}, \lambda_1, \lambda_2 > 0, \quad (3.2)$$

where  $b_1 = (1 - e^{-\beta a_1})$ ,  $b_2 = (1 - e^{-\beta a_2})$ ,  $\beta > 0$ . This produces the unconditional probability mass function

$$p(x_1, x_2) = b_1 b_2 e^{-\beta(a_1 x_1 + a_2 x_2)}, \quad x_1, x_2 = 0, 1, \dots, \quad (3.3)$$

In the limiting case, as  $\beta \rightarrow 0$ ,  $p(x_1, x_2)$  becomes essentially uniform over any finite set of points  $(x_1, x_2)$ .

The actual minimization of  $E_G\{t(X)\}$  may, in principle, be accomplished by finding the ordering which minimizes the contribution to  $E_G$  among all orderings of length  $N$  where  $N$  may be arbitrarily large. This may be done systematically by starting at the origin  $(0,0)$  and constructing a tree whose nodes at each stage are characterized by a candidate point newly adjoined to the ordering and the corresponding value of  $\sum p(x)t(x)$ , where the sum is taken over all  $x$ 's occurring in the path leading to the node, including the one just adjoined. At the  $N$ -th stage the node having the smallest accumulated sum identifies the optimal ordering of length  $N$ . This process may be facilitated by eliminating duplicate nodes and discontinuing branches when a node is reached whose value exceeds that known to be attainable in  $N$  stages. Nevertheless, because of the rapid increase in the number of nodes considered per stage, only the first forty or so points in the optimal orderings could be determined even using a very large computer facility.

In order to obtain examples of admissible orderings with which to compare the results of other methods, this computation was performed for two cases using an IBM 370/168. For both cases the

values  $\alpha = .10$  and  $a_1 = .30$  were used. For the first case the probabilities given in (3.3) with  $\beta = 1$  were used and the first 41 points of the optimal ordering were obtained. The forty-first stage of the computation produced 4557 nodes. For the second case the limiting situation as  $\beta \rightarrow 0$  where the  $p(x)$ 's are all equal was used and the first 43 points of the optimal ordering were obtained. The number of nodes produced at the forty-third stage was 6478.

A comparison of these results with the corresponding results obtained using Methods (i), (iii), and (v) is indicated in Figure 1. The horizontal axis indexes the first 50 points in the ordering produced by Method (v), the two-step prospective sequential procedure. The values of  $t(x)$  for these indexed points for the five methods are indicated by the plotted symbols. Values of  $t(x)$  for methods other than (v) are shown only when they differ from those produced by that method. Based on this evidence it appears that Method (iv) and Method (v) differ very little and that both are better than the other methods. In fact, Method (iv) for the uniform case ( $\beta = 0$ ) differs only trivially from Method (v). Since the use of Method (iv) for the construction of tables is now and probably always will be impractical, we are led to the choice for this purpose of the more tractable and virtually equivalent Method (v). Prospective sequential methods looking ahead more than two steps might be feasible, although the complexity of the computations increases rapidly with the number of steps. However, such procedures would be expected to produce only minute improvement over the two-step method.

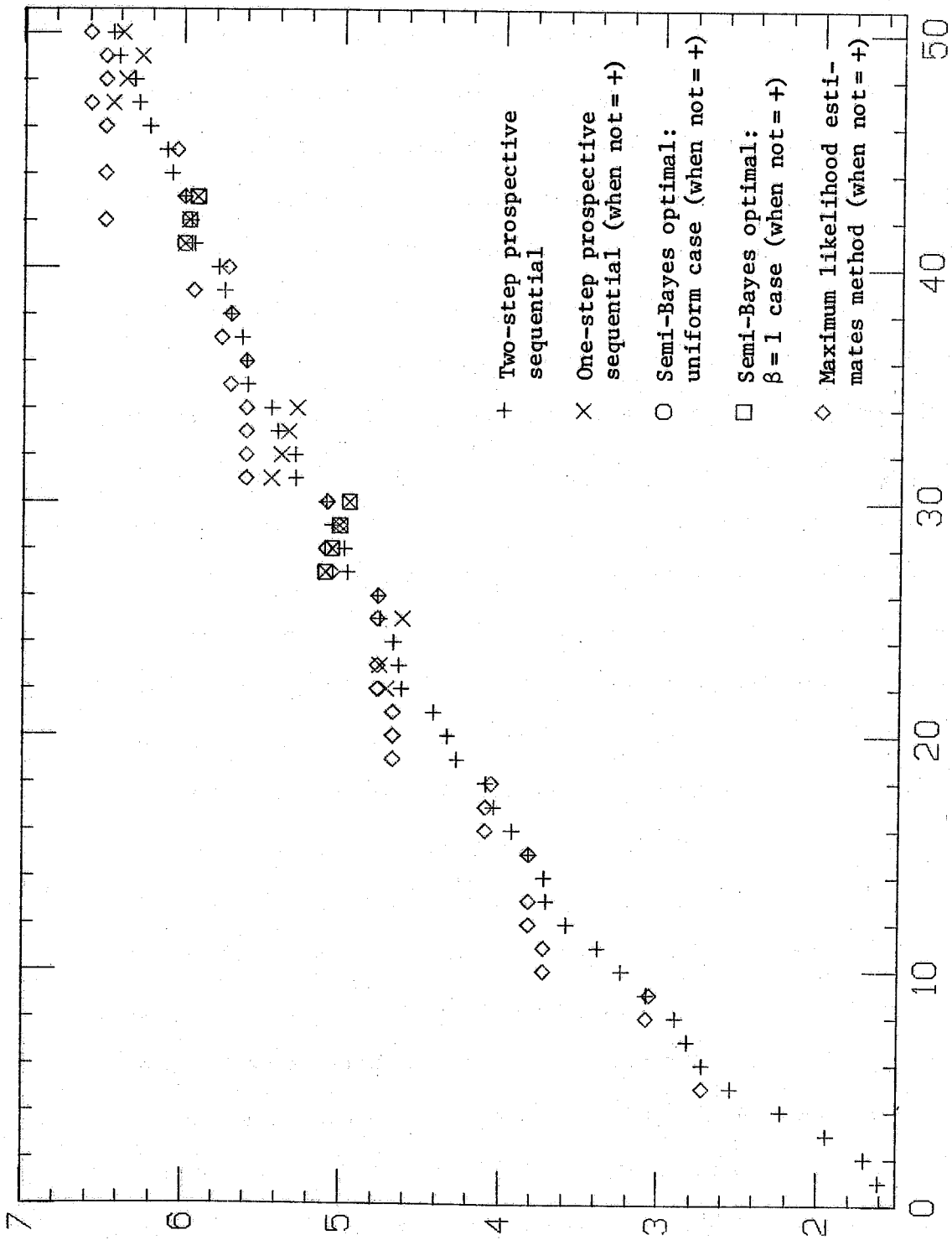


Figure 1. Comparison of confidence bounds for five ordering methods:  $\alpha = .10$ ,  $a_1 = .30$

Table 1 gives values of the upper confidence bound  $t(x)$  computed using (2.7) with the sample orderings generated by the two-step prospective sequential Method (v). The values are given for the first 100 points in each ordering for  $\alpha = .01, .05, .10$ , for  $a_1 = .05(.05).45$ . It is assumed that the components of the system are indexed so that  $a_1 < a_2$  which implies  $a_1 < .50$ . For convenience the values of  $(x_1, x_2)$  are listed systematically rather than in the order generated by the two-step procedure. This table provides a basis for computing accurate confidence bounds for the case  $k = 2$  using only simple interpolation. If values of  $a_1$  greater than .45 but (necessarily) less than .50 are required, the bound for  $a_1 = .5$  (corresponding to  $n_1 = n_2$ ) may be used for interpolation. This bound is obtained by simply multiplying the ordinary upper confidence bound for a single Poisson parameter based on  $x = x_1 + x_2$  failures by .50 (see, e.g., Pearson and Hartley 1958 for tables). The use of Table 1 is illustrated by the following two examples:

Example 1. Suppose that the two components of a series system are tested independently using sample sizes  $n_1 = 300$  and  $n_2 = 100$  respectively with the corresponding observed numbers of failures  $X_1 = 3$  and  $X_2 = 4$ . Then  $c = (1/300 + 1/100) = 4/300$  and  $a_1 = 1/cn_1 = .25 = 1 - a_2$ . If we wish to find a 95 percent confidence interval, we take  $\alpha = .05$ , and from Table 1 we find the confidence bound  $t(x)$  for  $\theta(\lambda)$  to be 7.333. Hence by (1.6) the 95 percent lower confidence bound for system reliability  $R$  is  $1 - (4/300)(7.333) = .902$ .

Table 1. The Confidence Bound  $t(x)$  for  $k = 2$  for the First 100 Points Generated by the Two-Stage Optimal Ordering Method for Each  $a_1$  and  $\alpha$ .

| alpha = 0.01 |    |          |    | alpha = 0.05 |          |    |    | alpha = 0.10 |    |    |          |
|--------------|----|----------|----|--------------|----------|----|----|--------------|----|----|----------|
| x1           | x2 | t(x1,x2) | x1 | x2           | t(x1,x2) | x1 | x2 | t(x1,x2)     | x1 | x2 | t(x1,x2) |
| 0            | 0  | 4.375    | 25 | 0            | 5.262    | 0  | 0  | 2.846        | 25 | 0  | 3.733    |
| 0            | 1  | 6.306    | 25 | 1            | 7.195    | 0  | 1  | 4.507        | 25 | 1  | 5.396    |
| 1            | 0  | 4.376    | 26 | 0            | 5.306    | 1  | 0  | 2.847        | 26 | 0  | 3.695    |
| 1            | 1  | 6.308    | 26 | 1            | 7.239    | 1  | 1  | 4.508        | 26 | 1  | 5.397    |
| 2            | 0  | 4.387    | 27 | 0            | 5.351    | 2  | 0  | 2.858        | 27 | 0  | 2.200    |
| 2            | 1  | 6.319    | 27 | 1            | 7.284    | 2  | 1  | 4.519        | 27 | 1  | 5.485    |
| 3            | 0  | 4.407    | 28 | 0            | 5.395    | 3  | 0  | 2.878        | 28 | 0  | 3.866    |
| 3            | 1  | 6.327    | 28 | 1            | 7.329    | 3  | 1  | 4.539        | 28 | 1  | 5.529    |
| 4            | 0  | 4.432    | 29 | 0            | 5.440    | 4  | 0  | 2.903        | 29 | 0  | 3.911    |
| 4            | 1  | 6.364    | 30 | 0            | 5.485    | 4  | 1  | 4.564        | 29 | 1  | 5.574    |
| 5            | 0  | 4.461    | 31 | 0            | 5.530    | 5  | 0  | 2.932        | 30 | 0  | 3.956    |
| 5            | 1  | 6.393    | 32 | 0            | 5.576    | 5  | 1  | 4.593        | 30 | 1  | 5.619    |
| 6            | 0  | 4.493    | 33 | 0            | 5.621    | 6  | 0  | 2.964        | 31 | 0  | 4.001    |
| 6            | 1  | 6.424    | 34 | 0            | 5.667    | 6  | 1  | 4.624        | 31 | 1  | 5.665    |
| 7            | 0  | 4.526    | 35 | 0            | 5.712    | 7  | 0  | 2.997        | 32 | 0  | 4.047    |
| 7            | 1  | 6.458    | 36 | 0            | 5.758    | 7  | 1  | 4.658        | 33 | 0  | 4.092    |
| 8            | 0  | 4.561    | 37 | 0            | 5.804    | 8  | 0  | 3.032        | 34 | 0  | 4.138    |
| 8            | 1  | 6.493    | 38 | 0            | 5.850    | 8  | 1  | 4.693        | 35 | 0  | 4.183    |
| 9            | 0  | 4.598    | 39 | 0            | 5.896    | 9  | 0  | 3.069        | 36 | 0  | 4.229    |
| 9            | 1  | 6.529    | 40 | 0            | 5.942    | 9  | 1  | 4.730        | 37 | 0  | 4.275    |
| 10           | 0  | 4.635    | 41 | 0            | 5.989    | 10 | 0  | 3.106        | 38 | 0  | 4.321    |
| 10           | 1  | 6.567    | 42 | 0            | 6.035    | 10 | 1  | 4.767        | 39 | 0  | 4.367    |
| 11           | 0  | 4.673    | 43 | 0            | 6.081    | 11 | 0  | 3.144        | 40 | 0  | 4.413    |
| 11           | 1  | 6.605    | 44 | 0            | 6.128    | 11 | 1  | 4.805        | 41 | 0  | 4.460    |
| 12           | 0  | 4.712    | 45 | 0            | 6.174    | 12 | 0  | 3.183        | 42 | 0  | 4.506    |
| 12           | 1  | 6.644    | 46 | 0            | 6.221    | 12 | 1  | 4.845        | 43 | 0  | 4.552    |
| 13           | 0  | 4.752    | 47 | 0            | 6.268    | 13 | 0  | 3.223        | 44 | 0  | 4.599    |
| 13           | 1  | 6.684    | 48 | 0            | 6.315    | 13 | 1  | 4.884        | 45 | 0  | 4.645    |
| 14           | 0  | 4.792    | 49 | 0            | 6.362    | 14 | 0  | 3.264        | 46 | 0  | 4.692    |
| 14           | 1  | 6.724    | 50 | 0            | 6.409    | 14 | 1  | 4.925        | 47 | 0  | 4.739    |
| 15           | 0  | 4.833    | 51 | 0            | 6.456    | 15 | 0  | 3.304        | 48 | 0  | 4.786    |
| 15           | 1  | 6.765    | 52 | 0            | 6.503    | 15 | 1  | 4.966        | 49 | 0  | 4.833    |
| 16           | 0  | 4.875    | 53 | 0            | 6.550    | 16 | 0  | 3.346        | 50 | 0  | 4.880    |
| 16           | 1  | 6.807    | 54 | 0            | 6.597    | 16 | 1  | 5.007        | 51 | 0  | 4.927    |
| 17           | 0  | 4.916    | 55 | 0            | 6.644    | 17 | 0  | 3.388        | 52 | 0  | 4.974    |
| 17           | 1  | 6.849    | 56 | 0            | 6.692    | 17 | 1  | 5.049        | 53 | 0  | 5.021    |
| 18           | 0  | 4.959    | 57 | 0            | 6.739    | 18 | 0  | 3.430        | 54 | 0  | 5.068    |
| 18           | 1  | 6.891    | 58 | 0            | 6.786    | 18 | 1  | 5.091        | 55 | 0  | 5.115    |
| 19           | 0  | 5.001    | 59 | 0            | 6.834    | 19 | 0  | 3.472        | 56 | 0  | 5.163    |
| 19           | 1  | 6.933    | 60 | 0            | 6.881    | 19 | 1  | 5.134        | 57 | 0  | 5.210    |
| 20           | 0  | 5.044    | 61 | 0            | 6.929    | 20 | 0  | 3.515        | 58 | 0  | 5.257    |
| 20           | 1  | 6.976    | 62 | 0            | 6.977    | 20 | 1  | 5.177        | 59 | 0  | 5.305    |
| 21           | 0  | 5.087    | 63 | 0            | 7.024    | 21 | 0  | 3.558        | 60 | 0  | 5.353    |
| 21           | 1  | 7.019    | 64 | 0            | 7.072    | 21 | 1  | 5.220        | 61 | 0  | 5.400    |
| 22           | 0  | 5.130    | 65 | 0            | 7.120    | 22 | 0  | 3.601        | 62 | 0  | 5.448    |
| 22           | 1  | 7.063    | 66 | 0            | 7.168    | 22 | 1  | 5.264        | 63 | 0  | 5.495    |
| 23           | 0  | 5.174    | 67 | 0            | 7.215    | 23 | 0  | 3.645        | 64 | 0  | 5.543    |
| 23           | 1  | 7.107    | 68 | 0            | 7.263    | 23 | 1  | 5.307        | 65 | 0  | 5.591    |
| 24           | 0  | 5.218    | 69 | 0            | 7.311    | 24 | 0  | 3.689        | 66 | 0  | 5.639    |
| 24           | 1  | 7.151    | 70 | 0            | 7.359    | 24 | 1  | 5.351        | 67 | 0  | 5.686    |

Table 1. (Continued)

| a1 = 0.1 |    |          | alpha = 0.01 |    |          | a1 = 0.1 |    |          | alpha = 0.05 |    |          | a1 = 0.1 |    |          | alpha = 0.10 |    |          |
|----------|----|----------|--------------|----|----------|----------|----|----------|--------------|----|----------|----------|----|----------|--------------|----|----------|
| x1       | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) | x1       | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) | x1       | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) |
| 0        | 0  | 4.145    | 17           | 0  | 5.384    | 0        | 0  | 2.696    | 16           | 1  | 5.426    | 0        | 0  | 2.072    | 15           | 1  | 4.568    |
| 0        | 1  | 5.975    | 17           | 1  | 7.221    | 0        | 1  | 4.269    | 16           | 2  | 6.827    | 0        | 1  | 3.501    | 15           | 2  | 5.862    |
| 0        | 2  | 7.565    | 18           | 0  | 5.477    | 0        | 2  | 5.666    | 17           | 0  | 3.936    | 0        | 2  | 4.790    | 16           | 0  | 3.221    |
| 1        | 0  | 4.151    | 18           | 1  | 7.314    | 0        | 3  | 6.978    | 17           | 1  | 5.519    | 0        | 3  | 6.013    | 16           | 1  | 4.659    |
| 1        | 1  | 5.981    | 19           | 0  | 5.569    | 1        | 0  | 2.702    | 17           | 2  | 6.920    | 1        | 0  | 2.078    | 16           | 2  | 5.954    |
| 1        | 2  | 7.571    | 19           | 1  | 7.407    | 1        | 1  | 4.275    | 18           | 0  | 4.028    | 1        | 1  | 3.507    | 17           | 0  | 3.312    |
| 2        | 0  | 4.184    | 20           | 0  | 5.662    | 1        | 2  | 5.672    | 18           | 1  | 5.612    | 1        | 2  | 4.796    | 17           | 1  | 4.752    |
| 2        | 1  | 6.014    | 20           | 1  | 7.501    | 2        | 0  | 2.735    | 19           | 0  | 4.121    | 1        | 3  | 6.019    | 17           | 2  | 6.048    |
| 2        | 2  | 7.605    | 21           | 0  | 5.756    | 2        | 1  | 4.309    | 19           | 1  | 5.706    | 2        | 0  | 2.111    | 18           | 0  | 3.404    |
| 3        | 0  | 4.235    | 21           | 1  | 7.596    | 2        | 2  | 5.706    | 20           | 0  | 4.214    | 2        | 1  | 3.540    | 18           | 1  | 4.850    |
| 3        | 1  | 6.065    | 22           | 0  | 5.850    | 3        | 0  | 2.787    | 20           | 1  | 5.800    | 2        | 2  | 4.829    | 19           | 0  | 3.497    |
| 3        | 2  | 7.656    | 22           | 1  | 7.691    | 3        | 1  | 4.360    | 21           | 0  | 4.307    | 2        | 3  | 6.052    | 19           | 1  | 4.940    |
| 4        | 0  | 4.297    | 23           | 0  | 5.945    | 3        | 2  | 5.757    | 21           | 1  | 5.895    | 3        | 0  | 2.163    | 20           | 0  | 3.590    |
| 4        | 1  | 6.127    | 23           | 1  | 7.786    | 4        | 0  | 2.849    | 22           | 0  | 4.402    | 3        | 1  | 3.590    | 20           | 1  | 5.035    |
| 4        | 2  | 7.718    | 24           | 0  | 6.040    | 4        | 1  | 4.422    | 22           | 1  | 5.991    | 3        | 2  | 4.881    | 21           | 0  | 3.684    |
| 5        | 0  | 4.366    | 24           | 1  | 7.862    | 4        | 2  | 5.819    | 23           | 0  | 4.496    | 3        | 3  | 6.104    | 21           | 1  | 5.130    |
| 5        | 1  | 6.196    | 25           | 0  | 6.135    | 5        | 0  | 2.917    | 23           | 1  | 6.087    | 4        | 0  | 2.225    | 22           | 0  | 3.778    |
| 5        | 2  | 7.786    | 25           | 1  | 7.978    | 5        | 1  | 4.491    | 24           | 0  | 4.591    | 4        | 1  | 3.654    | 22           | 1  | 5.227    |
| 6        | 0  | 4.439    | 26           | 0  | 6.230    | 5        | 2  | 5.889    | 24           | 1  | 6.183    | 4        | 2  | 4.943    | 23           | 0  | 3.873    |
| 6        | 1  | 6.270    | 26           | 1  | 8.075    | 6        | 0  | 2.991    | 25           | 0  | 4.687    | 5        | 0  | 2.293    | 23           | 1  | 5.323    |
| 6        | 2  | 7.861    | 27           | 0  | 6.326    | 6        | 1  | 4.565    | 25           | 1  | 6.280    | 5        | 1  | 3.723    | 24           | 0  | 3.968    |
| 7        | 0  | 4.516    | 27           | 1  | 8.172    | 6        | 2  | 5.963    | 26           | 0  | 4.782    | 5        | 2  | 5.013    | 24           | 1  | 5.423    |
| 7        | 1  | 6.348    | 28           | 0  | 6.422    | 7        | 0  | 3.068    | 26           | 1  | 6.380    | 6        | 0  | 2.367    | 25           | 0  | 4.064    |
| 7        | 2  | 7.939    | 28           | 1  | 8.270    | 7        | 1  | 4.643    | 27           | 0  | 4.878    | 6        | 1  | 3.797    | 25           | 1  | 5.520    |
| 8        | 0  | 4.596    | 29           | 0  | 6.519    | 7        | 2  | 6.040    | 27           | 1  | 6.478    | 6        | 2  | 5.087    | 26           | 0  | 4.160    |
| 8        | 1  | 6.428    | 29           | 1  | 8.369    | 8        | 0  | 3.148    | 28           | 0  | 4.975    | 7        | 0  | 2.444    | 26           | 1  | 5.619    |
| 8        | 2  | 8.020    | 30           | 0  | 6.616    | 8        | 1  | 4.723    | 28           | 1  | 6.577    | 7        | 1  | 3.875    | 27           | 0  | 4.257    |
| 9        | 0  | 4.679    | 30           | 1  | 8.467    | 8        | 2  | 6.121    | 29           | 0  | 5.073    | 7        | 2  | 5.165    | 27           | 1  | 5.718    |
| 9        | 1  | 6.510    | 31           | 0  | 6.713    | 9        | 0  | 3.230    | 29           | 1  | 6.676    | 8        | 0  | 2.524    | 28           | 0  | 4.354    |
| 9        | 2  | 8.102    | 31           | 1  | 8.565    | 9        | 1  | 4.806    | 30           | 0  | 5.170    | 8        | 1  | 3.955    | 28           | 1  | 5.817    |
| 10       | 0  | 4.763    | 32           | 0  | 6.810    | 9        | 2  | 6.204    | 30           | 1  | 6.776    | 8        | 2  | 5.246    | 29           | 0  | 4.451    |
| 10       | 1  | 6.595    | 32           | 1  | 8.665    | 10       | 0  | 3.314    | 31           | 0  | 5.268    | 9        | 0  | 2.606    | 29           | 1  | 5.916    |
| 10       | 2  | 8.187    | 33           | 0  | 6.908    | 10       | 1  | 4.891    | 31           | 1  | 6.878    | 9        | 1  | 4.038    | 30           | 0  | 4.548    |
| 11       | 0  | 4.848    | 34           | 0  | 7.006    | 10       | 2  | 6.289    | 32           | 0  | 5.366    | 9        | 2  | 5.329    | 30           | 1  | 6.016    |
| 11       | 1  | 6.681    | 35           | 0  | 7.104    | 11       | 0  | 3.400    | 33           | 0  | 5.464    | 10       | 0  | 2.690    | 31           | 0  | 4.646    |
| 11       | 2  | 8.273    | 36           | 0  | 7.202    | 11       | 1  | 4.977    | 34           | 0  | 5.563    | 10       | 1  | 4.123    | 31           | 1  | 6.116    |
| 12       | 0  | 4.935    | 37           | 0  | 7.309    | 11       | 2  | 6.376    | 35           | 0  | 5.661    | 10       | 2  | 5.414    | 32           | 0  | 4.744    |
| 12       | 1  | 6.768    | 38           | 0  | 7.399    | 12       | 0  | 3.487    | 36           | 0  | 5.760    | 11       | 0  | 2.776    | 33           | 0  | 4.843    |
| 12       | 2  | 8.361    | 39           | 0  | 7.498    | 12       | 1  | 5.065    | 37           | 0  | 5.859    | 11       | 1  | 4.209    | 34           | 0  | 4.950    |
| 13       | 0  | 5.023    | 40           | 0  | 7.600    | 12       | 2  | 6.464    | 38           | 0  | 5.959    | 11       | 2  | 5.501    | 35           | 0  | 5.049    |
| 13       | 1  | 6.857    | 41           | 0  | 7.700    | 13       | 0  | 3.575    | 39           | 0  | 6.058    | 12       | 0  | 2.863    | 36           | 0  | 5.149    |
| 13       | 2  | 8.450    | 42           | 0  | 7.799    | 13       | 1  | 5.154    | 40           | 0  | 6.158    | 12       | 1  | 4.297    | 37           | 0  | 5.249    |
| 14       | 0  | 5.112    | 43           | 0  | 7.899    | 13       | 2  | 6.553    | 41           | 0  | 6.258    | 12       | 2  | 5.590    | 38           | 0  | 5.349    |
| 14       | 1  | 6.947    | 44           | 0  | 8.000    | 14       | 0  | 3.664    | 42           | 0  | 6.359    | 13       | 0  | 2.951    | 39           | 0  | 5.450    |
| 14       | 2  | 8.540    | 45           | 0  | 8.100    | 14       | 1  | 5.244    | 43           | 0  | 6.459    | 13       | 1  | 4.386    | 40           | 0  | 5.550    |
| 15       | 0  | 5.202    | 46           | 0  | 8.200    | 14       | 2  | 6.644    | 44           | 0  | 6.559    | 13       | 2  | 5.679    | 41           | 0  | 5.651    |
| 15       | 1  | 7.037    | 47           | 0  | 8.301    | 15       | 0  | 3.754    | 45           | 0  | 6.660    | 14       | 0  | 3.040    | 42           | 0  | 5.752    |
| 15       | 2  | 8.631    | 48           | 0  | 8.402    | 15       | 1  | 5.334    | 46           | 0  | 6.761    | 14       | 1  | 4.476    | 43           | 0  | 5.853    |
| 16       | 0  | 5.293    | 49           | 0  | 8.503    | 15       | 2  | 6.735    | 47           | 0  | 6.862    | 14       | 2  | 5.770    | 44           | 0  | 5.955    |
| 16       | 1  | 7.129    | 50           | 0  | 8.604    | 16       | 0  | 3.844    | 48           | 0  | 6.963    | 15       | 0  | 3.130    | 45           | 0  | 6.056    |

Table 1. (Continued)

| alpha = 0.01 |    |          | alpha = 0.05 |    |          | alpha = 0.15 |    |          | alpha = 0.15 |    |          | alpha = 0.10 |    |          |
|--------------|----|----------|--------------|----|----------|--------------|----|----------|--------------|----|----------|--------------|----|----------|
| x1           | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) |
| 0            | 0  | 3.914    | 13           | 1  | 7.143    | 0            | 0  | 2.546    | 0            | 0  | 3.883    | 0            | 0  | 1.957    |
| 0            | 1  | 5.643    | 13           | 2  | 8.659    | 0            | 1  | 4.032    | 0            | 1  | 5.403    | 0            | 1  | 3.306    |
| 0            | 2  | 7.145    | 14           | 0  | 5.538    | 0            | 2  | 5.351    | 0            | 2  | 6.728    | 0            | 2  | 4.524    |
| 0            | 3  | 8.538    | 14           | 1  | 7.291    | 0            | 3  | 6.591    | 0            | 3  | 8.026    | 0            | 3  | 5.679    |
| 1            | 0  | 3.929    | 14           | 2  | 8.809    | 0            | 4  | 7.780    | 0            | 4  | 5.560    | 0            | 4  | 6.795    |
| 1            | 1  | 5.658    | 15           | 0  | 5.684    | 1            | 0  | 2.561    | 1            | 0  | 6.881    | 1            | 0  | 1.972    |
| 1            | 2  | 7.160    | 15           | 1  | 7.443    | 1            | 1  | 4.047    | 1            | 1  | 4.171    | 1            | 1  | 3.321    |
| 1            | 3  | 8.553    | 15           | 2  | 8.964    | 1            | 2  | 5.367    | 1            | 2  | 5.717    | 1            | 2  | 4.530    |
| 2            | 0  | 3.994    | 16           | 0  | 5.831    | 1            | 3  | 6.606    | 1            | 3  | 7.034    | 1            | 3  | 5.694    |
| 2            | 1  | 5.723    | 16           | 1  | 7.594    | 1            | 4  | 7.796    | 1            | 4  | 4.319    | 1            | 4  | 6.810    |
| 2            | 2  | 7.225    | 16           | 2  | 9.117    | 2            | 0  | 2.626    | 1            | 5  | 5.874    | 2            | 0  | 2.037    |
| 2            | 3  | 8.619    | 17           | 0  | 5.979    | 2            | 1  | 4.113    | 2            | 2  | 7.187    | 2            | 1  | 3.397    |
| 3            | 0  | 4.087    | 17           | 1  | 7.747    | 2            | 2  | 5.432    | 2            | 2  | 4.468    | 2            | 2  | 4.605    |
| 3            | 1  | 5.816    | 17           | 2  | 9.272    | 2            | 3  | 6.671    | 1            | 6  | 6.032    | 2            | 3  | 5.759    |
| 3            | 2  | 7.319    | 18           | 0  | 6.128    | 3            | 0  | 2.719    | 1            | 7  | 4.617    | 1            | 6  | 6.875    |
| 3            | 3  | 8.712    | 18           | 1  | 7.901    | 3            | 1  | 4.206    | 2            | 8  | 6.190    | 1            | 7  | 2.129    |
| 4            | 0  | 4.194    | 18           | 2  | 9.427    | 3            | 2  | 5.525    | 1            | 9  | 7.496    | 2            | 8  | 3.480    |
| 4            | 1  | 5.924    | 19           | 0  | 6.278    | 3            | 3  | 6.765    | 1            | 10 | 4.96     | 2            | 9  | 4.698    |
| 4            | 2  | 7.427    | 19           | 1  | 8.056    | 4            | 0  | 2.826    | 1            | 11 | 6.348    | 3            | 3  | 5.854    |
| 4            | 3  | 8.821    | 19           | 2  | 9.583    | 4            | 1  | 4.314    | 1            | 12 | 7.651    | 3            | 4  | 6.970    |
| 5            | 0  | 4.311    | 20           | 0  | 6.429    | 4            | 2  | 5.635    | 1            | 13 | 4.919    | 4            | 0  | 2.237    |
| 5            | 1  | 6.042    | 20           | 1  | 8.212    | 4            | 3  | 6.875    | 1            | 14 | 6.306    | 4            | 1  | 3.589    |
| 5            | 2  | 7.546    | 21           | 0  | 6.588    | 5            | 0  | 2.943    | 1            | 15 | 7.506    | 4            | 2  | 4.808    |
| 5            | 3  | 8.941    | 21           | 1  | 8.381    | 5            | 1  | 4.433    | 1            | 16 | 8.807    | 4            | 3  | 5.964    |
| 6            | 0  | 4.435    | 22           | 0  | 6.741    | 5            | 2  | 5.754    | 2            | 17 | 5.072    | 4            | 4  | 2.354    |
| 6            | 1  | 6.167    | 22           | 1  | 8.538    | 5            | 3  | 6.995    | 2            | 18 | 6.665    | 5            | 1  | 3.708    |
| 6            | 2  | 7.672    | 23           | 0  | 6.896    | 6            | 0  | 3.067    | 2            | 19 | 5.225    | 5            | 2  | 4.928    |
| 6            | 3  | 9.067    | 23           | 1  | 8.696    | 6            | 1  | 4.559    | 2            | 20 | 6.824    | 5            | 3  | 6.085    |
| 7            | 0  | 4.564    | 24           | 0  | 7.050    | 6            | 2  | 5.881    | 2            | 21 | 5.378    | 6            | 0  | 2.478    |
| 7            | 1  | 6.297    | 24           | 1  | 8.855    | 6            | 3  | 7.123    | 2            | 22 | 6.982    | 6            | 1  | 3.834    |
| 7            | 2  | 7.803    | 25           | 0  | 7.206    | 7            | 0  | 3.196    | 2            | 23 | 5.532    | 6            | 2  | 5.056    |
| 7            | 3  | 9.199    | 25           | 1  | 9.014    | 7            | 1  | 4.690    | 2            | 24 | 7.141    | 6            | 3  | 6.216    |
| 8            | 0  | 4.696    | 26           | 0  | 7.362    | 7            | 2  | 6.013    | 2            | 25 | 5.687    | 6            | 4  | 2.606    |
| 8            | 1  | 6.435    | 26           | 1  | 9.173    | 7            | 3  | 7.256    | 2            | 26 | 7.300    | 7            | 1  | 3.966    |
| 8            | 2  | 7.939    | 27           | 0  | 7.519    | 8            | 0  | 3.328    | 2            | 27 | 5.843    | 7            | 2  | 5.189    |
| 8            | 3  | 9.336    | 27           | 1  | 9.333    | 8            | 1  | 4.825    | 2            | 28 | 7.459    | 7            | 3  | 6.347    |
| 9            | 0  | 4.831    | 28           | 0  | 7.675    | 8            | 2  | 6.150    | 2            | 29 | 5.998    | 8            | 0  | 2.739    |
| 9            | 1  | 6.569    | 28           | 1  | 9.492    | 8            | 3  | 7.394    | 2            | 30 | 7.617    | 8            | 1  | 4.102    |
| 9            | 2  | 8.078    | 29           | 0  | 7.833    | 9            | 0  | 3.463    | 2            | 31 | 6.154    | 8            | 2  | 5.327    |
| 9            | 3  | 9.476    | 30           | 0  | 7.991    | 9            | 1  | 4.964    | 2            | 32 | 7.776    | 8            | 3  | 6.485    |
| 10           | 0  | 4.969    | 31           | 0  | 8.146    | 9            | 2  | 6.290    | 2            | 33 | 6.310    | 8            | 4  | 2.874    |
| 10           | 1  | 6.709    | 32           | 0  | 8.307    | 9            | 3  | 7.536    | 2            | 34 | 8.467    | 9            | 1  | 4.241    |
| 10           | 2  | 8.223    | 33           | 0  | 8.465    | 10           | 0  | 3.601    | 2            | 35 | 6.624    | 9            | 2  | 5.469    |
| 11           | 0  | 5.109    | 34           | 0  | 8.624    | 10           | 1  | 5.105    | 2            | 36 | 7.818    | 9            | 3  | 6.627    |
| 11           | 1  | 6.852    | 35           | 0  | 8.783    | 10           | 2  | 6.433    | 2            | 37 | 9.096    | 10           | 0  | 3.012    |
| 11           | 2  | 8.364    | 36           | 0  | 8.942    | 10           | 3  | 7.681    | 2            | 38 | 7.096    | 10           | 1  | 4.384    |
| 12           | 0  | 5.251    | 37           | 0  | 9.101    | 11           | 0  | 3.741    | 2            | 39 | 7.253    | 10           | 2  | 5.610    |
| 12           | 1  | 6.997    | 38           | 0  | 9.260    | 11           | 1  | 5.249    | 2            | 40 | 8.411    | 10           | 3  | 6.772    |
| 12           | 2  | 8.510    | 39           | 0  | 9.420    | 11           | 2  | 6.579    | 2            | 41 | 9.569    | 11           | 0  | 3.152    |
| 13           | 0  | 5.594    | 40           | 0  | 9.580    | 11           | 3  | 7.829    | 2            | 42 | 7.727    | 11           | 1  | 4.529    |

Table 1. (Continued)

| a1 = 0.2 |    |          |  | alpha = 0.01 |    |          |  | a1 = 0.2 |    |          |  | alpha = 0.05 |    |          |  | a1 = 0.2 |    |          |  | alpha = 0.10 |    |          |  |
|----------|----|----------|--|--------------|----|----------|--|----------|----|----------|--|--------------|----|----------|--|----------|----|----------|--|--------------|----|----------|--|
| x1       | x2 | t(x1,x2) |  | x1           | x2 | t(x1,x2) |  | x1       | x2 | t(x1,x2) |  | x1           | x2 | t(x1,x2) |  | x1       | x2 | t(x1,x2) |  | x1           | x2 | t(x1,x2) |  |
| 0        | 0  | 3.684    |  | 10           | 2  | 8.377    |  | 0        | 0  | 2.397    |  | 9            | 3  | 7.664    |  | 0        | 0  | 1.842    |  | 9            | 1  | 4.572    |  |
| 0        | 1  | 5.311    |  | 10           | 3  | 9.716    |  | 0        | 1  | 3.795    |  | 10           | 0  | 3.982    |  | 0        | 1  | 3.112    |  | 9            | 2  | 5.716    |  |
| 0        | 2  | 6.725    |  | 11           | 0  | 5.469    |  | 0        | 2  | 5.037    |  | 10           | 1  | 5.440    |  | 0        | 2  | 4.258    |  | 9            | 3  | 6.820    |  |
| 0        | 3  | 8.036    |  | 11           | 1  | 7.153    |  | 0        | 3  | 6.203    |  | 10           | 2  | 6.697    |  | 0        | 3  | 5.345    |  | 10           | 0  | 3.438    |  |
| 0        | 4  | 9.284    |  | 11           | 2  | 8.590    |  | 0        | 4  | 7.323    |  | 10           | 3  | 7.873    |  | 0        | 4  | 6.395    |  | 10           | 1  | 4.786    |  |
| 1        | 0  | 3.714    |  | 11           | 3  | 9.938    |  | 0        | 5  | 8.410    |  | 11           | 0  | 4.187    |  | 0        | 5  | 7.420    |  | 10           | 2  | 5.925    |  |
| 1        | 1  | 5.341    |  | 12           | 0  | 5.676    |  | 1        | 0  | 2.427    |  | 11           | 1  | 5.652    |  | 1        | 0  | 1.872    |  | 10           | 3  | 7.030    |  |
| 1        | 2  | 6.755    |  | 12           | 1  | 7.381    |  | 1        | 1  | 3.825    |  | 11           | 2  | 6.908    |  | 1        | 1  | 3.142    |  | 11           | 0  | 3.646    |  |
| 1        | 3  | 8.067    |  | 12           | 2  | 8.805    |  | 1        | 2  | 5.067    |  | 11           | 3  | 8.084    |  | 1        | 2  | 4.288    |  | 11           | 1  | 5.001    |  |
| 1        | 4  | 9.314    |  | 12           | 3  | 10.155   |  | 1        | 3  | 6.233    |  | 12           | 0  | 4.395    |  | 1        | 3  | 5.375    |  | 11           | 2  | 6.135    |  |
| 2        | 0  | 3.821    |  | 13           | 0  | 5.885    |  | 1        | 4  | 7.353    |  | 12           | 1  | 5.665    |  | 1        | 4  | 6.425    |  | 11           | 3  | 7.261    |  |
| 2        | 1  | 5.449    |  | 13           | 1  | 7.598    |  | 1        | 5  | 8.441    |  | 12           | 2  | 7.120    |  | 1        | 5  | 7.450    |  | 12           | 0  | 3.856    |  |
| 2        | 2  | 6.864    |  | 13           | 2  | 9.020    |  | 2        | 0  | 2.534    |  | 12           | 3  | 8.295    |  | 2        | 0  | 1.979    |  | 12           | 1  | 5.215    |  |
| 2        | 3  | 8.176    |  | 14           | 0  | 6.096    |  | 2        | 1  | 3.934    |  | 13           | 0  | 4.605    |  | 2        | 1  | 3.251    |  | 12           | 2  | 6.345    |  |
| 2        | 4  | 9.424    |  | 14           | 1  | 7.815    |  | 2        | 2  | 5.176    |  | 13           | 1  | 6.158    |  | 2        | 2  | 4.398    |  | 12           | 3  | 7.471    |  |
| 3        | 0  | 3.965    |  | 14           | 2  | 9.236    |  | 2        | 3  | 6.343    |  | 13           | 2  | 7.332    |  | 2        | 3  | 5.486    |  | 13           | 0  | 4.067    |  |
| 3        | 1  | 5.595    |  | 15           | 0  | 6.308    |  | 2        | 4  | 7.464    |  | 13           | 3  | 8.506    |  | 2        | 4  | 6.536    |  | 13           | 1  | 5.430    |  |
| 3        | 2  | 7.011    |  | 15           | 1  | 8.034    |  | 2        | 5  | 8.552    |  | 14           | 0  | 4.816    |  | 2        | 5  | 7.561    |  | 13           | 2  | 6.555    |  |
| 3        | 3  | 8.324    |  | 15           | 2  | 9.452    |  | 3        | 0  | 2.677    |  | 14           | 1  | 6.320    |  | 3        | 0  | 2.123    |  | 13           | 3  | 7.681    |  |
| 3        | 4  | 9.573    |  | 16           | 0  | 6.522    |  | 3        | 1  | 4.081    |  | 14           | 2  | 7.544    |  | 3        | 1  | 3.399    |  | 14           | 0  | 4.279    |  |
| 4        | 0  | 4.128    |  | 16           | 1  | 8.252    |  | 3        | 2  | 5.325    |  | 15           | 0  | 5.072    |  | 3        | 2  | 4.547    |  | 14           | 1  | 5.644    |  |
| 4        | 1  | 5.761    |  | 16           | 2  | 9.669    |  | 3        | 3  | 6.493    |  | 15           | 1  | 6.536    |  | 3        | 3  | 5.636    |  | 14           | 2  | 6.766    |  |
| 4        | 2  | 7.179    |  | 17           | 0  | 6.776    |  | 3        | 4  | 7.614    |  | 15           | 2  | 7.757    |  | 3        | 4  | 6.688    |  | 15           | 0  | 4.491    |  |
| 4        | 3  | 8.494    |  | 17           | 1  | 8.471    |  | 4        | 0  | 4.248    |  | 16           | 0  | 5.288    |  | 3        | 5  | 7.714    |  | 15           | 1  | 5.859    |  |
| 4        | 4  | 9.744    |  | 17           | 2  | 9.845    |  | 4        | 1  | 5.496    |  | 16           | 1  | 6.752    |  | 4        | 0  | 2.286    |  | 15           | 2  | 6.930    |  |
| 5        | 0  | 4.302    |  | 18           | 0  | 6.996    |  | 4        | 2  | 6.664    |  | 16           | 2  | 7.970    |  | 4        | 1  | 3.568    |  | 16           | 0  | 4.705    |  |
| 5        | 1  | 5.940    |  | 18           | 1  | 8.691    |  | 4        | 3  | 7.787    |  | 17           | 0  | 5.506    |  | 4        | 2  | 4.719    |  | 16           | 1  | 6.073    |  |
| 5        | 2  | 7.361    |  | 18           | 2  | 10.060   |  | 4        | 4  | 8.681    |  | 17           | 1  | 6.967    |  | 4        | 3  | 5.810    |  | 16           | 2  | 7.139    |  |
| 5        | 3  | 8.678    |  | 19           | 0  | 7.216    |  | 5        | 0  | 3.015    |  | 17           | 2  | 8.183    |  | 4        | 4  | 6.864    |  | 17           | 0  | 4.919    |  |
| 5        | 4  | 9.926    |  | 19           | 1  | 8.910    |  | 5        | 1  | 4.430    |  | 18           | 0  | 5.723    |  | 5        | 0  | 2.460    |  | 17           | 1  | 6.287    |  |
| 6        | 0  | 4.485    |  | 19           | 2  | 10.276   |  | 5        | 2  | 5.681    |  | 18           | 1  | 7.183    |  | 5        | 1  | 3.751    |  | 17           | 2  | 7.349    |  |
| 6        | 1  | 6.129    |  | 20           | 0  | 7.436    |  | 5        | 3  | 6.850    |  | 18           | 2  | 8.397    |  | 5        | 2  | 4.904    |  | 18           | 0  | 5.133    |  |
| 6        | 2  | 7.552    |  | 20           | 1  | 9.129    |  | 5        | 4  | 7.975    |  | 19           | 0  | 5.941    |  | 5        | 3  | 5.999    |  | 18           | 1  | 6.502    |  |
| 6        | 3  | 8.872    |  | 21           | 0  | 7.657    |  | 6        | 0  | 3.198    |  | 19           | 1  | 7.399    |  | 5        | 4  | 7.056    |  | 18           | 2  | 7.559    |  |
| 6        | 4  | 10.118   |  | 21           | 1  | 9.349    |  | 6        | 1  | 4.640    |  | 20           | 0  | 6.104    |  | 6        | 0  | 2.643    |  | 19           | 0  | 5.348    |  |
| 7        | 0  | 4.674    |  | 22           | 0  | 7.877    |  | 6        | 2  | 5.890    |  | 20           | 1  | 7.614    |  | 6        | 1  | 3.944    |  | 19           | 1  | 6.716    |  |
| 7        | 1  | 6.339    |  | 22           | 1  | 9.569    |  | 6        | 3  | 7.044    |  | 21           | 0  | 6.376    |  | 6        | 2  | 5.099    |  | 20           | 0  | 5.563    |  |
| 7        | 2  | 7.750    |  | 23           | 0  | 8.098    |  | 6        | 4  | 8.172    |  | 21           | 1  | 7.830    |  | 6        | 3  | 6.198    |  | 20           | 1  | 6.978    |  |
| 7        | 3  | 9.075    |  | 23           | 1  | 9.886    |  | 7        | 0  | 3.366    |  | 22           | 0  | 6.594    |  | 6        | 4  | 7.228    |  | 21           | 0  | 5.778    |  |
| 7        | 4  | 10.316   |  | 24           | 0  | 8.319    |  | 7        | 1  | 4.855    |  | 22           | 1  | 8.046    |  | 7        | 0  | 2.832    |  | 21           | 1  | 7.189    |  |
| 8        | 0  | 4.867    |  | 24           | 1  | 10.103   |  | 7        | 2  | 6.066    |  | 23           | 0  | 6.812    |  | 7        | 1  | 4.146    |  | 22           | 0  | 5.994    |  |
| 8        | 1  | 6.557    |  | 25           | 0  | 8.540    |  | 7        | 3  | 7.244    |  | 23           | 1  | 8.262    |  | 7        | 2  | 5.299    |  | 22           | 1  | 7.401    |  |
| 8        | 2  | 7.954    |  | 26           | 0  | 8.761    |  | 7        | 4  | 8.375    |  | 24           | 0  | 7.029    |  | 7        | 3  | 6.402    |  | 23           | 0  | 6.209    |  |
| 8        | 3  | 9.285    |  | 27           | 0  | 8.982    |  | 8        | 0  | 3.579    |  | 24           | 1  | 8.477    |  | 7        | 4  | 7.422    |  | 23           | 1  | 7.613    |  |
| 9        | 0  | 5.064    |  | 28           | 0  | 9.203    |  | 8        | 1  | 5.019    |  | 25           | 0  | 7.247    |  | 8        | 0  | 3.025    |  | 24           | 0  | 6.425    |  |
| 9        | 1  | 6.729    |  | 29           | 0  | 9.425    |  | 8        | 2  | 6.269    |  | 26           | 0  | 7.465    |  | 8        | 1  | 4.356    |  | 25           | 0  | 6.641    |  |
| 9        | 2  | 8.161    |  | 30           | 0  | 9.646    |  | 8        | 3  | 7.449    |  | 27           | 0  | 7.683    |  | 8        | 2  | 5.509    |  | 26           | 0  | 6.857    |  |
| 9        | 3  | 9.500    |  | 31           | 0  | 9.802    |  | 9        | 0  | 3.776    |  | 28           | 0  | 7.901    |  | 8        | 3  | 6.612    |  | 27           | 0  | 7.073    |  |
| 10       | 0  | 5.264    |  | 32           | 0  | 10.023   |  | 9        | 1  | 5.230    |  | 29           | 0  | 8.119    |  | 8        | 4  | 7.627    |  | 28           | 0  | 7.289    |  |
| 10       | 1  | 6.940    |  | 33           | 0  | 10.244   |  | 9        | 2  | 6.474    |  | 30           | 0  | 8.337    |  | 9        | 0  | 3.224    |  | 29           | 0  | 7.506    |  |



Table 1. (Continued)

| alpha = 0.01 |    |          |    | alpha = 0.05 |          |    |    | alpha = 0.10 |    |    |          |    |    |          |
|--------------|----|----------|----|--------------|----------|----|----|--------------|----|----|----------|----|----|----------|
| x1           | x2 | t(x1,x2) | x1 | x2           | t(x1,x2) | x1 | x2 | t(x1,x2)     | x1 | x2 | t(x1,x2) | x1 | x2 | t(x1,x2) |
| 0            | 0  | 3.224    | 7  | 2            | 7.907    | 0  | 0  | 2.097        | 7  | 1  | 5.319    | 0  | 0  | 1.612    |
| 0            | 1  | 4.647    | 7  | 3            | 9.143    | 0  | 1  | 3.321        | 7  | 2  | 6.465    | 0  | 1  | 2.723    |
| 0            | 2  | 5.884    | 7  | 4            | 10.206   | 0  | 2  | 4.407        | 7  | 3  | 7.427    | 0  | 2  | 3.726    |
| 0            | 3  | 7.032    | 7  | 5            | 11.097   | 0  | 3  | 5.428        | 7  | 4  | 8.440    | 0  | 3  | 4.676    |
| 0            | 4  | 8.123    | 8  | 0            | 5.549    | 0  | 4  | 6.407        | 7  | 5  | 9.386    | 0  | 4  | 5.597    |
| 0            | 5  | 9.176    | 8  | 1            | 7.104    | 0  | 5  | 7.359        | 8  | 0  | 4.446    | 0  | 5  | 6.492    |
| 0            | 6  | 10.199   | 8  | 2            | 8.257    | 0  | 6  | 8.290        | 8  | 1  | 5.751    | 0  | 6  | 7.373    |
| 0            | 7  | 11.200   | 8  | 3            | 9.476    | 0  | 7  | 9.204        | 8  | 2  | 6.810    | 0  | 7  | 8.240    |
| 1            | 0  | 3.315    | 8  | 4            | 10.545   | 1  | 0  | 2.189        | 8  | 3  | 7.667    | 1  | 0  | 1.703    |
| 1            | 1  | 4.741    | 9  | 0            | 5.910    | 1  | 1  | 3.416        | 8  | 4  | 8.620    | 1  | 1  | 2.818    |
| 1            | 2  | 5.979    | 9  | 1            | 7.465    | 1  | 2  | 4.504        | 9  | 0  | 4.801    | 1  | 2  | 3.823    |
| 1            | 3  | 7.128    | 9  | 2            | 8.607    | 1  | 3  | 5.525        | 9  | 1  | 6.006    | 1  | 3  | 4.776    |
| 1            | 4  | 8.220    | 9  | 3            | 9.816    | 1  | 4  | 6.506        | 9  | 2  | 6.987    | 1  | 4  | 5.696    |
| 1            | 5  | 9.274    | 9  | 4            | 10.885   | 1  | 5  | 7.459        | 9  | 3  | 8.164    | 1  | 5  | 6.594    |
| 1            | 6  | 10.298   | 10 | 0            | 6.273    | 1  | 6  | 8.390        | 9  | 4  | 8.953    | 1  | 6  | 7.475    |
| 2            | 0  | 3.552    | 10 | 1            | 7.727    | 1  | 7  | 9.305        | 10 | 0  | 5.155    | 1  | 7  | 8.337    |
| 2            | 1  | 4.992    | 10 | 2            | 8.959    | 2  | 0  | 2.425        | 10 | 1  | 6.354    | 2  | 0  | 1.940    |
| 2            | 2  | 6.232    | 10 | 3            | 9.989    | 2  | 1  | 3.666        | 10 | 2  | 7.487    | 2  | 1  | 3.074    |
| 2            | 3  | 7.386    | 10 | 4            | 11.248   | 2  | 2  | 4.765        | 10 | 3  | 8.341    | 2  | 2  | 4.096    |
| 2            | 4  | 8.484    | 11 | 0            | 6.637    | 2  | 3  | 5.798        | 10 | 4  | 9.284    | 2  | 3  | 5.060    |
| 2            | 5  | 9.543    | 11 | 1            | 8.081    | 2  | 4  | 6.785        | 11 | 0  | 5.510    | 2  | 4  | 5.932    |
| 2            | 6  | 10.573   | 11 | 2            | 9.311    | 2  | 5  | 7.695        | 11 | 1  | 6.699    | 2  | 5  | 6.830    |
| 3            | 0  | 3.840    | 11 | 3            | 10.332   | 2  | 6  | 8.627        | 11 | 2  | 7.825    | 2  | 6  | 7.711    |
| 3            | 1  | 5.310    | 12 | 0            | 2.714    | 3  | 0  | 2.714        | 11 | 3  | 8.772    | 2  | 7  | 8.579    |
| 3            | 2  | 6.538    | 12 | 1            | 8.435    | 3  | 1  | 3.970        | 12 | 0  | 5.865    | 3  | 0  | 2.228    |
| 3            | 3  | 7.699    | 12 | 2            | 9.665    | 3  | 2  | 5.082        | 12 | 1  | 7.108    | 3  | 1  | 3.384    |
| 3            | 4  | 8.804    | 12 | 3            | 10.675   | 3  | 3  | 6.044        | 12 | 2  | 7.999    | 3  | 2  | 4.340    |
| 3            | 5  | 9.870    | 13 | 0            | 7.367    | 3  | 4  | 7.026        | 12 | 3  | 9.104    | 3  | 3  | 5.294    |
| 3            | 6  | 10.909   | 13 | 1            | 8.788    | 3  | 5  | 8.065        | 13 | 0  | 6.221    | 3  | 4  | 6.212    |
| 4            | 0  | 4.154    | 13 | 2            | 10.089   | 3  | 6  | 8.907        | 13 | 1  | 7.289    | 3  | 5  | 7.111    |
| 4            | 1  | 5.663    | 13 | 3            | 11.018   | 4  | 0  | 3.027        | 13 | 2  | 8.502    | 3  | 6  | 7.992    |
| 4            | 2  | 6.865    | 14 | 0            | 7.826    | 4  | 1  | 4.294        | 13 | 3  | 9.636    | 4  | 0  | 2.542    |
| 4            | 3  | 8.028    | 14 | 1            | 9.133    | 4  | 2  | 5.412        | 14 | 0  | 6.577    | 4  | 1  | 3.710    |
| 4            | 4  | 9.042    | 14 | 2            | 10.438   | 4  | 3  | 6.341        | 14 | 1  | 7.629    | 4  | 2  | 4.641    |
| 4            | 5  | 10.097   | 15 | 0            | 8.186    | 4  | 4  | 7.322        | 14 | 2  | 8.841    | 4  | 3  | 5.590    |
| 4            | 6  | 11.212   | 15 | 1            | 9.496    | 4  | 5  | 8.275        | 15 | 0  | 6.934    | 4  | 4  | 6.517    |
| 5            | 0  | 4.483    | 15 | 2            | 10.785   | 4  | 6  | 9.213        | 15 | 1  | 8.131    | 4  | 5  | 7.415    |
| 5            | 1  | 6.024    | 16 | 0            | 8.547    | 5  | 0  | 3.360        | 15 | 2  | 9.179    | 4  | 6  | 8.425    |
| 5            | 2  | 7.214    | 16 | 1            | 9.850    | 5  | 1  | 4.637        | 16 | 0  | 7.387    | 5  | 0  | 2.892    |
| 5            | 3  | 8.371    | 16 | 2            | 11.131   | 5  | 2  | 5.658        | 16 | 1  | 8.310    | 5  | 1  | 4.047    |
| 5            | 4  | 9.382    | 17 | 0            | 8.909    | 5  | 3  | 6.670        | 16 | 2  | 9.518    | 5  | 2  | 4.962    |
| 5            | 5  | 10.433   | 17 | 1            | 10.203   | 5  | 4  | 7.777        | 17 | 0  | 7.735    | 5  | 3  | 5.992    |
| 6            | 0  | 4.837    | 18 | 0            | 9.270    | 5  | 5  | 8.728        | 17 | 1  | 8.652    | 5  | 4  | 6.898    |
| 6            | 1  | 6.384    | 18 | 1            | 10.557   | 6  | 0  | 3.742        | 18 | 0  | 8.008    | 5  | 5  | 7.787    |
| 6            | 2  | 7.559    | 19 | 0            | 9.631    | 6  | 1  | 4.977        | 18 | 1  | 8.993    | 5  | 6  | 8.661    |
| 6            | 3  | 8.712    | 19 | 1            | 10.911   | 6  | 2  | 6.129        | 19 | 0  | 8.426    | 6  | 0  | 3.235    |
| 6            | 4  | 9.714    | 20 | 0            | 9.994    | 6  | 3  | 7.148        | 19 | 1  | 9.333    | 6  | 1  | 4.427    |
| 6            | 5  | 10.763   | 21 | 0            | 10.356   | 6  | 4  | 7.969        | 20 | 0  | 8.770    | 6  | 2  | 5.437    |
| 7            | 0  | 5.192    | 22 | 0            | 10.718   | 6  | 5  | 9.056        | 21 | 0  | 9.117    | 6  | 3  | 6.311    |
| 7            | 1  | 6.744    | 23 | 0            | 11.080   | 7  | 0  | 4.094        | 22 | 0  | 9.462    | 6  | 4  | 7.215    |

Table 1. (Continued)

| alpha = 0.01 |    |          |    | alpha = 0.05 |          |    |    | alpha = 0.10 |    |    |          |    |    |          |
|--------------|----|----------|----|--------------|----------|----|----|--------------|----|----|----------|----|----|----------|
| x1           | x2 | t(x1,x2) | x1 | x2           | t(x1,x2) | x1 | x2 | t(x1,x2)     | x1 | x2 | t(x1,x2) | x1 | x2 | t(x1,x2) |
| 0            | 0  | 2.993    | 6  | 2            | 7.650    | 0  | 0  | 1.947        | 6  | 0  | 3.687    | 6  | 0  | 4.547    |
| 0            | 1  | 4.315    | 6  | 3            | 8.678    | 0  | 1  | 3.084        | 6  | 1  | 4.528    | 6  | 1  | 5.636    |
| 0            | 2  | 5.464    | 6  | 4            | 9.681    | 0  | 2  | 4.092        | 6  | 2  | 5.636    | 6  | 2  | 6.490    |
| 0            | 3  | 6.529    | 6  | 5            | 10.649   | 0  | 3  | 5.040        | 6  | 3  | 6.490    | 6  | 3  | 7.323    |
| 0            | 4  | 7.543    | 6  | 6            | 11.583   | 0  | 4  | 5.950        | 6  | 4  | 7.323    | 6  | 4  | 8.149    |
| 0            | 5  | 8.520    | 6  | 7            | 12.483   | 0  | 5  | 6.834        | 6  | 5  | 8.149    | 6  | 5  | 8.949    |
| 0            | 6  | 9.471    | 6  | 8            | 13.346   | 0  | 6  | 7.698        | 6  | 6  | 8.949    | 6  | 6  | 9.724    |
| 0            | 7  | 10.400   | 6  | 9            | 14.174   | 0  | 7  | 8.546        | 6  | 7  | 9.724    | 6  | 7  | 10.487   |
| 0            | 8  | 11.312   | 6  | 10           | 14.964   | 0  | 8  | 9.383        | 6  | 8  | 10.487   | 6  | 8  | 11.247   |
| 1            | 1  | 4.478    | 7  | 3            | 10.094   | 1  | 1  | 3.243        | 7  | 3  | 7.653    | 7  | 1  | 4.947    |
| 1            | 2  | 5.672    | 7  | 4            | 11.059   | 1  | 2  | 4.256        | 7  | 4  | 8.524    | 7  | 2  | 5.962    |
| 1            | 3  | 6.692    | 7  | 5            | 11.936   | 1  | 3  | 5.207        | 7  | 5  | 9.190    | 7  | 3  | 6.946    |
| 1            | 4  | 7.708    | 7  | 6            | 12.736   | 1  | 4  | 6.119        | 7  | 6  | 9.855    | 7  | 4  | 7.761    |
| 1            | 5  | 8.687    | 7  | 7            | 13.464   | 1  | 5  | 7.005        | 7  | 7  | 10.502   | 7  | 5  | 8.333    |
| 1            | 6  | 9.639    | 7  | 8            | 14.120   | 1  | 6  | 7.871        | 7  | 8  | 11.119   | 7  | 6  | 8.844    |
| 1            | 7  | 10.569   | 7  | 9            | 14.708   | 1  | 7  | 8.721        | 7  | 9  | 11.708   | 7  | 7  | 9.313    |
| 1            | 8  | 11.482   | 7  | 10           | 15.229   | 1  | 8  | 9.559        | 7  | 10 | 12.264   | 7  | 8  | 9.748    |
| 2            | 0  | 3.485    | 8  | 1            | 7.828    | 2  | 0  | 2.439        | 8  | 1  | 6.486    | 8  | 1  | 5.048    |
| 2            | 1  | 4.871    | 8  | 2            | 8.937    | 2  | 1  | 3.587        | 8  | 2  | 7.435    | 8  | 2  | 5.749    |
| 2            | 2  | 5.971    | 8  | 3            | 9.938    | 2  | 2  | 4.603        | 8  | 3  | 8.221    | 8  | 3  | 6.249    |
| 2            | 3  | 7.043    | 8  | 4            | 10.923   | 2  | 3  | 5.557        | 8  | 4  | 9.321    | 8  | 4  | 6.749    |
| 2            | 4  | 8.056    | 8  | 5            | 11.888   | 2  | 4  | 6.471        | 8  | 5  | 10.069   | 8  | 5  | 7.249    |
| 2            | 5  | 9.037    | 8  | 6            | 12.826   | 2  | 5  | 7.359        | 8  | 6  | 10.793   | 8  | 6  | 7.749    |
| 2            | 6  | 9.990    | 8  | 7            | 13.745   | 2  | 6  | 8.123        | 8  | 7  | 11.493   | 8  | 7  | 8.249    |
| 2            | 7  | 10.921   | 8  | 8            | 14.644   | 2  | 7  | 8.971        | 8  | 8  | 12.164   | 8  | 8  | 8.749    |
| 3            | 0  | 3.881    | 9  | 1            | 11.470   | 3  | 0  | 2.834        | 9  | 1  | 7.008    | 9  | 1  | 6.149    |
| 3            | 1  | 5.203    | 9  | 2            | 12.557   | 3  | 1  | 3.971        | 9  | 2  | 8.071    | 9  | 2  | 7.210    |
| 3            | 2  | 6.361    | 9  | 3            | 13.621   | 3  | 2  | 4.984        | 9  | 3  | 9.037    | 9  | 3  | 7.852    |
| 3            | 3  | 7.440    | 9  | 4            | 14.668   | 3  | 3  | 5.818        | 9  | 4  | 10.037   | 9  | 4  | 8.470    |
| 3            | 4  | 8.439    | 9  | 5            | 15.694   | 3  | 4  | 6.724        | 9  | 5  | 11.037   | 9  | 5  | 9.060    |
| 3            | 5  | 9.419    | 9  | 6            | 16.697   | 3  | 5  | 7.743        | 9  | 6  | 12.037   | 9  | 6  | 9.624    |
| 3            | 6  | 10.373   | 9  | 7            | 17.678   | 3  | 6  | 8.608        | 9  | 7  | 13.037   | 9  | 7  | 10.174   |
| 3            | 7  | 11.179   | 9  | 8            | 18.634   | 3  | 7  | 9.312        | 9  | 8  | 14.037   | 9  | 8  | 10.704   |
| 4            | 0  | 4.304    | 10 | 1            | 11.911   | 4  | 0  | 3.301        | 10 | 1  | 7.612    | 10 | 1  | 6.924    |
| 4            | 1  | 5.615    | 10 | 2            | 13.047   | 4  | 1  | 4.402        | 10 | 2  | 8.748    | 10 | 2  | 7.996    |
| 4            | 2  | 6.803    | 10 | 3            | 14.153   | 4  | 2  | 5.399        | 10 | 3  | 9.437    | 10 | 3  | 8.633    |
| 4            | 3  | 7.869    | 10 | 4            | 15.234   | 4  | 3  | 6.342        | 10 | 4  | 10.318   | 10 | 4  | 9.244    |
| 4            | 4  | 8.859    | 10 | 5            | 16.291   | 4  | 4  | 7.140        | 10 | 5  | 11.164   | 10 | 5  | 9.834    |
| 4            | 5  | 9.834    | 10 | 6            | 17.324   | 4  | 5  | 8.241        | 10 | 6  | 12.037   | 10 | 6  | 10.404   |
| 4            | 6  | 10.783   | 10 | 7            | 18.334   | 4  | 6  | 9.078        | 10 | 7  | 12.937   | 10 | 7  | 10.974   |
| 4            | 7  | 11.719   | 10 | 8            | 19.319   | 4  | 7  | 9.718        | 10 | 8  | 13.864   | 10 | 8  | 11.524   |
| 5            | 0  | 4.758    | 11 | 1            | 10.437   | 5  | 0  | 3.736        | 11 | 1  | 8.456    | 11 | 1  | 7.336    |
| 5            | 1  | 6.103    | 11 | 2            | 11.437   | 5  | 1  | 4.817        | 11 | 2  | 9.190    | 11 | 2  | 8.149    |
| 5            | 2  | 7.226    | 11 | 3            | 12.437   | 5  | 2  | 5.998        | 11 | 3  | 10.069   | 11 | 3  | 8.849    |
| 5            | 3  | 8.291    | 11 | 4            | 13.437   | 5  | 3  | 6.860        | 11 | 4  | 10.969   | 11 | 4  | 9.549    |
| 5            | 4  | 9.267    | 11 | 5            | 14.437   | 5  | 4  | 7.528        | 11 | 5  | 11.864   | 11 | 5  | 10.249   |
| 5            | 5  | 10.239   | 11 | 6            | 15.437   | 5  | 5  | 8.401        | 11 | 6  | 12.764   | 11 | 6  | 10.949   |
| 5            | 6  | 11.374   | 11 | 7            | 16.437   | 5  | 6  | 9.460        | 11 | 7  | 13.664   | 11 | 7  | 11.649   |
| 6            | 0  | 5.313    | 12 | 1            | 10.703   | 6  | 0  | 4.174        | 12 | 1  | 9.385    | 12 | 1  | 8.282    |
| 6            | 1  | 6.533    | 12 | 2            | 11.746   | 6  | 1  | 5.174        | 12 | 2  | 10.385   | 12 | 2  | 8.927    |

Table 1. (Continued)

| a1 = 0.4 |    |          | alpha = 0.01 |    |          | a1 = 0.4 |    |          | alpha = 0.05 |    |          | a1 = 0.4 |    |          | alpha = 0.10 |    |          |
|----------|----|----------|--------------|----|----------|----------|----|----------|--------------|----|----------|----------|----|----------|--------------|----|----------|
| x1       | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) | x1       | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) | x1       | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) |
| 0        | 0  | 2.763    | 5            | 4  | 9.059    | 0        | 0  | 1.797    | 5            | 3  | 6.770    | 0        | 0  | 1.382    | 5            | 3  | 6.053    |
| 0        | 1  | 3.983    | 5            | 5  | 9.926    | 0        | 1  | 2.846    | 5            | 4  | 7.580    | 0        | 1  | 2.334    | 5            | 4  | 6.792    |
| 0        | 2  | 5.044    | 5            | 6  | 10.889   | 0        | 2  | 3.778    | 5            | 5  | 8.373    | 0        | 2  | 3.193    | 5            | 5  | 7.349    |
| 0        | 3  | 6.027    | 6            | 0  | 5.902    | 0        | 3  | 4.652    | 5            | 6  | 9.153    | 0        | 3  | 4.008    | 5            | 6  | 8.414    |
| 0        | 4  | 6.963    | 6            | 1  | 6.748    | 0        | 4  | 5.492    | 5            | 7  | 9.913    | 0        | 4  | 4.796    | 5            | 7  | 8.821    |
| 0        | 5  | 7.865    | 6            | 2  | 7.943    | 0        | 5  | 6.308    | 6            | 0  | 4.737    | 0        | 5  | 5.565    | 6            | 0  | 4.213    |
| 0        | 6  | 8.742    | 6            | 3  | 8.651    | 0        | 6  | 7.114    | 6            | 1  | 5.576    | 0        | 6  | 6.319    | 6            | 1  | 4.951    |
| 0        | 7  | 9.600    | 6            | 4  | 9.763    | 0        | 7  | 7.889    | 6            | 2  | 6.421    | 0        | 7  | 7.063    | 6            | 2  | 5.739    |
| 0        | 8  | 10.442   | 6            | 5  | 10.521   | 0        | 8  | 8.661    | 6            | 3  | 7.241    | 0        | 8  | 7.797    | 6            | 3  | 6.305    |
| 0        | 9  | 11.270   | 6            | 6  | 11.370   | 0        | 9  | 9.423    | 6            | 4  | 8.047    | 0        | 9  | 8.524    | 6            | 4  | 7.564    |
| 1        | 0  | 3.022    | 7            | 0  | 6.442    | 1        | 0  | 2.057    | 6            | 5  | 8.836    | 1        | 0  | 1.641    | 6            | 5  | 8.016    |
| 1        | 1  | 4.270    | 7            | 1  | 7.258    | 1        | 1  | 3.080    | 6            | 6  | 9.577    | 1        | 1  | 2.563    | 6            | 6  | 9.033    |
| 1        | 2  | 5.368    | 7            | 2  | 8.456    | 1        | 2  | 4.103    | 7            | 0  | 5.259    | 1        | 2  | 3.416    | 7            | 0  | 4.708    |
| 1        | 3  | 6.258    | 7            | 3  | 9.370    | 1        | 3  | 4.876    | 7            | 1  | 6.065    | 1        | 3  | 4.228    | 7            | 1  | 5.427    |
| 1        | 4  | 7.190    | 7            | 4  | 10.260   | 1        | 4  | 5.713    | 7            | 2  | 6.899    | 1        | 4  | 5.014    | 7            | 2  | 6.446    |
| 1        | 5  | 8.090    | 7            | 5  | 11.006   | 1        | 5  | 6.528    | 7            | 3  | 7.712    | 1        | 5  | 5.782    | 7            | 3  | 6.973    |
| 1        | 6  | 8.966    | 7            | 6  | 11.703   | 1        | 6  | 7.324    | 7            | 4  | 8.523    | 1        | 6  | 6.535    | 7            | 4  | 7.796    |
| 1        | 7  | 9.822    | 8            | 0  | 6.980    | 1        | 7  | 8.107    | 7            | 5  | 9.301    | 1        | 7  | 7.277    | 7            | 5  | 8.251    |
| 1        | 8  | 10.663   | 8            | 1  | 7.797    | 1        | 8  | 8.878    | 8            | 0  | 5.774    | 1        | 8  | 8.011    | 8            | 0  | 5.198    |
| 1        | 9  | 11.491   | 8            | 2  | 8.967    | 1        | 9  | 9.640    | 8            | 1  | 6.552    | 1        | 9  | 8.737    | 8            | 1  | 5.891    |
| 2        | 0  | 3.514    | 8            | 3  | 9.646    | 2        | 0  | 2.549    | 8            | 1  | 7.377    | 2        | 0  | 2.133    | 8            | 2  | 6.833    |
| 2        | 1  | 4.784    | 8            | 4  | 10.635   | 2        | 1  | 3.503    | 8            | 2  | 8.183    | 2        | 1  | 2.974    | 8            | 2  | 7.216    |
| 2        | 2  | 5.843    | 8            | 5  | 11.490   | 2        | 2  | 4.415    | 8            | 3  | 8.983    | 2        | 2  | 3.818    | 8            | 3  | 8.467    |
| 2        | 3  | 6.675    | 9            | 0  | 7.519    | 2        | 3  | 5.282    | 8            | 4  | 9.751    | 2        | 3  | 4.626    | 8            | 4  | 8.702    |
| 2        | 4  | 7.604    | 9            | 1  | 8.317    | 2        | 4  | 6.118    | 9            | 0  | 6.282    | 2        | 4  | 5.409    | 9            | 0  | 5.682    |
| 2        | 5  | 8.502    | 9            | 2  | 9.248    | 2        | 5  | 6.932    | 9            | 1  | 7.037    | 2        | 5  | 6.175    | 9            | 1  | 6.148    |
| 2        | 6  | 9.376    | 9            | 3  | 10.145   | 2        | 6  | 7.728    | 9            | 2  | 7.854    | 2        | 6  | 6.927    | 9            | 2  | 7.386    |
| 2        | 7  | 10.227   | 9            | 4  | 11.126   | 2        | 7  | 8.502    | 9            | 3  | 8.653    | 2        | 7  | 7.668    | 9            | 3  | 7.674    |
| 2        | 8  | 11.072   | 10           | 0  | 8.058    | 2        | 8  | 9.272    | 9            | 4  | 9.615    | 2        | 8  | 8.358    | 9            | 4  | 8.878    |
| 2        | 9  | 4.077    | 10           | 1  | 8.835    | 2        | 9  | 10.033   | 10           | 0  | 6.785    | 2        | 9  | 9.127    | 9            | 5  | 9.153    |
| 3        | 0  | 5.169    | 10           | 2  | 9.986    | 3        | 0  | 3.158    | 10           | 1  | 7.521    | 3        | 0  | 2.681    | 10           | 1  | 6.282    |
| 3        | 1  | 6.394    | 10           | 3  | 10.751   | 3        | 1  | 3.982    | 10           | 2  | 8.330    | 3        | 1  | 3.532    | 10           | 1  | 6.615    |
| 3        | 2  | 7.287    | 10           | 4  | 11.784   | 3        | 2  | 5.086    | 10           | 3  | 9.123    | 3        | 2  | 4.352    | 10           | 2  | 7.889    |
| 3        | 3  | 8.193    | 11           | 0  | 8.596    | 3        | 3  | 5.831    | 10           | 4  | 9.985    | 3        | 3  | 5.144    | 10           | 3  | 8.131    |
| 3        | 4  | 9.169    | 11           | 1  | 9.476    | 3        | 4  | 6.651    | 11           | 0  | 7.283    | 3        | 4  | 5.917    | 11           | 0  | 6.752    |
| 3        | 5  | 10.038   | 11           | 2  | 10.370   | 3        | 5  | 7.452    | 11           | 1  | 8.002    | 3        | 5  | 6.672    | 11           | 1  | 7.080    |
| 3        | 6  | 10.780   | 11           | 3  | 11.248   | 3        | 6  | 8.238    | 11           | 2  | 8.804    | 3        | 6  | 7.485    | 11           | 2  | 8.301    |
| 3        | 7  | 11.611   | 12           | 0  | 9.128    | 3        | 7  | 9.013    | 11           | 3  | 9.397    | 3        | 7  | 8.154    | 11           | 3  | 8.587    |
| 3        | 8  | 4.655    | 12           | 1  | 9.863    | 3        | 8  | 9.778    | 12           | 0  | 7.777    | 3        | 8  | 8.948    | 12           | 0  | 7.236    |
| 4        | 0  | 5.687    | 12           | 2  | 10.868   | 4        | 0  | 3.679    | 12           | 1  | 8.482    | 4        | 0  | 3.197    | 12           | 1  | 7.539    |
| 4        | 1  | 6.909    | 12           | 3  | 11.634   | 4        | 1  | 4.581    | 12           | 2  | 9.524    | 4        | 1  | 3.996    | 12           | 2  | 8.758    |
| 4        | 2  | 7.760    | 13           | 0  | 9.656    | 4        | 2  | 5.450    | 12           | 3  | 9.868    | 4        | 2  | 4.814    | 12           | 3  | 9.042    |
| 4        | 3  | 8.773    | 13           | 1  | 10.489   | 4        | 3  | 6.289    | 13           | 0  | 8.267    | 4        | 3  | 5.598    | 13           | 0  | 7.701    |
| 4        | 4  | 9.539    | 13           | 2  | 11.365   | 4        | 4  | 7.104    | 13           | 1  | 8.960    | 4        | 4  | 6.510    | 13           | 1  | 8.005    |
| 4        | 5  | 10.397   | 14           | 0  | 10.178   | 4        | 5  | 7.911    | 13           | 2  | 10.059   | 4        | 5  | 7.117    | 13           | 2  | 9.213    |
| 4        | 6  | 11.239   | 14           | 1  | 10.992   | 4        | 6  | 8.694    | 14           | 0  | 8.755    | 4        | 6  | 7.865    | 14           | 0  | 8.163    |
| 4        | 7  | 11.239   | 14           | 2  | 11.811   | 4        | 7  | 9.461    | 14           | 1  | 9.229    | 4        | 7  | 8.593    | 14           | 1  | 8.460    |
| 5        | 0  | 5.325    | 15           | 0  | 10.697   | 5        | 0  | 4.209    | 15           | 0  | 9.357    | 5        | 0  | 3.710    | 15           | 0  | 8.624    |
| 5        | 1  | 6.210    | 15           | 1  | 11.497   | 5        | 1  | 4.971    | 15           | 1  | 9.703    | 5        | 1  | 4.478    | 15           | 1  | 8.925    |
| 5        | 2  | 7.628    | 16           | 0  | 11.212   | 5        | 2  | 5.942    | 15           | 2  | 9.833    | 5        | 2  | 5.276    | 15           | 2  | 9.084    |
| 5        | 3  | 8.277    | 17           | 0  | 11.724   | 5        | 3  | 6.942    | 16           | 0  | 9.833    | 5        | 3  | 6.084    | 16           | 0  | 9.084    |

Table 1. (Continued)

| alpha = 0.01 |    |          |    | alpha = 0.05 |          |    |    | alpha = 0.45 |    |    |          | alpha = 0.10 |    |          |    |    |          |
|--------------|----|----------|----|--------------|----------|----|----|--------------|----|----|----------|--------------|----|----------|----|----|----------|
| x1           | x2 | t(x1,x2) | x1 | x2           | t(x1,x2) | x1 | x2 | t(x1,x2)     | x1 | x2 | t(x1,x2) | x1           | x2 | t(x1,x2) | x1 | x2 | t(x1,x2) |
| 0            | 0  | 2.533    | 4  | 7            | 10.825   | 0  | 0  | 1.648        | 4  | 6  | 8.441    | 0            | 0  | 1.266    | 4  | 6  | 7.795    |
| 0            | 1  | 3.651    | 4  | 8            | 11.574   | 0  | 1  | 2.609        | 4  | 7  | 9.139    | 0            | 1  | 2.139    | 4  | 7  | 8.490    |
| 0            | 2  | 4.623    | 5  | 1            | 5.899    | 0  | 2  | 3.490        | 4  | 8  | 9.829    | 0            | 2  | 2.964    | 4  | 8  | 9.116    |
| 0            | 3  | 5.525    | 5  | 1            | 6.666    | 0  | 3  | 4.264        | 5  | 0  | 4.809    | 0            | 3  | 3.674    | 5  | 0  | 4.233    |
| 0            | 4  | 6.382    | 5  | 2            | 7.491    | 0  | 4  | 5.034        | 5  | 1  | 5.270    | 0            | 4  | 4.419    | 5  | 1  | 4.540    |
| 0            | 5  | 7.217    | 5  | 3            | 8.300    | 0  | 5  | 5.782        | 5  | 2  | 6.124    | 0            | 5  | 5.101    | 5  | 2  | 5.470    |
| 0            | 6  | 8.014    | 5  | 4            | 9.107    | 0  | 6  | 6.513        | 5  | 3  | 6.848    | 0            | 6  | 5.793    | 5  | 3  | 6.152    |
| 0            | 7  | 8.800    | 5  | 5            | 9.870    | 0  | 7  | 7.232        | 5  | 4  | 7.591    | 0            | 7  | 6.474    | 5  | 4  | 6.856    |
| 0            | 8  | 9.571    | 5  | 6            | 10.634   | 0  | 8  | 7.939        | 5  | 5  | 8.283    | 0            | 8  | 7.147    | 5  | 5  | 7.505    |
| 0            | 9  | 10.331   | 5  | 7            | 11.387   | 0  | 9  | 8.656        | 5  | 6  | 8.984    | 0            | 9  | 7.813    | 5  | 6  | 8.173    |
| 0            | 10 | 11.092   | 6  | 0            | 6.557    | 0  | 10 | 9.329        | 5  | 7  | 9.677    | 0            | 10 | 8.474    | 5  | 7  | 8.834    |
| 0            | 11 | 11.833   | 6  | 1            | 7.129    | 0  | 11 | 10.033       | 6  | 0  | 5.397    | 0            | 11 | 9.129    | 6  | 0  | 4.791    |
| 1            | 0  | 3.021    | 6  | 2            | 8.099    | 1  | 0  | 2.135        | 6  | 1  | 5.843    | 1            | 0  | 1.750    | 6  | 1  | 5.198    |
| 1            | 1  | 4.073    | 6  | 3            | 8.894    | 1  | 1  | 3.011        | 6  | 2  | 6.692    | 1            | 1  | 2.528    | 6  | 2  | 6.010    |
| 1            | 2  | 5.020    | 6  | 4            | 9.693    | 1  | 2  | 3.839        | 6  | 3  | 7.409    | 1            | 2  | 3.290    | 6  | 3  | 6.685    |
| 1            | 3  | 5.908    | 6  | 5            | 10.444   | 1  | 3  | 4.626        | 6  | 4  | 8.141    | 1            | 3  | 4.024    | 6  | 4  | 7.385    |
| 1            | 4  | 6.757    | 6  | 6            | 11.180   | 1  | 4  | 5.391        | 6  | 5  | 8.672    | 1            | 4  | 4.742    | 6  | 5  | 8.027    |
| 1            | 5  | 7.578    | 6  | 7            | 11.929   | 1  | 5  | 6.134        | 6  | 6  | 9.521    | 1            | 5  | 5.437    | 6  | 6  | 8.691    |
| 1            | 6  | 8.377    | 6  | 7            | 12.688   | 1  | 6  | 6.861        | 6  | 7  | 10.055   | 1            | 6  | 6.124    | 6  | 6  | 9.344    |
| 1            | 7  | 9.160    | 7  | 1            | 7.803    | 1  | 7  | 7.569        | 7  | 0  | 5.977    | 1            | 7  | 6.802    | 7  | 0  | 5.703    |
| 1            | 8  | 9.928    | 7  | 2            | 8.721    | 1  | 8  | 8.273        | 7  | 1  | 6.371    | 1            | 8  | 7.472    | 7  | 1  | 6.547    |
| 1            | 9  | 10.685   | 7  | 3            | 9.506    | 1  | 9  | 8.969        | 7  | 2  | 7.252    | 1            | 9  | 8.136    | 7  | 2  | 7.055    |
| 1            | 10 | 11.432   | 7  | 4            | 10.286   | 1  | 10 | 9.659        | 7  | 3  | 7.961    | 1            | 10 | 8.794    | 7  | 3  | 7.913    |
| 2            | 0  | 3.783    | 7  | 5            | 11.016   | 2  | 0  | 2.833        | 7  | 4  | 8.825    | 2            | 0  | 2.395    | 7  | 4  | 8.391    |
| 2            | 1  | 4.725    | 7  | 6            | 11.773   | 2  | 1  | 3.411        | 7  | 5  | 9.365    | 2            | 1  | 2.887    | 7  | 5  | 9.207    |
| 2            | 2  | 5.633    | 8  | 0            | 7.916    | 2  | 2  | 4.405        | 8  | 0  | 6.550    | 2            | 2  | 3.830    | 8  | 0  | 5.891    |
| 2            | 3  | 6.496    | 8  | 1            | 8.410    | 2  | 3  | 5.173        | 8  | 1  | 6.974    | 2            | 3  | 4.346    | 8  | 1  | 6.279    |
| 2            | 4  | 7.141    | 8  | 2            | 9.312    | 2  | 4  | 5.930        | 8  | 2  | 7.809    | 2            | 4  | 5.260    | 8  | 2  | 7.215    |
| 2            | 5  | 8.145    | 8  | 3            | 10.090   | 2  | 5  | 6.653        | 8  | 3  | 8.510    | 2            | 5  | 5.940    | 8  | 3  | 7.736    |
| 2            | 6  | 8.936    | 8  | 4            | 10.866   | 2  | 6  | 7.372        | 8  | 4  | 9.234    | 2            | 6  | 6.622    | 8  | 4  | 8.547    |
| 2            | 7  | 9.712    | 8  | 5            | 11.613   | 2  | 7  | 8.080        | 8  | 5  | 9.903    | 2            | 7  | 7.295    | 8  | 5  | 8.943    |
| 2            | 8  | 10.476   | 9  | 0            | 8.530    | 2  | 8  | 8.589        | 9  | 0  | 7.118    | 2            | 8  | 7.961    | 9  | 0  | 6.432    |
| 2            | 9  | 11.021   | 9  | 1            | 9.011    | 2  | 9  | 9.472        | 9  | 1  | 7.533    | 2            | 9  | 8.622    | 9  | 1  | 6.814    |
| 2            | 10 | 11.759   | 9  | 2            | 9.902    | 2  | 10 | 9.969        | 9  | 2  | 8.362    | 2            | 10 | 9.277    | 9  | 2  | 7.611    |
| 3            | 0  | 4.520    | 9  | 3            | 10.672   | 3  | 0  | 3.611        | 9  | 3  | 9.056    | 3            | 0  | 3.097    | 9  | 3  | 8.267    |
| 3            | 1  | 5.375    | 9  | 4            | 11.443   | 3  | 1  | 4.067        | 9  | 4  | 9.776    | 3            | 1  | 3.511    | 9  | 4  | 9.057    |
| 3            | 2  | 6.247    | 10 | 0            | 9.137    | 3  | 2  | 4.975        | 10 | 0  | 7.680    | 3            | 2  | 4.651    | 10 | 0  | 6.971    |
| 3            | 3  | 7.334    | 10 | 1            | 9.607    | 3  | 3  | 5.724        | 10 | 1  | 8.087    | 3            | 3  | 5.074    | 10 | 1  | 7.345    |
| 3            | 4  | 7.918    | 10 | 2            | 10.490   | 3  | 4  | 6.325        | 10 | 2  | 8.911    | 3            | 4  | 5.628    | 10 | 2  | 8.139    |
| 3            | 5  | 8.700    | 10 | 3            | 11.251   | 3  | 5  | 7.183        | 10 | 3  | 9.599    | 3            | 5  | 6.314    | 10 | 3  | 8.789    |
| 3            | 6  | 9.480    | 11 | 0            | 9.737    | 3  | 6  | 7.895        | 10 | 4  | 10.129   | 3            | 6  | 6.990    | 11 | 0  | 7.505    |
| 3            | 7  | 10.246   | 11 | 1            | 10.197   | 3  | 7  | 8.795        | 11 | 0  | 8.238    | 3            | 7  | 7.659    | 11 | 1  | 7.874    |
| 3            | 8  | 11.229   | 11 | 2            | 11.074   | 3  | 8  | 9.293        | 11 | 1  | 8.638    | 3            | 8  | 8.322    | 11 | 2  | 8.665    |
| 4            | 0  | 5.222    | 11 | 3            | 11.826   | 3  | 8  | 10.174       | 11 | 2  | 9.457    | 3            | 8  | 8.982    | 11 | 3  | 9.310    |
| 4            | 1  | 6.028    | 12 | 0            | 10.331   | 4  | 0  | 4.211        | 12 | 0  | 8.791    | 4            | 0  | 3.666    | 12 | 0  | 8.036    |
| 4            | 2  | 6.878    | 12 | 1            | 10.783   | 4  | 1  | 4.689        | 12 | 1  | 9.185    | 4            | 1  | 4.099    | 12 | 1  | 8.407    |
| 4            | 3  | 7.703    | 12 | 2            | 11.654   | 4  | 2  | 5.555        | 12 | 2  | 10.001   | 4            | 2  | 4.925    | 12 | 2  | 9.188    |
| 4            | 4  | 8.516    | 13 | 0            | 10.921   | 4  | 3  | 6.478        | 13 | 0  | 9.341    | 4            | 3  | 5.788    | 13 | 0  | 8.564    |
| 4            | 5  | 9.293    | 13 | 1            | 11.365   | 4  | 4  | 7.037        | 13 | 1  | 9.730    | 4            | 4  | 6.450    | 13 | 1  | 8.924    |
| 4            | 6  | 10.065   | 14 | 0            | 11.506   | 4  | 5  | 7.735        | 14 | 0  | 9.887    | 4            | 5  | 7.156    | 14 | 0  | 9.089    |

A somewhat more complicated case making use of the possibility of combining component data when some sample sizes are equal, as described in Section 1, is illustrated by

Example 2. Consider a five-component series system with test data  $X_1 = 0, X_2 = 4, X_3 = 2, X_4 = 1, X_5 = 3$ , based on corresponding sample sizes  $n_1 = n_2 = n_3 = 300, n_4 = n_5 = 200$ . Here  $c = (3/300 + 2/200) = .02, a_1 = a_2 = a_3 = 1/300(.02) = 1.6, a_4 = a_5 = 1/200(.02) = 1/4$ . For the equivalent  $k^* = 2$  problem we divide the two distinct values of the  $a_i$ 's by their sum  $c_0 = 1/6 + 1/4 = 5/12$ , obtaining  $a_1^* = .40, a_2^* = .60$ . The corresponding numbers of failures are  $X_1^* = X_1 + X_2 + X_3 = 6$  and  $X_2^* = X_4 + X_5 = 4$ . For the reduced problem we consult Table 1 for  $\alpha = .10$  to find the 90 percent confidence bound 7.564. This must be multiplied by  $c_0$  to obtain the bound for the original  $\theta(\lambda)$ . The 90 percent lower bound  $r(x)$  for the system reliability given by (1.6) thus becomes  $1 - c_0 t(x) = 1 - (.02)(5/12)(7.564) = .937$ .

If the observed numbers of failures fall outside the limits of Table 1, the approximate methods discussed in the next section may be employed.

#### 4. Approximations

In this section approximate confidence bounds for systems with more than two components having distinct sample sizes are developed. The use of the maximum likelihood ratio confidence bound for cases falling outside the scope of Table 1 is also discussed.

For  $k > 2$  it is impractical to generate the optimal ordering and the corresponding values of the upper confidence bound for more than a few illustrative cases. Thus, some method of approximating solutions for  $k > 2$  with acceptable precision is required. The approach to be followed here is to find a  $k = 2$  problem which is sufficiently similar in structure to the given  $k > 2$  problem so that the confidence bounds for the two problems are essentially the same except for a normalizing factor. This method may be thought of as an extension and refinement of the Lindstrom-Madden procedure (see Lloyd and Lipow 1977, and cf. Harris and Soms 1980).

The Lindstrom-Madden method first estimates the reliability by maximum likelihood and then uses the  $k = 1$  confidence bound solution for the component with the smallest sample size and a fictitious number of failures determined so as to reproduce the estimated system reliability. The procedure proposed here is to estimate two quantities, the value of  $\theta$  and the variance of the maximum likelihood estimate of  $\theta$ , and to use these estimates to find a  $k = 2$  problem based on two of the original  $a_i$ 's with a pair of corresponding fictitious observation values chosen to reproduce the estimated quantities. The two  $a_i$ 's are chosen to be as large as possible (corresponding to sample sizes as small as possible) subject to two constraints. The

first constraint guarantees that the resulting fictitious observations are non-negative. The second constraint requires that  $a_i$ 's corresponding to zero failures in the original problem not be considered unless all but one of the components have zero failures. These considerations lead to a unique  $k = 2$  problem whose solution provides a very good approximation to the solution for the original  $k > 2$  problem.

The elimination of  $a_i$ 's for components exhibiting zero failures is justified by the fact that the maximum likelihood ratio confidence bound (discussed later) is invariant under such transformations. That is, if the dimension  $k$  is reduced by the elimination of all  $a_i$ 's corresponding to zero failures, then the value of the maximum likelihood ratio bound remains unchanged.

The choice of the two quantities whose estimates are used to determine the pair of fictitious observations is supported by analogy with Lindstrom-Madden (in the case of  $\theta$ ) and by the fact that the two estimates are the ingredients of the asymptotic maximum likelihood ratio confidence bound thereby insuring asymptotic optimality. The details of the approximation algorithm are as follows:

(a) First  $\theta(\lambda) = \sum_{i=1}^k a_i \lambda_i$  is estimated by

$$\hat{\theta} = \sum_{i=1}^k a_i X_i, \quad (4.1)$$

and the quantity  $\text{Var}(\hat{\theta}) = \sum_{i=1}^k a_i^2 \lambda_i$  is estimated by

$$\hat{v} = \sum_{i=1}^k a_i^2 X_i. \quad (4.2)$$

(b) Next if at least one component exhibits one or more failures, the pair  $(a_i, a_j)$ ,  $i < j$ , is selected so that

$$(i) \quad a_i \leq \hat{v}/\hat{\theta} \quad , \quad \text{and}$$

$$(ii) \quad a_j \geq \hat{v}/\hat{\theta} \quad .$$

Subject to these conditions,  $a_i$  and  $a_j$  are taken to be the largest available values associated with at least one failure. If all  $a_i$ 's satisfying (i) correspond to zero failures, then  $a_i$  is taken to be the largest in that group. If all the  $a_j$ 's satisfying (ii) correspond to zero failures, then  $a_j$  is taken to be  $a_k$ . If all components exhibit zero failures,  $\hat{v}/\hat{\theta}$  is indeterminate and  $a_i$  and  $a_j$  are taken to be  $a_l$  and  $a_k$  respectively.<sup>1</sup>

(c) The pseudo-observations  $x_1^*$  and  $x_2^*$  are computed by the formulae

$$x_1^* = \frac{a_j \hat{\theta} - \hat{v}}{a_i (a_j - a_i)} \quad , \quad (4.3)$$

$$x_2^* = \frac{\hat{v} - a_i \hat{\theta}}{a_j (a_j - a_i)} \quad .$$

These values will be non-negative by conditions (i) and (ii) of (b), and when associated with  $a_i$  and  $a_j$  respectively they

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<sup>1</sup>For this case the resulting confidence bound is exact and is the same as would be obtained by multiplying  $a_k$  times the upper confidence bound for a single Poisson parameter when zero failures are observed, i.e.,  $t(0) = a_k \log(1/\alpha)$ .

reproduce the values of  $\hat{\theta}$  and  $\hat{v}$  provided all other observations are replaced by zeros.

- (d) The  $k = 2$  problem with  $x_1^*$  and  $x_2^*$  associated with  $a_1^* = a_i / (a_i + a_j)$  and  $a_2^* = a_j / (a_i + a_j)$  respectively may now be treated using Table 1 to yield  $t(x_1^*, x_2^*)$ . Since  $x_1^*$  and  $x_2^*$  are not necessarily integers, it may be necessary to interpolate with respect to these arguments as well as the value of  $a_1 = a_1^*$ .

- (e) The approximate upper confidence bound  $t^*$  for  $\theta$  for the original  $k > 2$  problem is then given by

$$t^*(x) = (a_i + a_j)t(x_1^*, x_2^*) \quad . \quad (4.4)$$

In order to check the validity of this approximation algorithm, the confidence bounds for the first 24 points in the optimal (two-stage prospective) ordering were computed for a typical  $k=3$  case. These results were obtained by a rather laborious method based on formula (2.7) and involving repeated interactive searches of the  $(\lambda_1, \lambda_2, \lambda_3)$  simplex. The approximation algorithm was applied to each of these points and the comparative results are shown in Table 2. The values of  $a_1$ ,  $a_2$ , and  $a_3$  for this example were chosen to be of roughly the same magnitude but not so close that the combination of any two would be indicated.

To check the algorithm for cases farther from the origin several additional examples were considered using a somewhat different method which avoids the necessity for sequentially generating the

Table 2. The Performance of the Approximation Algorithm for the First 24 Ordered Points for a Typical Example With  $k = 3$ .

| $\alpha = .05; a_1 = .2, a_2 = .3, a_3 = .5$ |       |       |       |                            |          |                   |
|----------------------------------------------|-------|-------|-------|----------------------------|----------|-------------------|
| Two-Step Prospective<br>Sequential           |       |       |       | Approximation<br>Algorithm |          |                   |
| n                                            | $x_1$ | $x_2$ | $x_3$ | $t(x)$                     | $t^*(x)$ | Relative<br>Error |
| 1                                            | 0     | 0     | 0     | 1.498                      | 1.498    | + 00.0 %          |
| 2                                            | 1     | 0     | 0     | 1.553                      | 1.555    | + 00.1            |
| 3                                            | 0     | 1     | 0     | 1.656                      | 1.663    | + 00.4            |
| 4                                            | 2     | 0     | 0     | 1.703                      | 1.705    | + 00.2            |
| 5                                            | 1     | 1     | 0     | 1.756                      | 1.540    | - 12.3            |
| 6                                            | 2     | 1     | 0     | 1.861                      | 1.752    | - 05.9            |
| 7                                            | 3     | 0     | 0     | 1.933                      | 1.891    | - 02.1            |
| 8                                            | 0     | 2     | 0     | 1.981                      | 1.995    | + 00.7            |
| 9                                            | 1     | 2     | 0     | 2.085                      | 2.052    | - 01.6            |
| 10                                           | 4     | 0     | 0     | 2.088                      | 2.094    | + 00.3            |
| 11                                           | 3     | 1     | 0     | 2.162                      | 1.991    | - 07.9            |
| 12                                           | 2     | 2     | 0     | 2.268                      | 2.208    | - 02.7            |
| 13                                           | 5     | 0     | 0     | 2.300                      | 2.309    | + 00.4            |
| 14                                           | 0     | 3     | 0     | 2.354                      | 2.397    | + 01.8            |
| 15                                           | 0     | 0     | 1     | 2.372                      | 2.372    | + 00.0            |
| 16                                           | 4     | 1     | 0     | 2.382                      | 2.291    | - 03.9            |
| 17                                           | 1     | 0     | 1     | 2.429                      | 2.431    | + 00.1            |
| 18                                           | 1     | 3     | 0     | 2.472                      | 2.438    | - 01.4            |
| 19                                           | 5     | 1     | 0     | 2.504                      | 2.486    | - 00.7            |
| 20                                           | 0     | 1     | 1     | 2.543                      | 2.529    | - 00.5            |
| 21                                           | 6     | 0     | 0     | 2.575                      | 2.551    | - 01.0            |
| 22                                           | 2     | 0     | 1     | 2.602                      | 2.589    | - 00.8            |
| 23                                           | 2     | 3     | 0     | 2.609                      | 2.641    | + 01.2            |
| 24                                           | 1     | 1     | 1     | 2.660                      | 2.668    | + 00.3            |

optimal ordering of the sample points. For these examples the ordering was generated by the values of the maximum likelihood ratio confidence bounds (see below) associated with the sample points. Since this ordering is asymptotically optimal (for large  $\lambda_i$ 's), it can be expected to produce good results for sample points well removed from the origin. The values of the confidence bounds calculated using (2.7) and the approximations obtained by the proposed algorithm are shown for these examples in Table 3.

In Table 3 the tendency of the computed values of  $t(x)$  to be slightly larger than the algorithm values may be due to the fact that the ordering used to compute the former is non-optimal. Formal application of the algorithm may occasionally result in the selection of nearly equal values  $a_i$  and  $a_j$  in step (b). When this happens, improved results may be obtained by first reducing the dimension  $k$  by combining the nearly equal  $a_i$ 's and then applying the algorithm. This method was used for the two cases in Table 3 marked by (+). The algorithm values for all cases where  $a = (.14, .16, .70)$  or  $a = (.15, .41, .44)$  do not differ substantially from the values which would be obtained by reducing to the  $k = 2$  case by combining the nearly equal  $a_i$ 's. Similarly, the values for the cases where  $a = (.32, .33, .35)$  can be nearly reproduced by multiplying the  $k = 1$  bound by  $\bar{a} = .333$ .

The number of sample points appearing in the ordering before a given level of  $t(x)$  is reached increases rapidly as  $k$  increases.

Table 3. Several Examples Comparing the Algorithm Values With the Exact Bounds Based on the Ordering Generated by the m&rb.

| $\alpha = .05$ |       |       |          |       |       |                      |        |                          |       |
|----------------|-------|-------|----------|-------|-------|----------------------|--------|--------------------------|-------|
| $a_1$          | $a_2$ | $a_3$ | $x_1$    | $x_2$ | $x_3$ | Position in Ordering | $t(x)$ | Algorithm Value $t^*(x)$ | m&rb  |
| .20, .30, .50  |       |       | 2, 2, 1  |       |       | 50                   | 3.329  | 3.273                    | 3.070 |
|                |       |       | 5, 0, 2  |       |       | 100                  | 4.068  | 3.957                    | 3.798 |
|                |       |       | 6, 6, 0  |       |       | 200                  | 4.887  | 4.789                    | 4.708 |
|                |       |       | 9, 1, 3  |       |       | 400                  | 5.980  | 5.892                    | 5.795 |
| .14, .16, .70  |       |       | 5, 4, 0  |       |       | 50                   | 3.026  | 3.084 <sup>†</sup>       | 2.218 |
|                |       |       | 2, 10, 0 |       |       | 100                  | 3.424  | 3.531 <sup>†</sup>       | 2.921 |
|                |       |       | 2, 5, 1  |       |       | 200                  | 4.407  | 4.086                    | 3.729 |
| .15, .41, .44  |       |       | 13, 0, 0 |       |       | 50                   | 3.245  | 3.223                    | 2.980 |
|                |       |       | 5, 2, 1  |       |       | 100                  | 4.028  | 3.860                    | 3.699 |
|                |       |       | 3, 3, 2  |       |       | 200                  | 4.861  | 4.738                    | 4.559 |
| .32, .33, .35  |       |       | 0, 3, 2  |       |       | 50                   | 3.492  | 3.518                    | 3.256 |
|                |       |       | 1, 5, 1  |       |       | 100                  | 4.339  | 4.332                    | 4.076 |
|                |       |       | 1, 5, 3  |       |       | 200                  | 5.198  | 5.234                    | 4.993 |

<sup>†</sup> Reduction to  $k=2$  case by combining  $a_1$  and  $a_2$ .

Hence for  $k > 2$  the algorithm may be applied to sample points positioned quite far along in the ordering without requiring values beyond the scope of Table 1, as is seen in the examples of Table 3.

The results detailed in Tables 2 and 3 suggest that the application of the proposed algorithm together with the combining of nearly equal  $a_i$ 's when indicated will nearly always produce confidence bounds for  $\theta(\lambda)$  subject to relative errors not exceeding 10 percent and often much less. Furthermore, the lower confidence bound for reliability  $r(x)$  given by (1.6) will exhibit a much smaller relative error since the relative error in approximations for  $t(x)$  applies only to the difference between the lower bound and one; a quantity which at worst is of the order of  $1/10$  in the contemplated applications.

In situations where very large sample sizes are available, it may happen that the observed numbers of failures exceed the limits of Table 1 even though the system reliability is high. For such cases the maximum likelihood ratio bound (m&rb) may be used. This approximate confidence bound is obtained in the usual way from the maximum likelihood ratio statistic for testing the hypothesis  $H_0 : \theta(\lambda) = \theta_0$  versus all alternatives. The corresponding one-sided confidence bound may be shown (see Johns 1975) to be determined as follows: Let  $\hat{\mu}$  be the positive real root less than  $1/(\text{largest } a_i \text{ for which } X_i > 0)$  of the equation

$$\chi_{1,2\alpha}^2 = 2 \sum_{i=1}^k X_i \left\{ \frac{a_i \hat{\mu}}{1 - a_i \hat{\mu}} + \log(1 - a_i \hat{\mu}) \right\}, \quad (4.5)$$

where  $\chi_{1,2\alpha}^2$  is the upper  $100(2\alpha)$ -th percentage point of the chi-squared distribution with one degree of freedom. Then the quantity

$$\hat{t}(X) = \sum_{i=1}^k a_i X_i / (1 - a_i \hat{\mu}) \quad (4.6)$$

is the approximate upper  $1 - \alpha$  level confidence bound for  $\theta(\lambda)$ . As  $\max(\lambda_1, \lambda_2, \dots, \lambda_n) \rightarrow \infty$ , the mlrb  $\hat{t}(X)$  may be shown (see Johns 1975) to be asymptotically equivalent to

$$\tilde{t}(X) = \sum_{i=1}^k a_i X_i + z_{\alpha} \left( \sum_{i=1}^k a_i^2 X_i \right)^{1/2}, \quad (4.7)$$

where  $z_{\alpha}$  is the  $100\alpha$ -th percentage point of the standard normal distribution.

Neither of these approximate bounds is useful for sample points near the origin in the usual orderings, but  $\hat{t}$  given by (4.6) becomes sufficiently precise for application to sample points beyond the scope of Table 1. A comparison of  $t(x_1, x_2)$  and the corresponding mlrb for the last (i.e., the 100-th) points in each of the optimal orderings for the cases covered in Table 1 is given in Table 4.

Table 4.  $m\ellrb/t(x_1, x_2)$  for the 100-th Point in Each Ordering

| $\alpha$ | $a_1$ |     |     |     |     |      |     |     |     |
|----------|-------|-----|-----|-----|-----|------|-----|-----|-----|
|          | .05   | .10 | .15 | .20 | .25 | .30  | .35 | .40 | .45 |
| .01      | .62   | .95 | .97 | .97 | .98 | .97  | .98 | .97 | .99 |
| .05      | .72   | .88 | .96 | .95 | .98 | .97  | .99 | .95 | .96 |
| .10      | .89   | .95 | .95 | .96 | .97 | 1.00 | .97 | .96 | .98 |

These results suggest that the  $m\ellrb$  possesses satisfactory precision for sample points beyond those listed in Table 1 for  $k = 2$  whenever  $a_1 \geq .10$ . The last column of Table 3 giving the  $m\ellrb$  values for the examples considered illustrates the fact that for  $k > 2$  the  $m\ellrb$  tends to underestimate the correct value of the bound for sample points within the range that can be dealt with using Table 1 and the algorithm.

Two potential sources of error for the lower confidence bound on system reliability remain to be discussed. They are (i) the Poisson approximation to the binomial distribution of the observed failures, and (ii) the approximation for reliability given in (1.2) and reflected in the formula (1.6) for the lower bound  $r(x)$ . It is intuitively clear from (1.1) ff. that the "worst case" for the Poisson approximation should occur when  $k = 1$ ; since to match a given  $k = 1$  level of reliability, say  $1 - p$ , by a  $k > 1$  case, we must have  $p \approx \sum_{i=1}^k p_i$ , so that the  $p_i$ 's must be smaller than  $p$  which tends to improve the Poisson approximation.

For the case  $k = 1$  the familiar upper confidence bounds for a single Poisson parameter apply and the actual coverage probabilities for the proposed method (3.1) can be computed for any  $n$  and  $p$  from tables of the binomial distribution. The results of several such calculations are shown in Table 5.

Table 5. "Worst Case" Analysis ( $k = 1$ ). Minimum Coverage Probabilities for  $r(x)$ .

| Reliability<br>$q = 1 - p$ | $1 - \alpha$ |      |      |
|----------------------------|--------------|------|------|
|                            | .90          | .95  | .99  |
| .95                        | .906         | .954 | .991 |
| .90                        | .912         | .958 | .992 |
| .80                        | .924         | .965 | .995 |
| .70                        | .936         | .972 | .997 |

These values suggest that the approximations operate to make the proposed confidence bounds slightly conservative. It is interesting to observe that the minimum coverage probabilities are not drastically different from the nominal values, even for a true reliability as low as .70.

##### 5. Acknowledgments and Remarks Concerning the Computations

The computations for Table 1 were performed on a Digital Equipment PDP-11/34 running under a UNIX operating system. The production program for these computations (506 lines) was written in the "C" language by the author. The method used for the

computation of the upper bounds was an implementation of (2.7); and as has been noted, the sample points were ordered by the two-step prospective sequential method. The tabled results were subjected to various checks to insure that the correct global maxima were found in each case.

The computations involved in obtaining the one-step look-ahead results and the tree analysis associated with the semi-Bayesian results discussed in Section 3 were done on an IBM 370/168 machine using Fortran programs developed by David Pasta. These programs compute the upper bounds by a somewhat more complicated method used in the earlier phases of the study and detailed in Johns (1975).

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generated sequentially by a two-stage, prospective optimization procedure. A generalization of the Lindstrom-Madden technique is proposed for using the tables to find confidence bounds for systems consisting of more than two components with differing sample sizes.