

AD-A104 474    JOHNS HOPKINS UNIV BALTIMORE MD DEPT OF MATHEMATICAL--ETC F/6 12/1  
AN UPPER BOUND ON THE PROBABILITY DENSITY FUNCTION IN TERMS OF --ETC(U)  
UNCLASSIFIED    APR 81    V K KLONIAS    N00014-79-C-0801  
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6) AN UPPER BOUND ON THE PROBABILITY  
DENSITY FUNCTION IN TERMS OF  
FISHER INFORMATION  
A Bound to the Density  
Function  
By  
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The Johns Hopkins University  
9) Technical Report No. 336  
ONR Technical Report No. 81-1  
11) April, 1981

12) 91

14) TR-336  
77-11-111

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15) Research supported in part by the Army, Navy and Air Force  
under Office of Naval Research Contract No. N00014-79-C-0861  
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Summary

An inequality for members of a Sobolev space of order one is demonstrated and as a corollary an upper bound, in terms of the Fisher information, is derived for a density function. Also two characterizations of the Laplace and the exponential distribution are indicated.

AN UPPER BOUND ON THE PROBABILITY DENSITY FUNCTION  
IN TERMS OF FISHER INFORMATION

A BOUND TO THE DENSITY FUNCTION

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Accession For	
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Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
Availability Codes	
Avail and/or	
Dist	Special

AMS 1970 subject classification. Primary 62B10, 46F99; Secondary 62E10, 60E15. Key words and phrases. Upper bound to the density, Fisher information, characterization of density functions, first distributional derivative.

1. Introduction. Let  $\|\cdot\|_\infty$ ,  $\|\cdot\|_2$  denote the  $\sup_{\mathbb{R}}$  and  $L_2(\mathbb{R})$  norms respectively, and let  $g'$  denote the first distributional derivative of the function  $g$  in  $L_2(\mathbb{R})$ . We will demonstrate inequalities

$$(1.1) \quad \|g g'\|_2^2 / \|g'\|_2^2 \leq \|g\|_\infty^2 \leq \|g\|_2 \|g'\|_2,$$

and a generalization thereof (see (2.3)), for members  $g$  of a Sobolev space of order one. Since  $\|g\|_2, \|g'\|_2 < +\infty$  imply  $\|g g'\|_2 < +\infty$ , through the relations  $\|g g'\|_2^2 \leq \|g\|_\infty^2 \|g'\|_2^2 \leq \|g\|_2 \|g'\|_2^3$ , and the lower bound on  $\|g\|_\infty^2$  in (1.1) is immediate we need derive only the upper bound. An interesting special case of inequalities (1.1), when  $f = g^2$  is a probability density function on the real line with finite Fisher information.  $I(f) = \int f'^2/f$ , is

$$(1.2) \quad \|f'\|_2^2 / I(f) \leq \|f\|_\infty \leq I(f)^{1/2},$$

with equality in the rightmost hand side if and only if  $f$  is the density or the Laplace distribution. A generalization of (1.2) is given by (3.1) below.

It should be noted that a weaker form of the upper bound in (1.1) appears in Tapia and Thompson (1978), page 116, for absolutely continuous functions. Their proof can be modified to achieve the upper bound in (1.1), but our proof allows slightly more general functions.

2. An inequality for members of a Sobolev space. Let  $\mathbb{R}$  denote the real line and let  $L_2 \equiv L_2(\mathbb{R})$  be the space of all Lebesgue measurable square integrable functions  $g : \mathbb{R} \rightarrow \mathbb{R}$ . It is well known that  $(L_2, \langle \cdot, \cdot \rangle_2)$  is a Hilbert space for the inner product  $\langle g, h \rangle_2 \equiv \int_{\mathbb{R}} g h$

and the induced norm  $\|g\|_2 \equiv (\int_{\mathbb{R}} g^2)^{1/2}$ . By a Sobolev space of order one on  $\mathbb{R}$ , denoted by  $H \equiv H^1(\mathbb{R})$ , is meant the space of all functions  $g: \mathbb{R} \rightarrow \mathbb{R}$  such that  $g, g' \in L_2$ , where  $g'$  denotes the distributional derivative of order one of  $g$ . Endowed with the inner product  $\langle g, h \rangle_{\underline{\alpha}} \equiv \alpha_0 \langle g, h \rangle_2 + \alpha_1 \langle g', h' \rangle_2$ , where  $\underline{\alpha} = (\alpha_0, \alpha_1)$  with  $\alpha_0, \alpha_1 > 0$ , and the induced norm  $\|g\|_{\underline{\alpha}} \equiv (\langle g, g \rangle_{\underline{\alpha}})^{1/2}$ ,  $H$  is a reproducing kernel Hilbert space (RKHS), i.e. there exists a unique real valued function  $k(x, y)$  on  $\mathbb{R} \times \mathbb{R}$ , called the reproducing kernel (RK), such that,

$$k_y(\cdot) \equiv k(\cdot, y) \in H \text{ for all } y \in \mathbb{R},$$

and

$$\langle g, k_y \rangle_{\underline{\alpha}} = g(y) \text{ for all } g \in H.$$

The kernel of  $H$  is given by

$$(1.1) \quad k_{\underline{\alpha}}(x, y) = (4\alpha_0\alpha_1)^{-1/2} \exp\{-\frac{\alpha_0}{\alpha_1} \frac{1}{2}|x-y|\}.$$

For a treatment of the RKHS and Sobolev spaces the reader is referred to Aronszajn (1950), Parzen (1967), Gel'fand and Shilov (1964) and Yosida (1974).

We next derive an inequality which is valid for all functions in  $H^1(\mathbb{R})$ . A discrete version of this inequality is now available also, due to Professor J. Keilson (personal communication), namely

$$\max_n g_n^2 \leq (\sum g_n^2)^{1/2} (\sum (g_n - g_{n-1})^2)^{1/2},$$

for all sequences  $(g_n)_{-\infty}^{+\infty}$  in  $l_2$ .

Proposition 2.1. For every  $g \in H$  we have

$$(2.2) \quad \|g\|_{\infty}^2 \leq \|g\|_2 \|g'\|_2 .$$

Proof: We endow  $H$  with the collection of norms  $\|\cdot\|_{\underline{\alpha}}$  induced by the inner products  $\langle \cdot, \cdot \rangle_{\underline{\alpha}}$ , defined above. For every  $\underline{\alpha} = (\alpha_0, \alpha_1)$  with  $\alpha_0, \alpha_1 > 0$ ,  $(H, \langle \cdot, \cdot \rangle_{\underline{\alpha}})$  is a RKBS with RK  $k_{\underline{\alpha}}$  given by (2.1).

Using the Cauchy-Schwarz inequality we have for every  $u \in H$

$$\begin{aligned} |g(x)| &= |\langle k_{\underline{\alpha}}(\cdot, x), g(\cdot) \rangle_{\underline{\alpha}}| \leq \|k_{\underline{\alpha}}(\cdot, x)\|_{\underline{\alpha}} \|g\|_{\underline{\alpha}} = \\ &((4\alpha_0\alpha_1)^{-\frac{1}{2}}(\alpha_0\|g\|_2^2 + \alpha_1\|g'\|_2^2))^{\frac{1}{2}} = ((\frac{1}{2})(r\|g\|_2^2 + r^{-1}\|g'\|_2^2))^{\frac{1}{2}} \end{aligned}$$

where  $r^2 = \alpha_0/\alpha_1$ , for all  $\alpha_0, \alpha_1 > 0$ . The right side is minimized by choosing  $r = \|g'\|_2/\|g\|_2$ , yielding (2.2).  $\square$

As a consequence of the necessary and sufficient condition for equality in the Cauchy-Schwarz inequality, equality in (2.2) is seen to hold if and only if  $g$  is proportional to  $\exp\{-\lambda|x-a|\}$ ,  $x, a \in \mathbb{R}$  and  $\lambda > 0$ .

As a corollary to Proposition 2.1 we derive an analogous result for functions defined on the half-line or on a finite interval. Denote by  $\|\cdot\|_{2,I}$  and  $\|\cdot\|_{\infty,I}$  the  $L_2(I)$ - and  $\sup_I$ -norms respectively where  $I = (a, b) \in \mathbb{R}$ .

Proposition 2.2. Let  $I$  be a possibly infinite interval  $(a, b)$ , and let  $g, g' \in L_2(I)$ . Then

$$(2.3) \quad \|g\|_{\infty, I}^2 \leq \|g\|_{2, I} \|g'\|_{2, I} + (g(a)^2 + g(b)^2) / 2 .$$

Proof: We first extend  $g$  to  $g_\lambda^*$  on  $\mathbb{R}$ , which under our assumptions is a member of  $H^1(\mathbb{R})$ , where

$$g_\lambda^*(x) \equiv g(a)g_{\lambda,a}^{\frac{1}{2}}(x) + g(x) + g(b)g_{\lambda,b}^u(x), \quad \lambda > 0,$$

with

$$g_{\lambda,a}^u(x) \equiv e^{-\lambda(x-a)} I_{[a,\infty)}(x) \quad \text{and} \quad g_{\lambda,a}^E \equiv e^{\lambda(x-a)} I_{(-\infty,a]}(x),$$

for  $a \in \mathbb{R}$  and  $\lambda > 0$ .

Then from Proposition (2.1), we have

$$\begin{aligned} \|g\|_{\infty, I}^2 &= \|g_\lambda^*\|_{\infty}^2 \leq \|g_\lambda^*\| \|g_\lambda^{*2}\|_2 \\ &= \left\{ \|g\|_2^2 + \frac{1}{\lambda} (g(a)^2 + g(b)^2)/2 \right\} \cdot \left\{ \|g'\|_2^2 + \lambda (g(a)^2 + g(b)^2)/2 \right\}^{\frac{1}{2}} \\ &= \left\{ \|g\|_2^2 \|g'\|_2^2 + ((g(a)^2 + g(b)^2)/2) (\|g\|_2^2 \lambda + \|g'\|_2^2/\lambda) \right. \\ &\quad \left. + ((g(a)^2 + g(b)^2)/2)^2 \right\}^{\frac{1}{2}}, \end{aligned}$$

for all  $\lambda > 0$ . Hence,

$$\|g\|_{\infty}^2 \leq \min_{\lambda > 0} (\|g_\lambda^*\|_2 \|g_\lambda^{*2}\|_2) = \|g\|_2 \|g'\|_2 + (g(a)^2 + g(b)^2)/2,$$

since the minimum is achieved for  $\lambda = \|g'\|_2/\|g\|_2$ .  $\square$

Note that inequality (2.3) is satisfied with equality if  $g$  is exponential on the half-line or constant on a finite interval  $(a,b)$ .

Also, if say  $g(a) = 0$ , then from (2.3) we have

$$\|g\|_{\infty}^2 \leq \|g\|_2 \|g'\|_2 + g(b)^2/2 \leq \|g\|_2 \|g'\|_2 + \|g\|_{\infty}^2/2.$$

which implies

$$(2.4) \quad \|g\|_{\infty}^2 \leq 2\|g\|_2 \|g'\|_2.$$

For functions  $g \in H^1((0, +\infty))$  inequality (2.4) can also be seen to hold by applying inequality (2.2) to  $g^*(x) \equiv g(x)I_{(a, \infty)} + g(2a-x)I_{(-\infty, a)}$ , where  $I_A$  denotes the indicator function of the set  $A$ . It is then clear that, as a consequence of the necessary and sufficient condition for equality in the Cauchy-Schwarz inequality, equality in (2.4) with  $g \in H^1((0, +\infty))$  holds if and only if  $g$  is exponential.

3. An application. Let  $f$  be a probability density function on a possibly infinite interval  $(a, b)$  and denote by  $I(f) \equiv \int_a^b (f'(x))^2/f(x) dx$  its Fisher information. Then  $I(f) = 4\|g'\|_{2, I}^2$ , where  $g = f^{1/2}$ . As a corollary to Proposition (2.2) (which covers Proposition (2.1)), we derive the following inequality involving Fisher information:

Corollary 3.1. Let  $g = f^{1/2}$  be as in Proposition 2.2. Then

$$(3.1) \quad \sup_{x \in (a, b)} f(x) \leq [I(f)^{1/2} + f(a) + f(b)]/2.$$

Inequality (3.1) may be modified to allow for discontinuities of  $f$  within  $(a, b)$ .

Remark 3.1. It might be of interest to note the following two characterizations of density functions with minimal Fisher information for a given maximum of the density function:

(1) For all  $f$  on  $\mathbb{R}$  with  $\sup_x f(x) = B$ ,  $I(f) \geq 4B^2$  with "=" iff  $f$  is the density of a Laplace (doubly exponential) distribution (see the remarks following Proposition (2.1)).

(ii) For all  $f$  on  $R^+$  with  $\sup_x f(x) = B$ ,  $I(f) \geq B^2$  with "=" if and only if  $f$  is exponential (see the remarks following inequality (2.4)).

Acknowledgments. The need for inequality (2.2) arose during the preparation of my Ph.D. dissertation, completed under the supervision of Professor W.J. Hall, to whom I would like to express my gratitude for his encouragement and for many helpful discussions and suggestions. I would also like to thank Professor Julian Keilson for many interesting comments and discussions. I would like to thank Professor R.J. Serfling for his encouragement and his comments on an earlier draft of this work. It is also a pleasure to acknowledge Professor R.H. Byrd for many helpful discussions.

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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1. REPORT NUMBER ONR No. 81-1	2. GOVT ACCESSION NO. AD-A204 474	3. RECIPIENT CATALOG NUMBER
4. TITLE AN UPPER BOUND ON THE PROBABILITY DENSITY FUNCTION IN TERMS OF FISHER INFORMATION	5. TYPE OF REPORT & PERIOD COVERED Technical Report	
	6. PERFORMING ORGANIZATION REPORT NO. Technical Report No. 336	
7. AUTHOR(s) V. K. Klonias	8. CONTRACT OR GRANT NUMBER(s) ONR No. N00014-79-C-0801	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematical Sciences The Johns Hopkins University Baltimore, MD 21218	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	
11. CONTROLLING OFFICE NAME & ADDRESS Office of Naval Research Statistics & Probability Program Arlington, VA 22217	12. REPORT DATE April 1981	
	13. NUMBER OF PAGES 6	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS (of this report) Unclassified	
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of this report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS Upper bound to the density, Fisher information, characterization of density functions, first distributional derivative.		
20. ABSTRACT An inequality for members of a Sobolev space of order one is demonstrated and as a corollary an upper bound, in terms of the Fisher information, is derived for a density function. Also two characterizations of the Laplace and the exponential distribution are indicated.		