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GENERALIZATION OF BILLINGSLEY'S INEQUALITIES.(U)

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(6) GENERALIZATION OF BILLINGSLEY'S INEQUALITIES

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ABSTRACT

GENERALIZATION OF BILLINGSLEY'S INEQUALITIES

Let $\xi_1, \xi_2, \dots, \xi_m$ be arbitrary random variables and define $S_k = \xi_1 + \xi_2 + \dots + \xi_k$ for $1 \leq k \leq m$, $S_0 = 0$, $M_m = \max_{0 \leq k \leq m} |S_k|$, $M'_m = \max_{0 \leq k \leq m} \min(|S_k|, |S_m - S_k|)$ and $M''_m = \max_{0 \leq i < j < k \leq m} \min(|S_j - S_i|, |S_k - S_j|)$. In this paper we establish bounds for the quantities $P(M_m \geq \lambda)$, $P(M'_m \geq \lambda)$ and $P(M''_m \geq \lambda)$ in terms of corresponding similar bounds assumed for the quantities $P(|S_j - S_i| \geq \lambda)$, $P(|S_k - S_j| \geq \lambda)$, all $0 \leq i \leq j \leq k \leq m$. The bounds explicitly involve a nonnegative function $f(i, j)$ which is quasi-superadditive, i.e., $f(i, j) + f(j, k) \leq f(i, k)$, all $0 \leq i \leq j \leq k \leq m$, for a fixed Q , $1 \leq Q < 2$. The results generalize theorems of Billingsley (1968) for the case $Q = 1$ and $f(i, j) = \sum_{1 \leq k \leq j} u_k$, where $u_1, \dots, u_m$ are nonnegative reals.

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1. Introduction. Let $\xi_1, \xi_2, \dots, \xi_m$ be arbitrary random variables. It is not assumed that the ξ_k 's are independent or identically distributed. The only restrictions on the dependence will be those imposed by the assumed bounds on $P(|S_j - S_i| \geq \lambda)$, $P(|S_j - S_1| \geq \lambda)$, $P(|S_k - S_j| \geq \lambda)$, where λ runs over an interval of the positive real line, and

$$S_j = \sum_{k=1}^j \xi_k \text{ for } 1 \leq j \leq m \text{ and } S_0 = 0.$$

Following Billingsley (1968, pp. 87-103) (we use the same notation that can be found there), define

$$M_m = \max_{0 \leq k \leq m} |S_k|,$$

$$M'_m = \max_{0 \leq k \leq m} \min(|S_k|, |S_m - S_k|),$$

$$M''_m = \max_{0 \leq i < j < k \leq m} \min(|S_j - S_i|, |S_k - S_j|),$$

and

$$N_m = \min_{1 \leq i < j \leq m} \max_{0 \leq k < l \leq m} |S_l - S_i|, \max_{0 \leq k < m} |S_m - S_i|.$$

We recall that

$$M'_m \leq M_m, M'_m \leq M''_m, \text{ and } M''_m \leq 2M'_m.$$

Our main goal is to establish bounds for the quantities

$$P(M_m \geq \lambda), P(M'_m \geq \lambda), P(M''_m \geq \lambda)$$

in terms of corresponding similar bounds assumed for the quantities listed above. The bounds will be related in specific ways to the variables $S_j - S_i$, $0 \leq i \leq j \leq m$, through some function $f(i, j)$ assumed to be nonnegative, nondecreasing in j for each fixed i , and Q -superadditive with $1 \leq Q < 2$. The latter property was introduced by Moricz, Serfling and Stout (1981). A function $f(i, j)$, $0 \leq i \leq j \leq m$, is said to be quasi-superadditive with index Q (or simply Q-superadditive) if

$$f(i, j) + f(j, k) \leq Qf(i, k), \text{ all } 1 \leq i \leq j \leq k \leq m.$$

The case $Q = 1$ corresponds to the usual notation of superadditivity.

We note that there is a slight difference in notation between this paper and the paper mentioned above. The relation between the function $f(i, j)$ occurring here and the function $g(i, j)$ used there is the following: $f(i, j) = g(i+1, j)$ (and similarly, $s_j - s_i = S(i+1, j)$, the latter being also used there).

For later reference we collect the assumed properties of $f(i, j)$ as follows

$$(1.1a) \quad f(i, j) \geq 0, \quad f(i, i) = 0, \quad \text{all } 0 \leq i \leq j \leq \lambda,$$

$$(1.1b) \quad f(i, j) \leq f(i, j+1), \quad \text{all } 0 \leq i \leq j < m,$$

$$(1.1c) \quad f(i, j) + f(j, k) \leq Qf(i, k), \quad \text{all } 0 \leq i \leq j \leq k \leq m.$$

In Billingsley's book (pp. 87-103) the case $f(i, j) = \sum_{1 \leq k \leq j} u_k$ is treated, where $u_1, u_2, \dots, u_m$ are nonnegative reals. This function $f(i, j)$ is clearly superadditive (even additive).

2. Main Results.

THEOREM 1. (The generalization of [1, Theorem 12.1].) Let $\alpha > 1/2$

be a given real. Suppose that there exist a function $f(i, j)$ satisfying (1.1)

with α , $1 \leq Q < 2^{(2\alpha-1)/2\alpha}$, and λ_0 , $0 < \lambda_0 \leq +\infty$, such that

$$(2.1) \quad P\{|s_j - s_i| \geq \lambda, |s_k - s_j| \geq \lambda\} \leq \frac{1}{\phi(\lambda)} f^{2\alpha}(i, k), \quad \text{all } 0 < \lambda < \lambda_0$$

and $0 \leq i \leq j \leq k \leq m$,

where $\phi(\lambda) > 0$ for $0 < \lambda < \lambda_0$ and for each constant C , $0 < C < 1$, we have

$$(2.2) \quad \inf_{0 < \lambda < \lambda_0} \frac{\phi(C)}{\phi(\lambda)} = X(C) > 0, \quad \lim_{C \rightarrow 1-0} X(C) = 1.$$

Then there exists a constant $K \geq 1$, depending on α , Q and X but not on m or $\{f_k\}$ or otherwise on f , such that

$$(2.3) \quad P\{W_m \geq \lambda\} \leq \frac{K}{\phi(\lambda)} f^{2\alpha}(0, m), \quad \text{all } 0 < \lambda < \lambda_0.$$

We note that $\phi(\lambda) = \lambda^Y$ satisfies condition (2.2) for each $Y \geq 0$.

Actually, Theorem 1 was proved by Billingsley (1968) in the special case that $\phi(\lambda) = \lambda^{2Y}$, $Y \geq 0$, and $f(i, j) = \sum_{1 \leq k \leq j} u_k$, $u_k \geq 0$. This remark pertains to the subsequent Theorem 2 and Corollaries 1 and 2.

The following corollary can be deduced from Theorem 1 in the same way that Theorem 12.2 is deduced from Theorem 12.1 in [1].

COROLLARY 1. (The generalization of [1, Theorem 12.2].) Let $\alpha > 1$ be a given real. Suppose that there exist a function $f(i, j)$ satisfying (1.1) with α , $1 \leq Q < 2^{(2\alpha-1)/\alpha}$, and λ_0 , $0 < \lambda_0 \leq +\infty$, such that

$$P\{|s_j - s_i| \geq \lambda\} \leq \frac{1}{\phi(\lambda)} f^\alpha(i, j), \quad \text{all } 0 < \lambda < \lambda_0 \text{ and } 0 \leq i \leq j \leq m,$$

where $\phi(\lambda) > 0$ for $0 < \lambda < \lambda_0$ and (2.2) is satisfied for each C , $0 < C < 1$. Then there exists a constant $K' \geq 1$, depending on α , Q and X but not on m or $\{f_k\}$ or otherwise on f , such that

$$P\{W_m \geq \lambda\} \leq \frac{K'}{\phi(\lambda)} f^\alpha(0, m), \quad 0 < \lambda < \lambda_0.$$

This result, using a direct proving method (and a somewhat different notation), was proved by Móricz, Serfling and Stout (1981, Theorem 3.1).

PROOF OF THEOREM 1. It goes along the same lines as the proof of [1, Theorem 12.1], i.e., by induction on m . The result is trivial for $m = 1$ and can be simply proved for $m = 2$.

Assume now as induction hypothesis that the result holds for each integer less than $m > 2$. We shall prove it for m itself. We may assume $f(0, m) > 0$.

By (1.1), there exists an integer h , $1 \leq h \leq m$, such that

$$(2.4) \quad f(0, h-1) \leq \frac{Q}{2} f(0, m) \leq f(0, h).$$

Then, by (1.1) and (2.4), we have

$$(2.5) \quad f(h, m) \leq \frac{Q}{2} f(0, m).$$

Following Billingsley's proof, consider the next four quantities:

$$U_1 = \max_{0 \leq i \leq h-1} \min(|S_i|, |S_{h-1} - S_i|),$$

$$U_2 = \max_{h \leq j \leq m} \min(|S_j - S_h|, |S_m - S_j|),$$

$$D_1 = \min(|S_{h-1}|, |S_m - S_{h-1}|),$$

$$D_2 = \min(|S_h|, |S_m - S_h|).$$

By (1, (12.37) and (12.38)),

$$(2.6) \quad P(U_1 \geq \lambda) \leq P(U_1 + D_1 \geq \lambda) + P(U_2 + D_2 \geq \lambda)$$

and

$$(2.7) \quad P(U_1 + D_1 \geq \lambda) \leq P(U_1 \geq p\lambda) + P(D_1 \geq q\lambda),$$

where p and q are positive reals and $p+q = 1$.

By the induction hypothesis and (2.4), we have

$$P(U_1 \geq p\lambda) \leq \frac{K}{\phi(p\lambda)} f^{2\alpha}(0, h-1) \leq \frac{K}{\phi(p\lambda)} \frac{Q^{2\alpha}}{2^{2\alpha}} f^{2\alpha}(0, m).$$

By (2.1),

$$P(D_1 \geq q\lambda) \leq \frac{1}{\phi(q\lambda)} f^{2\alpha}(0, m).$$

Taking (2.2) into account, from (2.7) it follows that

$$P(U_1 + D_1 \geq \lambda) \leq \frac{K f^{2\alpha}(0, m)}{\phi(t)} \left(\frac{1}{\chi(p)} \frac{Q^{2\alpha}}{2^{2\alpha}} + \frac{1}{K\chi(q)} \right).$$

The same inequality holds for $U_2 + D_2$ (using (2.5) instead of (2.4)).

By (2.6), therefore,

$$P(U_1 \geq \lambda) \leq \frac{K f^{2\alpha}(0, m)}{\phi(t)} \left(\frac{1}{\chi(p)} \frac{Q^{2\alpha}}{2^{2\alpha-1}} + \frac{2}{K\chi(q)} \right).$$

By assumption $Q^{2\alpha}/2^{2\alpha-1} < 1$. Thus, thanks to (2.2), we can define p ,

$0 < p < 1$, in such a way that

$$\frac{1}{\chi(p)} \frac{Q^{2\alpha}}{2^{2\alpha-1}} < 1.$$

Then let $q = 1-p$ and define K by the condition

$$\frac{1}{\chi(p)} \frac{Q^{2\alpha}}{2^{2\alpha-1}} + \frac{2}{K\chi(q)} \leq 1.$$

This completes the induction step and the proof of Theorem 1.

3. Further Inequalities

THEOREM 2. (The generalization of [1, Theorem 12.5]) Let $\alpha > 1/2$ be a given real. Suppose that there exist a function $f(i, j)$ satisfying

(1.1) with $a = 0$, $1 \leq Q < 2^{(2\alpha-1)/2}$, and $a\lambda_0' = 0 < \lambda_0 \leq +\infty$, such that

condition (2.1) holds, where $\phi(\lambda) > 0$ for $0 < \lambda < \lambda_0$ and (2.2) is satisfied for each C , $0 < C < 1$. Then there exists a constant $K'' \geq 1$, depending on α, Q and χ but not on m or $\{f_k\}$ or otherwise on f , such that we have both

$$P(N_m'' \geq \lambda) \leq \frac{K''}{\phi(\lambda)} f^{2\alpha}(0, m), \quad \text{all } 0 < \lambda < \lambda_0,$$

and

$$P(N_m \geq \lambda) \leq \frac{K''}{\phi(\lambda)} f^{2\alpha}(0, m), \quad \text{all } 0 < \lambda < \lambda_0.$$

The proof of Theorem 2 closely follows that of [1, Theorem 12.5], using the same modifications we performed in the proof of Theorem 1. We do not enter into details.

COROLLARY 2. (The generalization of [1, Theorem 12.6]) Let $\alpha > 1/2$

be a given real. Suppose that there exist a function $f(i, j)$ satisfying

(1.1) with $Q = 1$ and $a\lambda_0' = 0 < \lambda_0 \leq +\infty$, such that

$$(3.1) \quad P(|S_j - S_i| \geq \lambda, |S_k - S_j| \leq \lambda) \leq \frac{1}{\phi(\lambda)} f^Q(i, j) f^Q(j, k),$$

all $0 < \lambda < \lambda_0$ and $0 \leq i \leq j \leq k \leq m$, where $\phi(\lambda) > 0$ for $0 < \lambda < \lambda_0$ and

(2.2) is satisfied for each C , $0 < C < 1$. Then there exists a constant

$K'' \geq 1$, depending on o and λ but not on m or $\{f_i\}$ or otherwise on f , such that we have both

$$(3.2) \quad P(N_m \geq \lambda) \leq \frac{K''}{\phi(\lambda)} f^{2Q}(0, m) \min_{1 \leq h \leq m} \left[1 - \frac{f(h-1, h)}{f(0, m)} \right]^a$$

and

$$(3.3) \quad P(N_m \geq \lambda) \leq \frac{K''}{\phi(\lambda)} f^{2Q}(0, m) \min_{1 \leq h \leq m} \left[1 - \frac{f(h-1, h)}{f(0, m)} \right]^0$$

for all $0 < \lambda < \lambda_0$.

PROOF OF COROLLARY 2. Choose h to minimize the final factor in (3.3).

Following (1, (12.70), (12.71) and (12.74)) write

$$A_1 = \min_{1 \leq i \leq h} \max_{0 \leq j < i} |s_j|, \quad \max_{L_1 \leq i < h} |s_{h-1} - s_i|,$$

$$A_2 = \min_{h < L_2 \leq m} \max_{h \leq i < L_2} |s_i - s_h|, \quad \max_{L_2 \leq i < m} |s_i - s_i|,$$

and

$$B = \max\{\mu(0, h-1, m); \mu(0, h-1, h); \mu(h-1, h, m); \mu(0, h, m)\},$$

where

$$\mu(1, i, h) = \min(|s_j - s_i|, |s_h - s_i|).$$

Since (3.1) implies (2.1), by virtue of Theorem 2 we have

$$P(A_1 \geq \lambda) \leq \frac{K''}{\phi(\lambda)} f^{2Q}(0, h-1)$$

and

$$P(A_2 \leq \lambda) \leq \frac{K''}{\phi(\lambda)} f^{2Q}(h, m).$$

Owing to (1.1) for $Q = 1$,

$$f(0, h-1) \leq f(0, h) - f(h-1, h) \leq f(0, m) - f(h-1, h)$$

and

$$f(h, m) \leq f(0, m) - f(0, h) \leq f(0, m) - f(h-1, h).$$

Combining the last two inequalities with the two preceding ones, we obtain

$$(3.4) \quad P(A_1 \geq \lambda) \leq \frac{K''}{\phi(\lambda)} f^{2Q}(0, m) \left[1 - \frac{f(h-1, h)}{f(0, m)} \right]^a$$

and

$$(3.5) \quad P(A_2 \geq \lambda) \leq \frac{K''}{\phi(\lambda)} f^{2Q}(0, m) \left[1 - \frac{f(h-1, h)}{f(0, m)} \right]^0$$

Finally, due to (3.1),

$$P(\mu(0, h-1, m) \geq \lambda) \leq \frac{1}{\phi(\lambda)} f^{2Q}(h-1, m) \leq \frac{f^{2Q}(0, m)}{\phi(\lambda)} \left[1 - \frac{f(h-1, h)}{f(0, m)} \right]^a,$$

and there is a similar inequality for each of the other μ 's occurring in B.

Therefore,

$$(3.6) \quad P(B \geq \lambda) \leq \frac{4f^{2Q}(0, m)}{\phi(\lambda)} \left[1 - \frac{f(h-1, h)}{f(0, m)} \right]^a.$$

By (1, (12.76)),

$$N_m \leq \max\{A_1, A_2\} + 2B;$$

consequently,

$$P(N_m \geq \lambda) \leq P(A_1 \geq \frac{1}{2}\lambda) + P(A_2 \geq \frac{1}{2}\lambda) + P(B \geq \frac{1}{4}\lambda).$$

Applying inequalities (3.4), (3.5) and (3.6), and taking (2.2) into account,

we have

$$P(N_m \geq \lambda) \leq \left(\frac{2K''}{\chi(\frac{1}{4})} + \frac{4}{\chi(\frac{1}{4})} \right) \frac{f^{2Q}(0, m)}{\phi(\lambda)} \left[1 - \frac{f(h-1, h)}{f(0, m)} \right]^0.$$

This is the desired inequality (3.3).

Hence (3.2) immediately follows since $N_m \leq 2N''_m$.

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- [1] Billingsley, P. (1968). Convergence of Probability Measures. Wiley, New York.
- [2] Móricz, F., Serfling, R. J. and Stout, W. (1981). Moment and probability bounds with quasi-superadditive structure for the maximum partial sum.

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20. ABSTRACT

Let $\xi_1, \xi_2, \dots, \xi_m$ be arbitrary random variables and define $S_k = \xi_1 + \xi_2 + \dots + \xi_k$ for $1 \leq k \leq m$, $S_0 = 0$, $M_m = \max_{0 \leq k \leq m} |S_k|$, $M'_m = \max_{0 \leq k \leq m} \min(|S_k|, |S_m - S_k|)$ and $M''_m = \max_{0 \leq i < j < k \leq m} \min(|S_j - S_i|, |S_k - S_j|)$. In this paper we establish bounds for the quantities $P(M_m \geq \lambda)$, $P(M'_m \geq \lambda)$ and $P(M''_m \geq \lambda)$ in terms of corresponding similar bounds assumed for the quantities $P(|S_j - S_i| \geq \lambda)$, $P(|S_j - S_i| \geq \lambda, |S_k - S_j| \geq \lambda)$, all $0 \leq i \leq j \leq k \leq m$. The bounds explicitly involve a nonnegative function $f(i, j)$ which is quasi-superadditive, i.e., $f(i, j) + f(j, k) \leq Q f(i, k)$, all $0 \leq i \leq j \leq k \leq m$, for a fixed Q , $1 \leq Q < 2$. The results generalize theorems of Billingsley (1968) for the case $Q = 1$ and $f(i, j) = \sum_{1 \leq k \leq j} u_k$, where $u_1, \dots, u_m$ are nonnegative reals.