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FINAL REPORT

ELECTROMAGNETIC DIFFRACTION BY A NARROW SLIT

IN AN IMPEDANCE SHEET--E-POLARIZATION

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I. INTRODUCTION

The Air Force has a vital interest protecting sensitive electronic equipment inside airplanes from electromagnetic pulse (EMP) penetration through composite skin panels. Composite skin panels are being used on airplanes today instead of aluminum because they are stronger and lighter than aluminum. Composite panels offer less shielding than conventional metal panels, however. Besides direct penetration (diffusion) through the composite panel walls, inadvertent penetration through seams, joints, and windows is possible.

The evaluation of the electrical (shielding) properties of composite materials has been studied by Casey (1976, 1977). For a graphite composite, he concludes that the electrical conductivity in the direction normal to the sheet surface is approximately zero. The tangential conductivity is anisotropic, but he shows that it can be satisfactorily modelled by an isotropic conductivity of approximately 1.5×10^4 mhos/meter (for graphite). He further shows that the graphite composite acts, in effect, like a low pass filter. This means that only lower frequencies usually need to be considered.

Because the frequencies of interest are low, a sheet of graphite composite material of thickness δ and effective tangential conductivity σ_t can be modelled by an infinitely thin sheet of sheet impedance $Z_s = 1/\sigma_t\delta$. Ohm's law requries that inside the sheet,

$$\vec{E}_{tan} = Z_s\vec{J}$$

where $\vec{E}_{tan}$ is the tangential component of electric field in Volts/meter and $\vec{J}$ is the sheet current in Amps per meter.

This report describes a method of solving for the deterioration of shielding due to slits (seams) in composite materials. This is done by solving for the quasi-static (low frequency) magnetic field diffracted by a slit in an impedance sheet. Only the E-polarization is studied. Three different electromagnetic concepts are involved in treating this problem. These are

(a) Low frequency or quasi-static techniques

(b) Impedance boundary conditions

and (c) Scattering by slits.

Previous efforts which combine these three topics have been few. Hurd (1979) has recently treated a similar problem for the H-polarization. He uses an impedance plane instead of an impedance sheet boundary condition. Impedance planes have a surface impedance boundary condition whereas impedance sheets have a jump discontinuity boundary condition. Kaden (1959, p. 212) uses a conformal transformation to find the penetration through a gap in a plane shield with finite conductivity and thickness. This work has been summarized by Butler, et al (1976).

The literature that describes one of the three subjects individually is reviewed briefly below.

(a) References on low-frequency techniques.

Low frequency scattering techniques are often used because analytical solutions can sometimes be obtained. Two review articles on low frequency techniques have been written by Kleinman (1967, 1978). Quasi-static techniques reduce the dynamic electromagnetics problem to a static problem by making suitable low frequency approximations. Latham and Lee (1968) develop quasi-static boundary conditions for inductive shields by neglecting displacement currents. Standard techniques for solving statics problems can then be applied. Such techniques are detailed by Sneddon (1966), among others.

(b) References on impedance boundary conditions.

Use of impedance sheet boundary conditions has been of recent interest. Harrington and Mautz (1975) use them to treat thin dielectric shells. Senior (1978) discusses them in connection with impedance half planes and Senior (1979) discusses them in conection with resistive strips. Casey (1977) gives conditions under which composites can be modelled by impedance sheet boundary conditions. Babinet's principle for impedance boundary conditions has been given by Baum and Singaraju (1974), Lang (1973), and Senior (1977).

(c) References on scattering by slits.

Scattering by slits in perfectly conducting planes has been studied exhaustively. Scattering by slits in finitely conducting planes or in impedance sheets has not received much attention. Slits in perfectly conducting planes have been studied by Clemmow (1966), Hongo (1972), Houlberg (1967), Nomura and Katsura (1957), Millar (1960), and Otsuki (1976), among others. Lam (1976) has studied the shielding effectiveness of seams or joints in perfectly conducting aircraft skins. Diffraction by slits or apertures in impedance planes has been treated by Neugebauer (1956), Zakharyev, Lemanski and Shcheglov (1970), and Hongo (1972). Neugebauer develops an approach to treating apertures in absorbing screens by using symmetry conditions. Zakharyev, et al, handle the case of a dipole antenna located in a slit in a finitely conducting plane. Finally, Hongo uses the Weber-Schafheitlin integral to formulate the problem of diffraction by a slit in a screen with a surface impedance. Apparently, no one has previously studied the probelm of interest here -- diffraction by a slit in an impedance sheet.

II. OBJECTIVES

The objective of this research is to find the H-field diffracted by a narrow slit in an impedance (composite) sheet. Equations which are functions of slit width and panel thickness and effective conductivity are to be developed and solved. The solutions can then be used to study the deterioration of shielding due to slits (seams) in composite panels. Three possibilities exist for each slit width and panel thickness and conductivity. One might find that penetration through the slit is the major component of coupling. On the other hand, one might find that penetration through the slit can be neglected compared to the direct penetration (diffusion) through the walls. Finally, it might turn out that both coupling through the slit and the walls has to be considered.

III. REFLECTION AND TRANSMISSION COEFFICIENTS OF AND THE CURRENT IN AN IMPEDANCE SHEET

The study of an impedance sheet without a slit will be a helpful start and of use later. A similar discussion can be found in Seshadri (1971). Figure 1 shows an E-polarized plane wave incident upon an impedance sheet at an angle θ. The sheet is in the $x = 0$ plane.

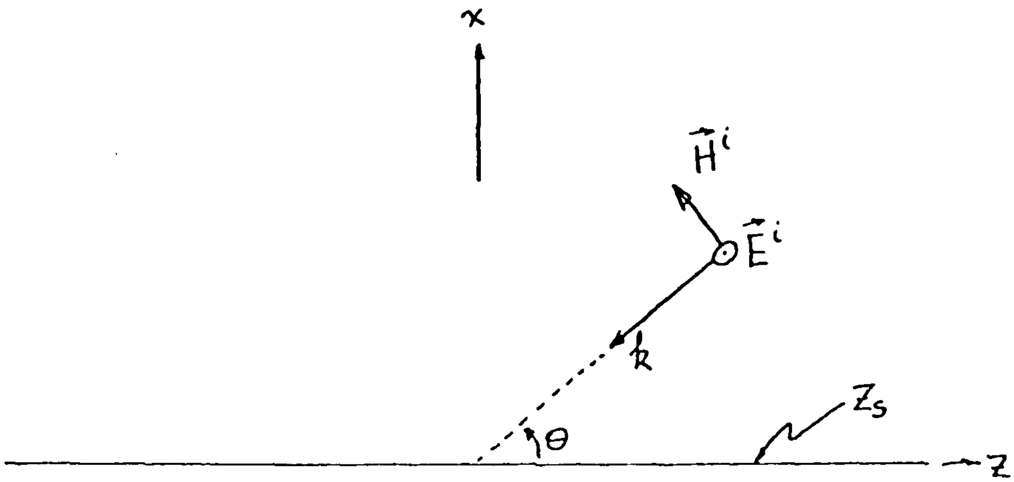

Figure 1. Plane wave incident on impedance sheet.

An $e^{-i\omega t}$ time dependence is assumed. The incident, reflected, and transmitted fields are denoted by superscripts i, r, and t, respectively. Let

$$\vec{k} = -k\sin\theta\,\hat{x} - k\cos\theta\,\hat{z} \qquad \vec{r} = x\hat{x} + z\hat{z} \qquad (3.1a,b)$$

then

$$\vec{H}^i = H_0\, e^{i\vec{k}\cdot\vec{r}}\,(\hat{x}\cos\theta - \hat{z}\sin\theta) \qquad (3.2a)$$

$$\vec{E}^i = Z_0 H_0\, e^{i\vec{k}\cdot\vec{r}}\,\hat{y} \qquad (3.2b)$$

$$\vec{H}^t = T H_0\, e^{i\vec{k}\cdot\vec{r}}\,(\hat{x}\cos\theta - \hat{z}\sin\theta) \qquad (3.3a)$$

$$\vec{E}^t = T Z_0 H_0\, e^{i\vec{k}\cdot\vec{r}}\,\hat{y} \qquad (3.3b)$$

Let

$$\vec{k}' = k\sin\theta\,\hat{x} - k\cos\theta\,\hat{z} \qquad (3.4)$$

then

$$\vec{H}^r = R' H_0\, e^{i\vec{k}'\cdot\vec{r}}\,(\hat{x}\cos\theta + \hat{z}\sin\theta) \qquad (3.5a)$$

$$\vec{E}^r = R' z_0 H_0 e^{ik'\cdot \vec{r}} \hat{y} \qquad (3.5b)$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$. There are two boundary conditions that must be satisfied across the impedance sheet [Casey (1977, p. 23)]. These are that $\vec{E}$ tangential ($\vec{E}_{tan}$) must be continuous and that

$$\vec{E}_{tan}(x=0) = Z_s \vec{J} = Z_s \left(\hat{x} \times \hat{z} \left[H_z(x=0^+) - H_z(x=0^-) \right] \right) \qquad (3.6)$$

Applying these conditions, one obtains

$$1 + R' = T \qquad (3.7)$$

$$\frac{Z_0}{\sin\theta} \frac{T}{Z_s} = 1 - R' - T \qquad (3.8)$$

The reflection and transmission coefficients become

$$R' = \frac{-Z_0}{Z_0 + 2Z_s \sin\theta} \qquad T = \frac{2Z_s \sin\theta}{Z_0 + 2Z_s \sin\theta} \qquad (3.9a,b)$$

The current in the sheet becomes

$$\vec{J} = \frac{\vec{E}_{tan}}{Z_s} = \frac{Z_0 H_0}{Z_s} \left(\frac{2Z_s \sin\theta}{Z_0 + 2Z_s \sin\theta} \right) e^{-ikz\cos\theta} \hat{y} \qquad (3.10)$$

This current satisfies the integral equation

$$\frac{kZ_0}{4} \int_{-\infty}^{\infty} I(z') H_0^{(1)}(k|z-z'|) dz' + Z_s I(z) = Z_0 H_0 e^{-ikz\cos\theta} \qquad (3.11)$$

as may be easily shown by a Fourier transformation.

IV. SLIT INTEGRAL EQUATIONS: DYNAMIC CASE

A standard approach for solving electromagnetics problems is the integral equation approach. Consider a two-dimensional slit of width 2b between two impedance sheets of impedances Z_{s_1} and Z_{s_2}, as shown in

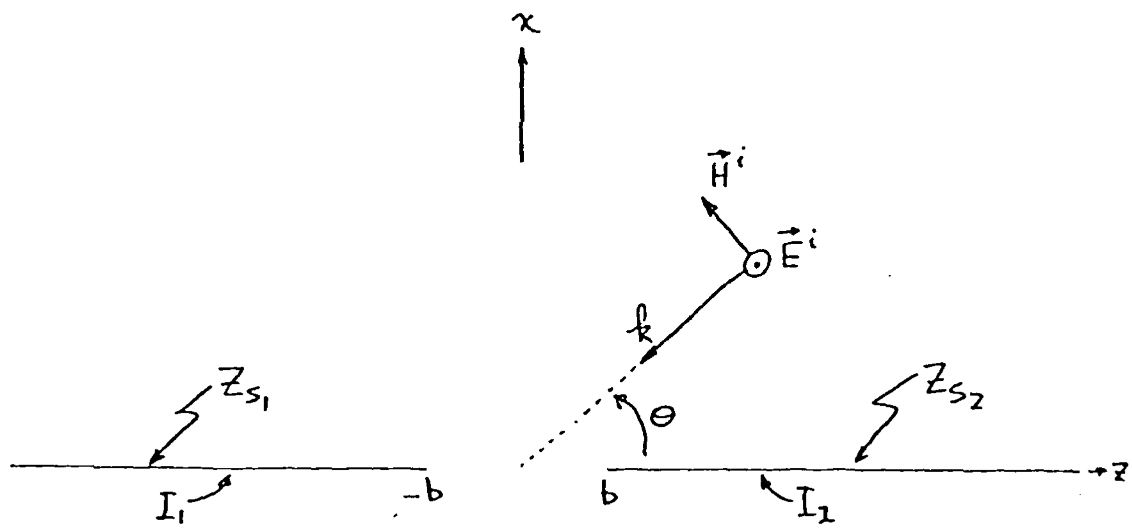

Figure 2. Dynamic plane wave incident on slit in impedance plane.

Figure 2. The vector potential for this problem is

$$\vec{A} = i\frac{\mu_0}{4}\hat{y}\int_{-\infty}^{-b} I_1(z')H_0^{(1)}\left(k\sqrt{x^2+(z-z')^2}\right)dz'$$

$$+ i\frac{\mu_0}{4}\hat{y}\int_{b}^{\infty} I_2(z')H_0^{(1)}\left(k\sqrt{x^2+(z-z')^2}\right)dz' \qquad (4.1)$$

The electric field scattered by the impedance sheets is given by

$$\vec{E}^s = i\omega\vec{A} \qquad (4.2)$$

Since the total electric field is equal to the incident field plus the scattered field and is also equal to $Z_{s_1} I_1$ over $-\infty < z < -b$ and to $Z_{s_2} I_2$ over $b < z < \infty$, both for $x=0$, one obtains the coupled integral equations

$$Z_0 H_0 e^{-ikz\cos\theta} - \frac{kZ_0}{4}\int_{-\infty}^{-b} I_1(z') H_0^{(1)}(k|z-z'|)\,dz' - \frac{kZ_0}{4}\int_{b}^{\infty} I_2(z') H_0^{(1)}(k|z-z'|)\,dz' = Z_{s_1} I_1(z) \qquad (4.3a)$$

$$z < -b$$

$$Z_0 H_0 e^{-ikz\cos\theta} - \frac{kZ_0}{4}\int_{-\infty}^{-b} I_1(z') H_0^{(1)}(k|z-z'|)\,dz' - \frac{kZ_0}{4}\int_{b}^{\infty} I_2(z') H_0^{(1)}(k|z-z'|)\,dz' = Z_{s_2} I_2(z) \qquad (4.3b)$$

$$z > b$$

These equations can be solved by using numerical techniques.

Since these equations are very complicated, it is desirable to simplify them. One possibility would be to try using a Babinet's principle of the types described by Baum and Singaraju (1974), Lang (1973), or Senior (1977). Unfortunately, this does not lead to a significant simplification of the equations and so is not considered further. Another possibility is to look at the equations for low frequencies. This also does not lead to a major simplification in the equations as they stand because the currents are supported over two semi-infinite regions and

and are therefore not directly tenable to low frequency approximations. A solution can be found using this method, however, by solving for the difference between the actual currents and the currents that would be present without the slit. This perturbation current goes to zero as one gets away from the slit. This approach is the one taken here.

V. QUASI-STATIC DUAL INTEGRAL EQUATION FORMULATION OF THE SLIT PROBLEM

The problem that will be studied here is less general than the one detailed in the last section. First of all, it will be treated on a quasi-static basis. Secondly, only the case of normal incidence will be studied. Finally, both parts of the sheet have the same impedance. This is shown in Figure 3.

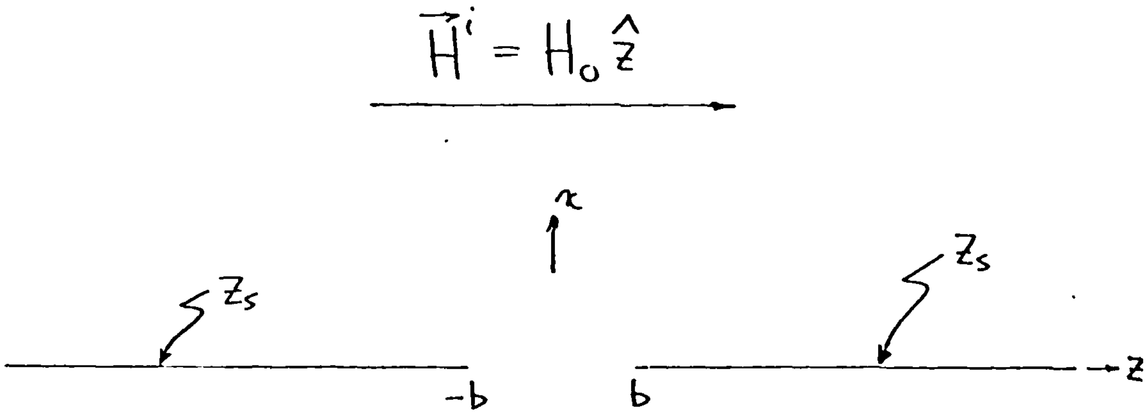

Figure 3. Quasi-static field incident on slit in impedance plane.

Let the incident field $\vec{H}^i = H_0\hat{z}$ be the field of a static current sheet of strength $2H_0\hat{y}$ located at infinity. It is of interest here to find the field above and below the impedance sheet and also in the gap. The basis for the quasi-static formulation used here is provided by Latham and Lee (1968). Maxwell's equations can be written as

$$\nabla \times \vec{H} = -i\omega\varepsilon\vec{E} + \vec{J} \qquad (5.1)$$

$$\nabla \times \vec{E} = i\omega\mu \vec{H} \qquad (5.2)$$

$$\nabla \cdot \vec{H} = 0 \qquad \nabla \cdot \vec{E} = 0 \qquad (5.3a,b)$$

where

$$\vec{E}_{tan} = Z_s \vec{J} \qquad (5.4)$$

Thus, these equations can be written as

$$\nabla \times \vec{H} = \vec{J} = \frac{\vec{E}_{tan}}{Z_s} \qquad (5.5)$$

$$\nabla \times \vec{E} = i\omega\mu \vec{H} \qquad (5.6)$$

provided that Z_s is small enough and the frequency is low enough so that the displacement current can be neglected. Off of the sheets, one has

$$\nabla \times \vec{H} = 0 \qquad (5.7)$$

and

$$\nabla \cdot \vec{H} = 0 \qquad (5.8)$$

The first condition implies that

$$\vec{H} = -\nabla\phi \qquad (5.9)$$

where ϕ is the magnetic scalar potential [Stratton (1941), pp. 225-267] while the second gives that

$$\nabla^2 \phi = 0 \qquad (5.10)$$

away from the sheets. The incident potential is related to the incident field by

$$\vec{H}^i = -\nabla \phi^i \tag{5.11}$$

so that

$$\phi^i = -H_0 z . \tag{5.12}$$

The arbitrary constant of integration is chosen to be zero so that $\phi^i(z=0) = 0$. This makes the incident potential an odd function--a property which will be very useful later.

It is now necessary to determine the boundary conditions which the potential must satisfy. Latham and Lee (1968) show that the surface divergence of $\vec{J}$ is zero, that is,

$$\nabla_s \cdot \vec{J} = 0 \tag{5.13}$$

in the quasi-static case for an impedance sheet. Thus, the current can be written as

$$\vec{J} = \hat{x} \times \nabla_s \psi \tag{5.14}$$

where ψ is a scalar function. This follows from equation 38, page 502 of Van Bladel (1964). Figure 4 shows the cross-secional view of the impedance sheet and a surface S of integration. Integrating equation (5.5)

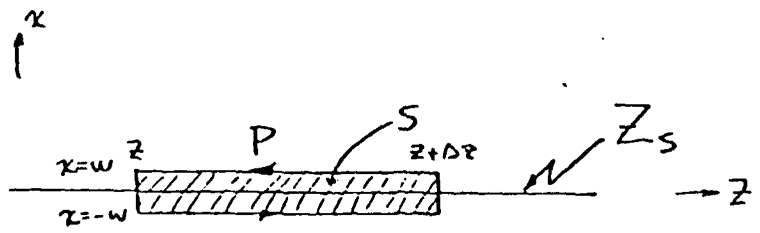

Figure 4. Surface and path of integration.

over the surface S gives

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_P \vec{H} \cdot d\vec{L} = \left[-H_z(0^+) + H_z(0^-) \right] \Delta z$$

$$= \int_S \vec{J} \cdot d\vec{S} = \left(-\nabla_s \psi \right) \Delta z \qquad (5.15)$$

After using (5.9), one obtains

$$\phi(0^+) - \phi(0^-) = -\psi . \qquad (5.16)$$

This relates the potential to ψ. Since $\mu \oint \vec{H} \cdot d\vec{S} = 0$ over a small "pillbox" enclosing the surface, normal $\vec{H}$ is continuous across the surface and

$$\left. \frac{\partial \phi}{\partial x} \right|_+ = \left. \frac{\partial \phi}{\partial x} \right|_- = \frac{\partial \phi}{\partial x} \qquad (5.17)$$

on the boundary. One last condition must be found. Latham and Lee (1968, p. 1749) show that

$$i \omega \mu H_x = \hat{x} \cdot \nabla_s \times \vec{E}_{tan} = \hat{x} \cdot \nabla_s \times \left(Z_s \vec{J} \right) \qquad (5.18)$$

so that

$$-i \omega \mu \, \hat{x} \cdot \nabla \phi = Z_s \, \hat{x} \cdot \left(\nabla_s \times \hat{x} \times \nabla_s \psi \right) \qquad (5.19)$$

Therefore, one has

$$-i \omega \mu \frac{\partial \phi}{\partial x} = Z_s \left(\nabla_s^2 \psi \right) \qquad (5.20)$$

after using identity 37, page 502 of Van Bladel (1964). Eliminating ψ between this equation and (5.16), one arrives at the second boundary condition that must be satisfied, namely,

$$i\omega\mu \left.\frac{\partial\phi}{\partial x}\right|_0 = Z_s\left(\nabla_s^{\,2}\left(\phi(0^+,z) - \phi(0^-,z)\right)\right) \qquad (5.21)$$

One might expect that a canonical problem to test these boundary conditions would be that of scattering by an impedance sheet without a slit. This does not lead to any useful result, however. Instead, one can use the results of section III for normal incidence. Thus, for k=0 and θ = 90°, one has that $\vec{H}^i = H_0\hat{z}$, $\vec{H}^t = TH_0\hat{z}$, and $\vec{H}^r = -R'H_0\hat{z}$ where $T = 2Z_s/(Z_0+2Z_s)$ and $R' = -Z_0/(Z_0+2Z_s)$. It can thus be assumed that for $\phi^i = -H_0 z$

$$\phi^t = -TH_0 z \quad \text{and} \quad \phi^r = -RH_0 z \qquad (5.22a,b)$$

where R = -R'. By using these reflected and transmitted potentials, one can formulate the problem in terms of a potential ϕ^s which will be significant only in the neighborhood of the slit. For convenience, define

$$\phi^s = \begin{cases} \phi_> & x>0 \\ \phi_< & x<0 \end{cases} \qquad (5.23)$$

This potential must satisfy Laplace's equation (5.10). Using separation of variables techniques and integrating over all possible separation constants, one can write

$$\phi_> = H_0\int_0^\infty F_1(\lambda)\sin\lambda z\, e^{-\lambda x}\frac{d\lambda}{\lambda} \qquad (5.24a)$$

$$\phi_< = H_0\int_0^\infty F_2(\lambda)\sin\lambda z\, e^{+\lambda x}\frac{d\lambda}{\lambda} \qquad (5.24b)$$

The total potential can be written as

$$\phi = \begin{cases} \phi^+ & x > 0 \\ \phi^- & x < 0 \end{cases} \tag{5.25}$$

where

$$\phi^+ = \phi_> + \phi^i + \phi^r \tag{5.26a}$$

and

$$\phi^- = \phi_< + \phi^t \tag{5.26b}$$

Applying the condition (5.17), one obtains

$$-F_1(\lambda) = +F_2(\lambda) \equiv -F(\lambda). \tag{5.27}$$

Substituting this, (5.24), (5.22) in (5.26), one arrives at

$$\phi^+ = H_0 \int_0^\infty F(\lambda) \sin\lambda z \, e^{-\lambda x} \frac{d\lambda}{\lambda} - H_0 z - R H_0 z \tag{5.28a}$$

$$\phi^- = -H_0 \int_0^\infty F(\lambda) \sin\lambda z \, e^{+\lambda x} \frac{d\lambda}{\lambda} - T H_0 z \tag{5.28b}$$

It is now necessary to find $F(\lambda)$. To do this, the previously derived boundary condition (5.21) can be applied on the impedance sheet. In the slit, the magnetic scalar potential must be continuous. The application of these two conditions gives dual integral equations which can be solved for $F(\lambda)$. Application of the first condition gives

$$Z_s \frac{d^2}{dz^2}\left(\phi^+(0,z) - \phi^-(0,z)\right) = -i\omega\mu H_0 \int_0^\infty F(\lambda) \sin\lambda z \, d\lambda \qquad z > b,\ z < -b \tag{5.29}$$

Since $1+R-T = \dfrac{2Z_0}{Z_0+2Z_s}$, we have

$$\phi^+(0,z) - \phi^-(0,z) = 2H_0\int_0^\infty F(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} - 2H_0\, z\left(\frac{Z_0}{Z_0+2Z_s}\right) \qquad (5.30)$$

and (5.29) becomes

$$2Z_s\frac{d^2}{dz^2}\left\{\int_0^\infty F(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} - \left(\frac{Z_0}{Z_0+2Z_s}\right)z\right\} + i\omega\mu\int_0^\infty F(\lambda)\sin\lambda z\,d\lambda = 0 \qquad |z|>b \qquad (5.31a)$$

Application of the second condition, continuity in the slit, yields

$$\int_0^\infty F(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} = \left(\frac{Z_0}{Z_0+2Z_s}\right)z \qquad |z|<b \qquad (5.31b)$$

These two equations are the dual integral equations that must be solved for the unknown $F(\lambda)$. Once $F(\lambda)$ is known, the potential can be determined from (5.28).

The following section discusses the evaluation of some integrals that we will need to solve the problem and the section after that solves the equations in the case of a perfectly conducting screen. Section VIII gives several possible formulations of the dual integral equations.

VI. GENERALIZED HANKEL TRANSFORMS

In the process of formulating and solving this problem, one finds it necessary to evaluate a great many integrals. Many of these integrals can be expressed as conventional Hankel transforms. If one denotes the class of all absolutely integrable functions $f(x)$ on the interval $(0, \infty)$ by $L_1(0,\infty)$, each $f(x) \in L_1(0,\infty)$ must satisfy

$$\int_0^\infty |f(x)|dx < \infty . \qquad (6.1)$$

If $f(x) \in L_1(0, \infty)$, then the conventional Hankel transform is given by

$$F(\lambda) = \int_0^\infty f(x)\sqrt{\lambda x}\, J_\mu(\lambda x)\,dx \qquad (6.2)$$

where $J_\mu(x)$ is the Bessel function of the first kind. If an integral can be cast in this form, it is easily looked up since several integral tables of conventional Hankel transforms are available.

In the following, it will occasionally be necessary to evaluate the Hankel transform integral when f(x) is not absolutely integrable. If one attempts to look such an integral up in a table, they will often find that the limits of applicability given next to the formula are violated. In other cases, one will find that the value of the integral is listed as infinity. This can be particularly disturbing if the integral was supposed to be a kernel or a right-hand side in an integral equation.

The solution of the slit problem at hand is based in part on the classical theory of dual integral equations presented by Sneddon (1966). This book is based on the conventional Hankel transform as is a later publication of his given in Ross (1975). The use of the generalized Hankel transform to formulate and solve dual integral equations has apparently not received a great deal of attention to date except by Zemanian (1968) and Walton (1973). The following discussion of the generalized Hankel transform is essentially a brief summary of that given by Zemanian (1968).

There are two equivalent ways in which a generalized function may be defined. First, a generalized function may be defined to be a continuous linear functional on some space of testing functions. Second, a generalized function by be defined as a limit of equivalent fundamental sequences of continuous functions. To each generalized function in the functional approach, there is one corresponding generalized function in the sequential approach. In order to follow Zemanian (1968), the functional approach is used here.

In order to define a functional, one must first define a space of testing functions. For the generalized Hankel transform, Zemanian chooses a class $\mathcal{H}_\mu$ of testing functions ϕ of rapid descent. A function $\phi(x)$ is in $\mathcal{H}_\mu$ if and only if

a. it is defined on $0 < x < \infty$

b. it is complex valued and smooth

c. all derivatives are of rapid descent.

For the exact mathematical meaning of these terms, the reader is referred to Zemanian.

DEFINITION: A generalized function is a continuous linear functional f on a testing function space (in this case $\mathcal{H}_\mu$).

A functional f on $\mathcal{H}_\mu$ is a rule that assigns a complex number $\langle f, \phi \rangle$ to every member $\phi \in \mathcal{H}_\mu$. A continuous linear functional f on $\mathcal{H}_\mu$ possesses

(1) linearity; that is, for any two testing functions ϕ_1 and ϕ_2 in $\mathcal{H}_\mu$ and any complex number α

$$\langle f, \phi_1 + \phi_2 \rangle = \langle f, \phi_1 \rangle + \langle f, \phi_2 \rangle$$

$$\langle f, \alpha \phi_1 \rangle = \alpha \langle f, \phi_1 \rangle$$

(2) continuity; that is, for any sequence $\{ \phi_\nu(t) \}_{\nu=1}^{\infty}$ in $\mathcal{H}_\mu$ that converges in $\mathcal{H}_\mu$ to ϕ

$$\{ \phi_\nu \}_{\nu \to \infty} \to \phi \quad \text{in} \quad \mathcal{H}_\mu \, ,$$

then

$$\lim_{\nu \to \infty} | \langle f, \phi \rangle - \langle f, \phi_\nu \rangle | \to 0$$

The collection of all continuous linear functionals on $\mathcal{H}_\mu$ is called the dual space of $\mathcal{H}_\mu$ and is denoted by $\mathcal{H}_\mu'$. The generalized functions $f \in \mathcal{H}_\mu'$ are sometimes called generalized functions of slow growth.

A locally integrable function f(x) is said to be a function of slow growth if

$$\lim_{t \to \infty} |t|^{-N} f(t) = 0 \qquad (6.3)$$

for some integer N. Let f(x) be a locally integrable function on $0 < x < \infty$ such that f(x) is of slow growth as $x \to \infty$ and $x^{\mu + 1/2} f(x)$ is absolutely integrable on $0 < x < 1$. Then, f(x) generates a regular generalized function f in $\mathcal{H}_\mu'$ by

$$\langle f, \phi \rangle = \int_0^\infty f(x) \, \phi(x) \, dx \qquad (6.4)$$

The generalized Hankel transform is based on Parseval's formula for conventional Hankel transforms. If f(x) and G(y) are in $L_1(0, \infty)$, if $\mu \geq -\frac{1}{2}$, if $F(y) = \mathfrak{h}_\mu[f(x)]$ and if $g(x) = \mathfrak{h}_\mu^{-1}[G(y)]$, then the Parseval formula states

$$\int_0^\infty f(x) g(x)\, dx = \int_0^\infty F(y) G(y)\, dy \qquad (6.5)$$

$\mathfrak{h}_\mu$ is the symbol for the Hankel transform. If we let G be an arbitrary $\Phi \in \mathcal{H}_\mu$ such that $\phi = \mathfrak{h}_\mu \Phi$ and let f be an arbitrary $f \in \mathcal{H}_\mu'$, we define the generalized Hankel transform $F = \mathfrak{h}_\mu f$ by

$$\langle F, \Phi \rangle = \langle f, \phi \rangle \qquad (6.6)$$

or

$$\langle \mathfrak{h}_\mu f, \Phi \rangle = \langle f, \mathfrak{h}_\mu \Phi \rangle \qquad (6.7)$$

By using this definition of the Hankel transform, Zemanian (1968) proves some useful properties in his Theorem 5.5-2. Although his results are applicable for any $\mu \geq -\frac{1}{2}$, they will be specialized here to sine and cosine functions, $\mu = \frac{1}{2}$ and $\mu = -\frac{1}{2}$, respectively.

By Theorem 5.5-2(5) with $\mu = -\frac{1}{2}$, one finds

$$\int_0^\infty x f(x) \sin(xy)\, dx = -\frac{d}{dy} \int_0^\infty f(x) \cos(xy)\, dx \qquad (6.8)$$

By Theorem 5.5-2(7) with $\mu = -\frac{1}{2}$, one finds

$$\int_0^\infty x^2 f(x) \cos(xy)\, dx = -\frac{d^2}{dy^2} \int_0^\infty f(x) \cos(xy)\, dx \qquad (6.9)$$

With $\mu = +\frac{1}{2}$, one obtains

$$\int_0^\infty x^2 f(x) \sin(xy)\, dx = -\frac{d^2}{dy^2} \int_0^\infty f(x) \sin(xy)\, dx \qquad (6.10)$$

Using Theorem 5.5-2(9) with $\mu = -\frac{1}{2}$, one finds

$$\int_0^\infty x f(x) \cos(xy)\, dx = \frac{d}{dy} \int_0^\infty f(x) \sin(xy)\, dx \qquad (6.11)$$

In each case, if the integral on the left does not exist in the conventional sense, a generalized function can often be defined for it through the expression on the right.

For example, consider the integral

$$\int_0^\infty \frac{\cos(ax)\cos(bx)}{x}\,dx \qquad (6.12)$$

Standard integral tables, such as Dwight (1961), give the value of the integral to be infinity. If we take the derivative with respect to b and use (6.8), we have

$$\frac{d}{db}\int_0^\infty \frac{\cos ax}{x}\cos bx\,dx = -\int_0^\infty x\,\frac{\cos ax}{x}\sin(bx)\,dx =$$

$$= -\int_0^\infty \cos ax\,\sin bx\,dx$$

$$= -\frac{1}{2}\int_0^\infty \sin(b+a)x\,dx + \frac{1}{2}\int_0^\infty \sin(a-b)x\,dx \qquad (6.13)$$

As Butkov (1968, p. 312) points out, these integrals are divergent. If they are considered to be generalized functions, however, Butkov states without proof that

$$\int_0^\infty \sin kx\,dx = \frac{1}{k} \qquad (6.14)$$

This can be proven if $\int_0^\infty \frac{\sin kx}{k}\,dk = \frac{\pi}{2} \quad x > 0$ is interpretted as an inverse Fourier transform. Thus, our integral becomes

$$-\frac{1}{2}\frac{1}{b+a} + \frac{1}{2}\frac{1}{a-b} = \frac{b}{a^2-b^2} \qquad (6.15)$$

Thus, since

$$\frac{d}{db}\int_0^\infty \frac{\cos ax \cos bx}{x}\,dx = \frac{b}{a^2-b^2} \qquad (6.16)$$

$$\int_0^\infty \frac{\cos ax \cos bx}{x}\,dx = -\frac{1}{2}\ln|a^2-b^2| + C \qquad (6.17)$$

where C is an arbitrary constant. This agrees with the generalized value for this integral given by Khadem and Keer (1974).

Another integral which we will have to evaluate is

$$\int_0^\infty \sin tx \sin ux\,dx = \int_0^\infty x\,\frac{\sin ux}{x}\sin tx\,dx \qquad (6.18)$$

Using (6.8), we find

$$\int_0^\infty \sin tx \sin ux\,dx = -\frac{d}{dt}\int_0^\infty \frac{\sin ux}{x}\cos tx\,dx$$

$$= -\frac{\pi}{2}\frac{d}{dt}1_+(u-t) \qquad u,t>0 \qquad (6.19)$$

Thus, we have

$$\int_0^\infty \sin tx \sin ux\,dx = \frac{\pi}{2}\delta(u-t) \qquad (6.20)$$

which agrees with Stakgold (1967, vol. I, p. 286) and Davis (1974). In the next section, the dual integral equation approach will be used to solve the problem for the perfectly conducting case $Z_s = 0$.

VII. QUASI-STATIC DUAL INTEGRAL EQUATION SOLUTION FOR A SLIT IN A PERFECTLY CONDUCTING PLANE

The problem of a slit in a perfectly conducting plane is of interest because for a typical graphite composite sheet, $Z_s = 0.045$ ohms. This means that one can consider $Z_s = 0$ for some applications. Upon setting $Z_s = 0$ in (5.31), one has the dual integral equations

$$\int_0^\infty F(\lambda) \sin \lambda z \frac{d\lambda}{\lambda} = z \qquad 0<z<b \qquad (7.1a)$$

$$\int_0^\infty F(\lambda) \sin \lambda z \; d\lambda = 0 \qquad z>b \qquad (7.1b)$$

A similar pair of equations is solved in another way by Clemmow (1966, p. 91). The solution to these dual integral equations is given by Sneddon (1966, p. 103) and is

$$F(\lambda) = b\, J_1(\lambda b)$$

Substituting this in equation (5.28), one obtains

$$\phi = \begin{cases} H_0 b \int_0^\infty J_1(\lambda b) \sin \lambda z \; e^{-\lambda x} \frac{d\lambda}{\lambda} - 2H_0 z & x>0 \\ -H_0 b \int_0^\infty J_1(\lambda b) \sin \lambda z \; e^{+\lambda x} \frac{d\lambda}{\lambda} & x<0 \end{cases} \qquad (7.2)$$

To evaluate this explicitly, one needs to evaluate integrals of the type

$$\int_0^\infty J_1(\lambda b) \sin \lambda z \; e^{-\lambda |x|} \frac{d\lambda}{\lambda} \qquad (7.3)$$

Integrals of this type have been evaluated by George (1962). The result is

$$\int_0^\infty J_1(\lambda b) \sin \lambda z \; e^{-\lambda|x|} \frac{d\lambda}{\lambda} =$$

$$= \frac{z}{b} - \frac{\operatorname{sgn}(z)}{b} \left\{ \frac{\sqrt{(x^2+b^2-z^2)^2 + 4x^2z^2} - (x^2+b^2-z^2)}{2} \right\}^{\frac{1}{2}} \qquad (7.4)$$

Note that for x=0, this becomes

$$\frac{1}{b}\left\{ z - \operatorname{sgn}(z) \sqrt{\frac{|b^2-z^2| - (b^2-z^2)}{2}} \right\} = \begin{cases} \frac{z}{b} & z<b \\ \frac{1}{b}\left(z - \sqrt{z^2-b^2}\right) & z>b \end{cases} \qquad (7.5)$$

If the expression for z b is multiplied and divided by $\left(z+\sqrt{z^2-b^2}\right)$, we find the same result as that given by Magnus and Oberhettinger (1949, p. 36).

$$\int_0^{\infty} J_1(\lambda b) \sin \lambda z \, \frac{d\lambda}{\lambda} = \begin{cases} \dfrac{z}{b} & |z| < b \\ \dfrac{b \, \mathrm{sgn}(z)}{|z| + \sqrt{z^2-b^2}} & |z| > b \end{cases} \tag{7.6}$$

The potential becomes

$$\phi = -H_0 \left\{ z + \mathrm{sgn}(xz) \left[\frac{\sqrt{(x^2+b^2-z^2)^2 + 4x^2z^2} - (x^2+b^2-z^2)}{2} \right]^{\frac{1}{2}} \right\} \tag{7.7}$$

The x-component of the H-field is given by equation (5.9).

$$H_x = -\frac{\partial \phi}{\partial x} \tag{7.8}$$

After performing this differentiation and simplifying, one obtains

$$H_x(x,z) = H_0 z \left[\frac{1}{2} \, \frac{\sqrt{(x^2+b^2-z^2)^2 + 4x^2z^2} + (x^2+b^2-z^2)}{(x^2+b^2-z^2)^2 + 4x^2z^2} \right]^{\frac{1}{2}}$$

$$- H_0 \, \mathrm{sgn}(z) |x| \left[\frac{1}{2} \, \frac{\sqrt{(x^2+b^2-z^2)^2 + 4x^2z^2} - (x^2+b^2-z^2)}{(x^2+b^2-z^2)^2 + 4x^2z^2} \right]^{\frac{1}{2}} \tag{7.9}$$

Special cases of interest are x=0, z=0.

$$H_x(x,0) = 0 \tag{7.10a}$$

$$H_x(0,z) = \begin{cases} \dfrac{H_0 z}{\sqrt{b^2-z^2}} & z < b \\ 0 & z > b \end{cases} \tag{7.10b}$$

The z-component of the H field is given by

$$H_z = -\frac{\partial \phi}{\partial z} \tag{7.11}$$

Performing this differentiation, we have

$$H_z = H_o + H_o x \left[\frac{1}{2} \frac{\sqrt{(x^2+b^2-z^2)^2 + 4x^2z^2} + (x^2+b^2-z^2)}{(x^2+b^2-z^2)^2 + 4x^2z^2} \right]^{\frac{1}{2}}$$

$$+ \operatorname{sgn}(x) H_o |z| \left[\frac{1}{2} \frac{\sqrt{(x^2+b^2-z^2)^2 + 4x^2z^2} - (x^2+b^2-z^2)}{(x^2+b^2-z^2)^2 + 4x^2z^2} \right]^{\frac{1}{2}} \tag{7.12}$$

For the special cases x=0 and z=0, one obtains

$$H_z(x,0) = H_o\left(1 + \frac{x}{\sqrt{x^2+b^2}}\right) \tag{7.13}$$

$$H_z(0,z) = H_o \qquad |z| < b \tag{7.14a}$$

$$\begin{aligned} H_z(x=0^+, z) &= 2H_o + \frac{H_o b^2}{\sqrt{z^2-b^2}\left(|z| + \sqrt{z^2-b^2}\right)} \\ &= H_o\left(1 + \frac{|z|}{\sqrt{z^2-b^2}}\right) \qquad |z| > b \end{aligned} \tag{7.14b}$$

$$\begin{aligned} H_z(x=0^-, z) &= \frac{-H_o b^2}{\sqrt{z^2-b^2}\left(|z| + \sqrt{z^2-b^2}\right)} \\ &= H_o\left(1 - \frac{|z|}{\sqrt{z^2-b^2}}\right) \qquad |z| > b \end{aligned} \tag{7.14c}$$

The equivalent current in the conductor can be written as

$$J_y = -2H_0 \frac{|z|}{\sqrt{z^2 - b^2}} \qquad |z| > b \qquad (7.15)$$

VIII. DUAL INTEGRAL EQUATION FORMULATIONS FOR A SLIT IN AN IMPEDANCE SHEET

The fundamental quasi-static dual integral equation for the problem of a slit in an impedance sheet is given by (5.31). As it stands, (5.31a) is not in a form whose solution appears in Sneddon (1966). There are several dual integral equation pairs that can be obtained from (5.31). The first results from direct differentiation in (5.31a). Using (6.10), we have

$$\int_0^\infty F(\lambda) \sin \lambda z \, \frac{d\lambda}{\lambda} = \left(\frac{Z_0}{Z_0 + 2Z_s}\right) z \qquad |z| < b \qquad (8.1a)$$

$$\int_0^\infty F(\lambda)\left\{-2Z_s\lambda + i\omega\mu\right\} \sin \lambda z \, d\lambda = 0 \qquad |z| > b \qquad (8.1b)$$

If the methods of Sneddon (1966) are used to reduce this to a different form for solution, one finds that the normalized integral equation to be solved is of the form

$$i\omega\mu h_2(t) + \frac{4Z_s}{\pi}\int_1^\infty h_2(u)\, u \frac{d}{du}\, u \frac{d}{du}\left\{\frac{K\left(\frac{2\sqrt{ut}}{t+u}\right)}{\sqrt{ut}\,\sqrt{u+t}}\right\} du$$

$$= -\frac{Z_0}{Z_0 + 2Z_s}\,\frac{4Z_s}{\sqrt{\pi}}\int_0^1 u^{\frac{1}{2}}\, \frac{d}{du}\, u \frac{d}{du}\left\{\frac{K\left(\frac{2\sqrt{ut}}{t+u}\right)}{\sqrt{ut}\,\sqrt{u+t}}\right\} du \qquad (8.2)$$

$$t > 1$$

K is the elliptic integral. In order to derive this result, heavy use of the generalized Hankel transformation had to be made. Most integrals that arise have value infinity when treated in the conventional sense. This equation appears to be too complicated to solve easily, and therefore, the dual integral above will be rejected.

Another dual integral equation can be obtained from (5.31) by manipulating the second integral in (5.31a). From (6.10), one has

$$\int_0^\infty x^2 f(x)\sin(xy)\,dx = -\frac{d^2}{dy^2}\int_0^\infty f(x)\sin(xy)\,dx \qquad (8.3)$$

By identifying a new function $F(x) = x^2 f(x)$, one finds that

$$\int_0^\infty F(\lambda)\sin\lambda z\,d\lambda = -\frac{d^2}{dz^2}\int_0^\infty F(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda^2} \qquad (8.4)$$

Substituting this for the second integral in (5.31a), one obtains

$$\frac{d^2}{dz^2}\left\{2Z_s\int_0^\infty F(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} - \left(\frac{2Z_sZ_0}{Z_0+2Z_s}\right)z - i\omega\mu\int_0^\infty F(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda^2}\right\} = 0 \qquad |z|>b \qquad (8.5)$$

or

$$\int_0^\infty F(\lambda)\left\{2Z_s - \frac{i\omega\mu}{\lambda}\right\}\sin\lambda z\,\frac{d\lambda}{\lambda} = \left(\frac{2Z_sZ_0}{Z_0+2Z_s}\right)z + Az+B + C\,\mathrm{sgn}(z) \qquad |z|>b \qquad (8.6)$$

where A, B, and C are constants of integration. The C sgn(z) term is included because the limits $|z|>b$ only allow such a term. Indeed, any term which has C sgn(z) behavior for $|z|>b$ and any behavior whatsoever for $|z|<b$ is allowed. The dual integral equation becomes

$$\int_0^\infty F(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} = \left(\frac{Z_0}{Z_0+2Z_s}\right)z \qquad |z|<b \qquad (8.7a)$$

$$\int_0^\infty F(\lambda)\left\{2Z_s - \frac{i\omega\mu}{\lambda}\right\}\sin\lambda z\,\frac{d\lambda}{\lambda} = \left\{A + \left(\frac{2Z_sZ_0}{Z_0+2Z_s}\right)\right\}z + B + C\,\mathrm{sgn}(z) \qquad |z|>b \qquad (8.7b)$$

This equation will be converted to a Fredholm equation of the second kind in the next section.

The constants A, B, and C in equation (8.7b) need to be evaluated. The evaluation of these constants is not straightforward and a unique result is not forthcoming. Values for A, B, and C will be assumed here based on certain assumptions. Since this problem has apparently not been solved before, there are no results to compare the results of this work to. This makes verification of the assumptions used to find A, B, and C impossible at this time.

To find a value for B, one can assume that since the left-hand side of equation (8.7b) is an odd function of z, the right-hand side must also be an odd function of z. Therefore, the constant B must be zero

The assumption that will be used to find a value for A is that the right-hand side should remain finite as z approaches infinity, thereby making

$$A = -2 Z_s \left(\frac{Z_0}{Z_0+2Z_s} \right) \tag{8.8}$$

Note that his is not the only assumption that could be made. If A, B, and C were all taken to be zero, for instance, the right-hand side would be proportional to z. If equation (8.7) were considered to be a conventional dual integral equation, this would almost certainly be disallowed because the right-hand side would blow up at infinity. If, on the other hand, we recall that we are allowing generalized function solutions, then since z is a distribution of slow growth over the interval $(1, \infty)$, a right-hand side proportional to z is not disallowed in the generalized sense.

There are at least two reasons to assume that A is given by (8.8). First, we formulated the problem by using an unknown function which is assumed to be small far from the slit. The fields far from the slit were already accounted for. Second, it gives agreement for the perfectly conducting case. Let $Z_s=0$ in equation (8.7b). Then we obtain

$$-i\omega\mu \int_0^\infty F(\lambda) \sin \lambda z \, \frac{d\lambda}{\lambda^2} = Az + C\,\mathrm{sgn}(z) \qquad |z| > b \tag{8.9}$$

where B has been taken to be zero. For the perfectly conducting case, we found (Section VII) that $F(\lambda) = b\, J_1(\lambda b)$. Substituting this in the above integral, one obtains

$$b\int_0^\infty J_1(b\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda^2} = \frac{\pi b^2}{4}\,\mathrm{sgn}(z) \qquad |z|>b \qquad (8.10)$$

Thus, one has

$$-\imath\omega\mu\left(\frac{\pi b^2}{4}\,\mathrm{sgn}(z)\right) \stackrel{?}{=} Az + C\,\mathrm{sgn}(z) \qquad |z|>b \qquad (8.11)$$

If we let z get large, we see that A=0. This agrees with our assumption that $A = -2Z_sZ_0/(Z_0+2Z_s)$ because $Z_s=0$ in this case. Assuming that (8.8) for A is correct, the dual integral equation (8.7) becomes

$$\int_0^\infty F(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} = \left(\frac{Z_0}{Z_0+2Z_s}\right)z \qquad |z|<b \qquad (8.12a)$$

$$\int_0^\infty F(\lambda)\left\{2Z_s - \frac{\imath\omega\mu}{\lambda}\right\}\sin\lambda z\,\frac{d\lambda}{\lambda} = C\,\mathrm{sgn}(z) \qquad |z|>b \qquad (8.12b)$$

C will be evaluated later. This equation will be used later.

There is another dual integral equation that one can obtain from (5.31) by using (6.8) through (6.11). Differentiating (5.31b) and using (6.11) with f(x) = F(x)/x, one finds

$$\int_0^\infty F(\lambda)\cos(\lambda z)\,d\lambda = \frac{Z_0}{Z_0+2Z_s} \qquad |z|<b$$

By using (6.8) and (6.11), (5.31a) can be written as

$$\int_0^\infty F(\lambda)\left[2Z_s - \frac{\imath\omega\mu}{\lambda}\right]\cos(\lambda z)\,d\lambda = \frac{2Z_sZ_0}{Z_0+2Z_s} + C_0 \qquad |z|>b$$

where C_0 is an arbitrary constant of integration. If we assume that C_0 is not proportional to Z_s, then for the perfectly conducting case when $Z_s=0$, $F(\lambda) = b\,J_1(b\lambda)$ and

$$-\imath\omega\mu b\int_0^\infty J_1(\lambda b)\cos(\lambda z)\,\frac{d\lambda}{\lambda} = C_0 \qquad |z|>b$$

Since

$$\int_0^\infty J_1(\lambda b)\cos(\lambda z)\frac{d\lambda}{\lambda} = \begin{cases} \sqrt{1-\left(\frac{z}{b}\right)^2} & |z|<b \\ 0 & |z|>b \end{cases}$$

one has $C_0=0$. The dual integral equations for this case then become

$$\int_0^\infty F(\lambda)\cos\lambda z\,d\lambda = \frac{Z_0}{Z_0+2Z_s} \qquad |z|<b$$

$$\int_0^\infty F(\lambda)\left[2Z_s - \frac{\imath\omega\mu}{\lambda}\right]\cos\lambda z\,d\lambda = \frac{2Z_sZ_0}{Z_0+2Z_s} \qquad |z|>b$$

If the techniques of Sneddon (1966) are applied to these equations, one finds that an auxiliary integral equation that needs to be solved is

$$2Z_s h\left(\frac{z}{b}\right) + \imath\omega\mu\frac{z}{\pi}\int_b^\infty h\left(\frac{t}{b}\right)\ln\left|t^2-z^2\right|dt$$

$$-\imath\omega\mu\frac{2b}{\pi}C_1\int_1^\infty h(t)\,dt$$

$$=\sqrt{\frac{2b}{\pi}}\left(\frac{2Z_sZ_0}{Z_0+2Z_s}\right)+$$

$$+\frac{\imath\omega\mu}{\pi}\sqrt{\frac{2b}{\pi}}\left(\frac{Z_0}{Z_0+2Z_s}\right)\left\{(z-b)\ln(z-b)-(z+b)\ln(z+b)+C_2\right\}$$

$$|z|>b$$

where C_1 and C_2 are constants of integration. In order to derive this result, considerable use of the generalized Hankel transformation of Zemanian (1968) had to be made. The kernel turns out to be (6.17) which is infinity if interpreted in the conventional sense. Grosjean (1972) briefly presents some identities for the $\ln\left|t^2 - z^2\right|$ kernel. Because of the fact that conventional mathematics breaks down in this case, the above dual integral equation will be avoided for further study. The dual integral equation (8.12) is the only equation we will choose for further study. The next section presents a method for changing this dual integral equation into a Fredholm integral equation of the second kind.

IX. REDUCTION OF THE DUAL INTEGRAL EQUATION TO A SINGLE FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

A. FORMULATION FOR LARGER VALUES OF SHEET IMPEDANCE

The dual integral equation of (8.13)

$$\int_0^\infty F(\lambda) \sin \lambda z \frac{d\lambda}{\lambda} = \frac{Z_0}{Z_0 + 2Z_s} z \qquad |z| < b \qquad (9.1a)$$

$$\int_0^\infty F(\lambda) \left\{ 2Z_s - \frac{i\omega\mu}{\lambda} \right\} \sin \lambda z \frac{d\lambda}{\lambda} = C \operatorname{sgn}(z) \qquad |z| > b \qquad (9.1b)$$

give the best results when Z_s is large. C will be assumed to be zero for convenience. For infinitely large Z_s, we have

$$\int_0^\infty F(\lambda) \sin \lambda z \frac{d\lambda}{\lambda} = 0 \qquad z < b \qquad (9.2a)$$

$$\int_0^\infty F(\lambda) \sin \lambda z \frac{d\lambda}{\lambda} = 0 \qquad z > b \qquad (9.2b)$$

or $F(\lambda) = 0$. This makes

$$\phi = \begin{cases} \phi^+ & x > 0 \\ \phi^- & x < 0 \end{cases} \qquad (9.3)$$

equal to $-H_0 z$, all x, as we expect. The dual integral equation , in this form, gives us a result which has $F(\lambda)$ smaller and smaller for larger and larger sheet impedances.

The dual integral equation (9.1) can be reduced to a single Fredholm integral equation of the second kind in a straight-forward manner if we let $C=0$. Let

$$F(\lambda) = G(\lambda) + H(\lambda) \qquad (9.4)$$

where $G(\lambda)$ is a function which satisfies (9.1a) and $H(\lambda)$ is such that the integral in (9.1a) is equal to zero, that is,

$$\int_0^\infty H(\lambda) \sin \lambda z \frac{d\lambda}{\lambda} = 0 \qquad |z| < b \qquad (9.5)$$

Sneddon (1966, p.113) gives formulas for determining $G(\lambda)$ and $H(\lambda)$. In the present case, his formulas give

$$G(\lambda) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{Z_0}{Z_0 + 2Z_s}\right) b (b\lambda)^{\frac{1}{2}} J_{3/2}(b\lambda) \qquad (9.6)$$

$$H(\lambda) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} b^{\frac{1}{2}} \lambda \int_1^\infty \sin(b\lambda t)\, h(t)\, dt \qquad (9.7)$$

where h(t) is an unknown function which will be solved for later. By substituting (9.4) into the dual integral equation (9.1), one obtains

$$\int_0^\infty H(\lambda) \sin \lambda z \frac{d\lambda}{\lambda} = \frac{Z_0}{Z_0 + 2Z_s} z - \int_0^\infty G(\lambda) \sin \lambda z \frac{d\lambda}{\lambda} \qquad |z| < b \qquad (9.8a)$$

$$\int_0^\infty H(\lambda)\left[2Z_s - \frac{i\omega\mu}{\lambda}\right] \sin \lambda z \frac{d\lambda}{\lambda} =$$

$$= - \int_0^\infty G(\lambda)\left[2Z_s - \frac{i\omega\mu}{\lambda}\right] \sin \lambda z \frac{d\lambda}{\lambda} \qquad |z| > b \qquad (9.8b)$$

The integrals involving $G(\lambda)$ on the right-hand sides can be found in standard integral tables. They are

$$\int_0^\infty G(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} = \begin{cases} \dfrac{Z_0}{Z_0+2Z_s}\, z & |z|<b \\ 0 & |z|>b \end{cases} \qquad (9.9)$$

$$\int_0^\infty G(\lambda)\left[2Z_s - \frac{i\omega\mu}{\lambda}\right]\sin\lambda z\,\frac{d\lambda}{\lambda} =$$

$$= -i\omega\mu\int_0^\infty G(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda^2} =$$

$$= +i\omega\mu\left(\frac{Z_0}{Z_0+2Z_s}\right)\frac{2b^2}{3\pi}\left\{Q_2\left(\frac{z}{b}\right)-Q_0\left(\frac{z}{b}\right)\right\} \qquad (9.10)$$

$$|z|>b$$

where $Q_n(z)$ is the Legendre function of the second kind as defined by Magnus, Oberhettinger, and Soni (1966, p. 176). In terms of more familiar functions, we have

$$Q_0(z)-Q_2(z) = \frac{3}{2}\left\{z + \frac{1-z^2}{2}\ln\left(\frac{z+1}{z-1}\right)\right\} \qquad z>1$$

$$= 3\int_z^\infty Q_1(t)\,dt \qquad (9.11)$$

$$= -\frac{3}{2}(z^2-1)\frac{dQ_1(z)}{dz}$$

Substituting these integrals of $G(\lambda)$ in (9.8), and simplifying yields the new dual integral equation

$$\int_0^\infty H(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} = 0 \qquad |z| < b \qquad (9.12a)$$

$$\int_0^\infty H(\lambda)\left[2Z_s - \frac{i\omega\mu}{\lambda}\right]\sin\lambda z\,\frac{d\lambda}{\lambda} =$$

$$= i\omega\mu\left(\frac{Z_0}{Z_0+2Z_s}\right)\frac{2b^2}{3\pi}\left\{\mathcal{D}_0\left(\frac{z}{b}\right) - \mathcal{S}_2\left(\frac{z}{b}\right)\right\} \qquad (9.12b)$$

$$|z| > b$$

Substituting the expression (9.7) for $H(\lambda)$ in the first equation, one finds

$$\int_0^\infty H(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} =$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} b^{\frac{1}{2}} \int_0^\infty \int_1^\infty \sin(b\lambda t)\,h(t)\,dt \;\; \sin\lambda z \;\; d\lambda\,dt$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} b^{\frac{1}{2}} \int_1^\infty h(t) \int_0^\infty \sin(b\lambda t)\sin\lambda z\,d\lambda \;\; dt \qquad (9.13)$$

This assumes that h(t) is such that the interchange of the order of integration is permissible. By using the theory of distributions, one can show that

$$\int_0^\infty \sin(b\lambda t)\sin\lambda z\,d\lambda = \frac{\pi}{2}\,\delta(bt - z) \qquad (9.14)$$

Substituting this in the above integral, one obtains

$$\int_0^\infty H(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} =$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} b^{\frac{1}{2}}\,\frac{\pi}{2}\int_1^\infty h(t)\,\delta(bt - z)\,dt =$$

$$= \begin{cases} 0 & |z| < b \\ \left(\frac{\pi}{2}\right)^{\frac{1}{2}} b^{-\frac{1}{2}} h\left(\frac{z}{b}\right) & |z| > b \end{cases} \qquad (9.15)$$

This shows that such an $H(\lambda)$ satisfies (9.12a).

Substituting (9.7) in the second equation, one has

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} b^{\frac{1}{2}} \int_0^\infty \lambda \int_1^\infty \sin(bt\lambda) h(t)\, dt \left\{2Z_s - \frac{i\omega\mu}{\lambda}\right\} \sin\lambda z \,\frac{d\lambda}{\lambda} =$$

$$= i\omega\mu \left(\frac{Z_0}{Z_0 + 2Z_s}\right) \frac{2b^2}{3\pi} \left\{ \mathcal{P}_0\left(\frac{z}{b}\right) - \mathcal{P}_2\left(\frac{z}{b}\right) \right\} \qquad z > b$$

(9.16)

Again assuming that h(t) is such that the order of integration can be changed, one has

$$\int_1^\infty h(t) \int_0^\infty \left\{2Z_s - \frac{i\omega\mu}{\lambda}\right\} \sin(bt\lambda) \sin z\lambda \; d\lambda \; dt =$$

$$= i\omega\mu \left(\frac{Z_0}{Z_0 + 2Z_s}\right) \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{b^{3/2}}{3} \left\{ \mathcal{P}_0\left(\frac{z}{b}\right) - \mathcal{P}_2\left(\frac{z}{b}\right) \right\} \qquad (9.17)$$

but

$$\int_0^\infty \left\{2Z_s - \frac{i\omega\mu}{\lambda}\right\} \sin bt\lambda \; \sin z\lambda \; d\lambda =$$

$$= 2Z_s \frac{\pi}{2} \delta(bt - z) - i\omega\mu \frac{1}{2} \ln\left(\frac{bt + z}{bt - z}\right) \qquad (9.18)$$

Substituting this in the above equation, one obtains

$$\int_1^{\infty} h(t)\left[2Z_s\frac{\pi}{2}\delta(bt-z) - \frac{i\omega\mu}{2}\ln\left(\frac{bt+z}{bt-z}\right)\right]dt =$$

$$= i\omega\mu\left(\frac{Z_0}{Z_0+2Z_s}\right)\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{b^{3/2}}{3}\left\{\mathcal{J}_0\left(\frac{z}{b}\right)-\mathcal{J}_2\left(\frac{z}{b}\right)\right\} \qquad (9.19)$$

$$z>b$$

Using the sifting property of the delta function in (9.15), one obtains the Fredholm integral equation of the second kind

$$\frac{\pi Z_s}{b}h\left(\frac{z}{b}\right) - \frac{i\omega\mu}{2}\int_1^{\infty} h(t)\ln\left(\frac{bt+z}{bt-z}\right)dt =$$

$$= i\omega\mu\left(\frac{Z_0}{Z_0+2Z_s}\right)\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{b^{3/2}}{3}\left(\mathcal{J}_0\left(\frac{z}{b}\right)-\mathcal{J}_2\left(\frac{z}{b}\right)\right) \qquad (9.20)$$

$$z>b$$

Using the change of variables τ = bt in the integral and $\omega\mu = kZ_0$, one finds

$$\frac{\pi Z_s}{ikbZ_0}h\left(\frac{z}{b}\right) - \frac{1}{2b}\int_b^{\infty} h\left(\frac{\tau}{b}\right)\ln\left|\frac{\tau+z}{\tau-z}\right|d\tau =$$

$$= b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{Z_0}{Z_0+2Z_s}\right)\frac{1}{3}\left\{\mathcal{J}_0\left(\frac{z}{b}\right)-\mathcal{J}_2\left(\frac{z}{b}\right)\right\} \qquad (9.21)$$

$$z>b$$

By using the fact that

$$\mathcal{P}_0\left(\frac{\tau}{z}\right) = \frac{1}{2}\ln\left|\frac{\tau+z}{\tau-z}\right| , \qquad (9.22)$$

the integral equation can also be written as

$$\frac{\pi Z_s}{ikbZ_0} h\left(\frac{z}{b}\right) - \frac{1}{b}\int_b^\infty h\left(\frac{\tau}{b}\right)\mathcal{P}_0\left(\frac{\tau}{z}\right)d\tau =$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{Z_0}{Z_0+2Z_s}\right)\frac{b^{3/2}}{3}\left\{\mathcal{P}_0\left(\frac{z}{b}\right) - \mathcal{P}_2\left(\frac{z}{b}\right)\right\} \qquad (9.23)$$

$$z > b$$

The dual integral equation has now been reduced to a single Fredholm integral equation of the second kind. If this equation can be solved for $h(\frac{z}{b})$, then the solution to the dual equation (9.1) can be written as

$$F(\lambda) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{Z_0}{Z_0+2Z_s}\right)b(b\lambda)^{\frac{1}{2}} J_{3/2}(b\lambda) +$$

$$+ \left(\frac{2}{\pi}\right)^{\frac{1}{2}} b^{-\frac{1}{2}}\lambda\int_b^\infty h\left(\frac{t}{b}\right)\sin(t\lambda)\,dt \qquad (9.24)$$

In order to show the necessity of using the theory of distributions, it is helpful to point out what happens if (8.4) is not used to simplify (5.31). Equation (5.31a) can be written as

$$2Z_s\frac{d^2}{dz^2}\int_0^\infty F(\lambda)\sin\lambda z\frac{d\lambda}{\lambda} + i\omega\mu\int_0^\infty F(\lambda)\sin\lambda z\,d\lambda = 0 \qquad (9.25)$$

$$z > b$$

Substituting (9.4) in the above, one finds

$$2Z_s \frac{d^2}{dz^2}\int_0^{\infty} H(\lambda) \sin \lambda z \frac{d\lambda}{\lambda} + i\omega\mu \int_0^{\infty} H(\lambda) \sin \lambda z \, d\lambda =$$

$$= -2Z_s \frac{d^2}{dz^2}\int_0^{\infty} G(\lambda) \sin \lambda z \frac{d\lambda}{\lambda} - i\omega\mu \int_0^{\infty} G(\lambda) \sin \lambda z \, d\lambda \qquad z > b \qquad (9.26)$$

From (9.9), the first integral on the right-hand side is equal to zero. The second integral can be reduced to

$$\int_0^{\infty} G(\lambda) \sin \lambda z \, d\lambda =$$

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{Z_0}{Z_0 + 2Z_s}\right) b \int_0^{\infty} (b\lambda)^{\frac{1}{2}} J_{3/2}(b\lambda) \sin \lambda z \, d\lambda \qquad (9.27)$$

This integral can be expressed in the form of a Weber-Schafheitlin integral as given by Watson (1944, p. 398-404). Unfortunately, using Watson's criterion, the convergence of this integral is not assured. Indeed, it is not listed in any standard integral tables. It would thus appear that the dual integral equation approach can not be used to solve this problem using classical mathematics. We will assume that all of the steps leading to the integral equation (9.23) are justifiable. The justification rests on a foundation of the theory of distributions.

B. FORMULATION FOR SMALLER VALUES OF SHEET IMPEDANCE

Since for the composite problem at hand, we know that Z_s is very small, typically 0.045 ohms and that the above formulation is best for large Z_s, it is necessary to find a formulation good for small values of sheet impedance. One possible way to do this would be to subtract the known solution $bJ_1(\lambda b)$ for the perfectly conducting case $Z_s = 0$ from the unknown function $F(\lambda)$ of (9.1) and try formulating a dual integral equation in terms of

$$F_s(\lambda) = F(\lambda) - bJ_1(b\lambda) \qquad (9.28)$$

where the s subscript means small values of Z_s. Adding and subtracting terms of $bJ_1(b\lambda)$ to $F(\lambda)$ in (9.1) gives the dual integral equations

$$\int_0^\infty F_s(\lambda) \sin \lambda z \frac{d\lambda}{\lambda} = \frac{Z_0}{Z_0 + 2Z_s} z - b\int_0^\infty J_1(b\lambda) \sin \lambda z \frac{d\lambda}{\lambda} \qquad z < b \tag{9.29a}$$

$$\int_0^\infty F_s(\lambda)\left\{2Z_s - \frac{i\omega\mu}{\lambda}\right\} \sin \lambda z \frac{d\lambda}{\lambda} =$$

$$= -b\int_0^\infty J_1(b\lambda)\left\{2Z_s - \frac{i\omega\mu}{\lambda}\right\} \sin \lambda z \frac{d\lambda}{\lambda} + C\,\mathrm{sgn}(z) \qquad z > b \tag{9.29b}$$

Erdelyi, et al (1954, p. 99) have evaluated the integrals on the right-hand sides.

$$b\int_0^\infty J_1(b\lambda) \sin \lambda z \frac{d\lambda}{\lambda} = \begin{cases} z & z < b \\ \dfrac{b^2\,\mathrm{sgn}(z)}{|z| + \sqrt{z^2 - b^2}} & z > b \end{cases} \tag{9.30}$$

$$b\int_0^\infty J_1(b\lambda) \sin \lambda z \frac{d\lambda}{\lambda^2} = \begin{cases} \dfrac{z}{2}\sqrt{b^2 - z^2} + \dfrac{b^2}{2}\sin^{-1}\left(\dfrac{z}{b}\right) & z < b \\ \dfrac{\pi b^2}{4} & z > b \end{cases} \tag{9.31}$$

Substituting these results in (9.29) gives the dual integral equations

$$\int_0^\infty F_s(\lambda)\sin\lambda z\,\frac{d\lambda}{\lambda} = -\,\frac{2Z_s}{Z_0+2Z_s}\,z \qquad z<b \qquad (9.32a)$$

$$\int_0^\infty F_s(\lambda)\left\{2Z_s - \frac{i\omega\mu}{\lambda}\right\}\sin\lambda z\,\frac{d\lambda}{\lambda} =$$

$$= \left\{\frac{ikZ_0\pi b^2}{4} + C\right\}\operatorname{sgn}(z) - 2Z_s\,\frac{b^2\operatorname{sgn}(z)}{|z|+\sqrt{z^2-b^2}} \qquad (9.32b)$$

$$z>b$$

Proceeding in the same manner as before, one finds that

$$F_s(\lambda) = G_s(\lambda)+H_s(\lambda) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{2Z_s}{Z_0+2Z_s}\right)b(b\lambda)^{\frac{1}{2}}J_{3/2}(b\lambda) +$$

$$+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}b^{\frac{1}{2}}\lambda\int_1^\infty \sin(bt\lambda)\,h_s(t)\,dt \qquad (9.33)$$

where $h_s(t)$ is the solution of the integral equation

$$\frac{\pi Z_s}{ikbZ_0}h_s\left(\frac{z}{b}\right) - \frac{1}{b}\int_b^\infty h_s\left(\frac{\tau}{b}\right)\mathcal{D}_0\left(\frac{\tau}{z}\right)d\tau =$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(\frac{2Z_s}{Z_0+2Z_s}\right)\frac{2b^{3/2}}{3\pi}\left\{\mathcal{D}_2\left(\frac{z}{b}\right) - \mathcal{J}_0\left(\frac{z}{b}\right)\right\}$$

$$+\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{\pi b^{3/2}}{4}\operatorname{sgn}(z) + \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{C}{ikZ_0 b^{1/2}}\operatorname{sgn}(z) +$$

$$-\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{2Z_s}{ikZ_0}\cdot\frac{b^{3/2}\operatorname{sgn}(z)}{|z|+\sqrt{z^2-b^2}} \qquad (9.34)$$

If C is chosen to be

$$C = -\ ikZ_0 \frac{\pi b^2}{4} \tag{9.35}$$

then one has to solve the integral equation

$$\frac{\pi Z_s}{ikbZ_0} h_s\left(\frac{z}{b}\right) - \frac{1}{b}\int_b^\infty h_s\left(\frac{\tau}{b}\right) \mathcal{P}_0\left(\frac{\tau}{z}\right) d\tau =$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{2Z_s}{Z_0 + 2Z_s}\right) \frac{2b^{3/2}}{3\pi} \left\{ \mathcal{P}_2\left(\frac{z}{b}\right) - \mathcal{P}_0\left(\frac{z}{b}\right) \right\}$$

$$- \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{2Z_s}{ikZ_0} \frac{b^{3/2}\, \mathrm{sgn}(z)}{|z| + \sqrt{z^2 - b^2}} \qquad z > b \tag{9.36}$$

It is not known if the above value for C is correct. It was chosen to allow an easy solution in terms of Legendre polynomials.

X. EXPRESSIONS FOR POTENTIALS AND FIELDS

The unknown function $F(\lambda)$ in the dual integral equation (8.12) has in the last section been specified in terms of a known function $G(\lambda)$ and an unknown function $H(\lambda)$. It is of interest to find the potentials and fields in all space once $F(\lambda)$ has been found. For the formulation of Section IX.A., the potential is given by equation (5.28) with $F(\lambda)$ replaced by $G(\lambda) + H(\lambda)$.

$$\phi^+ = H_0 \int_0^\infty [G(\lambda) + H(\lambda)] \sin \lambda z\, e^{-\lambda x} \frac{d\lambda}{\lambda} - 2H_0 \left(\frac{Z_0 + Z_s}{Z_0 + 2Z_s}\right) z \tag{10.1a}$$

$$\phi^- = -H_0 \int_0^\infty [G(\lambda) + H(\lambda)] \sin \lambda z\, e^{+\lambda x} \frac{d\lambda}{\lambda} - 2H_0 \left(\frac{Z_s}{Z_0 + 2Z_s}\right) z \tag{10.1b}$$

$G(\lambda)$ and $H(\lambda)$ are given by (9.6) and (9.7), respectively. For the formulation of Section IX.B., the potentials are again given by (5.28) with $F(\lambda)$ given by equation (9.28)

$$F(\lambda) = F_s(\lambda) + b\, J_1(b\lambda) \tag{10.2}$$

where $F_s(\lambda) = G_s(\lambda) + H_s(\lambda)$ is given by (9.33). In both formulations, one has to do potential integrals involving functions of the form $G_0(\lambda) = b(b\lambda)^{\frac{1}{2}} J_{3/2}(b\lambda)$. Substituting this in (10.1), one finds integrals of the form [Gradshteyn and Ryzhik (1980, pp. 491-492)]

$$f_G = \int_0^\infty G_0(\lambda) \sin\lambda z\; e^{-\lambda|x|} \frac{d\lambda}{\lambda} =$$

$$= b\int_0^\infty (b\lambda)^{\frac{1}{2}} J_{3/2}(b\lambda) \sin\lambda z\; e^{-\lambda|x|} \frac{d\lambda}{\lambda} =$$

$$= \frac{z}{2}\tan^{-1}\left(\frac{2|x|\,b}{x^2+z^2-b^2}\right) + \frac{|x|}{4}\ln\left(\frac{x^2+(z-b)^2}{x^2+(z+b)^2}\right) - s\frac{\pi}{2}b \tag{10.3}$$

where if $x^2 < (z^2 - b^2)$ then s=1 else s=0. The $H(\lambda)$ and $H_s(\lambda)$ contributions can be reduced to integrals of the form f_H and f_{H_s} where

$$f_H = \int_0^\infty H(\lambda) \sin\lambda z\; e^{-\lambda|x|} \frac{d\lambda}{\lambda} =$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} b^{\frac{1}{2}} \int_0^\infty \lambda \int_1^\infty \sin(b\lambda t)\, h(t)\, dt\; \sin\lambda z\; e^{-\lambda|x|} \frac{d\lambda}{\lambda} =$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} b^{\frac{1}{2}} \int_1^\infty h(t) \int_0^\infty \sin(b\lambda t) \sin\lambda z\; e^{-\lambda|x|} d\lambda\; dt \tag{10.4}$$

The inside integral can be evaluated analytically [Erdelyi et al (1954, p. 159)] as

$$\int_0^\infty \sin(b\lambda t)\,\sin\lambda z\; e^{-\lambda|x|}\,d\lambda = \frac{2|x|z\,bt}{\left[x^2+(bt-z)^2\right]\left[x^2+(bt+z)^2\right]} \tag{10.5}$$

If for x=0, this integral is interpretted as a distribution, then

$$\int_0^\infty \sin(bt\lambda)\,\sin\lambda z\; e^{-\lambda|x|}\,d\lambda = \frac{\pi}{2}\,\delta(bt-z) \quad \text{for} \quad x=0 \tag{10.6}$$

where δ is the delta distribution. The integral f_H then becomes

$$f_H = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} b^{\frac{1}{2}} \int_1^\infty h(t)\,\frac{2|x|z\,bt}{\left[x^2+(bt-z)^2\right]\left[x^2+(bt+z)^2\right]}\,dt \tag{10.7}$$

f_{H_s} is given by this integral with h(t) replaced by $h_s(t)$. For the special case x=0, one obtains

$$f_H = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} b^{\frac{1}{2}} \int_1^\infty h(t)\,\delta(bt-z)\,dt = \begin{cases} 0 & z<b \\ \left(\frac{\pi}{2b}\right)^{\frac{1}{2}} h\left(\frac{z}{b}\right) & z>b \end{cases} \quad x=0 \tag{10.8}$$

It is interesting to note that h(t) is just the value of f_H evaluated on the impedance sheet. The final potential integral that needs to be evaluated is

$$\int_0^\infty b\,J_1(b\lambda)\,\sin\lambda z\; e^{-\lambda|x|}\,\frac{d\lambda}{\lambda}$$

This integral has already been evaluated in Section VII.

The complete expressions for the potential when using the formulation of Section IX.A. is

$$\phi^{+} = H_0\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{Z_0}{Z_0+2Z_s}\right)f_G + H_0 f_H - 2H_0\left(\frac{Z_0+Z_s}{Z_0+2Z_s}\right)z \qquad (10.9a)$$

$$\phi^{-} = -H_0\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{Z_0}{Z_0+2Z_s}\right)f_G - H_0 f_H - 2H_0 \frac{Z_s}{Z_0+2Z_s} z \qquad (10.9b)$$

The potential when using the formulation of Section IX.B. is

$$\phi^{+} = -H_0\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{2Z_s}{Z_0+2Z_s}\right)f_G + H_0 f_{H_s} + \phi^{+}_{PEC} - 2H_0\left(\frac{Z_0+Z_s}{Z_0+2Z_s}\right)z \qquad (10.10a)$$

$$\phi^{-} = H_0\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{2Z_s}{Z_0+2Z_s}\right)f_G + -H_0 f_{H_s} + \phi^{-}_{PEC} - 2H_0 \frac{Z_s}{Z_0+2Z_s} z \qquad (10.10b)$$

where PEC is the abbreviation for Perfect Electrical Conductor.

The x and z components of the H-field may be obtained from

$$H_x = -\frac{\partial\phi}{\partial x} \qquad H_z = -\frac{\partial\phi}{\partial z} \qquad (10.11)$$

XI. REDUCTION OF THE FREDHOLM INTEGRAL EQUATIONS TO INFINITE SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS IN AN INFINITE NUMBER OF UNKNOWNS

Fredholm integral equations similar to (9.20) and (9.36) have often been solved by expanding the unknown and the right-hand side in series of orthogonal polynomials such as Chebyshev or Legendre polynomials. Morar' and Popov (1970), Popov (1972), and earlier untranslated works of Popov (in Russian) use this solution procedure. England and Shail (1977) use the technique for solving some elasticity problems. Aleksandrov and Kovalenko (1977) also solve elasticity problems.

A. FORMULATION FOR LARGER VALUES OF SHEET IMPEDANCE

The Fredholm integral equation that needs to be solved for this case is given by (9.20). Substituting z=ub, one obtains

$$\frac{2\pi Z_s}{ikbZ_0} h(u) - \int_1^\infty h(t) \ln\left|\frac{t+u}{t-u}\right| dt = \qquad (11.1)$$

$$= b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{Z_0}{Z_0+2Z_s}\right)\left\{u + \left(\frac{1-u^2}{2}\right)\ln\left|\frac{u+1}{u-1}\right|\right\} \qquad u>1$$

By using symmetry (the right-hand side is an odd function) and two changes of variables, one obtains the equation

$$\frac{2\pi Z_s}{ikbZ_0} h\left(\frac{1}{s}\right) + \int_{-1}^{1} \frac{1}{v^2} h\left(\frac{1}{v}\right) \ln|s-v|\, dv = \qquad (11.2)$$

$$= b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{Z_0}{Z_0+2Z_s}\right)\frac{1}{s^2}\left\{s + \frac{s^2-1}{2}\ln\left|\frac{1+s}{1-s}\right|\right\} \qquad s<1$$

Let $h^*(t) = t^{-2}\, h\left(\frac{1}{t}\right)$. Then the above equation becomes

$$\frac{2\pi Z_s}{ikbZ_0} s^2 h^*(s) + \int_{-1}^{1} h^*(v) \ln|s-v|\, dv = \qquad (11.3)$$

$$= b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{Z_0}{Z_0+2Z_s}\right)\frac{1}{s^2}\left\{s + \frac{s^2-1}{2}\ln\left|\frac{1+s}{1-s}\right|\right\} \qquad s<1$$

The right-hand side can be expressed as a power series.

$$\frac{1}{t^2}\left\{t + \frac{t^2-1}{2}\ln\left|\frac{1+t}{1-t}\right|\right\} = 2\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)(2n+3)} \qquad (11.4)$$

This is an odd function and can also be expanded in odd order Chebyshev polynomials T_{2k+1} and Legendre polynomials P_{2n-1}. Only the Legendre polynomial solution will be attempted here because the Chebyshev solution apparently gives fields which are more singular than they should be near the edge. Since by Luke (1969, vol. I, p. 277)

$$t^{2n+1} = \sum_{m=0}^{n} b_m(n) P_{2m+1}(t) \qquad (11.5a)$$

$$b_m(n) = \frac{(2n+1)!\,(n+m+2)!\,(2m+\frac{3}{2})\,2^{2m+3}}{(2[n+m+2])!\,(n-m)!}. \qquad (11.5b)$$

The power series can be written in terms of Legendre polynomials.

$$2\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)(2n+3)} = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+3)}\sum_{m=0}^{n} b_m(n) P_{2m+1}(t) \qquad (11.6)$$

It can easily be shown that

$$\sum_{n=0}^{\infty} a(n)\sum_{m=0}^{n} b(n,m) P_{2m+1}(t) =$$

$$= \sum_{n=0}^{\infty} P_{2n+1}(t)\left\{\sum_{m=n}^{\infty} a(m)\, b(m,n)\right\} \qquad (11.7)$$

Using this identity, one finds

$$\frac{1}{t^2}\left\{t + \frac{t^2-1}{2}\ln\left|\frac{1+t}{1-t}\right|\right\} = \sum_{n=1}^{\infty} d_n P_{2n-1}(t) \qquad (11.8a)$$

where

$$d_n = \sum_{m=n-1}^{\infty} \frac{(2m+1)!\,(m+n+1)!\,(2n-\frac{1}{2})\,2^{2n+2}}{(2m+1)(2m+3)\left[2(m+n+1)\right]!\,(m-n+1)!} \qquad (11.8b)$$

Assume that the unknown $h^*(t)$ can be expanded in a Legendre polynomial series.

$$h^*(t) = \sum_{n=1}^{\infty} b_n P_{2n-1}(t) \qquad (11.9)$$

In order to use orthogonality, $t^2 h^*(t)$ must be expressed as a series of Legendre polynomials. Identity 8.915 5., page 1026 of Gradshteyn and Ryzhik (1980) can be used to do this. One finds that

$$t^2 P_{2n-1}(t) = \sum_{p=0}^{2} a_p(n) P_{2(n-p)+1}(t) \qquad (11.10)$$

$$n = 1, 2, \cdots, \infty$$

$$a_0(k) = \frac{(2k+1)(2k)}{(4k+1)(4k-1)} \qquad (11.11a)$$

$$a_1(k) = \frac{1}{3}\left\{1 + 2\,\frac{2k(2k-1)}{(4k-3)(4k+1)}\right\} \qquad (11.11b)$$

$$a_2(k) = \frac{(2k-1)(2k-2)}{(4k-1)(4k-3)} \qquad (11.11c)$$

Note that $a_2(1) = 0$. Substituting (11.8), (11.9), and (11.10) in (11.3), one arrives at

$$\sum_{n=1}^{\infty} b_n \int_{-1}^{1} P_{2n-1}(v) \ln|s-v|\,dv \; +$$

$$+ \frac{2\pi Z_s}{ikb Z_0} \sum_{n=1}^{\infty} b_n \sum_{p=0}^{2} a_p(n) P_{2(n-p)+1}(s) =$$

$$= b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{Z_0}{Z_0 + 2Z_s}\right) \sum_{n=1}^{\infty} d_n P_{2n-1}(s) \qquad (11.12)$$

It has been found by Davis (1970) that

$$I_{m,n} = \int_{-1}^{1}\int_{-1}^{1} P_{2n-1}(t) P_{2m-1}(\tau) \ln|t-\tau|\,dt\,d\tau =$$

$$= \frac{2}{(m+n)(m+n-1)\left[4(m-n)^2-1\right]} \qquad (11.13)$$

Multiplying through by $P_{2m-1}(s)$, integrating from -1 to 1, and simplifying, one obtains

$$\left(\frac{4m-1}{2}\right) \sum_{n=1}^{\infty} I_{m,n} b_n + \frac{2\pi Z_s}{ikb Z_0} \sum_{p=0}^{2} a_p(m+p-1) b_{m+p-1} =$$

$$= b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{Z_0}{Z_0 + 2Z_s}\right) d_m \qquad (11.14)$$

This is an infinite set of equations in an infinite number of unknowns which must be truncated and solved on the computer. This will be done in a later section.

B. FORMULATION FOR SMALLER VALUES OF SHEET IMPEDANCE

The Fredholm integral equation that needs to be solved for this case is given by (9.36). Substituting $z=ub$ and $\tau=tb$ in this equation gives the equation

$$\frac{2\pi Z_s}{ikbZ_0} h_s(u) - \int_1^\infty h_s(t) \ln\left|\frac{t+u}{t-u}\right| dt =$$

$$= -b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{2Z_s}{Z_0+2Z_s}\right)\left\{u + \frac{1-u^2}{2}\ln\left(\frac{u+1}{u-1}\right)\right\} \qquad (11.15)$$

$$- b^{3/2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{4Z_s}{ikbZ_0} \frac{\mathrm{sgn}(u)}{|u|+\sqrt{u^2-1}} \qquad |u|>1$$

The last term in this equation can be simplified if numerator and denominator are multiplied by $|u|-\sqrt{u^2-1}$. One finds

$$\frac{\mathrm{sgn}(u)}{|u|+\sqrt{u^2-1}} = \mathrm{sgn}(u)\left(|u|-\sqrt{u^2-1}\right) \qquad (11.16)$$

By using symmetry and two changes of variables, one obtains

$$\frac{2\pi Z_s}{ikbZ_0} h_s\left(\frac{1}{s}\right) + \int_{-1}^{1} \frac{1}{v^2} h_s\left(\frac{1}{v}\right) \ln|s-v|\, dv =$$

$$= -b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{2Z_s}{Z_0+2Z_s}\right)\frac{1}{s^2}\left\{s + \frac{s^2-1}{2}\ln\left(\frac{1+s}{1-s}\right)\right\}$$

$$- b^{3/2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{4Z_s}{ikbZ_0} \frac{1}{s}\left(1-\sqrt{1-s^2}\right) \qquad (11.17)$$

Let $h_s^*(t) = t^{-2}h_s\left(\frac{1}{t}\right)$. Substituting this in the above equation, one obtains

$$\frac{2\pi Z_s}{ikbZ_0} s^2 h_s^*(s) + \int_{-1}^{1} h_s^*(v) \ln|s-v| \, dv =$$

$$= -b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{2Z_s}{Z_0+2Z_s}\right)\frac{1}{s^2}\left\{s + \frac{s^2-1}{2}\ln\left(\frac{1+s}{1-s}\right)\right\}$$

$$-b^{3/2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{4Z_s}{ikbZ_0} \frac{1}{s}\left(1-\sqrt{1-s^2}\right) \qquad s<1 \tag{11.18}$$

Except for the last term on the right-hand side, this equation is similar to (11.3). By using Luke (1969, vol. I, p. 38), the last term becomes

$$\frac{1}{t}\left\{1-(1-t^2)^{\frac{1}{2}}\right\} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{(n+1)!} t^{2n+1} \qquad |t|<1 \tag{11.19}$$

$(\frac{1}{2})_n$ is the Pochhammer's Symbol. Substituting (11.5) in this equation and using (11.7), one obtains

$$\frac{1}{t}\left\{1-(1-t^2)^{\frac{1}{2}}\right\} = \frac{1}{2}\sum_{k=1}^{\infty} e_k P_{2k-1}(t) \tag{11.20a}$$

where

$$e_k = \sum_{m=k-1}^{\infty} \frac{(\frac{1}{2})_m (2m+1)!\,(m+k+1)!\,(2k-\frac{1}{2})\, 2^{2k+1}}{(m+1)!\,(2[m+k+1])!\,(m-k+1)!} \tag{11.20b}$$

Expanding $h_s^*(t)$ in a Legendre polynomial series and proceeding as before, one obtains the infinite set of equations in an infinite number of unknowns

$$\left(\frac{4m-1}{2}\right)\sum_{n=1}^{\infty} I_{m,n} b_n + \frac{2\pi Z_s}{ikbZ_0}\sum_{p=0}^{2} a_p (m+p-1) b_{m+p-1} =$$

$$= -b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{2Z_s}{Z_0+2Z_s}\right)d_m - b^{3/2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{2Z_s}{ikbZ_0} e_m \qquad m=1,2,\cdots\infty \tag{11.21}$$

Note that if $Z_s=0$, then $b_n=0$ for $n=1,2,\ldots,\infty$ as expected.

XII. NUMERICAL SOLUTION OF EQUATIONS (11.14) and (11.21)

The simultaneous equations (11.14) and (11.21) were solved on the computer by truncating the limit on the summations to N, a variable in the computer programs. Doing this for (11.14) yields the result

$$\left(\frac{4m-1}{2}\right)\sum_{n=1}^{N} I_{m,n}\, b_n + \frac{2\pi Z_s}{ikbZ_0}\sum_{p=0}^{2} a_p(m+p-1)\,b_{m+p-1} = \qquad (12.1)$$

$$= b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{Z_0}{Z_0+2Z_s}\right) d_m \qquad m=1,2,\cdots,N$$

d_m presents a computational difficulty. The infinite series (11.8b) turns out to be very slowly converging. Let

$$d_n = \left(2n-\tfrac{1}{2}\right)2^{2n+2}\sum_{m=n-1}^{I=\infty} D_m(n) \qquad (12.2)$$

where

$$D_m(n) = \frac{(2m+1)!\,(m+n+1)!}{(2m+1)(2m+3)[2(m+n+1)]!\,(m-n+1)!} \qquad (12.3)$$

The next term in the series can be computed from

$$D_{m+1}(n) = \frac{(2m+1)(2m+2)(2m+3)}{2(2m+5)[2(m+n+1)+1](m-n+2)}\, D_m(n) \qquad (12.4)$$

The first term in the d_n series is given by $D_{n-1}(n)$. The first term in the next series (d_{n+1}) is given by $D_n(n+1)$.

$$D_n(n+1) = \frac{(2n-1)\,n\,(2n+1)}{2(2n+3)(4n+3)(4n+1)}\, D_{n-1}(n) \qquad (12.5a)$$

$$D_0(1) = \frac{1}{36} \qquad (12.5b)$$

The numerical values for the d_n's are given in Table 12.1 along with the upper index of summation I in (12.2). As can be seen, a large number of terms

Table 12.1. Computed values of d_n

n	I	d_n
1	209	0.7988899796
2	523	0.1205192866
3	892	0.0382577779
4	1308	0.0165125030
5	1765	0.0085110354
6	2256	0.0049279487
7	2776	0.0030974268
8	3324	0.0020692245
9	3896	0.0014488469
10	4491	0.0010530191
11	5106	0.0007888795
12	5740	0.0006060148
13	6393	0.0004754722
14	7062	0.0003798273
15	7748	0.0003081716
16	8449	0.0002534407
17	9165	0.0002109242
18	9895	0.0001773995
19	10638	0.0001506120
20	11395	0.0001289524
21	12164	0.0001112509
22	12945	0.0000966439
23	13737	0.0000844842
24	14000	0.0000742759
25	14000	0.0000656441
26	14000	0.0000582972
27	14000	0.0000520049
28	14000	0.0000465846

are required to give seven digit accuracy. If more than 14000 terms are needed, the summation is truncated there. As a check, the value of the Legendre series in (11.8a) was evaluated at ten equally spaced points between zero and one and compared to the values obtained by other methods. The two values so obtained matched for all points to approximately four digits of accuracy. While this was only a spot check, it did show that the Legendre series in (11.8a) is a proper representation for the function on the left in (11.8a), and that the d_n's in Table 12.1 are correct to at least four decimal places.

The simultaneous equations (12.1) were solved on the computer for various matrix dimensions N, wavelengths λ, slit half-widths b, and sheet impedances Z_s. It was hoped that b_1 would be much larger than the other b_n's and so would provide a one term approximation. Unfortunately, this was not the case. For (λ, b, Z_s) = (3m, 0.5cm, 0.045) and N from 10 to 28, most of the b_n's were of the same order of magnitude (10^{-4} to 10^{-6}). Convergence was quite good in that the values of the b_n's did not vary much with changing N. For this particular choice of (λ, b, Z_s), it does not appear that a one term solution could be used to give suitable results. One thing should be kept in mind though. This is simply the possibility that in equation (10.9), f_H could be significantly lower in magnitude than the f_G term. As they stand, the equations are in such a form that it is difficult to evaluate the relative importance of the f_G and f_H terms. The individual terms when h(t) in (10.7) is replaced by the Legendre series are of the form

$$\frac{|X|}{2}\int_0^1 P_{2n-1}(t)\left\{\frac{1}{\left(\frac{b}{t}-z\right)^2+X^2}-\frac{1}{\left(\frac{b}{t}+z\right)^2+X^2}\right\}dt \qquad (12.6)$$

A general closed form expression for any n for this integral has not yet been found.

The wavelength λ was made larger to λ = 30m. The solution for this case, (λ, b, Z_s) = (30m, 0.5cm, 0.045), is given in Table 12.2. From the results, one can see that only the first and second terms probably need to be retained. The cases (30m, 0.05cm, 0.045) and (3m, 0.5cm, 0.45) were similar in that the first three terms were dominant. These three cases were fairly special, however. In cases such as (3m, 0.5cm, 1,000,000), (30m, 0.05cm, 0.045), (300m, 0.5cm, 0.045), and (3m, 0.5cm, 0) at least ten of the b_n's were of comparable magnitudes. The decimal values ranged from 2.5×10^{-4} for (3m, 0.5cm, 0) to 4×10^{-18} for (3m, 0.5cm, 1,000,000). Due to the very small size of the coefficients for large Z_s, it could be argued that f_H could be neglected in equation (10.9).

Table 12.2. Values of b_n

N= 28 LAMBDA= 30.00000000 B= 0.00500000 ZS= 0.04500000

I,	SOLUTION(I)	
1	-0.0001286771	0.0000523017
2	-0.0000292388	0.0001135824
3	0.0000412771	0.0000374772
4	0.0000097743	-0.0000068537
5	-0.0000035659	0.0000054503
6	0.0000039863	0.0000043682
7	0.0000009835	-0.0000000319
8	0.0000000985	0.0000016002
9	0.0000008883	0.0000007096
10	0.0000001430	0.0000003934
11	0.0000002808	0.0000005342
12	0.0000002166	0.0000002411
13	0.0000001093	0.0000002722
14	0.0000001443	0.0000001977
15	0.0000000804	0.0000001525
16	0.0000000823	0.0000001421
17	0.0000000645	0.0000001045
18	0.0000000520	0.0000000969
19	0.0000000478	0.0000000784
20	0.0000000375	0.0000000678
21	0.0000000348	0.0000000600
22	0.0000000280	0.0000000498
23	0.0000000272	0.0000000461
24	0.0000000195	0.0000000391
25	0.0000000237	0.0000000332
26	0.0000000129	0.0000000363
27	0.0000000188	0.0000000172
28	0.0000000156	0.0000000417

Equation (11.21) has also been solved on the computer. Truncating the limit on the summation to N, one obtains

$$\left(\frac{4m-1}{2}\right)\sum_{n=1}^{N} I_{m,n}\, b_n + \frac{2\pi Z_s}{ikbZ_0}\sum_{p=0}^{2} a_p(m+p-1)\, b_{m+p-1} =$$

$$= -b^{3/2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{2Z_s}{Z_0+2Z_s}\right) d_m - b^{3/2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{2Z_s}{ikbZ_0}\, e_m \qquad (12.7)$$

$$m = 1, 2, \dots, N$$

The d_n's have been calculated and are presented in Table 12.1. Only the e_k's need to be calculated. Let

$$e_k = (2k - \tfrac{1}{2}) 2^{2k+1} \sum_{m=k-1}^{I=\infty} E_m(k) \tag{12.8}$$

where

$$E_m(k) = \frac{\left(\frac{1}{2}\right)_m (2m+1)!\,(m+k+1)!}{(m+1)!\,(2[m+k+1])!\,(m-k+1)!} \tag{12.9}$$

The next term in the series can be computed from

$$E_{m+1}(k) = \frac{(2m+1)(2m+2)(2m+3)}{4(m+2)(2m+2k+3)(m-k+2)} E_m(k) \tag{12.10}$$

The first term in the series e_k is given by $E_{k-1}(k)$. The first term in the next series (e_{k+1}) is given by $E_k(k+1)$. The relation between the two is

$$E_k(k+1) = \frac{(k-\frac{1}{2})(2k+1)k}{2(k+1)(4k+3)(4k+1)} E_{k-1}(k) \tag{12.11a}$$

$$E_0(1) = \frac{1}{24} \tag{12.11b}$$

The numerical values for the e_k's are given in Table 12.3 along with the value of the upper index I in equation (12.8). As can be seen, the e_k's are extremely slowly convergent.

Equation (12.7) was solved on the computer for various values of matrix dimension N, wavelength λ, slit half-width b, and sheet impedance Z_s. It was hoped that a one term result would be obtained. For (λ, b, Z_s)=(3m, 0.5cm, 0.045), the first ten b_n coefficients were of comparable amplitude as shown in Table 12.4. The trend in the data for (30m, 0.5cm, 0.045) is similar to that in Table 12.2.

Table 12.3. Computed values of e_k

k	I	e_k
1	532	0.6437826847
2	1361	0.1438840258
3	2282	0.0620739102
4	3276	0.0342583375
5	4327	0.0216270308
6	5425	0.0148645553
7	6563	0.0108335385
8	7735	0.0082413892
9	8938	0.0064777034
10	10169	0.0052241416
11	11425	0.0043016205
12	12705	0.0036031900
13	14000	0.0030618054
14	14000	0.0026333372
15	14000	0.0022886738
16	14000	0.0020073083
17	14000	0.0017746398
18	14000	0.0015800448
19	14000	0.0014156449
20	14000	0.0012754983
21	14000	0.0011550553
22	14000	0.0010507849
23	14000	0.0009599128
24	14000	0.0008802349
25	14000	0.0008099822
26	14000	0.0007477220
27	14000	0.0006922837
28	14000	0.0006427035

The magnitudes of the first three coefficients are a factor of ten larger than the other coefficients. For (3m, 0.5cm, 10^{-6}) all the coefficients are of the same order of magnitude and were in the neighborhood of $3\text{x}10^{-10}$. It does not appear that it can be predicted on the basis of an analysis of the data when more or fewer coefficients will be needed. The relationship appears quite complex. It does not appear that any simplification occurs when the "small sheet impedance" formulation of XI.B. is used.

Table 12.4. Values of b_n

N= 26 LAMBDA= 3.00000000 B= 0.00500000 ZS= 0.04500000

I,	SOLUTION(I)	
1	-0.0000004882	-0.0000052007
2	-0.0000022124	-0.0000072143
3	-0.0000040522	-0.0000067656
4	-0.0000052469	-0.0000051073
5	-0.0000055239	-0.0000030881
6	-0.0000050141	-0.0000013140
7	-0.0000040616	-0.0000000909
8	-0.0000030213	0.0000005533
9	-0.0000021297	0.0000007617
10	-0.0000014770	0.0000007240
11	-0.0000010475	0.0000005925
12	-0.0000007802	0.0000004546
13	-0.0000006134	0.0000003437
14	-0.0000005033	0.0000002635
15	-0.0000004246	0.0000002069
16	-0.0000003646	0.0000001663
17	-0.0000003171	0.0000001360
18	-0.0000002784	0.0000001129
19	-0.0000002466	0.0000000948
20	-0.0000002199	0.0000000804
21	-0.0000001974	0.0000000687
22	-0.0000001782	0.0000000591
23	-0.0000001618	0.0000000512
24	-0.0000001477	0.0000000442
25	-0.0000001346	0.0000000379
26	-0.0000001229	0.0000000354
27	-0.0000001246	0.0000000321
28	-0.0000001183	-0.0000000114

XIII. CONCLUSIONS

This report provides a numerical-analytic solution to the problem of penetration of a composite skin panel with a slit of width 2b by a quasi-static magnetic field at normal incidence. The panel is modelled by an impedance sheet of impedance Z_s. The basis for the quasi-static formulation used is provided by Latham and Lee (1968). The solution of the resulting dual integral equations is proqided by Sneddon (1966). For this particular problem it is found that distribution theory and the generalized Hankel transform had to be used to interpret some integrals in Sneddon's solution procedure.

A convincing argument for determining the values of two constants, A and C of (8.7), arising in the formulation needs to be found. The values for these constants have been chosen so as to allow an easy solution in terms of Legendre polynomials. C has been chosen to be one of two different values depending on whether the large or small Z_s formulation was used. For the special case $Z_s=0$, an analytic solution was obtained which gives the same boundary values as have been obtained by other methods. This shows that the basic formulation is correct.

The numerical results for the b_n coefficients of (11.14) and (11.21) were disappointing. It had been hoped that b_1 would be significantly larger than the other b_n's so that a one term numerical solution could be found. Unfortunately, this was not generally the case and problems in evaluating equation (12.6) for $n > 1$ did not allow computation of the potentials in (10.9) and (10.10) for any practical cases.

We have seen that the Legendre polynomial solution procedure in general did not adequately model the physical behavior for a one term solution. Perhaps another solution procedure could be found which would give a one term solution, but it is not clear how one would determine such a solution procedure.

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