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SOLUTION METHODS ON ALGEBRA PROBLEMS WITH SIMULTANEOUS EQUATION--ETC(U)
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Solution Methods on Algebra

Problems with Simultaneous Equations

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Running head: Solution Methods

[1]This article is not based on research conducted by The Rand Corporation, and the views expressed herein are those of the author and should not be interpreted as representing those of Rand or any of the government agencies sponsoring its research.

Abstract

Problem solving approaches often find a path from the givens to the unknown or from the unknown to the givens. This study explores those approaches using written protocols of Navy subjects while solving for the numerical value of an unknown from several knowns and a system of equations. For a subset of the items, solutions require subjects to retain expressions with both literals and numerals. Successful and unsuccessful students were similar in their preference to work backward and to use literals rather than numerals and in the efficacy of their solutions. Students acquired successful solution methods during instruction and used them consistently.

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Solution Methods on Algebra Problems with Simultaneous Equations

This research concerns solution methods of algebraic problem solving. Previous studies of problem-solving approaches have generally used problems and systems of equations which yield solution paths linking the givens to the unknown, using intermediate variables through one or more stages or steps. Hayes (1965) used binary relationships indicating communication among spies. Malin (1979), Kieras and Greeno (1975) and Mayer and Greeno (1975) used algebraic equations similar to those in the present study.

In the selection of a solution method for simple problems with equations, at each step a subject typically chooses between the method of working forward toward the unknown using the given values, or working backward from the unknown toward the given values, using the appropriate statements drawn from the system of equations. As more operations are required in order to arrive at the solution, it may be advantageous to combine these methods in organizing the problem. Thus, the subject can employ a number of different strategies, focusing on all aspects simultaneously or using a smaller set of attributes: the goal, the givens, or a subset of the givens.

To actually compute the solution of such problems, the subject must determine the series of steps and transform the equations. Kieras and Greeno (1975) asked the subjects simply to indicate computability and Malin (1973) asked for the solution path via verbal protocols. Both of these involved memory load. Only Mayer and Greeno (1975) required the subjects to actually compute the value of the unknown.

Malin (1979) found variability of preferred strategy between subjects. Her "grouped" condition is most similar to the procedure in this study. In that condition she found that four subjects preferred to use a backward strategy, two forward, ten mixed (a single backward step before going forward), and two another strategy. Solution paths, not calculated answers, were all that she required.

Subjects' verbal protocols do not necessarily reflect all information processing, as shown in two studies investigating blind alleys (steps that are not progressing toward the solution), as well as length of solution. Both Hayes (1965) and Malin (1979) suggest that certain inconsistent outcomes of these two variables are likely due to covert steps not reported by the subjects, such as scanning ahead to eliminate certain paths. Thus, although written protocols generally lack some of the information processing of the subject, verbal protocols do not obtain all processing. Furthermore, it has been shown that subjects do not determine a solution path before using external memory (paper and pencil). Mayer and Greeno (1975), asking subjects to calculate answers on paper, found unusually long response times on unanswerable items. They conclude that subjects proceed to carry out calculations at the beginning of problem solving rather than first working out a solution method to see if an answer is obtainable. Thus, analyses of solution strategies from written protocols may yield some additional insight into problem solving behaviors.

The efficiency of working foward or working backward at any given step depends on the relative branching of the search spaces. For example, if only one equation could apply to givens to produce the value of an unknown intermediate variable, but two or more could apply to the goal, the forward approach may be applied more frequently. Furthermore, if one of the two equations with the goal as a variable also has a given as a variable, the third variable becomes the new subgoal. But it is unknown whether this new subgoal is computable from the givens. The preference to use the forward or backward choices may not be fully identified from written protocols if some information processing occurs outside of the written work. However, an analysis of written protocols can add to an understanding of problem solving.

An additional aspect which may affect written solutions is the necessity to transform equations. For example, given the equation C = A * B, it is easier to find C from A and B than it is to find A from C and B, since in the latter case the equation must be transformed. Thus, the selection of a step, even to a blind alley, may depend on the directness of the application. An application which requires no transformation is easier to perform than one which requires a transformation. If ease of computation is a criterion, selection of alternatives at any choice point may be a function of the directness of the solution.

This research focuses on solution strategies of well-structured algebraic problems. Greeno (1976) defines a well-structured problem in terms of three criteria:

1. The problem occurs in an environment containing a specified set of elements and a set of rules for combining the elements to form objects.

2. A set of operators for transforming one problem state into another is given.

3. The initial state and problem goal are specified as simple conjunctions of features that must be present for the goal to be achieved.

Each problem in this study presented one unknown and the value of two or three given variables. Subjects were to solve for the unknown using a system of ten equations which specify the relationship between variables. Solving these problems, the subject searches for a method by which he can get from the initial state (the givens) to the goal (the unknown).

Polya (1971) recommended the working backward strategy for geometric proofs. This strategy has certain inherent disadvantages when solving written algebraic problems. Working backward does yield a solution method. However, the subject must then apply the given values in the solution path, working in a forward direction, to obtain the value of the unknown. Thus, backward strategies may be efficient if the search space is smaller or if only solution paths are needed; forward strategies may be more efficient if computed values are required. The conclusion by Mayer and Greeno (1975) that subjects tend to carry out calculations before working out a solution method supports this contention.

Purpose

This study was designed to analyze written protocols of solutions to two types of algebra problems. One type requires only sequential application of rules--students can successfully solve these problems working forward or backward on paper by substituting values and transforming equations. The search space for the forward and backward approaches is approximately equal in each step. In the context of this study, working backward is defined as paper and pencil manipulations of an equation with an unknown (the goal or a subgoal from the goal) and working forward as manipulation of the givens or a derived value from the givens. It is possible that students omitted writing portions of their thinking processes. The other type requires simultaneous application: neither forward nor backward methods directly yield a solution; students must retain expressions with both literals (letters) and numerals and use two equations simultaneously. Given that a student can work on paper, one purpose of this study was to determine whether demonstrated approaches on problems requiring sequential applications of equations relate to ability to use equations simultaneously. Another purpose was to investigate differing solution strategies before and after classroom instruction. In particular, this study was designed to explore the use of forward and backward solution methods, and of literals within the context of the types of problems described here, distinguishing approaches of more able students from those less able students. This was done in two ways--by comparing two groups before instruction and by comparing the same students before and after instruction.

Method

Sample

Seventeen students, who recently completed a Navy Basic Electricity and Electronics (BE&E) Course, participated in the study. During all portions of the BE&E course, including exams, students were permitted to use formula sheets which include the expression of each variable in terms of pairs of other variables. Twelve of the equations could be derived from two basic equations. The students were required during the BE&E course to plug in values for the variables, but not to derive one of the twelve equations from other equations.

Pretest

Students were given a pretest requiring them to solve algebraic problems based on the following system of equations;

$$A_1 = B_1 C_1 \qquad A_2 = B_2 C_2 \qquad A_3 = B_3 C_3 \qquad A_3 = A_1 + A_2$$

$$C_1 = B_1 D_1 \qquad C_2 = B_2 D_2 \qquad C_3 = B_3 D_3 \qquad C_3 = C_1 + C_2$$

$$D_3 = D_1 + D_2$$

$$B_3 = B_1 = B_2$$

The system was identical to a subset of equations on series circuits, except that the system was disguised by replacing the canonical notation used in electronics, namely, P, I, E, and R, with a different notation, namely A, B, C, and D. The students were asked to show their work.

The system of equations used in the pretest may be described as consisting of three types of equations; two operands with addition (type A), two operands with multiplication (type M), and equality of three variables (type E). There were three equations of type A, six of type M, and one of type E.

The pretest consisted of 53 items which could be differentiated by the following variables:

1. Minimum number of equations required to solve the problem in either a forward or backward solution path (ranging from one-step to four-step problems).

2. Sequence of applications in a forward solution of types A, M, and E, above (e.g., MM, MA, AM, ME, EM, MMA, MME, AMM, AME, etc.). Note: Certain sequences were not used when the resulting problems were not solvable, e.g., AA or AE.

3. Whether at each step the solution of the equation used an indirect (I) application (whether the unknown was one of the operands) or direct (D) application (e.g., DD, DI, ID, IDD, etc.).

Nine problems required the application of only one equation. Nine problems could not be solved by a simple forward or backward solution path, but instead required manipulation of two equations simultaneously (noted by *). The two simultaneous equations were A = BC and C = BD. These problems required solving for B or C, given A and D. The remaining 35 problems could be solved by successively applying rules of substitution and transformation to at least two single equations, one at a time. The students were given integer values of two or three variables and were asked to find the value of another variable. For example, a problem read: $B_2 = 5$, $D_2 = 5$, $A_1 = 5$. Find A_3. No thinking aloud protocols were taken.

Instruction

For three days following the pretest, the students were given instruction for three hours per day on systems of equations (total time = 9 hours). Each day the students were taught to solve problems with a new system of equations, differing in the operations used and the letters by which variables were referenced. No system used the letters with which they were familiar (P, I, E, R).

Instruction was similar for all three days. Students were given a system of equations and from 37 to 53 problems to solve. The items were sequenced in approximate order of difficulty, based on the minimum number of successive equations which must be used in a correct forward solution, the operations involved, and the directness of the application of each equation.

After students attempted to solve the problems, the instructor worked through all problems on the blackboard. When more than one step was required in the solution, the instructor demonstrated how to look at the problem to determine the data and the unknown (which are clearly set forth in the problem) and the relevant conditions (which equations are useful). It was stressed that the selection of equations should be such that they eventually link the data to the unknown. Consistent focuses were placed on two orientations simultaneously: (1) What can be derived from the given data and (2) what would be useful to know to solve for the unknown. Alternative methods for solution suggested by students were also used for every problem, including substitution, transitivity of equality, and addition, subtraction, multiplication, and division of equations where applicable. The former two methods were more frequently suggested and thus received greater stress in instruction. The nine* problems were more difficult and took more instruction time per problem. Roughly one-third of the instruction period was devoted to these problems.

The system of equations given on the first day of instruction (day 2) was similar to that of the pretest. However, all operations of addition were replaced by multiplication, and all operations of multiplication were replaced by addition. All equality relationships remained. On the second day of instruction (day 3), the students were given the original set of equations, with each multiplication operand replaced by division and each addition operand replaced by subtraction. In addition, the original differentiation of variables by letters was replaced by subscripts, and the differentiation by subscripts was replaced by varying the alphabetic letters. On the third day of instruction (day 4), addition was replaced by division and multiplication by subtraction.

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*Requiring the use of two equations simultaneously.

Posttest

On the fifth day of the sessions, the students were given a posttest. All problems on the pretest missed by at least three students were presented as the posttest. This included all nine* problems (requiring the use of two equations simultaneously) and seven sequential application problems.

No feedback was ever given concerning the responses on the pretest, and that system was not employed during instruction. The posttest employed the original set of equations, but replacing subscripted variable differentiation with alphabetic differentiation, and replacing the alphabetic letter differentiation with subscripts.

Method of Analysis

Two of the 17 students demonstrated an inability to transpose equations. Their tests were omitted from the sample, leaving 15 tests for analysis. The students were divided into two groups, eight successful in solving the * items (simultaneous equations) and seven unsuccessful.

Analyses were performed on three sets of items: The 35 multi-step pretest items requiring only sequential rule application, the seven items appearing on both the pretest and posttest requiring only sequential applications, and the nine* times appearing on both the pretest and posttest requiring simultaneous rule application. Different sets of research questions were addressed with each item set.

Successful and unsuccessful students were compared on the 35 pretest items which required the use of at least two equations, excluding * items, in terms of forward vs. backward approaches, transposing before replacing values vs.

replacing before transposing, and consistency of approach.

It was expected that

1. Few students would work backwards on paper although they may mentally process solution paths from a working backward approach.

2. Some students would use both forward and backward approaches in their solutions.

3. Those who worked backwards on paper would be successful on the * problems.

4. Those who transposed before replacing values would be successful on the * problems.

Comparisons were made of the successful and unsuccessful students on the pretest and posttest of the seven items solvable with sequential rule application. It was hypothesized that successful students on the pretest would produce more efficient solutions, omit implicit steps, and use literals more often (substitute expressions more often and replace values less often). It was also hypothesized that, as a result of instruction, students on the posttest would produce more efficient solutions, omit implicit steps, and use literals more often.

The measures used to test these hypotheses were:

1. Number of lines needed for solution ($a = b = c$ is one line),
2. Number of equations needed for solution ($a = b = c$ is two equations),
3. Frequency of implicit operations,
4. Frequency of replacing variables with given values,
5. Frequency of replacing variables with computed values, and
6. Frequency of substituting an expression for a variable.

The nonparametric Wilcoxon Rank Statistic was used for making comparisons between groups.

Analyses on the nine * items were made to determine consistency of approach by each student on the pretest. Also a comparison was made between pretest and posttest on consistency. It was expected that students would retain successful methods from the pretest, regardless of alternative methods presented in instruction.

Identification of Successful Students on * Problems

Incorrect answers among these 15 student papers are due to one of five causes: Computational error, transcription error, not solving for (misreading) the unknowns, algebraic error, or inability to arrive at a solution method. Errors of the first three types could be considered "measurement error" since they are not related to the skill tested and were not considered incorrect in identifying successful students on the nine * problems.

Results and Discussion

Pretest - 35 Items

It was not expected that students solving problems in written form would work backward on paper, although their selection of a solution path may be guided from unwritten information processed by working backwards. This was not the case. Two students worked backward on a majority of the 35 written problems and four others worked backward occasionally. There is a considerable amount of flexibility demonstrated by the students in solution methods. Although they all showed preference for either a forward or backward approach, 40% used both backward and forward solution methods at least once in their solutions to the 35 pretest problems. The percentage with mixed methods here is not as great as that found in Malin's (1979) study, in which 63% prefer mixed strategies.

Given that some students do work backwards, are they more likely able to solve the simultaneous * problems? As shown in Table 1, there were only slightly more students in the successful group who worked backward at least once compared with the unsuccessful group. Thus, working backward on paper does not indicate a much greater facility with solving equations.

Working forward or backward, the student who solves one equation at a time can first transpose the equation to one with the subproblem unknown on one side and then plug in values or he can plug in the values first and then transpose an equation with only a single variable, namely the unknown. Students also demonstrated variability on this aspect of solution methods, with 46% using both approaches on the 35 item pretest.

It was expected that students who transpose first, using equations with literals, would have greater facilitation with literals and would be able to solve the nine * problems with relative ease. The data in Table 1 indicate that this was not the case. Approximately the same number of subjects in both groups demonstrated the ability to transform equations using only literals and, in fact, slightly more successful students always preferred to plug in before transforming. Thus, a decision to manipulate equations with literals does not imply that a student is more likely to deal with two equations simultaneously requiring use of literals. Furthermore, a preference for working with numerals does not indicate less ability in working with literals.

Table 1

Frequency of Solution Methods

for Successful and Unsuccessful Students

	Successful (n = 8)	Unsuccessful (n = 7)
Forward all	4	5
Forward most	3	1
Backward most	1	1
Plug in, then transpose all	2	2
Plug in, then transpose most	4	0
Transpose, plug in all	1	3
Transpose, then plug in most	1	2

Pretest and Posttest - 7 items

Seven items which could be solved with a forward or backward solution appeared on both the pretest and the posttest. One student left the last two pretest items blank and another only wrote answers without solution steps to several items on the posttests. They were both omitted from parts of the analysis. There were no differences between successful and unsuccessful problems solvers on either the pretest or posttest on any of the six variables studied: Number of lines needed for solution, number of equations needed for solution, frequency of implicit operations, frequency of replacing variables with given values, frequency of replacing variables with computed values, or frequency of substituting an expression for a variable. The medians are shown in Table 2.

Table 2

Medians of Six Variables on Seven Items Appearing

on the Pretest and Posttest

	Pretest		Posttest	
	Unsuccessful Group	Successful Group	Unsuccessful Group	Successful Group
Number of lines	63	56	52	40
Number of equations	65	60	55	44
Frequency of implicit operations	4	7	2	5
Frequency of replacing with givens	19	19	11	10
Frequency of replacing with computed values	13	13	2	3
Frequency of substituting expressions	0	0	9	9

All frequencies are per student.

However, combining groups, problem solving behavior changed from the pretest to the posttest. Comparisons were made using a binomial test on dichotomous variables representing greater or less frequency of a given behavior on the posttest than on the pretest. On these seven problems, which all students could solve prior to instruction, a significant proportion used fewer lines ($p = .80$, $\alpha < .05$), fewer equations ($p = .93$, $\alpha < .01$), fewer replacements of given values ($p = .85$, $\alpha < .05$), fewer replacements of computed values ($p = .93$, $\alpha < .01$), and more substitutions of expressions for variables ($p = .92$, $\alpha < .01$) on the posttest than the pretest. There were no significant differences on number of implicit operations. These measures are not independent. In fact, students tended after instruction to prefer substituting expressions rather than values. Although all problems in this set could be solved without replacing expressions, that method proved effective for all problems, including the * problems in instruction. Therefore, students adopted an approach from the instruction which could be applied successfully and used it often. By using this approach of substituting expressions, solutions were shortened, minimizing the number of lines and equations needed and reducing the frequency of replaced values.

Pretest and Posttest - 9 * Items

The eight students who successfully solved the pretest * problems did so with great variability both between and within students. The reader may recall that the two simultaneous equations were A = BC and C = BD. The students had to find B or C, given A and D. Three students used a consistent solution method for all nine problems, three for eight of the nine problems, and the remaining two varied the methods across all nine problems. Thus, one cannot conclude that students employ a single method on either easy or difficult problems. After a successful method was found for solving a given type of problem, students did not always use that solution method for every other problem of the same type. Instead, they appeared to approach each problem anew, using the solution method which seemed appropriate at the moment.

Of the unsuccessful students, two realized after the first of the nine problems that they did not know how to solve them and they left the remaining eight blank. One always worked forward until confronted with A and D and had to solve for B or C, at which point he quit. Another worked forward writing every equation he needed, including those for steps after the simultaneous equations, but he never put them together. The others wrote one or two expressions for the unknown, but each equation included one given and one variable; the two equations were never put together using any of the possible methods. Thus, the unsuccessful students either quit when they realized they could not determine the numerical value of the unknown or they continued as best they could, leaving variables in the expression for the unknown.

This behavior is an example of Maier's (1945) concept of reproductive and productive thinking. The other problems are very similar to problems previously mastered by the students; solutions require only reproductive thinking. However, on the * items students must restructure their approach. They can no longer use one equation at a time. Those unable to approach the problem differently lack the knowledge of a more complex strategy, or lack the ability to develop a more complex strategy.

On the posttest, the solution behavior of all students changed due to instruction. Interestingly, only three of the eight successful students generally maintained the same approach as they had used on the pretest. The other five students, although having successful methods before instruction, changed their solution method. All used a single method for at least eight of the nine problems. Teachers often assume that instruction has no effect on students who are already proficient in solving the problems in the curriculum, unless instruction focuses on teaching a new method which students are instructed to use. However, this analysis shows that training which is not designed to change behavior, but merely to show successful alternative solution methods, can change problem solving behavior even when a skill is present.

All students who were unsuccessful on the pretest were successful on the posttest. All selected one approach and used it throughout the nine * problems. As did most originally successful students, most originally unsuccessful students chose transitivity of equality, the most frequent solution method offered by students during instruction.

Thus, for both groups, the effect of introducing several solution methods in instruction was for students to choose one method and use it throughout the problem set. What likely occurred was that on the pretest, students had to determine a solution strategy for each item and apply it, whereas on the posttest, the solution method had become automatic, causing little within student variability among solutions.

Conclusion

In this study, we have shown that written techniques for solving relatively easy problems (solvable by successfully using one equation at a time) do not differ between students who are successful and those who are unsuccessful in solving more difficult problems (requiring the application of two equations simultaneously). Neither preference to work backward nor preference to transpose literal equations rather than numeral equations differed between the two groups. Thus, the preference to use literals with either a backward solution or transposing literal equations does not indicate greater ability to solve equations requiring the use of literals in simultaneous equations. Conversely, preference for numerals does not indicate inability to use literals. Both groups showed considerable within-student variability for these preferences.

Additionally, the two groups showed the same written solution behavior on efficiency of solutions, omission of implicit operations, and use of literals.

However, after instruction, fewer replacements of variables with values were made in favor of substituting expressions. This resulted in more efficient solutions.

On the nine problems requiring simultaneous applications of equations, the successful group demonstrated some variability in approach on the pretest but stronger consistency on the posttest. Additionally, most students did not maintain the same solution methods from the pretest to posttest. The unsuccessful group also maintained consistency on the posttest. It appears that each problem was approached separately on the pretest with a solution strategy that seemed appropriate to the student for the given problem. This is an example of productive thinking. Students felt confronted by a new and unique problem with each * problem and they had to develop a solution method from an unusual application of known techniques. After nine hours of instruction, students had already chosen the solution method they intended to use and performance on the posttest was merely an application of those preferred methods. This exemplifies reproductive thinking, where students merely need apply old techniques to a familiar situation. Thus, the instruction was extremely effective in assisting the students to develop an easy, efficient successful approach to solving these kinds of problems. Students were not given specific instruction on which solution method to use for the * problems. Rather, they were exposed to several methods. They apparently acquired a successful method from among several presented during instruction, and then used it consistently.

These findings have implications for training. Ability to use literals on relatively easy problems does not always transfer to their continued use with more complex problems. Thus, to design instruction which encourages the use of literals in one situation may not transfer to a greater facility with literals in other situations. Furthermore, if students demonstrate the capability of using literals, preference for numerals should not be discouraged as leading to noncreative or unproductive problem solving methods. Ability rather than preference to use literals is the key to more advanced applications which require literal usage.

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