Geometric Properties of Dynamic Nonlinear Networks: Transversality, Local-Solvability and Eventual Passivity

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Abstract—This paper gives several basic results on dynamic nonlinear networks from a geometric point of view. One of the main advantages of a geometric approach is that it is coordinate-free, i.e., results obtained by a geometric method do not depend on the particular choices of a tree, a loop matrix, state variables, etc. Therefore, the method is suitable for studying intrinsic properties of networks.

It is shown that *transversality* of resistor constitutive relations and Kirchhoff space is a sufficient condition for the configuration space to be a submanifold. Main result of the paper states that a network is *locally solvable*, i.e., the dynamics of a network is well defined in the sense of *Definition 3, if and only if, capacitor charges and inductor fluxes serve as a local coordinate system for the configuration space*. In other words, if all the variables in a network are determined as functions of capacitor charges and inductor fluxes, at least locally, then the dynamics is well defined. Conversely, if the dynamics is well defined, then all the variables in a network are determined as functions of capacitor charges and inductor fluxes. Because of its coordinate-free property, the main result also says that if the dynamics is well defined in terms of some coordinate system, then it must be well defined in terms of capacitor charges and inductor fluxes. Conversely, if the dynamics is not well-defined in terms of capacitor charges and inductor fluxes, then there is no choice of variables in terms of which the dynamics is well defined in the sense of *Definition 3*. This shows that capacitor charges and inductor fluxes are the fundamental quantities in describing the dynamics of networks. Perturbation results are given which guarantee transversality and local solvability. Finally, several other perturbation results are given which guarantee eventual strict passivity of dynamic nonlinear networks. They explain why the voltage and current waveforms of almost every network of practical importance are eventually uniformly bounded.

I. INTRODUCTION

THIS PAPER gives several basic results on dynamic nonlinear networks from a geometric point of view. One of the main advantages of a geometric approach is that it is coordinate-free, i.e., the results obtained by a geometric method do not depend on the particular choices of a tree, a loop matrix, state variables, etc. Also, the

geometric approach allows us to choose a convenient coordinate system and use it to derive general conclusions which hold with respect to any other coordinate system. Therefore, this approach is suitable for studying intrinsic properties of networks and it enables us to resolve and clarify a number of subtle paradoxes and perplexing questions which lie at the very foundation of nonlinear circuit theory. In particular, several basic questions involving the formulation of state equations for nonlinear networks are hereby resolved in a rigorous manner.

In Section II we will describe nonlinear networks in a coordinate-free manner. In Section III we will discuss transversality of the resistor constitutive relations and the Kirchhoff space. Transversality is important in that it guarantees the configuration space to be a submanifold. We will give two perturbation results which guarantee transversality. One involves element perturbation, i.e., perturbing the existing resistor constitutive relations. The other involves network perturbations, i.e., augmenting the network with capacitors and inductors. In Section IV we will discuss local solvability which is a condition for the dynamics to be well defined. Main result (*Theorem 1*) says that a network is locally solvable, i.e., the dynamics is well defined in the sense of *Definition 3, if and only if*, capacitor charges and inductor fluxes serve as a local coordinate system for the configuration space of the network. This means that if all the variables in a network are determined as functions of capacitor charges and inductor fluxes, at least locally at each point, then the network is locally solvable, i.e., the dynamics is well defined. Conversely, if a network is locally solvable, then all the variables in the network are necessarily determined as functions of capacitor charges and inductor fluxes. Because of its coordinate-free property, the main result also says that if the network is locally solvable in terms of some coordinate system, then it must be locally solvable in terms of capacitor charges and inductor fluxes. Conversely, if the network is not locally solvable in terms of capacitor charges and inductor fluxes, then there is no choice of variables in the network in terms of which it is locally solvable. This says that capacitor charges and inductor fluxes are the fundamental quantities in describing the dynamics of networks. One of the interesting implications (*Corollary 5*) of this result is that if a network is locally solvable, then the capacitors are

necessarily locally charge controlled and inductors are necessarily locally flux controlled. This seems to explain why almost every capacitor (resp. inductor) of practical importance is locally charge (resp. flux) controlled. After proving the main result, we will give a network perturbation technique which guarantees local solvability. In Section V we will give several perturbation results which guarantee eventual strict passivity of dynamic nonlinear networks. *Theorem 2*, another important result of the paper, explains why the voltage and current wave forms of almost every network of practical importance are eventually uniformly bounded.

General Remark: For simplicity, we will usually delete the superscript T denoting the "transpose" of a vector or matrix whenever no confusion arises.

II. COORDINATE-FREE DESCRIPTION OF NONLINEAR NETWORKS

Throughout the paper, we need to use the fact that transversality, local solvability and eventual passivity are coordinate-free properties, i.e., they are independent of the choices of a tree, a loop matrix, a cut set matrix, state variables, etc. Here we will explain how nonlinear networks are described in a coordinate-free manner.

Consider a nonlinear network $\mathfrak{N}$ containing n_R resistors, n_C capacitors and n_I inductors. Let $v = (v_R, v_C, v_L)$ and $i = (i_R, i_C, i_L)$ be the branch voltages and branch currents, respectively, and let q and ϕ be the capacitor charges and inductor fluxes, respectively, where R, C, and L denote resistors, capacitors, and inductors, respectively. The following are the standing assumptions of this paper:

(a) The linear graph $\mathfrak{G}$ which defines the topology of $\mathfrak{N}$ is connected.

(b) $\mathfrak{N}$ is time invariant.

(c) The resistor constitutive relations are characterized by

$$(v_R, i_R) \in \Lambda_R \subset \mathbf{R}^{2n_R} \qquad (1)$$

where Λ_R is an n_R-dimensional C^2 submanifold.

(d) Capacitors are characterized by

$$(v_C, q) \in \Lambda_C \subset \mathbf{R}^{2n_C} \qquad (2)$$

and

$$\frac{dq}{dt} = i_C \qquad (3)$$

where Λ_C is an n_C-dimensional C^2 submanifold. Inductors are characterized by

$$(i_L, \phi) \in \Lambda_L \subset \mathbf{R}^{2n_I} \qquad (4)$$

and

$$\frac{d\phi}{dt} = v_L \qquad (5)$$

where Λ_L is an n_I-dimensional C^2 submanifold.

(e) There are no capacitor-only loops and no inductor-only cut sets.

Remarks: 1) There is no loss of generality in assuming (a) since disconnected subgraphs can be hinged together.

Connectedness is necessary for a tree to exist.

2) Most of the results of this paper can be easily generalized to include the time-varying case under appropriate conditions. We make this assumption simply to avoid introducing complicated notations.

3) Under assumption (c) resistors can be coupled to each other and they need not be voltage or current controlled. This includes virtually all modes of representation, including the hybrid and transmission representations. In particular, a broad class of nonlinear dependent sources are allowed in this formulation. We regard independent sources as uncoupled resistors. All multi-terminal elements are represented as coupled two-terminal elements.

4) Under the present formulation, capacitors need not be voltage or charge controlled. Similarly, inductors need not be current or flux controlled. Notice that capacitors can be coupled to each other and inductors can be coupled to each other.

5) Coupling among elements of different kinds are not allowed. For example, dependent sources controlled by variables of reactive elements are not allowed.

6) We need C^2 property of Λ_R, Λ_C, and Λ_I rather than C^1 because we would like to define C^1 vector fields on the configuration space. (See Section IV.)

7) Assumption (e) was introduced only for simplicity. This involves no loss of generality in view of the results of Chua and Green [1] and Sangiovani-Vincentelli and Wang [2].

Now let $b = n_R + n_C + n_I$ and let

$$\Lambda \triangleq \{(v, i, q, \phi) | (v_R, i_R) \in \Lambda_R, (v_C, q) \in \Lambda_C, (i_I, \phi) \in \Lambda_I \}. \qquad (6)$$

Then it follows from (c) and (d) that Λ is a $(b + n_C + n_I)$-dimensional C^2 submanifold. Let

$$K \triangleq \{(v, i, q, \phi) | (v, i) \text{ satisfies Kirchhoff Laws} \}. \qquad (7)$$

It is well known that K is a $(b + n_C + n_I)$-dimensional linear subspace. This space is called the *Kirchhoff space* and is independent of the particular choices of a tree, a loop matrix, a cut set matrix etc. Since (v, i, q, ϕ) must satisfy the constitutive relations and the Kirchhoff laws simultaneously, the operating points are restricted to within the following subset:

$$\Sigma = \Lambda \cap K. \qquad (8)$$

The set Σ is called the *configuration space* of $\mathfrak{N}$ since this is where the dynamics takes place.

III. TRANSVERSALITY

Since the dynamics takes place on the configuration space Σ, the object Σ should be well behaved at least to the extent that we can write down differential equations on it. For that purpose, it suffices to require Σ to be a differentiable submanifold. A little problem is that even if Λ and K are perfectly well-defined differentiable submanifolds, their intersection Σ may or may not be another submanifold. A

sufficient condition for Σ to be a submanifold is the transversality [3] of Λ and K, which is abbreviated by $\Lambda \pitchfork K$. It is shown in [4] that if $\Lambda \pitchfork K$, then Σ is an $(n_C + n_L)$-dimensional submanifold. This is true for any C^r submanifolds, $r \geq 1$. We first give a formula for checking transversality of Λ and K.

Since Λ_R is a C^2 submanifold of dimension n_R, for each point $(v_{R_0}, i_{R_0}) \in \Lambda_R$, there is a neighborhood $U_R \subset \mathbf{R}^{2n_R}$ of this point and there is a C^2 function $f_R: U_R \rightarrow \mathbf{R}^{n_R}$ such that

$$\Lambda_R \cap U_R = f_R^{-1}(0) \tag{9}$$

and

$$\mathrm{rank}(Df_R)_{(v_R, i_R)} = n_R, \quad \text{for all } (v_R, i_R) \in \Lambda_R \cap U_R \tag{10}$$

where $(Df_R)_{(v_R, i_R)}$ is the derivative of f_R at (v_R, i_R). Similarly, for each point $(v_{C_0}, q_0) \in \Lambda_C$ (resp. $(i_{L_0}, \phi_0) \in \Lambda_L$), there is a neighborhood $U_C \subset \mathbf{R}^{2n_C}$ (resp. $U_L \subset \mathbf{R}^{2n_L}$) of this point and there is a C^2 function $f_C: U_C \rightarrow \mathbf{R}^{n_C}$ (resp. $f_L: U_L \rightarrow \mathbf{R}^{n_L}$) such that

$$\Lambda_C \cap U_C = f_C^{-1}(0) \left(\text{resp. } \Lambda_L \cap U_L = f_L^{-1}(0)\right) \tag{11}$$

and

$$\mathrm{rank}(Df_C)_{(v_C, q)} = n_C, \quad \text{for all } (v_C, q) \in \Lambda_C \cap U_C \tag{12}$$

$$\left(\text{resp. } \mathrm{rank}(Df_L)_{(i_L, \phi)} = n_L, \quad \text{for all } (i_L, \phi) \in \Lambda_L \cap U_L\right).$$

It follows from (e) that there is a proper tree $\mathcal{T}$. Let $\hat{\mathcal{C}}$ be its associated cotree and let v and i be partitioned accordingly:

$$v = \left(v_C \;\vdots\; v_{\hat{\mathcal{T}}}\right) = \left(v_{R_C}, v_L \;\vdots\; v_{R_q}, v_C\right) \tag{13}$$

$$i = \left(i_C \;\vdots\; i_{\hat{\mathcal{T}}}\right) = \left(i_{R_C}, i_L \;\vdots\; i_{R_q}, i_C\right). \tag{14}$$

Let B be the fundamental loop matrix associated with $\mathcal{T}$. Then

$$B = \left[\mathbf{1} \;\vdots\; B_{\hat{\mathcal{T}}}\right]. \tag{15}$$

Set $x \triangleq (v, i, q, \phi)$.

Proposition 1: $\Lambda \pitchfork K$ if and only if

$$\mathrm{rank}\, \mathcal{F}(x) = b, \quad \text{for all } x \in \Sigma \tag{16}$$

where

$\mathcal{F}(x) \triangleq$

$$\begin{array}{cccccc}
v_{R_q} & v_C & i_{R_C} & i_L & q & \phi \\
\left[D_{v_{R_q}} f_R - (D_{v_C} f_R) B_{\hat{\mathcal{T}}} \;\vdots\; D_{i_C} f_R + (D_{i_q} f_R) B_{\hat{\mathcal{T}}}^T \;\vdots\; & & & & \cdot & \cdot \right. \\
\cdot \;\vdots\; D_{v_C} f_C \;\vdots\; & \cdot & & & D_q f_C \;\vdots\; & \cdot \\
\left. \cdot \;\vdots\; & & \cdot \;\vdots\; D_{i_L} f_L & \cdot \;\vdots\; & \cdot \;\vdots\; & D_\phi f_L \right]_x
\end{array}$$

$$\tag{17}$$

$$\mathcal{F}_R^1(v_R, i_R) \triangleq \left[D_{v_{R_q}} f_R - (D_{v_{R_C}} f_R) B_{RR} \;\vdots\; (D_{v_{R_C}} f_R) B_{RC} \;\vdots\; D_{i_{R_C}} f_R + (D_{i_{R_q}} f_R) B_{RR}^T \;\vdots\; (D_{i_{R_q}} f_R) B_{LR}^T\right]_{(v_R, i_R)}. \tag{25}$$

Here $D_{v_{R_q}} f_R$ denotes partial derivative of f_R with respect to v_{R_q} and $\cdot$ denotes a zero submatrix. Other symbols have similar meanings.

Proof: It follows from a similar argument to the proof of *Proposition 1* of [3] that $\Lambda \pitchfork K$ if and only if for each $x \in \Sigma$,

$$\mathrm{rank}\begin{array}{cccc}
v & i & q & \phi \\
\left[B & \cdot & \cdot & \cdot \right. \\
\cdot & Q & \cdot & \cdot \\
D_v f_R & D_i f_R & \cdot & \cdot \\
\cdot \;\vdots\; D_{v_C} f_C & \cdot & D_q f_C & \cdot \\
\left. \cdot \;\vdots\; & D_{i_L} f_L & \cdot & D_\phi f_L \right]_x
\end{array} = 2b$$

$$\tag{18}$$

where B is as in (15) and Q is the fundamental cut set matrix. Since $Q = [-B_{\hat{\mathcal{T}}}^T \;\vdots\; \mathbf{1}]$, one can show, by elementary operations, that (18) holds if and only if (16) holds. $\square$

Remark: $\mathcal{T}$ need not be a proper tree. One simply has two more nonzero submatrices in (17).

If Λ_C and Λ_L admit special forms, then we can give more explicit formulas.

Definition 1: Capacitor constitutive relations Λ_C is said to be *locally voltage* (resp. *charge*) *controlled* if

$$\mathrm{rank}(D_q f_C)_{(v_C, q)} = n_C, \quad \text{for all } (v_C, q) \in \Lambda_C \cap U_C' \tag{19}$$

$$\left(\text{resp. } \mathrm{rank}(D_{v_C} f_C)_{(v_C, q)} = n_C, \quad \text{for all } (v_C, q) \in \Lambda_C \cap U_C\right)$$

where U_C is as in (11) and (12). Similarly, inductor constitutive relations Λ_L are said to be *locally current* (resp. *flux*) *controlled* if

$$\mathrm{rank}(D_\phi f_L)_{(i_L, \phi)} = n_L, \quad \text{for all } (i_L, \phi) \in \Lambda_L \cap U_L$$

$$\left(\text{resp. } \mathrm{rank}(D_{i_L} f_L)_{(i_L, \phi)} = n_L, \quad \text{for all } (i_L, \phi) \in \Lambda_L \cap U_L\right)$$

where U_L is as in (11) and (12).

Let $\pi_R': \mathbf{R}^{2b + n_C + n_L} \rightarrow \mathbf{R}^{2n_R}$ be the projection map defined by

$$\pi_R'(v, i, q, \phi) = (v_R, i_R). \tag{20}$$

Let $\iota: \Sigma \rightarrow \mathbf{R}^{2b + n_C + n_L}$ be the inclusion map defined by

$$\iota(v, i, q, \phi) = (v, i, q, \phi) \tag{21}$$

and set

$$\pi_R = \pi_R' \circ \iota. \tag{22}$$

We next decompose B of (15) according to (13) and (14):

$$B_{\hat{\mathcal{T}}} = \begin{bmatrix} B_{RR} & B_{RC} \\ B_{LR} & B_{LC} \end{bmatrix}. \tag{23}$$

Corollary 1: Let Λ_C (resp. Λ_L) be locally voltage (resp. current) controlled. Then $\Lambda \pitchfork K$ if and only if

$$\mathrm{rank}\, \mathcal{F}_R^1(v_R, i_R) = n_R, \quad \text{for all } (v_R, i_R) \in \pi_R(\Sigma) \tag{24}$$

where

Proof: It follows from (17) that if Λ_C (resp. Λ_L) is locally voltage (resp. current) controlled, then (16) holds if and only if

$$\mathrm{rank}\left[D_{v_q}f_R - (D_{v_q}f_R)B_{\cdot} \; \vdots \; D_{i_c}f_R + (D_{i_q}f_R)B_{\cdot}^I\right]_{(v,i)} = n_R.$$

$$(26)$$

Since

$$D_{v_q}f_R = \left[D_{v_{R_q}}f_R \; \vdots \; \cdot\right], \qquad D_{v_c}f_R = \left[D_{v_{R_c}}f_R \; \vdots \; \cdot\right]$$

$$D_{i_q}f_R = \left[D_{i_{R_q}}f_R \; \vdots \; \cdot\right], \qquad D_{i_c}f_R = \left[D_{i_{R_c}}f_R \; \vdots \; \cdot\right]$$

substituting these and (23) into (26), we have (25). Since $(v,i) \in \Sigma$, the vector (v_R, i_R) must belong to $\pi_R(\Sigma)$. $\qquad \square$

Corollary 2: Let Λ_C (resp. Λ_L) be locally charge (resp. flux) controlled. Then $\Lambda \pitchfork K$ if and only if

$$\mathrm{rank}\,\mathscr{F}^2(x) = n_R, \qquad \text{for all } x \in \Sigma \qquad (27)$$

where

$$\mathscr{F}^2(x) \triangleq$$

$$\left[D_{v_{R_q}}f_R - (D_{v_{R_c}}f_R)B_{RR} \; \vdots \; D_{i_{R_c}}f_R + (D_{i_{R_q}}f_R)B_{RR}^T \; \vdots \; (D_{v_{R_c}}f_R)B_{RC}(D_{v_c}f_c)^{-1}D_q f_c \; \vdots \; (D_{i_{R_q}}f_R)B_{IR}^I(D_{i_l}f_l)^{-1}D_\phi f_l\right]_x. \qquad (28)$$

Proof: If Λ_C (resp. Λ_L) is locally charge (resp. flux) controlled, then $(D_{v_c}f_c)$ (resp. $(D_{i_l}f_l)$) is nonsingular. Therefore, by elementary operations, one can show that (18) holds if and only if (27) holds. $\qquad \square$

Example 1: Consider the circuit of Fig. 1(a), where Λ_C is described by $q = g_c(v_c)$ as in Fig. 1(b). Then $\mathcal{T} = \{C\}$ is a proper tree, $B_{RC} = 1$ and

$$\mathscr{F}_R^1(v_R, i_R) = \left[-D_{i_R}f_R \; \vdots \; D_{i_R}f_R\right]_{(v_R,i_R)}$$

which has rank 1 because of (10). Therefore, $\Lambda \pitchfork K$ for any Λ_R as long as it is a C^2 submanifold. Suppose, now, that $v_c = g_c(q)$ as in Fig. 1(c). Since g_c is not injective, Λ_C is not locally voltage controlled and *Corollary 1* does not apply. In order to apply *Corollary 2*, we compute

$$\mathscr{F}^2(x) = \left[D_{i_R}f_R \; \vdots \; (D_{i_R}f_R)(Dg_c)\right]_x$$

which may or may not have rank 1 depending on f_R and g_c. If $i_R = f_R(v_R)$, however, the above matrix always has rank 1 and $\Lambda \pitchfork K$.

Next suppose that Λ_R admits a *generalized port coordinate* [3], i.e., Λ_R is represented by

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} v_R \\ i_R \end{bmatrix}, \qquad \xi = F(\eta) \qquad (29)$$

where α, β, γ, and δ are $n_R \times n_R$ matrices and $F: \mathbf{R}^{n_R} \to \mathbf{R}^{n_R}$ is a C^2 function. Recall the partition $v_R = (v_{R_c} \; \vdots \; v_{R_q})$, $i_R = (i_{R_c} \; \vdots \; i_{R_q})$ and partition α, β, γ, and δ accordingly:

$$\alpha = \left[\alpha_1 \; \vdots \; \alpha_2\right], \quad \beta = \left[\beta_1 \; \vdots \; \beta_2\right]$$

$$\gamma = \left[\gamma_1 \; \vdots \; \gamma_2\right], \quad \delta = \left[\delta_1 \; \vdots \; \delta_2\right]. \qquad (30)$$

Also recall that Λ_R is said to be *globally voltage controlled* [3] if $\xi = i_R$, $\eta = v_R$ and *globally current controlled* if $\xi = v_R$, $\eta = i_R$.

Corollary 3: Let Λ_R admit a generalized port coordinate representation and let Λ_C (resp. Λ_L) be locally voltage
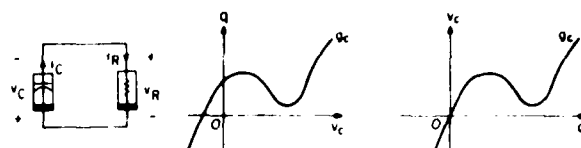

Fig. 1. A nonlinear circuit with $\Lambda \pitchfork K$. (a) The circuit diagram. (b) Capacitor constitutive relation is locally voltage controlled. (c) Capacitor constitutive relation is locally charge controlled.

(resp. current) controlled. Then

$$\mathscr{F}_R^1(v_R, i_R) = \left[(\alpha_2 - (DF)\gamma_2) - (\alpha_1 - (DF)\gamma_1)B_{RR} \; \vdots \right.$$

$$(\alpha_1 - (DF)\gamma_1)B_{RC} \; \vdots \; (\beta_1 - (DF)\delta_1)$$

$$\left. + (\beta_2 - (DF)\delta_2)B_{RR}^T \; \vdots \; (\beta_2 - (DF)\delta_2)B_{IR}^I\right]_{(v_R,i_R)}$$

$$(31)$$

In particular, if Λ_R is globally voltage controlled, then

$$\mathscr{F}_R^1(v_R, i_R)$$

$$= \left[(DF)\begin{bmatrix} B_{RR} \\ 1 \end{bmatrix} \; \vdots \; (DF)\begin{bmatrix} B_{RC} \\ \cdot \end{bmatrix} \; \vdots \; \begin{bmatrix} 1 \\ B_{RR}^I \end{bmatrix} \; \vdots \; \begin{bmatrix} \cdot \\ B_{IR}^I \end{bmatrix}\right]_{v_R}$$

$$(32)$$

and if Λ_R is globally current controlled, then

$$\mathscr{F}_R^1(v_R, i_R) = \left[\begin{bmatrix} -B_{RR} \\ 1 \end{bmatrix} \; \vdots \; \begin{bmatrix} B_{RC} \\ \cdot \end{bmatrix} \; \vdots \; (DF)\begin{bmatrix} 1 \\ B_{RR}^I \end{bmatrix} \; \vdots \right.$$

$$\left. (DF)\begin{bmatrix} \cdot \\ B_{IR}^T \end{bmatrix}\right]_{i_R}. \qquad (33)$$

We can also give similar formulas for $\mathscr{F}^2$ when Λ_C (resp. Λ_L) is locally charge (resp. flux) controlled. We omit them, however.

Now suppose that Λ_R is *globally parametrizable* [3], i.e., Λ_R is globally diffeomorphic to $\mathbf{R}^{n_R}$ and write

$$(v_R(\rho_R), i_R(\rho_R)) = \psi_R^{-1}(\rho_R), \qquad \rho_R \in \mathbf{R}^{n_R} \qquad (34)$$

where $\psi_R: \Lambda_R \to \mathbf{R}^{n_R}$ is a global coordinate system.

Definition 2: Λ_C (resp. Λ_L) is said to be *globally parametrizable* if $(v_c, q) \in \Lambda_C$ (resp. $(i_l, \phi) \in \Lambda_L$) is represented by

$$(v_c(\rho_c), q(\rho_c)) = \psi_c^{-1}(\rho_c), \qquad \rho_c \in \mathbf{R}^{n_c} \qquad (35)$$

$$(\text{resp. } (i_l(\rho_l), \phi(\rho_l)) = \psi_l^{-1}(\rho_l), \rho_l \in \mathbf{R}^{n_l})$$

where $\psi_c: \Lambda_C \to \mathbf{R}^{n_c}$ (resp. $\psi_l: \Lambda_L \to \mathbf{R}^{n_l}$) is a global coordinate system.

If Λ_R, Λ_C and Λ_L are globally parametrized, then Λ of (6) is parametrized by $\rho = (\rho_R, \rho_c, \rho_l)$ and (i_c, v_l). We write this as $x = x(\rho, i_c, v_l) \in \Lambda$.

Proposition 2: Let Λ_R, Λ_C, and Λ_I be globally parametrized. Let $\mathcal{T}$ be a proper tree and let $\mathcal{C}$ be its associated cotree. Then $\Lambda \pitchfork K$ if and only if for each $\rho \in \mathbb{R}^b$, with $x(\rho, i_C, v_L) \in \Sigma$:

$$\text{rank } \mathcal{F}^*(\rho) = n_R \qquad (36)$$

where

$$\mathcal{F}^*(\rho) = \begin{bmatrix} Dv_{R_E} + B_{RR}(Dv_{R_\pi}) & B_{RC}(Dv_C) & \cdot \\ Di_{R_\pi} - B_{RR}^T(Di_{R_E}) & \cdot & \cdot & B_{IR}^T(Di_L) \end{bmatrix}_\rho$$

$$\cdot \qquad (37)$$

Proof: First, observe that for any $x \in \Lambda$

$$T_x \Lambda = \text{Im} \begin{bmatrix} Dv \\ Di \\ Dq \\ D\phi \end{bmatrix}_{(\rho, i_C, v_L)} \qquad (38)$$

Recall that the Kirchhoff space K is parametrized by (v, i, q, ϕ):

$$v = Q^T v_C, \quad i = B^T i_C, \quad q \in \mathbb{R}^{n_C}, \quad \phi \in \mathbb{R}^{n_I}.$$

This implies that

$$T_x K = \text{Im} \begin{bmatrix} Q^T & \cdot & \cdot & \cdot \\ \cdot & B^T & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}. \qquad (39)$$

By definition [3], $\Lambda \pitchfork K$ if and only if

$$\text{Im} \begin{bmatrix} Dv \\ Di \\ Dq \\ D\phi \end{bmatrix}_{(\rho, i_C, v_L)} + \text{Im} \begin{bmatrix} Q^T & \cdot & \cdot & \cdot \\ \cdot & B^T & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} = \mathbb{R}^{2b+n_C+n_I} \qquad (40)$$

which is equivalent to

$$\text{rank} \left[\begin{bmatrix} Dv \\ Di \\ Dq \\ D\phi \end{bmatrix} \vdots \begin{bmatrix} Q^T & \cdot & \cdot & \cdot \\ \cdot & B^T & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \right]_{(\rho, i_C, v_L)} = 2b + n_C + n_I. \qquad (41)$$

More explicitly, this matrix has the following form:

By elementary operations, one can show that (41) holds if and only if (36) holds.

Remark: *Proposition 2* holds even when Λ_R, Λ_C, and Λ_I are locally parametrized at each point, and it includes [5] as a special case.

Suppose now that $\Lambda \pitchfork K$. Then it would be helpful if one can perturb $\mathcal{N}$ in an appropriate way such that the resulting network satisfies transversality. In the following we give two perturbation results. The first method involves element perturbation and consists of perturbing the existing resistor constitutive relations Λ_R. The second method involves network perturbation and consists of augmenting $\mathcal{N}$ by adding arbitrarily small linear inductors and arbitrarily large linear capacitors by pliers-type entry, and by adding arbitrarily large linear inductors and arbitrarily small linear capacitors by soldering-iron entry. Therefore, in the limit we recover the original network. Notice that this procedure consists of adding parasitic capacitors and inductors at appropriate locations.

In order to give a transversalization result via element perturbation, let us first define a C^2 perturbation of Λ_R. Let M be a C^2 submanifold of $\mathbb{R}^n$ and let $C^2(M, \mathbb{R}^n)$ be the set of all C^2 maps from M into $\mathbb{R}^n$. Let $F \in C^2(M, \mathbb{R}^n)$ and consider

$$\mathcal{N}^2(F; \epsilon(\cdot))$$

$$= \left\{ G: M \to \mathbb{R}^n \left| \begin{array}{l} G \in C^2(M, \mathbb{R}^n) \\ \|F(x) - G(x)\| + \|(dF)_x - (dG)_x\| \\ + \|(d^2 F)_x - (d^2 G)_x\| < \epsilon(x), \\ \text{for all } x \in M \end{array} \right. \right\}$$

where $\epsilon(x)$ is an arbitrary continuous function from M into the set of positive numbers and $d^2 F$ and $d^2 G$ are the second derivatives. These sets generate the strong C^2 topology for $C^2(M, \mathbb{R}^n)$ [6]. The set $\text{Emb}^2(M, \mathbb{R}^n)$ of all C^2 embeddings of M into $\mathbb{R}^n$ is open with respect to this topology [6]. Let $\mathcal{N}^2(\iota_M)$ be a neighborhood of the inclusion map such that all elements of $\mathcal{N}^2(\iota_M)$ are embeddings. Then a C^2 perturbation $\hat{M}$ of M is defined by $\hat{M} = G(M)$, where $G \in \mathcal{N}^2(\iota_M)$. The following is our first transversalization result via element perturbation. Although the proof is similar to that of *Theorem 3* of [3], there is a technical

	ρ_R	ρ_C	ρ_I	i_C	v_L	v_{R_π}	v_C	i_{R_E}	i_I	q	ϕ
Dv_{R_E}	·	·	·	·	·	$-B_{RR}$	B_{RC}	·	·	·	·
	·	·	·	1	·	B_{IR}	$-B_{IC}$	·	·	·	·
Dv_{R_π}	·	·	·	·	·	1	·	·	·	·	·
	·	Dv_C	·	·	·	·	1	·	·	·	·
Di_{R_E}	·	·	·	·	·	·	·	1	·	·	·
	·	·	Di_I	·	·	·	·	·	1	·	·
Di_{R_π}	·	·	·	·	·	·	·	B_{RR}^I	B_{IR}^I	·	·
	·	·	·	1	·	·	·	B_{RC}^I	B_{IC}^I	·	·
	·	Dq	·	·	·	·	·	·	·	1	·
	·	·	$D\phi$	·	·	·	·	·	·	·	1

$$(42)$$

difference because of the C^2 perturbations instead of C^1 perturbations. Proof is given in the Appendix.

In the sequel, ̂ denotes variables, functions, sets, etc., associated with a perturbed network.

Proposition 3: Given a nonlinear network $\mathfrak{N}$ let $\Lambda \cap K$ $\neq \varnothing$ and $\Lambda \pitchfork K$. Suppose that Λ_C (resp. Λ_L) is locally voltage (resp. current) controlled. Then there is a perturbation $\hat{\Lambda}_R$ of Λ_R arbitrarily close to Λ_R in the strong C^2 topology such that $\hat{\Lambda} \cap \hat{K} \neq \varnothing$ and $\hat{\Lambda} \pitchfork \hat{K}$, where $\hat{\Lambda}$ is defined by (6) in which Λ_R is replaced by $\hat{\Lambda}_R$.

Remark: Recall *Corollary 1* where Λ_C (resp. Λ_L) is locally voltage (resp. current) controlled and observe that $\mathfrak{T}_R^1$ depends only on (v_R, i_R). This is the reason why one can transversalize Λ and K by perturbing Λ_R only.

The next result gives a transversalization procedure via network perturbation.

Proposition 4: Given a nonlinear network $\mathfrak{N}$ let $\Lambda \cap K$ $\neq \varnothing$ and $\Lambda \pitchfork K$. Let $\mathfrak{T}$ be a proper tree for $\mathfrak{N}$ and let $\mathfrak{L}$ be its associated cotree. Partition $\mathfrak{T}$ and $\mathfrak{L}$ as $\mathfrak{T} = R_\mathfrak{T} \cup C$ and $\mathfrak{L} = R_\mathfrak{L} \cup L$, respectively, where R, C, and L denote resistors, capacitors and inductors, respectively. Insert an arbitrarily small linear capacitor in parallel with each branch of $R_\mathfrak{T}$ and insert an arbitrarily small linear inductor in series with each branch of $R_\mathfrak{L}$. Then the perturbed network $\mathfrak{N}$ satisfies the following properties: (i) $\hat{\Lambda} \cap \hat{K} \neq \varnothing$, (ii) $\hat{\Lambda} \pitchfork \hat{K}$.

Proof: Let C_1 denote the branches representing the capacitors added in parallel with $R_\mathfrak{T}$ and let L_1 denote the branches representing the inductors added in series with $R_\mathfrak{L}$. Then $\hat{\mathfrak{T}} = C \cup C_1 \cup R_\mathfrak{T}$ is a proper tree for $\mathfrak{N}$ and $\hat{\mathfrak{L}} = L \cup L_1 \cup R_\mathfrak{L}$ is its associated cotree. Let

$$\hat{v} = \left(v_{R_\mathfrak{T}}, v_L, v_{L_1} : v_{R_\mathfrak{L}}, v_C, v_{C_1} \right) \Bigg|$$

$$\hat{i} = \left(i_{R_\mathfrak{T}}, i_L, i_{L_1} : i_{R_\mathfrak{L}}, i_C, i_{C_1} \right) \Bigg| \qquad (43)$$

$$\hat{q} = (q, q_1), \hat{\phi} = (\phi, \phi_1) \Bigg|$$

we have

$$(\hat{v}_0, \hat{i}_0, \hat{q}_0, \hat{\phi}_0) \in \hat{K} \qquad (47)$$

where C_1 and L_1 are the capacitance matrix and inductance matrix, respectively, of the added elements. Since $(\hat{v}_0, \hat{i}_0, \hat{q}_0, \hat{\phi}_0)$ corresponds to open-circuiting branches of C_1 and short-circuiting branches of L_1 and since such a situation is contained in $\hat{K}$, we have (47). Next, since no resistors are added, we have

$$\hat{\Lambda} = \left\{ (\hat{v}, \hat{i}, \hat{q}, \hat{\phi}) | (v, i, q, \phi) \in \Lambda \right\}. \qquad (48)$$

This implies that

$$(\hat{v}_0, \hat{i}_0, \hat{q}_0, \hat{\phi}_0) \in \hat{\Lambda} \qquad (49)$$

which together with (47) implies (i).

(ii) In order to prove $\hat{\Lambda} \pitchfork \hat{K}$, we compute $\hat{\mathfrak{F}}(\hat{x})$ of (17) for $\mathfrak{N}$. Observe that fundamental loop matrix $\hat{B}$ for $\mathfrak{N}$ associated with the tree $\hat{\mathfrak{T}}$ assumes the following form:

$$
\begin{array}{cccccc}
v_{R_\mathfrak{T}} & v_L & v_{L_1} & v_{R_\mathfrak{L}} & v_C & v_{C_1}
\end{array}
$$
$$
\left[
\begin{array}{cccc:cc}
 & & & \cdot & \cdot & -1 \\
\mathbf{1} & & & \cdot & B_{LC} & B_{LR} \\
 & & & 1 & B_{RC} & B_{RR}
\end{array}
\right] \qquad (50)
$$

where the submatrices in (50) correspond to those of B for $\mathfrak{N}$ (see (23)). The sign of the identity matrices in (50) are chosen merely for convenience and involves no loss of generality. Next, notice that

$$\hat{f}_R = f_R, \quad (\hat{v}_R, \hat{i}_R) \quad (v_R, i_R)$$

$$D_{\hat{v}_{R_\mathfrak{T}}}\hat{f}_R = D_{v_{R_\mathfrak{T}}}f_R, \quad D_{\hat{v}_{R_\mathfrak{L}}}\hat{f}_R \quad D_{v_{R_\mathfrak{L}}}f_R$$

$$D_{\hat{i}_{R_\mathfrak{T}}}\hat{f}_R = D_{i_{R_\mathfrak{T}}}f_R, \quad D_{\hat{i}_{R_\mathfrak{L}}}\hat{f}_R \quad D_{i_{R_\mathfrak{L}}}f_R.$$

Substituting these and (50) into (17) we have

$$
\hat{\mathfrak{F}}(\hat{x}) \quad
\left[
\begin{array}{cccc:c:ccc:c}
D_{v_{R_\mathfrak{L}}}f_R & \cdot & D_{v_{R_\mathfrak{T}}}f_R & D_{i_{R_\mathfrak{T}}}f_R & \cdot & D_{i_{R_\mathfrak{L}}}f_R & \cdot & \cdot & \cdot \\
\cdot & D_{v_C}f_C & \cdot & \cdot & \cdot & \cdot & D_q f_C & \cdot & \cdot \\
\cdot & \cdot & -C_1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
D_{i_L}f_L & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_\phi f_L & \cdot \\
\cdot & \cdot & \cdot & L_1 & \cdot & \cdot & \cdot & \cdot & 1
\end{array}
\right]_{\hat{x}} \qquad (51)
$$

be the variables of $\mathfrak{N}$. Let

$$(v_0, i_0, q_0, \phi_0) \in \Lambda \cap K \neq \varnothing \qquad (44)$$

$$\begin{aligned} v_0 & \quad (v_{R_\mathfrak{T},0}, v_{L,0} : v_{R_\mathfrak{L},0}, v_{C,0}) \Bigg| \\ i_0 & \quad (i_{R_\mathfrak{T},0}, i_{L,0} : i_{R_\mathfrak{L},0}, i_{C,0}) \Bigg| \end{aligned} \qquad (45)$$

We first claim that with

$$\hat{v}_0 \quad (v_{R_\mathfrak{T},0}, v_{L,0}, 0 : v_{R_\mathfrak{L},0}, v_{C,0}, v_{R_\mathfrak{T},0}) \Bigg|$$

$$\hat{i}_0 \quad (i_{R_\mathfrak{T},0}, i_{L,0}, i_{R_\mathfrak{L},0} : i_{R_\mathfrak{L},0}, i_{C,0}, 0) \Bigg| \qquad (46)$$

$$\hat{q}_0 \quad (q_0, C_1 v_{R_\mathfrak{T},0}), \hat{\phi}_0 \quad (\phi_0, L_1 i_{R_\mathfrak{L},0}) \Bigg|$$

It follows from (10) and (12) that

$$\text{rank}[D_{v_{R_\mathfrak{L}}}f_R : D_{v_{R_\mathfrak{T}}}f_R : D_{i_{R_\mathfrak{T}}}f_R : D_{i_{R_\mathfrak{L}}}f_R]_{(v_R, i_R)} \quad n_R$$

$$\text{rank}[D_{v_C}f_C : D_q f_C]_{(v_C, q)} = n_C$$

$$\text{rank}[D_{i_L}f_L : D_\phi f_L]_{(i_L, \phi)} \quad n_L$$

for all $(v_R, i_R) \in \Lambda_R$, $(v_C, q) \in \Lambda_C$ and $(i_L, \phi) \in \Lambda_L$, respectively. It follows from (51) and (48) that

$$\text{rank}\,\hat{\mathfrak{F}}(\hat{x}) \quad n_R + n_C + n_L + n_{C_1} + n_{L_1}$$

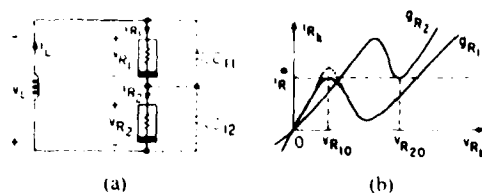for all $\hat{x} \in \hat{\Sigma}$, where n_{C_1} and n_{L_1} are the number of capaci-

Fig. 2. A nonlinear circuit with $\Lambda \bar{\pitchfork} K$. (a) The circuit diagram. (b) Resistor constitutive relations.
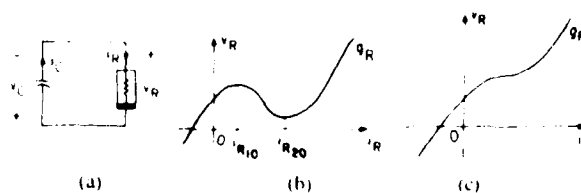

Fig. 3. A nonlinear circuit whose local solvability depends on resistor constitutive relations. (a) The circuit diagram. (b) Resistor constitutive relation where the circuit is not locally solvable. (c) Resistor constitutive relation where the circuit is locally solvable.

tors and inductors added, respectively. It follows from *Proposition 1* that $\Lambda \bar{\pitchfork} K$. □

Example 2: Consider the circuit of Fig. 2(a) where the resistor constitutive relations are given in Fig. 2(b) with $i_{R_k} = f_{R_k}(v_{R_k})$, $k = 1, 2$. Then $\mathcal{T} = \{R_1, R_2\}$ is our proper tree and $B_{RR} = B_{RI} = B_{II} = \varnothing$,[1] $B_{LR} = [1 \quad 1]$,

$$D_{v_{R_n}} f_R = \begin{bmatrix} -Df_{R_1} & \cdot \\ \cdot & -Df_{R_2} \end{bmatrix}, \qquad D_{v_{R_e}} f_R = \varnothing$$

$$D_{i_{R_n}} f_R = \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix}, \qquad D_{i_{R_e}} f_R = \varnothing.$$

Therefore (17) is given by

$$\mathcal{F}(v, i, \phi) = \begin{bmatrix} -Df_{R_1} & \cdot & 1 & \cdot \\ \cdot & -Df_{R_2} & 1 & \cdot \\ \cdot & \cdot & -L & 1 \end{bmatrix}_{(v_{R_1}, v_{R_2})}.$$

Now for the value i_R^* in Fig. 2(b), we have $i_R^* = f_R(v_{R_{10}}) = f_R(v_{R_{20}})$ and $(Df_{R_1})_{v_{R_{10}}} = (Df_{R_2})_{v_{R_{20}}} = 0$. Let $(v_{R_0}, i_{R_0}) = (v_{R_{10}}, v_{R_{20}}, i_R^*, i_R^*)$, $v_{L_0} = -v_{R_{10}} - v_{R_{20}}$, $i_{L_0} = i_R^*$ and $\phi_0 = Li_{L_0}$. Then $(v_{R_0}, v_L, i_{R_0}, i_{L_0}, \phi_0) = (v_0, i_0, \phi_0) \in \Sigma$ and rank $\mathcal{F}(v_0, i_0, \phi_0) = 2 < 3$ and hence $\Lambda \bar{\pitchfork} K$. Insert, now, C_{11} and C_{12} as in Fig. 2(a). Then *Proposition 4* tells us that $\Lambda \bar{\pitchfork} K$.

The transversalization procedure is simplified if Λ_R is locally voltage controlled [3], i.e., (9) holds and

$$\text{rank}(D_{i_R} f_R)_{(v_R, i_R)} = n_R, \qquad \text{for all } (v_R, i_R) \in \Lambda_R \cap U_R$$
$$(52)$$

or locally current controlled, i.e., (9) holds and

$$\text{rank}(D_{v_R} f_R)_{(v_R, i_R)} = n_R, \qquad \text{for all } (v_R, i_R) \in \Lambda_R \cap U_R.$$
$$(53)$$

Proposition 5: Consider the situation of *Proposition 4* and assume that Λ_R is locally voltage controlled. Insert a small linear capacitor in parallel with each branch of R. Then the perturbed network $\hat{\mathcal{N}}$ satisfies the following properties: (i) $\hat{\Lambda} \cap \hat{K} \neq \varnothing$; and (ii) $\hat{\Lambda} \bar{\pitchfork} \hat{K}$.

Proof: (i) can be proved in a manner similar to that of *Proposition 4*. It is clear that $\hat{\mathcal{T}} = C \cup C_t$ is a proper tree for $\hat{\mathcal{N}}$ and $\hat{\mathcal{C}} = R \cup R_t \cup L$ is its associated cotree, where C_t represents the branches of the capacitors added. To compute $\hat{\mathcal{F}}(x)$, observe that the fundamental loop

matrix for $\hat{\mathcal{N}}$ is given by

$$\begin{array}{ccccc} v_{R_c} & v_{R_n} & v_l & v_c & v_c \\ \begin{bmatrix} & & & \vdots & & \\ & & & \vdots & \cdot & 1 \\ & 1 & & \vdots & B_{Rc} & B_{RR} \\ & & & \vdots & B_{Ic} & B_{LR} \end{bmatrix} \end{array}$$

where the submatrices are those of $\mathcal{N}$. Therefore,

$$D_{\hat{v}_{R_n}} f_R = \varnothing, \quad D_{\hat{v}_{R_e}} f_R = D_{v_R} f_R$$

$$D_{\hat{i}_{R_n}} f_R = \varnothing, \quad D_{\hat{i}_{R_e}} f_R = D_{i_R} f_R, \quad (\hat{v}_R, \hat{i}_R) = (v_R, i_R).$$

Substituting these into (17) we have

$$\hat{\mathcal{F}}(\hat{x}) = \begin{bmatrix} (D_{v_R} f_R)\hat{B}_{Rc} & \vdots & D_{i_R} f_R & \vdots & \cdot & \vdots & \cdot \\ D_{v_c} f_c & \vdots & & \vdots & & D_q f_c & \cdot \\ -C_t & \vdots & & \vdots & & & 1 \\ & \vdots & D_{i_l} f_l & \vdots & \cdot & \cdot & D_\phi f_l \end{bmatrix},$$

By using (12), (52) and elementary operations, one can show that rank $\hat{\mathcal{F}}(\hat{x}) = n_R + n_c + n_{c_t} + n_l$ for all $\hat{x} \in \hat{\Sigma}$, where n_{c_t} is the number of capacitors added. Therefore, *Proposition 1* implies that $\hat{\Lambda} \bar{\pitchfork} \hat{K}$.

A dual argument shows the following:

Proposition 6: Under the same setting as that of *Proposition 4*, assume that Λ_R is locally current controlled. Insert a small linear inductor in series with each branch of R. Then the perturbed network $\hat{\mathcal{N}}$ has the following properties: (i) $\hat{\Lambda} \cap \hat{K} \neq \varnothing$, (ii) $\hat{\Lambda} \bar{\pitchfork} \hat{K}$.

IV. LOCAL SOLVABILITY

Recall that transversality of Λ and K is a *static* condition in the sense that it has nothing to do with the dynamics of $\mathcal{N}$. In order to motivate the discussion of this section we first consider the following example.

Example 3: Consider the circuit of Fig. 3(a) where Λ_R is given by Fig. 3(b) with $v_R = g_R(i_R)$. If we choose $\mathcal{T} = \{C\}$ to be our tree, then $B_{Rc} = 1$, $D_{v_{R_c}} f_R = 1$, $D_{i_{R_c}} f_R = Dg_R$ and rank $\mathcal{F}_R^{-1}(v_R, i_R) = \text{rank}[1 \quad Dg_R]_{i_R} = 1$. It follows from *Corollary 1* that $\Lambda \bar{\pitchfork} K$ and Σ is a perfectly well-defined one-dimensional submanifold. The dynamics, however, has points where it is not well defined. To show this observe that i_R serves as a global coordinate for Σ, i.e., $(v_R, v_c, i_R, i_c, q) = (g_R(i_R), g_R(i_R), i_R, i_R, Cg_R(i_R))$, where C is the capacitance. (Notice that v_c cannot serve as a coordinate.) In terms of this coordinate, the dynamics is

[1] We denote a 0×0 matrix by $\cdot$.

given by

$$C(Dg_R)_{i_R}\frac{di_R}{dt} = -i_R. \tag{54}$$

Since $(Dg_R)_{i_{R_{10}}} = (Dg_R)_{i_{R_{20}}} = 0$, differential equation (54) is undefined at $i_R = i_{R_{10}}$ and $i_R = i_{R_{20}}$. Then one might like to choose another coordinate and check if the dynamics is well defined everywhere in terms of it. If it fails, then one may try to choose another coordinate and repeat the same procedure. The problem, however, is that there are, in general, infinitely many coordinates. Therefore, well-definedness of network dynamics should be defined in a coordinate-free manner and there should be a method of checking that porperty in a coordinate independent manner. That is exactly the problem of *local solvability* discussed in this section.

We will first show how the dynamics of a network is described in a coordinate-free manner. Let $\pi': \mathbb{R}^{2b+n_c+n_l} \to \mathbb{R}^{n_c+n_l}$ be the projection map defined by

$$\pi'(v, i, q, \phi) = (q, \phi) \tag{55}$$

and let

$$\pi \doteq \pi' \circ \iota \tag{56}$$

where ι is defined by (21). This map is the same as the restriction $\pi'|\Sigma$ of π' to Σ. Consider the following symmetric two-tensor G on $\mathbb{R}^{n_c+n_l}$:

$$G \doteq \sum_{k=1}^{n_c} dq_k \otimes dq_k - \sum_{k=1}^{n_l} d\phi_k \otimes d\phi_k \tag{57}$$

and the following one-form on $\mathbb{R}^{2b+n_c+n_l}$:

$$\eta \doteq \sum_{k=1}^{n_c} i_{C_k} dq_k - \sum_{k=1}^{n_l} v_{L_k} d\phi_k. \tag{58}$$

Remark: A simple explanation of one-forms is given in [3]. A symmetric two-tensor G on $\mathbb{R}^2$ is a collection of functions: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given at each point $(x_1, x_2) \in \mathbb{R}^2$ by

$$G_{(x_1, x_2)} = \sum_{m,n=1}^{2} f_{mn}(x_1, x_2) \, dx_m \otimes dx_n$$

where f_{mn} are real-valued functions, $f_{mn} = f_{nm}$ and

$$dx_1 \otimes dx_1 = \begin{bmatrix} 1 & \cdot \\ \cdot & \cdot \end{bmatrix}, \quad dx_1 \otimes dx_2 = \begin{bmatrix} \cdot & 1 \\ \cdot & \cdot \end{bmatrix}$$

$$dx_2 \otimes dx_1 = \begin{bmatrix} \cdot & \cdot \\ 1 & \cdot \end{bmatrix}, \quad dx_2 \otimes dx_2 = \begin{bmatrix} \cdot & \cdot \\ \cdot & 1 \end{bmatrix}.$$

Consequently,

$$G_{(x_1, x_2)}([1 \ 0]^T, [1 \ 0]^T) = f_{11}(x_1, x_2)$$

where we look at a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as a map: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}((x_1, y_1)^T, (x_2, y_2)^T) = (x_1, y_1)\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

Therefore, G can be thought of as the matrix-valued function $\{f_{mn}\}$. One needs to be careful, however, in defining

two-tensors on a general manifold since manifolds generally are nonlinear.

Let π^* be the induced map [6] of π. Then π^* pulls G back to Σ by the following formula:

$$(\pi^*G)_x(\xi_1, \xi_2) \doteq G_{\pi(x)}((d\pi)_x \xi_1, (d\pi)_x \xi_2) \tag{59}$$

where $\xi_1, \xi_2 \in T_x\Sigma$. Similarly, let ι^* be the induced map of ι. Then

$$\omega \doteq \iota^*\eta \tag{60}$$

is a one-form on Σ defined by

$$\omega_x(\xi) \doteq \eta_{\iota(x)}((d\iota)_x\xi). \tag{61}$$

A coordinate-free description of the dynamics is given by the following:

Proposition 7: Let Σ be an (n_c+n_l)-dimensional C^2 submanifold. Then the vector field X^2 describing the dynamics of network satisfies the following:

$$(\pi^*G)_x(X_x, \xi) = \omega_x(\xi), \quad \text{for all } \xi \in T_x\Sigma. \tag{62}$$

Remark: An easy way of understanding (62) is the following: Let A be a symmetric $n \times n$ nonsingular matrix. Then (Ax, y) is a symmetric bilinear function on $\mathbb{R}^n \cdot \mathbb{R}^n$, where $(\cdot, \cdot)$ is the inner product, i.e., $(Ax, y) = (x, A^t y)$, and $(Ax, \cdot)$ and $(A\cdot, y)$ are linear. For a vector $\omega \in \mathbb{R}^n$, the formula

$$(Ax, y) = (\omega, y), \quad \text{for all } y \in \mathbb{R}^n$$

uniquely defines the vector $x = A^{-1}\omega$. If the network is reciprocal and if P denotes the mixed potential, then ω of (62) is given by $\omega = dP$, differential of P. If (q, ϕ) serves as a global coordinate system for Σ, then

$$X_x = \left(\frac{dq}{dt}, \frac{d\phi}{dt} \right)$$

$$\omega_x = \sum_{k=1}^{n_c} F_{C_k}(q, \phi) \, dq_k - \sum_{k=1}^{n_l} F_{L_k}(q, \phi) \, d\phi_k$$

where F_{C_k} and F_{L_k} are determined by

$$(i_C, -v_L)$$
$$= (F_{C_1}(q, \phi), \cdots, F_{C_{n_c}}(q, \phi), F_{L_1}(q, \phi), \cdots, F_{L_{n_l}}(q, \phi)).$$

Therefore, (62) is reduced to

$$\begin{cases} \dfrac{dq_k}{dt} = F_{C_k}(q, \phi), & k = 1, \cdots, n_c \\ -\dfrac{d\phi_k}{dt} = F_{L_k}(q, \phi), & k = 1, \cdots, n_l. \end{cases}$$

Proof of Proposition 7: Let

$$(d\pi)_x \xi = \zeta = (\zeta_{q_1}, \cdots, \zeta_{q_{n_c}}, \zeta_{\phi_1}, \cdots, \zeta_{\phi_{n_l}}). \tag{63}$$

A vector field X on manifold Σ is a function such that the value X_x at $x \in \Sigma$ belongs to $T_x\Sigma$, the tangent space of Σ at x. The vector field X naturally generates a flow $x(t)$ such that $dx(t)/dt = X_{x(t)}$.

Then, by definition, the left-hand side of (62) reads

$$(\pi^*G)_x(X_x, \xi) = G_{\pi(x)}((d\pi)_x X_x, (d\pi)_x\xi)$$

$$= G_{\pi(x)}\left(\left(\frac{dq}{dt}, \frac{d\phi}{dt}\right)_{\pi(x)}, \xi\right)$$

$$= \left(\sum_{k=1}^{n_c} dq_k \otimes dq_k - \sum_{k=1}^{n_l} d\phi_k \otimes d\phi_k\right)$$

$$\cdot \left(\left(\frac{dq}{dt}, \frac{d\phi}{dt}\right)_{\pi(x)}, \xi\right)$$

$$= \sum_{k=1}^{n_c} \frac{dq_k}{dt}\zeta_{q_k} - \sum_{k=1}^{n_l} \frac{d\phi_k}{dt}\zeta_{\phi_k}. \tag{64}$$

The right-hand side of (62) is given by

$$(\iota^*\eta)_x(\xi) = \left(\sum_{k=1}^{n_c} \iota_{c_k} dq_k - \sum_{k=1}^{n_l} v_{l_k} d\phi_k\right)((d\iota)_x\xi)$$

$$= \left(\sum_{k=1}^{n_c} \iota_{c_k} dq_k - \sum_{k=1}^{n_l} v_{l_k} d\phi_k\right)((d\pi)_x\xi)$$

$$= \sum_{k=1}^{n_c} \iota_{c_k}\zeta_{q_k} - \sum_{k=1}^{n_l} v_{l_k}\zeta_{\phi_k}. \tag{65}$$

Equations (64) and (65) together with (3) and (5) imply (62). □

Proposition 7 and Example 3 naturally lead to the following definition:

Definition 3: Given a nonlinear network $\mathfrak{N}$ assume that Σ is an $(n_c + n_l)$-dimensional C^2 submanifold. Then $\mathfrak{N}$ is said to be *locally solvable* if (62) uniquely defines a C^1 vector $X_x \in T_x\Sigma$ at each point $x \in \Sigma$.

Remark: Local solvability defined above is a coordinate-free version of the one in Chua and Wang [7]. If Λ_C (resp. Λ_L) is globally voltage (resp. current) controlled, this definition coincides with regularity of Smale [8].

We are now ready to state the main result of this paper.

Theorem 1: Given a nonlinear network $\mathfrak{N}$ suppose that Σ is an $(n_c + n_l)$-dimensional C^2 submanifold. Then $\mathfrak{N}$ is locally solvable if and only if, at each point $x \in \Sigma$, (q, ϕ) serves as a local coordinate system for Σ.

For proof we need a lemma. Recall $(\pi^*G)_x(\cdot, \cdot)$ defined by (59) is a bilinear function on $T_x\Sigma \times T_x\Sigma$.

Lemma 1: Suppose that Σ is an $(n_c + n_l)$-dimensional C^2 submanifold. Then $\mathfrak{N}$ is locally solvable if and only if at each point $x \in \Sigma$, $(\pi^*G)_x$ is nonsingular, i.e.,

$$(\pi^*G)_x(\xi_1, \xi_2) = 0, \quad \text{for all } \xi_1 \in T_x\Sigma \text{ implies } \xi_2 = 0.$$
$$\tag{66}$$

Proof: We look at $(\pi^*G)_x(\cdot, \cdot)$ in a slightly different manner. Consider the map J_x defined by

$$J_x: \xi_1 \mapsto (\pi^*G)_x(\xi_1, \cdot). \tag{67}$$

To each ξ_1, the map J_x assigns the linear functional[3] $(\pi^*G)_x(\xi_1, \cdot)$ on $T_x\Sigma$. A linear functional on $T_x\Sigma$ belongs to its dual $T_x^*\Sigma$. This means that $J_x = (\pi^*G)_x(\cdot, \cdot)$ maps $T_x\Sigma$ into $T_x^*\Sigma$. It is clear that (66) implies that J_x is an isomorphism and, therefore, it is invertible. It follows from

[3]A linear functional is a real-valued linear function.

(62) that a vector field X_x is uniquely determined by

$$X_x = (J_x)^{-1}\omega_x. \tag{68}$$

In order to show that X is C^1, recall definition (59) of π^*G. Since Σ is C^2, the map $d\pi$ is C^1. Therefore, $(J_x)^{-1}$ is C^1. Similarly ω is C^1. This implies that X determined by (68) is C^1. Conversely, if J_x is not an isomorphism (62) cannot determine a unique vector field.

Remark: In Example 3, in terms of the coordinate i_R, we have $(\pi^*G)_x = C(Dg_R)_{i_R} di_R \otimes di_R$ which becomes singular when $(Dg_R)_{i_{R_0}} = (Dg_R)_{i_{R_0}} = 0$.

Proof of Theorem 1: Recall definition (59) of π^*G. Since $G_{\pi(x)}$ defined by (57) is always nonsingular in the sense of *Lemma 1*, we see that $(\pi^*G)_x$ is nonsingular if and only if the following map is an isomorphism:

$$(d\pi)_x: T_x\Sigma \to T_{\pi(x)}\mathbb{R}^{n_c + n_l} \tag{69}$$

i.e., π is a local diffeomorphism at x. But this precisely means that (q, ϕ) serves as a local coordinate system for Σ at x.

Remarks: 1) Because of its coordinate-free property, *Theorem 1* is of fundamental importance. It says that if all the variables of a network are expressible in terms of capacitor charges and inductor fluxes, at least locally, then the network is locally solvable, i.e., the dynamics is well defined. Conversely, if the network is locally solvable, then (q, ϕ) necessarily determines all the variables in the network. Another important interpretation of *Theorem 1* is that if (62) uniquely defines a C^1 vector field with respect to one coordinate system, then it defines a unique C^1 vector field with respect to capacitor charges and inductor fluxes also. Conversely, if (62) fails to specify a unique C^1 vector field with respect to (q, ϕ), then there is no choice of variables in the network in terms of which (62) specifies a unique C^1 vector field. These observations show that capacitor charges and inductor fluxes are the fundamental quantities in describing the dynamics of a network.

2) Let us explain why Λ must be C^2 in order to define a C^1 vector field by using a simple example. Consider the circuit of Fig. 3(a) where Λ_R is given by Fig. 3(c). Assume that g_R is a global C^1 diffeomorphism. Therefore $i_R = h_R(v_R)$, where $h_R = g_R^{-1}$ and h_R is also a global C^1 diffeomorphism. The sets Λ and Σ are C^1 submanifolds. Capacitor voltage v_C serves as a global coordinate for Σ and the dynamics is given by

$$\frac{dv_C}{dt} = \frac{h_R(-v_C)}{C}. \tag{70}$$

The right-hand side is C^1. Now it is clear that i_R is another global coordinate for Σ and the dynamics is given by

$$\frac{di_R}{dt} = \frac{i_R}{C(Dg_R)_{i_R}}. \tag{71}$$

Since g_R is C^1, the right-hand side is only C^0. This gives rise to a problem because a C^0 vector field cannot guarantee uniqueness of solutions. If we assume, however, that g_R is a C^2 global diffeomorphism, then the right-hand side of (70) and (71) is *at least* C^1. Therefore, C^1-ness does not depend on the choices of coordinates. More generally, let X be a vector field on Σ and let $(U \cap \Sigma, \psi)$ be a local chart at

$x \in \Sigma$. Then a natural coordinate representation is

$$(X(\psi))_{\psi(x)} = (d\psi)_x X_x.$$

If $(V \cap \Sigma, \phi)$ is another chart, then for $x \in U \cap V \cap \Sigma$ one has

$$(X(\phi))_{\phi(x)} = \left(d(\phi \circ \psi^{-1} \circ \psi) \right)_x X_x$$

$$= \left(d(\phi \circ \psi^{-1}) \right)_{\psi(x)} (d\psi)_x X_x$$

$$= \left(d(\phi \circ \psi^{-1}) \right)_{\psi(x)} (X(\psi))_{\psi(x)}.$$

Therefore, if we want $X(\psi)$ to be C^r independent of the choice of coordinates, we must require the change of coordinates $d(\phi \circ \psi^{-1})$ to be C^r. This requires $\phi \circ \psi^{-1}$ to be at least C^{r+1}. But this is exactly the condition required for Σ to be at least C^{r+1}. Therefore, a C^r vector field can possibly be well-defined only on C^s manifolds with $s > r$.

3) Observe that C^1-ness of vector field is required in order to guarantee uniqueness of flows because a C^0 vector field, in general, does not suffice for generating a unique flow. Roska [9] obtained several uniqueness results in terms of the network topology and resistor constitutive relations.

4) If $\dim \Sigma \neq n_C + n_L$, then $(d\pi)_x$ of (69) is always singular and (62) cannot determine a unique vector field. Therefore, transversality of Λ and K is one of the important conditions for local solvability.

The following shows that the results of [5], [8], and [10] are a special case of *Theorem 1*.

Corollary 4: Suppose that Λ_C (resp. Λ_L) is represented by $q = g_C(v_C)$ (resp. $\phi = g_L(i_L)$) and suppose that $(Dg_C)_{v_C}$ (resp. $(Dg_L)_{i_L}$) is nonsingular. Let Σ be an $(n_C + n_L)$-dimensional C^2 submanifold. Then $\mathcal{N}$ is locally solvable if and only if at each point $x \in \Sigma$, (v_C, i_L) serves as a local coordinate system for Σ.

Proof: If $(Dg_C)_{v_C}$ (resp. $(Dg_L)_{i_L}$) is nonsingular, g_C (resp. g_L) is a local diffeomorphism. Therefore, (v_C, i_L) serves as a local coordinate system for Σ if and only if (q, ϕ) serves as a local coordinate system for Σ. $\square$

Remark: In [5], [8], and [10], $(Dg_C)_{v_C}$ and $(Dg_L)_{i_L}$ are symmetric and positive definite. Therefore, they are nonsingular.

The following is an example of a locally solvable circuit whose capacitor is *not* voltage controlled.

Example 4: Consider the circuit of Fig. 1(a) where Λ_C is given by Fig. 1(c) and Λ_R is given by $i_R = g_R(v_R)$. Capacitor charge q is a global coordinate for Σ and the dynamics is described by $\dot{q} = g_R(-g_C(q))$. Clearly, this circuit is locally solvable but the dynamics cannot be described in terms of v_C.

We will next show that if the assumptions of *Corollary 4* are satisfied, then (62) is reduced to a formula in [10]. To this end let

$$\Sigma^\dagger = \Lambda^\dagger \cap K^\dagger \qquad (72)$$

where

$$\Lambda^\dagger = \left\{ (v, i) \in \mathbf{R}^{2b} | (v_R, i_R) \in \Lambda_R \right\} \qquad (73)$$

$$K^\dagger = \left\{ (v, i) \in \mathbf{R}^{2b} | (v, i) \text{ satisfies KVL, KCL} \right\}. \qquad (74)$$

Let Λ_C and Λ_L be characterized as in *Corollary 4* and define

$$G^\dagger = \sum_{m,n=1}^{n_C} C_{mn}(v_C) dv_{C_m} \otimes dv_{C_n} - \sum_{m,n=1}^{n_L} L_{mn}(i_L) di_{L_m} \otimes di_{L_n}$$

where

$$\left[C_{mn}(v_C) \right] = (Dg_C)_{v_C}, \quad \left[L_{mn}(i_L) \right] = (Dg_L)_{i_L}. \qquad (75)$$

Let $\pi^\dagger : \Sigma^\dagger \to \mathbf{R}^{n_C + n_L}$ be the projection map defined by

$$\pi^\dagger(v, i) = (v_C, i_L) \qquad (76)$$

and let $\iota^\dagger : \Sigma^\dagger \to \mathbf{R}^{2b}$ be the inclusion map. Finally, let

$$\eta^\dagger = \sum_{k=1}^{n_R} v_{R_k} di_{R_k} + d \left(\sum_{k=1}^{n_C} v_{C_k} i_{C_k} \right).$$

Then the vector field $X^\dagger$ which describes the dynamics is given by [10]

$$\left(\pi^{\dagger *} G^\dagger \right)_{(v,i)} \left(X^\dagger_{(v,i)}, \xi^\dagger \right) = \omega^\dagger_{(v,i)}(\xi^\dagger), \text{ for all } \xi^\dagger \in T_{(v,i)}\Sigma^\dagger \qquad (77)$$

where $\omega^\dagger = \iota^{\dagger *}\eta^\dagger$. Let $F: \Sigma^\dagger \to \Sigma$ be the (global) diffeomorphism defined by

$$F(v, i) = (v, i, g_C(v_C), g_L(i_L)). \qquad (78)$$

Proposition 8: Suppose that the assumptions of *Corollary 4* are satisfied. If $\mathcal{N}$ is locally solvable, then (62) is reduced to (77), i.e.,

$$X^\dagger_{(v,i)} = (dF^{-1})_x X_{F(v,i)}. \qquad (79)$$

Proof: Let $X_{F(v,i)}$ be the vector field determined by (62). It follows from (78) that

$$X_{F(v,i)} = (dF)_{(v,i)} X^\dagger_{(v,i)} \qquad (80)$$

for some $X^\dagger_{(v,i)} \doteq (X^\dagger_v, X^\dagger_i) \in T_{(v,i)}\Sigma^\dagger$. Let us write

$$X^\dagger_v = \left(X^\dagger_{v_R}, X^\dagger_{v_C}, X^\dagger_{v_L} \right), X^\dagger_i = \left(X^\dagger_{i_R}, X^\dagger_{i_C}, X^\dagger_{i_L} \right).$$

Then

$$(dF)_{(v,i)} X^\dagger_{(v,i)} = \left(X^\dagger_v, X^\dagger_i, (Dg_C)_{v_C} X^\dagger_{v_C}, (Dg_L)_{i_L} X^\dagger_{i_L} \right).$$

We will show that $X^\dagger_{(v,i)}$ is the same as the one determined from (77). To this end note that for $\xi \in T_x\Sigma$, there is a $\xi^\dagger \in T_{(v,i)}\Sigma^\dagger$ such that

$$\xi = (dF)_{(v,i)} \xi^\dagger = \left(\xi^\dagger_v, \xi^\dagger_i, (Dg_C)_{v_C} \xi^\dagger_{v_C}, (Dg_L)_{i_L} \xi^\dagger_{i_L} \right). \qquad (81)$$

We substitute (80) and (81) into (62). Then the left-hand side reads

$$\left(\sum_{k=1}^{n_C} dq_k \otimes dq_k - \sum_{k=1}^{n_L} d\phi_k \otimes d\phi_k \right) \left((d(\pi \circ F))_{(v,i)} X_{(v,i)}, (d(\pi \circ F))_{(v,i)} \xi^\dagger \right)$$

$$= \sum_{m=1}^{n_C} \left(\sum_{n=1}^{n_C} C_{mn}(v_C) X^\dagger_{v_{C_n}} \right) \left(\sum_{n=1}^{n_C} C_{mn}(v_C) \xi^\dagger_{v_{C_n}} \right) - \sum_{m=1}^{n_L} \left(\sum_{n=1}^{n_L} L_{mn}(i_L) X^\dagger_{i_{L_n}} \right) \left(\sum_{n=1}^{n_L} L_{mn}(i_L) \xi^\dagger_{i_{L_n}} \right) \qquad (82)$$

where we used (75). The right-hand side turns out to be

$$\sum_{m=1}^{n_c} r_{c_m} \left(\sum_{n=1}^{n_c} C_{mn}(r_c) \xi_{1_{c_n}}^1 \right) \sum_{m=1}^{n_l} r_{l_m} \left(\sum_{n=1}^{n_l} L_{mn}(i_l) \xi_{i_{l_n}}^1 \right).$$

(83)

Since $\zeta = (d(\pi \circ F))_{(c,i)} \xi$, since $(dF)_{(c,i)}$ is nonsingular and since $\mathcal{K}$ is locally solvable, i.e., $(d\pi)_{F(c,i)}$ is nonsingular, there is a ξ^1 such that

$$\sum_{n=1}^{n_c} C_{mn}(r_c) \xi_{1_{c_n}}^1 = \delta_{mn}, \quad \sum_{n=1}^{n_l} L_{mn}(i_l) \xi_{i_{l_n}}^1 = 0 \qquad (84)$$

where $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ if $m \ne n$.
Substitution of (84) into (82) and (83) gives

$$\sum_{n=1}^{n_c} C_{mn}(r_c) X_{r_{c_n}} = r_{c_m}. \qquad (85)$$

Similarly, there is another vector $\hat{\xi}^{\dagger}$ such that

$$\sum_{n=1}^{n_c} C_{mn}(r_c) \hat{\xi}_{1_{c_n}}^{\dagger} = 0, \quad \sum_{n=1}^{n_l} L_{mn}(i_l) \hat{\xi}_{i_{l_n}}^{\dagger} = \delta_{mn}. \qquad (86)$$

Substituting (86) into (82) and (83) we have

$$\sum_{n=1}^{n_l} L_{mn}(i_l) X_{i_{l_n}} = r_{l_m}. \qquad (87)$$

The vector field $X_{(c,i)}$ determined by (85) and (87) is exactly the same as the one obtained by (77).   □

We will next give a simple formula for checking local solvability of $\mathcal{K}$. Let f_R, f_c, and f_l be as in (9)–(12).

Proposition 9: Let Σ be an $(n_c + n_l)$-dimensional C^2 submanifold. Pick a proper tree $\mathcal{T}$ and let $\mathcal{L}$ be its associated cotree. Then $\mathcal{K}$ is locally solvable if and only if

$$\det \mathcal{K}(x) \ne 0, \qquad \text{for all } x \in \Sigma \qquad (88)$$

where

$$\mathcal{K}(x) = \begin{bmatrix} D_{r_R} f_R + (D_{v_c} f_R)B & : & D_{i_c} f_R + (D_{i_R} f_R)B^t \\ \cdot & : & D_{v_c} f_c & : & \cdot \\ & & : & & : & D_{i_l} f_l \end{bmatrix}_x. \qquad (89)$$

Proof: Let $(\psi, \Sigma \cap U)$ be a local chart for Σ at x. Then $(d\pi)_x$ is an isomorphism if and only if $(D(\pi \circ \psi^{-1}))_{\psi(x)}$ is a nonsingular matrix. Since $\pi \circ \psi^{-1} = \pi' \circ \iota \circ \psi^{-1}$, we have

$$\left(D(\pi \circ \psi^{-1}) \right)_{\psi(x)} = (D\pi')_x (d\iota)_x (D\psi^{-1})_{\psi(x)}. \qquad (90)$$

Since $(d\iota)_x$ is a linear inclusion map, the matrix of (90) is nonsingular if and only if

$$\text{Ker}(D\pi')_x \cap \text{Im}(D\psi^{-1})_{\psi(x)} = \{0\}. \qquad (91)$$

Let $G: U \to \mathbb{R}^{2b}$ be defined by

$$G(x) = \begin{bmatrix} Br \\ Qi \\ f_R(r_R, i_R) \\ f_c(r_c, q) \\ f_l(i_l, \phi) \end{bmatrix} \qquad (92)$$

where B, Q, etc., are as in (18). Since $\Sigma \cap U = G^{-1}(0)$, we have [3]

$$T_x \Sigma = \text{Im}(D\psi^{-1})_{\psi(x)} = \text{Ker}(DG)_x. \qquad (93)$$

It follows from (91) and (93) that the matrix of (90) is nonsingular if and only if

$$\text{Ker}(D\pi')_x \cap \text{Ker}(DG)_x = 0 \qquad (94)$$

which is equivalent to

$$\text{rank} \begin{bmatrix} DG \\ D\pi' \end{bmatrix}_x = 2b + n_c + n_l. \qquad (95)$$

Computing the matrix of (95), one can show that it has rank $2b + n_c + n_l$ if and only if the following matrix has rank $2b$:

$$\begin{array}{cccc} r_c & v_c & i_l & i \\ \begin{bmatrix} 1 & B & : & \cdot & \cdot \\ \cdot & \cdot & : & B^t & 1 \\ D_{v_c} f_R & D_{v_R} f_R & : & D_{i_l} f_R & D_{i_R} f_R \\ \cdot & : D_{v_c} f_c & : & \cdot & \\ & \cdot & : & : D_{i_l} f_l & \cdot \end{bmatrix}_x \end{array} \qquad (96)$$

By elementary operations, one can show that this matrix has rank $2b$ if and only if (88) holds.

This result has an interesting consequence. Let $\pi_c: \Sigma \to \mathbb{R}^{2n_c}$ (resp. $\pi_l: \Sigma \to \mathbb{R}^{2n_l}$) be the projection map defined by $\pi_c(x) = (r_c, q)$, (resp. $\pi_l(x) = (i_l, \phi)$)

Corollary 5: If $\mathcal{K}$ is locally solvable, then

$$\det(D_{v_c} f_c)_{(c,q)} \ne 0, \qquad \text{for all } (r_c, q) \in \pi_c(\Sigma) \qquad (97)$$

and

$$\det(D_{i_l} f_l)_{(i_l, \phi)} \ne 0, \qquad \text{for all } (i_l, \phi) \in \pi_l(\Sigma). \qquad (98)$$

Proof: Since $(D_{v_c} f_c)_{(c,q)}$ and $(D_{i_l} f_l)_{(i_l, \phi)}$ are square matrices, (88) forces (97) and (98) to hold.

Remark: The above result says that if $\mathcal{K}$ is locally solvable, then *capacitors must be locally charge controlled* on $\pi_c(\Sigma)$ and *inductors must be locally flux controlled* on $\pi_l(\Sigma)$. In other words, if capacitors are not locally charge controlled at some point $(r_c, q_0) \in \pi_c(\Sigma)$ or inductors are not locally flux controlled at some point $(i_l, \phi_0) \in \pi_l(\Sigma)$, then there is *no choice* of local coordinate system in terms of which the network is locally solvable. This seems to explain why we do not find capacitors (resp. inductors) of practical importance which are not locally charge (resp. flux) controlled.

Importance of local charge (resp. flux) controlledness is further emphasized by the following:

Corollary 6: Suppose that Λ_c (resp. Λ_l) is described by $q = g_c(r_c)$ (resp. $\phi = g_l(i_l)$) and suppose that (r_c, i_l) serves as a global coordinate system for Σ. If $\det(Dg_c)_{r_c} = 0$ for some $r_c \in \mathbb{R}^{n_c}$ or $\det(Dg_l)_{i_l} = 0$ for some $i_l \in \mathbb{R}^{n_l}$, then $\mathcal{K}$ is not locally solvable.

Proof: If (r_c, i_l) serves as a global coordinate system for Σ, then $(r_c, g_c(r_c)) \in \pi_c(\Sigma)$ for all $r_c \in \mathbb{R}^{n_c}$ and $(i_l, g_l(i_l)) \in \pi_l(\Sigma)$ for all $i_l \in \mathbb{R}^{n_l}$. Therefore, Corollary 5 implies the result.

Observe that the above result says that *even if* (v_c, i_l) *serves as a global coordinate system for* Σ, a network may not be locally solvable. The following example shows the case in point.

Example 5: Consider the circuit of Fig. 1(a) where Λ_c is given by Fig. 1(c). As was shown in Example 1, $\Lambda \bar{\pi} K$ always holds for any Λ_R as long as it is a C^2 submanifold. Suppose that Λ_R is described by $i_R = g_R(v_R)$. Then v_c serves as a global coordinate for Σ. Since there are points where $(Dg_c)_{v_c} = 0$, this circuit is not locally solvable.

If we know that Λ_c (resp. Λ_l) is locally charge (resp. flux) controlled, then the formula in *Proposition 9* is simplified as follows:

Corollary 7: In the same setting as in *Proposition 9* assume that Λ_c (resp. Λ_l) is locally charge (resp. flux) controlled. Then Σ is locally solvable if and only if

$$\det \mathcal{K}_R(v_R, i_R) \neq 0, \quad \text{for all } (v_R, i_R) \in \pi_R(\Sigma) \quad (99)$$

where π_R is defined by (22) and

$$\mathcal{K}_R(v_R, i_R)$$
$$\left[Dv_{R_c} f_R \quad (D_{v_{c}} f_R) B_{RR} \quad D_{i_{c}} f_R \cdot (D_{i_{R}} f_R) B_{RR}^I \right]_{(v_R, i_R)} \quad (100)$$

Proof: If the above hypothesis is satisfied, $(D_{i_c} f_c)_{(v_c, \phi)}$ and $(D_{i_l} f_l)_{(i_l, \phi)}$ are nonsingular. Then one can show that (99) is equivalent to (88).

Remark: In [5], [8], and [10], Λ_c (resp. Λ_l) is represented by $q = g_c(v_c)$ (resp. $\phi = g_l(i_l)$) and $(Dg_c)_{v_c}$ (resp. $(Dg_l)_{i_l}$) is positive definite. Therefore, Λ_c (resp. Λ_l) is locally charge (resp. flux) controlled.

Example 6: Consider the circuit of Example 3. Since $\mathcal{K}_R(v_R, i_R) = -(Dg_R)_{i_R}$, it fails to have rank 1 at $i_R = i_R$ and $i_R = i_{R_2}$ and, therefore, this circuit is not locally solvable.

Example 7: Consider the circuit of Example 2, where Λ_R is given in Fig. 4(a) with $i_{R_k} = g_{R_k}(v_{R_k})$, $k = 1, 2$. Since

$$\mathcal{F}_R^I(v_R, i_R) \left[\begin{array}{ccc} Dg_{R_1} & \cdot & \vdots & 1 \\ \cdot & Dg_{R_2} & \vdots & 1 \end{array} \right]_{v_R}$$

and since Dg_{R_1} and Dg_{R_2} never vanish simultaneously, rank $\mathcal{F}_R^I(v_R, i_R) = 2$. Consequently $\Lambda \bar{\pi} K$ and Σ is a one-dimensional submanifold. Since

$$\mathcal{K}_R(v_R, i_R) \left[\begin{array}{ccc} -Dg_{R_1} & \cdot \\ \cdot & -Dg_{R_2} \end{array} \right]_{v_R}$$

there are points where $\det \mathcal{K}_R(v_R, i_R) = 0$. Therefore, the circuit is not locally solvable. If we use *Corollary 4*, we can see this more clearly. Consider the projection $\mathcal{R}$ of Σ onto the (v_l, i_l)-space given in Fig. 4(b). If we further project $\mathcal{R}$ onto the i_l-axis, we see that i_l cannot be a local coordinate where the curve intersects itself. Therefore, $\mathcal{R}$ is not locally solvable.

Corollary 8: Let Λ_R admit a generalized port coordinate representation and let Λ_c (resp. Λ_l) be locally charge (resp. flux) controlled. Then

$$\mathcal{K}_R(v_R, i_R) \left[(\alpha_2 \quad (DF)\gamma_2) - (\alpha_1 \quad (DF)\gamma_1) B_{RR} \quad \vdots \quad (\beta_1 \quad (DF)\delta_1) + (\beta_2 \quad (DF)\delta_2) B_{RR}^I \right]_{(v_R, i_R)} \quad (101)$$
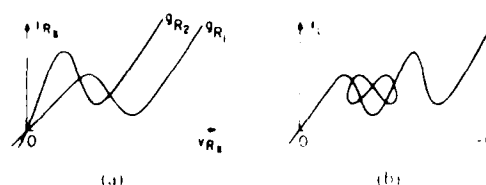

Fig. 4. A nonlinear circuit which is not locally solvable. (a) Resistor constitutive relations. (b) Projection of Σ onto the (v_l, i_l)-space

where α, β, γ, and δ are as in (29). In particular, if Λ_R is globally voltage controlled, then

$$\mathcal{K}_R(v_R, i_R) \left[(DF) \left[\begin{array}{c} B_{RR} \\ 1 \end{array} \right] \quad \vdots \quad \left[\begin{array}{c} 1 \\ B_{RR}^I \end{array} \right] \right]_{i_c} \quad (102)$$

and if Λ_R is globally current controlled, then

$$\mathcal{K}_R(v_R, i_R) \left[\left[\begin{array}{c} B_{RR} \\ 1 \end{array} \right] \quad \vdots \quad (DF) \left[\begin{array}{c} 1 \\ B_{RR}^I \end{array} \right] \right]_{i_c} \quad (103)$$

Recall (34), (35) and the notation used in *Proposition 2*.

Proposition 10: Let Λ_R, Λ_c, and Λ_l be globally parametrized and let Σ be an $(n_c + n_l)$-dimensional C^2 submanifold. Pick a proper tree $\mathcal{T}$ and let c be its associated cotree. Then $\mathcal{R}$ is locally solvable if and only if for each $\rho \in \mathbb{R}^n$ with $x(\rho, i_c, v_l) \in \Sigma$.

$$\det \mathcal{K}^*(\rho) \neq 0 \quad (104)$$

where

$$\mathcal{K}^*(\rho)$$
$$= \left[\begin{array}{ccccc} Dv_{R_c} + B_{RR}(Dv_{R_c}) & \vdots & B_{Rc}(Dv_c) & \vdots & \cdot \\ Di_{R_c} & B_{RR}^I(Di_{R_c}) & \vdots & \cdot & \vdots & B_{lR}^I(Di_l) \\ \cdot & \vdots & \cdot & Dq & \vdots & \cdot \\ \cdot & \vdots & \cdot & \vdots & D\phi \end{array} \right]_\rho \quad (105)$$

Proof: Substitute (34) and (35) into KVL and KCL:

$$\left[\begin{array}{cccc} 1 & \cdot & \vdots & B_{RR} B_{Rc} \\ \cdot & 1 & \vdots & B_{lR} B_{lc} \end{array} \right] \left[\begin{array}{c} v_{R_c}(\rho_R) \\ v_l \\ v_{R_c}(\rho_R) \\ v_c(\rho_c) \end{array} \right] \quad 0 \quad (106)$$

$$\left[\begin{array}{cccc} \cdot & B_{RR}^I & B_{lR}^I & \vdots & 1 & \cdot \\ \cdot & B_{Rc}^I & B_{lc}^I & \vdots & \cdot & 1 \end{array} \right] \left[\begin{array}{c} i_{R_c}(\rho_R) \\ i_l(\rho_l) \\ i_{R_c}(\rho_R) \\ i_c \end{array} \right] \quad 0. \quad (107)$$

Let us write (106) and (107) as

$$H(\rho, i_c, v_l) \quad 0. \quad (108)$$

Then Σ is diffeomorphic to $H^{-1}(0)$. By a similar argument to that of the proof of *Proposition 9*, one sees that (90) is
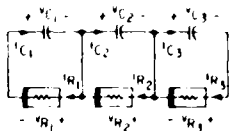
Fig. 5. A nonlinear circuit which is locally solvable, yet Σ is not diffeomorphic to $\mathbf{R}^n$

nonsingular if and only if

$$\text{rank} \begin{bmatrix} DH \\ D\pi' \end{bmatrix}_{(\rho, i, e_l)} = b + n_C + n_L. \qquad (109)$$

Computing the matrix of (109) and using elementary operations, one can show that (109) is equivalent to (104). □

Remark: The above result holds even when Λ_R, Λ_C and Λ_L are locally parametrized.

Now, observe that $\mathcal{K}(x)$ of (89) is a submatrix of $\mathfrak{F}(\cdot)$ defined by (17). This implies the following:

Proposition 11: If (88) holds, then $\Lambda \bar{\pitchfork} K$ and $\mathcal{H}$ is locally solvable.

Remark: A similar result holds for *Proposition 10* and *Proposition 2.* Observe that while *Proposition 9* assumes that Σ is an $(n_C + n_L)$-dimensional C^2 submanifold, *Proposition 11* does not.

In many practical networks, π is a global diffeomorphism, i.e., all variables in the network can be globally expressed as a function of (q, ϕ) and hence Σ is globally diffeomorphic to $\mathbf{R}^{n_C + n_L}$. Of course $\mathcal{H}$ is locally solvable. In Example 7, $\mathcal{H}$ is not locally solvable and Σ is not diffeomorphic to $\mathbf{R}$. A question arises: Are there networks such that Σ is a submanifold not diffeomorphic to $\mathbf{R}^{n_C + n_L}$, yet they are locally solvable? The answer is affirmative as the following example shows.

Example 8: Consider the map $F: \mathbf{R}^3 \to \mathbf{R}^6$ defined by $F(x, y, z) = (e^x \cos x, e^x \sin x, z, y, \cos x, \sin x)$. For $x, x' \in \mathbf{R}$, define the equivalence relation $x \sim x'$ by $x - x' = 2k\pi$ where k is an integer. Clearly, then, the quotient space of $\mathbf{R}$ with respect to this equivalence relation can be regarded as the unit circle S^1 in $\mathbf{R}^2$; $\mathbf{R}/\sim = S^1$. Let $[x]$ denote the equivalence class. Then F naturally induces the map $\tilde{F}$: $S^1 \times \mathbf{R}^2 \to \mathbf{R}^6$ by

$$\tilde{F}([x], y, z) \triangleq F(x, y, z). \qquad (110)$$

Since

$$(d\tilde{F})_{([x], y, z)} = \begin{bmatrix} e^x(-\sin x) & e^x \cos x & \cdot \\ e^x(\cos x) & e^x \sin x & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \\ -\sin x & \cos x & \cdot \end{bmatrix}_{([x], y, z)}$$

and since

$$\det \begin{bmatrix} e^x(-\sin x) & e^x \cos x & \cdot \\ e^x(\cos x) & e^x \sin x & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} = e^{2x} \neq 0$$

we have rank $(d\tilde{F})_{([x], y, z)} = 3$ for all $([x], y, z) \in S^1 \times \mathbf{R}^2$. Therefore, $\tilde{F}$ is an immersion [6]. Clearly $\tilde{F}$ is injective. Since $\|\tilde{F}([x], y, z)\|^2 = e^{2x} + z^2 + y^2 + 1$, we have

$\|\tilde{F}([x], y, z)\| \to \infty$ as $\|(y, z)\| \to \infty$. Hence $\tilde{F}$ is proper [6]. Consequently it is an embedding [6]. Define

$$\Lambda_R = \tilde{F}(S^1 \times \mathbf{R}^2)$$

$$v_{R_1} = e^x \cos x, \quad v_{R_2} = e^x \sin x, \quad v_{R_3} = z$$

$$i_{R_1} = y, \quad i_{R_2} = \cos x, \quad i_{R_3} = \sin x. \qquad (111)$$

This is a parametric representation of Λ_R. Consider the circuit of Fig. 5 where Λ_R is described by (111). It follows from the above argument that Λ_R is a three-dimensional submanifold diffeomorphic to $S^1 \times \mathbf{R}^2$. It is clear that Σ is diffeomorphic to Λ_R and therefore diffeomorphic to $S^1 \times \mathbf{R}^2$. Notice that $\rho_R \triangleq (x, y, z)$ always serves as a local coordinate system for Λ_R (not a global coordinate system, however). As we remarked earlier, *Proposition 10* holds even when Λ_R is locally parametrized. The matrix of (105) is given by

$$\mathcal{K}^*(\rho)$$

$$= \begin{bmatrix} -e^x \sin x & e^x \cos x & \cdot & \cdot & C_1 & \cdot & \cdot \\ e^x \cos x & e^x \sin x & \cdot & \cdot & \cdot & -C_2 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -C_3 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}_\rho$$

Since $\det \mathcal{K}^*(\rho) = -e^{2x} \neq 0$, $\mathcal{H}$ is locally solvable. Consequently, the dynamics of $\mathcal{H}$ is perfectly well defined on Σ, yet there is no global coordinate system in terms of which the dynamics admits a global state equation because $\Sigma \approx S^1 \times \mathbf{R}^2 \not\approx \mathbf{R}^3$.

Next, we will give two more examples that are of interest.

Example 9: This example shows that there is a nontrivial locally solvable circuit whose inductor is locally flux controlled, but not locally current controlled. Consider the circuit of Fig. 6(a) which consists of a 1-Ω linear resistor and a Josephson Junction device characterized by $i_J = k_1 \sin k_2 \phi$, where k_1 and k_2 are constants (Fig. 6(b)). This is a flux controlled inductor which is not locally current controlled. One can easily show that transversality and local solvability are satisfied.

Example 10: Here we will illustrate a power of geometric approach using an interesting example of Göcknar [11]. Consider the circuit of Fig. 7(a) where the resistor is linear 1-Ω and Λ_C is characterized by

$$v_C = (q - Q)^3 + E \triangleq g_C(q) \qquad (112)$$

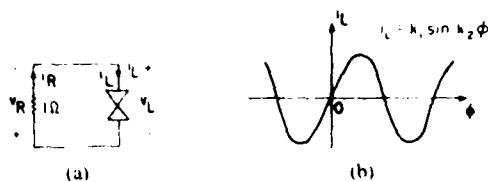where Q and $E \neq 0$ are constants. Since $\mathcal{F}(x)$ of (17) is

Fig. 6. A locally solvable circuit whose inductor constitutive relation is not locally current controlled. (a) The circuit diagram. (b) Inductor constitutive relation.
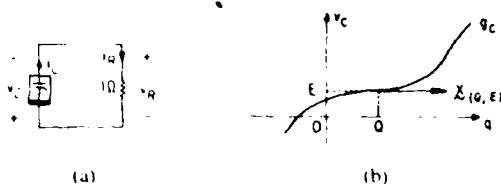

Fig. 7. A circuit where v_C is not a C^2 coordinate. (a) The circuit diagram. (b) Capacitor constitutive relation.

given by

$$\mathcal{G}(x) = \left[\begin{array}{ccc} 1 & -1 & \\ 1 & & 3(q-Q)^2 \end{array} \right]_x$$

transversality holds. Since

$$\mathcal{K}(x) = \left[\begin{array}{cc} -1 & -1 \\ 1 & \cdot \end{array} \right]$$

local solvability holds and the dynamics in terms of q is given by

$$\frac{dq}{dt} = -(q-Q)^3 - E. \qquad (113)$$

A problem arises, however, if one argues as follows: Differentiating (112) with respect to t one has

$$\frac{dv_C}{dt} = (Dg_C)_q \frac{dq}{dt} = (Dg_C)_q i_C = (Dg_C)_q i_R$$

$$= (Dg_C)_q v_R = -(Dg_C)_q v_C. \qquad (114)$$

Since $q = g_C^{-1}(v_C) = (v_C - E)^{1/3} + Q$ and since $(Dg_C)_q = 3(q-Q)^2$, one has

$$(Dg_C)_{q=g_C^{-1}(v_C)} = 3(v_C - E)^{2/3}.$$

This and (114) imply

$$\frac{dv_C}{dt} = -3(v_C - E)^{2/3} v_C. \qquad (115)$$

Is (115) another differential equation describing the dynamics of the same circuit as (113)? The answer is no! If it were, (113) and (115) must have the same qualitative properties. So if (115) is the differential equation describing the dynamics of the circuit of Fig. 7, then $v_C = E$ is an equilibrium of the dynamics. But the corresponding value $q = g_C^{-1}(E) = Q$ is not an equilibrium point of (113). Since the existence of an equilibrium point must be a coordinate-free property, there should be something wrong with saying that (115) is the differential equation describing the dynamics of the circuit of Fig. 7. The point here is that v_C is not a C^2 coordinate for Σ. Although the map: $v_C \mapsto (v_R, v_C, i_R, i_C, q) = (-v_C, v_C, -v_C, -v_C, (v_C - E)^{1/3} + Q)$ is a homeomorphism, it is not a C^2 diffeomorphism be-

cause $(v_C - E)^{1/3} + Q$ is not differentiable. (It is not even a C^1 diffeomorphism.) On the other hand the map:

$$q \mapsto (v_R, v_C, i_R, i_C, q)$$
$$= (-(q-Q)^3 - E, (q-Q)^3 + E,$$
$$-(q-Q)^3 - E, -(q-Q)^3 - E, q)$$

is a C^2 diffeomorphism and (113) describes the dynamics. A more geometric way of looking at the situation is as follows. The configuration space Σ is diffeomorphic to the graph of g_C. The vector field $X_{(Q,E)}$ at (Q, E) is contained in the tangent space $T_{(Q,E)}\Sigma$, i.e., $X_{(Q,E)}$ is in parallel with the q-axis. Therefore, if we look at $X_{(Q,E)}$ from the q-axis, we can observe the direction and the length of $X_{(Q,E)}$. On the other hand, if we look at $X_{(Q,E)}$ from the v_C-axis, we can detect neither the direction nor the length of $X_{(Q,E)}$. Finally, let us remark that even though (115) does not qualify as the differential equation describing the dynamics, (115) is true in the sense that for the flow $x(t)$ on Σ,

$$\frac{dv_C(x(t))}{dt} = -3(v_C(x(t)) - E)^{2/3} v_C(x(t)).$$

Our geometric approach seems to be the right tool to explain what is happening in this example.

We will next give a perturbation result on local solvability. Recall that Λ_R is said to be locally hybrid [3], if (9) holds and

$$\det((Df_R)A)_{(v_R, i_R)} \neq 0, \quad \text{for all } (v_R, i_R) \in \Lambda_R \cap U_R \qquad (116)$$

for some fixed $2n_R \times n_R$ matrix A, where each column of A has either of the following forms:

$$(0, \cdots, 0, 1, 0, \cdots, 0, 0, \cdots \quad \cdots \quad \cdots 0$$

$$\underbrace{(0, \cdots \quad \cdots \quad \cdots, 0, 0}_{n_R}, \underbrace{\cdots, 0, 1, 0, \cdots, 0)}_{n_R}.$$

Let

$$((Df_R)A)_{(v_R, i_R)} = [F_1, \cdots, F_{n_R}] \qquad (117)$$

and suppose that F_k corresponds to i_{R_k} (resp., v_{R_k}). Then that particular resistor is said to be locally voltage controlled (resp., locally current controlled).

Remark: Observe that in (52) and (53), local controlledness is defined for Λ_R, whereas in the above definition, local controlledness is defined for each resistor provided that Λ_R is locally hybrid.

Proposition 12: Given a nonlinear network $\mathcal{X}$, assume the following:

(i) Λ_R is locally hybrid and Λ_C (resp. Λ_I) is locally charge (resp. flux) controlled.

(ii) $\Lambda \cap K \neq \varnothing$.

Then, by adding small linear capacitors and small linear inductors appropriately we can obtain a new network $\hat{\mathcal{X}}$ such that (1) $\hat{\Lambda} \cap \hat{K} \neq \varnothing$, (2) $\hat{\Lambda} \hat{\pitchfork} \hat{K}$, (3) $\hat{\mathcal{X}}$ is locally solvable.

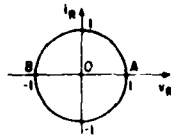*Proof:* Pick a proper tree $\mathcal{T}$ containing a maximum number of locally voltage controlled resistors and a mini-

Fig. 8. Resistor constitutive relation for the circuit of Example 13.

(v_C, v_{C_1}, i_{I_1}) serves as a global coordinate system for $\hat{\Sigma}$. Since $(Dg_C)_{v_C} = 0$ somewhere, we see that $\det \mathcal{K}(\hat{x}) = 0$ there. If the resistor belongs to $\hat{\mathcal{L}}$, then a similar argument shows that $\mathcal{K}$ is not locally solvable.

We will next discuss relationship between local solvability of $\mathcal{K}$ and transversality of the $(n_C + n_I)$-port N derived from $\mathcal{K}$, under certain excitations. Replace capacitors and inductors of $\mathcal{K}$ with ports. The resulting network is called the $(n_C + n_I)$-port N derived from $\mathcal{K}$. For the purpose of convenience we will keep the same notation for N as $\mathcal{K}$. Drive the capacitor ports by independent voltage sources v_a^* and drive the inductor ports by independent current sources i_b^*. Define

$$\Lambda^\dagger(v_a^*, i_b^*) \triangleq \{(v,i) \in \mathbf{R}^{2b} | (v_R, i_R) \in \Lambda_R, v_C = v_a^*, i_I = i_b^*\}. \tag{121}$$

This set represents the internal constitutive relations of N under the excitation (v_a^*, i_b^*). Clearly $\Lambda^\dagger(v_a^*, i_b^*)$ is a b-dimensional submanifold. Recall $\Sigma^\dagger$, $K^\dagger$, and $\pi^\dagger$ defined by (72), (74), and (76), respectively.

Proposition 13: Given a nonlinear network $\mathcal{K}$, assume that Λ_C (resp. Λ_L) is represented by $q = g_C(v_C)$ (resp. $\phi = g_I(i_I)$) and that $(Dg_C)_{v_C}$ (resp. $(Dg_L)_{i_L}$) is symmetric and positive definite. Assume also that Σ is an $(n_C + n_I)$-dimensional C^2 submanifold. Then $\mathcal{K}$ is locally solvable if and only if for the $(n_C + n_I)$-port N derived from $\mathcal{K}$, the following holds:

$$\Lambda^\dagger(v_a^*, i_b^*) \pitchfork K^\dagger, \quad \text{for all } (v_a^*, i_b^*) \in \pi^\dagger(\Sigma^\dagger). \tag{122}$$

Proof: It follows from the hypothesis and *Corollary 4* that $\mathcal{K}$ is locally solvable if and only if $\pi^\dagger$ is a local diffeomorphism. To prove sufficiency, let $(v_a^*, i_b^*) \in \pi^\dagger(\Sigma^\dagger)$ and define

$$G(v,i) \triangleq \begin{bmatrix} f_R(v_R, i_R) \\ \pi^{\dagger\prime}(v,i) - \begin{bmatrix} v_a^* \\ i_b^* \end{bmatrix} \end{bmatrix} \tag{123}$$

where $\pi^{\dagger\prime}: \mathbf{R}^{2b} \to \mathbf{R}^{n_C + n_I}$ is the projection map

$$\pi^{\dagger\prime}(v,i) = (v_C, i_L)$$

and f_R is defined by (9). By the definition of $\Lambda^\dagger(v_a^*, i_b^*)$, for each $(v_0, i_0) \in \Lambda^\dagger(v_a^*, i_b^*)$, there is a neighborhood $V \subset \mathbf{R}^{2b}$ of (v_0, i_0) such that

$$\Lambda^\dagger(v_a^*, i_b^*) \cap V = G^{-1}(0)$$

$$\text{rank}(DG)_{(v,i)} = b, \quad \text{for all } (v,i) \in \Lambda^\dagger(v_a^*, i_b^*) \cap V. \tag{124}$$

Using (122) and (123) and an argument similar to the proof of *Proposition 1*, one sees that for each $(v_a^*, i_b^*) \in$

$\pi^\dagger(\Sigma^\dagger)$ and for $(v,i) \in \Lambda^\dagger(v_a^*, i_b^*) \cap K^\dagger$,

$$\text{rank} \begin{bmatrix} B & \cdot \\ \cdot & Q \\ D_v f_R & D_i f_R \\ D_v \pi^{\dagger\prime} & D_i \pi^{\dagger\prime} \end{bmatrix}_{(v,i)} = 2b. \tag{125}$$

By an argument similar to that of the proof of *Proposition 9*, one can show that $\pi^\dagger$ is a local diffeomorphism if and only if (125) holds. Finally, since

$$\Sigma^\dagger = \Lambda^\dagger \cap K^\dagger = \bigcup_{(v_a^*, i_b^*) \in \pi^\dagger(\Sigma^\dagger)} \Lambda^\dagger(v_a^*, i_b^*) \cap K^\dagger \tag{126}$$

it follows from (125) that for each $(v,i) \in \Sigma^\dagger$, $\pi^\dagger$ is a local diffeomorphism. Therefore, $\mathcal{K}$ is locally solvable. Conversely, if $\pi^\dagger$ is a local diffeomorphism at each $(v,i) \in \Sigma^\dagger$, then (125) and (126) imply (122).

V. EVENTUAL STRICT PASSIVITY

Eventual strict passivity is an important qualitative property of electrical networks, because it guarantees that all trajectories eventually approach a fixed compact subset of the configuration space [12], [14]. Roughly speaking, the results of this section say the following: Suppose that the resistors are eventually strictly passive and that every capacitor is in parallel with a large linear resistor and every inductor is in series with a small linear resistor. Then all trajectories approach a fixed compact subset of the configuration space. Since the above assumption is satisfied by most practical networks, the results guarantee that the voltage and current waveforms are bounded in most networks of practical interest.

Consider the following one-form on $\mathbf{R}^{2b + n_C + n_I}$:

$$\theta = \sum_{k=1}^{n_C} v_{C_k} dq_k + \sum_{k=1}^{n_I} i_{I_k} d\phi_k \tag{127}$$

and suppose that capacitors and inductors are reciprocal:[4]

$$d\theta = \sum_{k=1}^{n_C} dv_{C_k} \wedge dq_k + \sum_{k=1}^{n_I} di_{I_k} \wedge d\phi_k = 0 \tag{128}$$

on $\mathbf{R}^{2n_R} \times \Lambda_C \times \Lambda_I$. If, in addition, Σ is simply connected, then [4] there is a unique real-valued function E on Σ such that for any two points x and x_0 of Σ

$$E(x) = E(x_0) + \int_\Gamma \iota^* \theta \tag{129}$$

where Γ is any smooth curve on Σ connecting x and x_0. If we fix an arbitrary $x_0 \in \Sigma$, then (129) is a well-defined function on Σ. Clearly, E is the *energy* stored in capacitors and inductors relative to the point $x_0 \in \Sigma$. Let $W_R': \mathbf{R}^{2n_R} \to \mathbf{R}$ be defined by

$$W_R'(v_R, i_R) \triangleq \sum_{k=1}^{n_R} v_{R_k} i_{R_k}. \tag{130}$$

[4] Reciprocity of capacitors and inductors is related with the existence of energy function, whereas reciprocity of a network [10] ($d\omega = 0$, where ω is defined by (60)) is related with the existence of mixed potential function
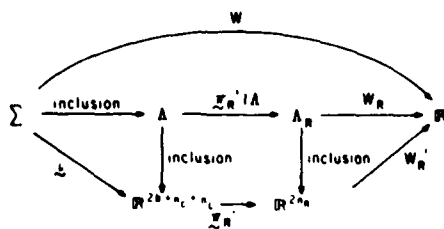
Fig. 9.   Diagram defining the two functions W and W_R.

Fig. 10.   A network which is not eventually strictly passive. (a) The circuit diagram. (b) Trajectories on the linear subspace $W = 0$.

Recall π'_R defined by (20) and let W_R and W be defined by Fig. 9. The function W is the *power* at resistors. It follows from Tellegen's theorem that

$$\frac{dE(x(t))}{dt} = -W(x(t)).\qquad(131)$$

Recall that a network $\mathfrak{N}$ is said to be *eventually strictly passive* [12]-[14] if there is a compact subset $\Omega \subset \Sigma$ such that

$$W(x)>0,\qquad \text{for all } x\in\Sigma-\Omega.\qquad(132)$$

The following two propositions show the importance of eventual strict passivity.

Proposition 14: [12]-[14] Let E be proper, i.e., for every $\alpha\in\mathbf{R}$, the set $\{x\in\Sigma|E(x)\leqslant\alpha\}$ is compact, and let $\mathfrak{N}$ be eventually strictly passive. Then the set defined by

$$\mathcal{B}\triangleq\{x\in\Sigma|E(x)\leqslant\alpha_1\}\qquad(133)$$

$$\alpha_1=\max_{x\in\Omega}E(x)\qquad(134)$$

is compact, and for any initial state $x(0)$, either one of the following happens:
(i) There is a $t_1>0$ such that $x(t)\in\mathcal{B}$ for all $t\geqslant t_1$.
(ii) $x(t)\notin\mathcal{B}$ for all $t\geqslant0$ but $\lim_{t\to\infty}x(t)\in\mathcal{B}$.

The set $\mathcal{B}$ contains many of the important information concerning the dynamics. In particular, the following holds:
Proposition 15: Under the same setting as *Proposition 14*, we have
(i) All periodic orbits and equilibria are in $\mathcal{B}$.
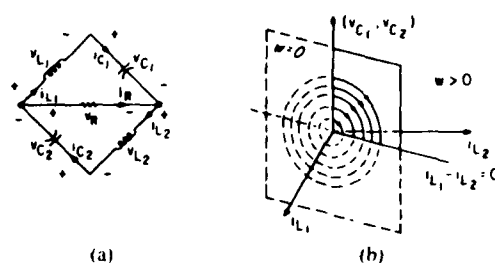(ii) In particular, equilibria lie in the set

$$\{x\in\Sigma|W(x)=0\}.\qquad(135)$$

Proof: (i) It follows from (131) and (132) that for any $x(t)\in\Sigma-\Omega$, the energy $E(x(t))$ is strictly decreasing with respect to t. This implies that for $x(0)\in\Sigma-\Omega$, the trajectory $x(t)$ cannot come back to $x(0)$. Similarly, $x(t)$ cannot remain at $x(0)$.
(ii) Since $E(x(t))$ is either strictly increasing or strictly decreasing outside the set defined by (135), the equilibria must be located in (135). □

The set $\mathcal{B}$ in (133) is called a *set of attraction* since it attracts all trajectories.

Eventual strict passivity is a property of W on Σ, while W_R is defined on Λ_R. These two functions may behave very differently depending on the properties of ι and π'_R. (See Fig. 9). The properties of W_R are much easier to check than

those of W because W_R depends only on Λ_R but not on K so that one does not have to worry about Kirchhoff laws. We need the following:

Definition 4: The resistor constitutive relations represented by Λ_R are said to be eventually strictly passive if there is a compact subset Ω_R of Λ_R such that

$$W_R(v_R,i_R)>0,\qquad\text{for all }(v_R,i_R)\in\Lambda_R-\Omega_R.\quad(136)$$

Eventual strict passivity of Λ_R is a physically meaningful condition because it simply says that the resistors dissipate positive power eventually. The condition is satisfied by a broad class of resistors. A natural question, then, arises: Does eventual strict passivity of Λ_R imply existence of a compact set of attraction? Another interesting question related to this one was raised by Smale [8]. In terms of our terminology, the problem is rephrased as follows: Suppose that there is a number $\beta>0$ satisfying

$$W_R(v_R,i_R)\geqslant\beta\sum_{k=1}^{n_R}\left(v_{R_k}^2+i_{R_k}^2\right)\qquad(137)$$

for all (v_R,i_R) with $\|(v_R,i_R)\|$ sufficiently large. Then, does the network have a compact set of attraction? The answer to both of the two questions is *no* as the following example shows.

Example 15: Consider the circuit of Fig. 10(a), where all elements are linear and element values are all equal to one. Since the resistor is linear and 1-Ω, W_R is positive everywhere except for the origin. Hence Λ_R is eventually strictly passive. Observe that

$$W_R(v_R,i_R)=v_Ri_R=v_R^2=i_R^2\geqslant\beta(v_R^2+i_R^2)$$

for $0<\beta\leqslant1/2$. Therefore, (137) is satisfied. We claim that this circuit does not have a compact set of attraction. To this end let us write the dynamics in terms of $(v_{C_1},v_{C_2},i_{L_1},i_{L_2})$:

$$C_1\frac{dv_{C_1}}{dt}=i_{L_1},\quad C_2\frac{dv_{C_2}}{dt}=i_{L_2}$$

$$L_1\frac{di_{L_1}}{dt}=-v_{C_1}+R(-i_{L_1}+i_{L_2})$$

$$L_2\frac{di_{L_2}}{dt}=-v_{C_2}-R(-i_{L_1}+i_{L_2}).\qquad(138)$$

Drawing trajectories, (Fig. 10(b)), one can show that there
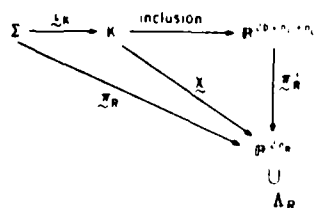
Fig. 11. Diagram defining the two functions χ and ι_K

is a linear subspace on which all the trajectories are concentric circles. More specifically, the element values satisfy the condition for this bridge circuit to be balanced at the angular frequency one. Therefore, for any $a \in \mathbf{R}$, the following is a solution to (138):

$$v_C(t) = v_C(t) = a\sin t$$

$$i_L(t) = i_L(t) = a\cos t.$$

Since $a \in \mathbf{R}$ is arbitrary, the solution can have an arbitrarily large magnitude. Therefore, there is no compact set of attraction. In terms of the above coordinate system, we have

$$W(v_C, i_L) = R(i_{L_2} - i_{L_1})^2$$

and hence it does not satisfy (132). Notice that any trajectory starting outside the linear subspace $W = 0$, approaches the origin.

Since (136) is satisfied by most resistors of practical interest, it is natural for us to seek conditions under which (136) implies (132). The following is a generalization of a recent result by Chua and Green [12] for a general manifold. We assume that Λ_R is closed for technical reasons. This is not a restrictive condition, however.

Lemma 2: Let Σ be an $(n_C + n_L)$-dimensional C^2 submanifold and assume the following:

(i) Λ_R is closed and eventually strictly passive,

(ii) $\|(q, \phi)\| \to \infty$ implies $\|(v_C, i_L)\| \to \infty$ on $\Lambda_C \times \Lambda_L$.

Then $\mathcal{R}$ is eventually strictly passive if the following *fundamental topological hypothesis* is satisfied:

There are no loops and no cut sets consisting only of capacitors and inductors, or equivalently

(1) there is a tree $\mathcal{T}(R)$ consisting only of resistors,

(2) there is a tree $\mathcal{T}(CL)$ containing all capacitors and inductors.

Proof: Recall the map π_R defined by (22). Suppose that Λ_R is eventually strictly passive and let Ω_R be as in (136). If π_R is proper, then the preimage $\pi_R^{-1}(\Omega_R)$ is compact because the preimage of a compact set under a proper map is compact. It is, then, clear that the inequality in (132) holds with respect to $\pi_R^{-1}(\Omega_R)$. So we show that the fundamental topological hypothesis implies that π_R is proper. To this end let

$$\iota_K : \Sigma \to K \qquad (139)$$

be the inclusion map and consider the map χ defined by Fig. 11. Since Λ_R is assumed to be closed, Λ is also closed.

Therefore, $\Sigma = \Lambda \cap K$ is a closed submanifold of K. Consequently, for any compact subset A of K, the preimage $\iota_K^{-1}(A)$ is compact. This shows that ι_K is proper. Therefore, we need only show that χ is proper. Since χ is obviously continuous, we need only show that the preimage of a bounded subset of $\mathbf{R}^{2n_K}$ is bounded. Suppose that the fundamental topological hypothesis holds and let $\hat{v}_{R_u}$ (resp., $\hat{i}_{R_c}$) be the tree branch voltages (resp., link currents) for $\mathcal{T}(R)$ (resp., links associated with $\mathcal{T}(CL)$). It follows from (15) that for $(v, i, q, \phi) \in K$,

$$v = \hat{Q}'\hat{v}_{R_u}, \quad i = \hat{B}'\hat{i}_{R_c} \qquad (140)$$

where $\hat{Q}$ and $\hat{B}$ are the fundamental cut set matrix and the fundamental loop matrix associated with $\mathcal{T}(R)$ and $\mathcal{T}(CL)$, respectively. Equation (140) and assumption (ii) imply that

$$\|x\| \to \infty, \quad x \in K \to \|(\hat{v}_{R_u}, \hat{i}_{R_c})\| \to \infty. \qquad (141)$$

Since $(\hat{v}_{R_u}, \hat{i}_{R_c})$ is a subvector of (v_R, i_R), we have

$$\|x\| \to \infty, \quad x \in K \to \|(v_R, i_R)\| \to \infty. \qquad (142)$$

This shows that the preimage of a bounded subset under χ is bounded. Since the properties of χ do not depend on a particular choice of a tree, χ is proper.

Remark: Observe that in the above proof we took full advantage of the coordinate free property, since in (140)–(142) we are using two different trees simultaneously.

Now, experiences tell us that most networks of practical interest have a compact set of attraction. We next justify this observation formally by carrying out a slight network perturbation. The perturbation we make is simply a formalization of the following hypothesis: "Every capacitor is in parallel with a large linear resistor and every inductor is in series with a small linear resistor." Before stating the results, we need the following:

Definition 5: A nonlinear network $\mathcal{R}$ is said to be *strongly locally solvable* if

$$\det \mathcal{K}(x) \neq 0, \qquad \text{for all } x \in \Lambda \qquad (143)$$

where $\mathcal{K}(x)$ is defined by (89) and Λ is defined by (6).

Remarks: 1) If Λ_C (resp. Λ_L) is locally charge (resp. flux) controlled, then $\mathcal{R}$ is strongly locally solvable if and only if

$$\det \mathcal{K}_R(v_R, i_R) \neq 0, \qquad \text{for all } (v_R, i_R) \in \Lambda_R \qquad (144)$$

where $\mathcal{K}_R(v_R, i_R)$ is defined by (100).

2) Condition (143) is stronger than (88) since for strong local solvability the determinant should be nonzero on Λ and since $\Sigma \subset \Lambda$. This condition is satisfied by many networks, however. For example, the circuit of Fig. 2 with capacitors added, satisfies this condition because

$$\mathcal{K}_R(v_R, i_R) = D_{i_R} f_R = \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix}.$$

The perturbed network $\hat{\mathcal{R}}$ of *Proposition 12* is strongly locally solvable because the matrix of (120) is nonsingular for all $(v_R, i_R) \in \Lambda_R$.

Proposition 16: Given a nonlinear network $\mathcal{R}$, assume the following:

(i) $\mathcal{R}$ is strongly locally solvable.

(ii) Λ_R is closed and eventually strictly passive,

(iii) $\|(q,\phi)\| \to \infty$ implies $\|(v_C, i_L)\| \to \infty$ on $\Lambda_C \times \Lambda_L$. Insert a large linear resistor g_k, $k = 1, \cdots, n_C$, in parallel with every capacitor and insert a small linear resistor r_k, $k = 1, \cdots, n_L$, in series with every inductor. Then

(1) $\hat{\Lambda} \cap \hat{K} \neq \varnothing$ and $\hat{\Sigma} = \hat{\Lambda} \cap \hat{K}$ is an $(n_C + n_L)$-dimensional C^2 submanifold,

(2) $\hat{\mathfrak{N}}$ is locally solvable,

(3) $\hat{\mathfrak{N}}$ is eventually strictly passive. Consequently $\hat{\mathfrak{N}}$ has a compact set of attraction.

Proof: (1) In terms of a proper tree $\mathfrak{T}$, the original network $\mathfrak{N}$ is described by

$$v_{R_e} + B_{RR}v_{R_q} + B_{RC}v_C = 0 \tag{145}$$

$$v_L + B_{LR}v_{R_q} + B_{LC}v_C = 0 \tag{146}$$

$$i_{R_q} - B_{RR}^T i_{R_e} - B_{LR}^T i_L = 0 \tag{147}$$

$$i_C - B_{RC}^T i_{R_e} - B_{LC}^T i_L = 0 \tag{148}$$

$$(v, i, q, \phi) \in \Lambda. \tag{149}$$

Let $\hat{\mathfrak{T}} = \mathfrak{T} \cup r_{\overline{q}}$ where $r_{\overline{q}}$ represents branches of r_k's. It is clear that $\hat{\mathfrak{T}}$ is a proper tree for $\hat{\mathfrak{N}}$. Decompose $\hat{v}$ and $\hat{i}$ as

$$\hat{v} = \left(v_{R_e}, v_g, v_L; v_{R_q}, v_r, v_C \right)$$

$$\hat{i} = \left(i_{R_e}, i_g, i_L; i_{R_q}, i_r, i_C \right)$$

where g and r denote the variables associated with g_k's and r_k's. Then $\hat{\mathfrak{N}}$ is described by

$$v_{R_e} + B_{RR}v_{R_q} + B_{RC}v_C = 0 \tag{150}$$

$$v_L + B_{LR}v_{R_q} + B_{LC}v_C + v_r = 0 \tag{151}$$

$$v_g + v_C = 0 \tag{152}$$

$$i_{R_q} - B_{RR}^T i_{R_e} - B_{LR}^T i_L = 0 \tag{153}$$

$$i_C - B_{RC}^T i_{R_e} - B_{LR}^T i_L - i_g = 0 \tag{154}$$

$$i_r - i_L = 0 \tag{155}$$

$$(v, i, q, \phi) \in \Lambda \tag{156}$$

$$i_g = g^{-1}v_g \tag{157}$$

$$v_r = r i_r \tag{158}$$

where

$$g = \mathrm{diag}(g_1, \cdots, g_{n_C}), \quad r = \mathrm{diag}(r_1, \cdots, r_{n_L}). \tag{159}$$

Eliminating v_g, v_r, i_g, and i_r, we see that $\hat{\mathfrak{N}}$ is described by (150), (153), (156), and

$$v_L + B_{LR}v_{R_q} + B_{LC}v_C + r i_L = 0 \tag{160}$$

$$i_C - B_{RC}^T i_{R_e} - B_{LR}^T i_L + g^{-1}v_C = 0 \tag{161}$$

$$v_g = -v_C \tag{162}$$

$$i_g = -g^{-1}v_C \tag{163}$$

$$v_r = r i_L \tag{164}$$

$$i_r = i_L. \tag{165}$$

Let us rewrite (145)–(149) and (160)–(165) more concisely. Let B and Q be the fundamental loop matrix and the fundamental cut set matrix for $\mathfrak{N}$, respectively. Then, of

course, (145)–(149) are written as

$$\begin{bmatrix} B^T & \cdot \\ \cdot & Q^T \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} = 0 \tag{166}$$

$$(v, i, q, \phi) \in \Lambda. \tag{167}$$

Comparing (145)–(149) with (150), (153), (156), (160)–(167), we see that the differences between $\mathfrak{N}$ and $\hat{\mathfrak{N}}$ are in the last two terms of (160), (161), and (162)–(165). Therefore, $\hat{\mathfrak{N}}$ is described by

$$\begin{bmatrix} B^T & \cdot \\ \cdot & Q^T \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} \cdot & F \\ G & \cdot \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} = 0 \tag{168}$$

$$(v, i, q, \phi) \in \Lambda \tag{169}$$

$$(v_g, v_r, i_g, i_r) = H \begin{bmatrix} v_C \\ i_L \end{bmatrix} \tag{170}$$

where

$$F = \begin{matrix} \begin{matrix} i_{R_e} & i_L & i_{R_q} & i_C \end{matrix} \\ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & r & \cdot & \cdot \end{bmatrix} \end{matrix}$$

$$G = \begin{matrix} \begin{matrix} v_{R_e} & v_L & v_{R_q} & v_C \end{matrix} \\ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & g^{-1} \end{bmatrix} \end{matrix} \tag{171}$$

$$H = \begin{bmatrix} -1 & \cdot \\ \cdot & r \\ -g^{-1} & \cdot \\ \cdot & 1 \end{bmatrix}. \tag{172}$$

Now let $x_0 \in \Lambda \cap K \neq \varnothing$ and let U be a bounded neighborhood of x_0 in $\mathbb{R}^{2b + n_C + n_L}$. Since the set $(\Lambda \cap K) \cap U$ is a bounded submanifold, small perturbations of $K \cap U$ do not destroy transversality of $\Lambda \cap U$ and $K \cap U$ and hence they do not destroy nonemptiness of $(\Lambda \cap U) \cap (K \cap U)$. Therefore, if g_k is large enough and if r_k is small enough, then $\|F\|$ and $\|G\|$ in (168) are small enough to guarantee nonemptiness of the intersection of (168) and (169). Since (170) does not destroy this nonemptiness, we have $\hat{\Lambda} \cap \hat{K} \neq \varnothing$. We show $\hat{\Lambda} \pitchfork \hat{K}$ later.

(2) Since $(D_{v_C} f_C)_{(v_C, q)}$ (resp. $(D_{i_L} f_L)_{(i_L, \phi)}$) is a square matrix, (89) implies that if $\mathfrak{N}$ is strongly locally solvable, then $(D_{v_C} f_C)_{(v_C, q)}$ (resp. $(D_{i_L} f_L)_{(i_L, \phi)}$) is nonsingular on Λ_C (resp. Λ_L), i.e., Λ_C (resp. Λ_L) is locally charge (resp. flux) controlled. Therefore, we can use *Corollary 7* to check local solvability of $\hat{\mathfrak{N}}$. Observe that

$$\hat{f}_R(\hat{v}_R, \hat{i}_R) = \begin{bmatrix} f_R(v_R, i_R) \\ i_g - g^{-1}v_g \\ v_r - r i_r \end{bmatrix} \tag{173}$$

and that the fundamental loop matrix $\hat{B}$ for $\hat{\mathfrak{N}}$ is given by

$$\begin{matrix} \begin{matrix} v_{R_e} & v_L & v_g & v_{R_q} & v_C & v_R \end{matrix} \\ \begin{bmatrix} & & & B_{RR} & B_{RC} & \cdot \\ & 1 & & B_{LR} & B_{LC} & 1 \\ & & & \cdot & 1 & \cdot \end{bmatrix} \end{matrix}. \tag{174}$$

It follows from this that

$$\hat{B}_{RR} = \begin{bmatrix} B_{RR} & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{matrix} v_{R_\xi} \\ v_g \end{matrix} , \qquad \hat{B}_{RC} = \begin{bmatrix} B_{RC} \\ 1 \end{bmatrix} \begin{matrix} v_{R_\xi} \\ v_g \end{matrix}$$

(175)

$$\hat{B}_{LR} = \begin{bmatrix} B_{LR} & 1 \end{bmatrix} v_L .$$

Let $\hat{v}_{R_\xi} = (v_R, v_g)$, $\hat{v}_{R_q} = (v_{R_q}, v_r)$, $\hat{i}_{R_\xi} = (i_{R_\xi}, i_g)$, and $\hat{i}_{R_q} = (i_{R_q}, i_r)$. Then

$$D_{\hat{v}_{R_q}} \hat{f}_R - (D_{\hat{v}_{R_\xi}} \hat{f}_R) \hat{B}_{RR}$$

$$= \begin{bmatrix} D_{v_{R_q}} f_R & \cdot \\ \cdot & \cdot \\ \cdot & 1 \end{bmatrix} - \begin{bmatrix} D_{v_{R_\xi}} f_R & \cdot \\ \cdot & -g \\ \cdot & \end{bmatrix} \begin{bmatrix} B_{RR} & \cdot \\ \cdot & \end{bmatrix}$$

$$= \begin{bmatrix} D_{v_{R_q}} f_R - (D_{v_{R_\xi}} f_R) B_{RR} & \cdot \\ \cdot & \cdot \\ \cdot & 1 \end{bmatrix}$$

(176)

$$D_{\hat{i}_{R_\xi}} \hat{f}_R + (D_{\hat{i}_{R_q}} \hat{f}_R) \hat{B}_{RR}^T$$

$$= \begin{bmatrix} D_{i_{R_\xi}} f_R & \cdot \\ \cdot & 1 \\ \cdot & \end{bmatrix} + \begin{bmatrix} D_{i_{R_q}} f_R & \cdot \\ \cdot & \cdot \\ \cdot & -r \end{bmatrix} \begin{bmatrix} B_{RR}^T & \cdot \\ \cdot & \end{bmatrix}$$

$$= \begin{bmatrix} D_{i_{R_\xi}} f_R + (D_{i_{R_q}} f_R) B_{RR}^T & \cdot \\ \cdot & \cdot \\ \cdot & 1 \end{bmatrix} .$$

(177)

It follows from (176), (177), and (100) that

$$\hat{\mathfrak{X}}_R(\hat{v}_R, \hat{i}_R) = \begin{bmatrix} D_{v_{R_q}} f_R - (D_{v_{R_\xi}} f_R) B_{RR} & \cdot & \vdots & D_{i_{R_\xi}} f_R + (D_{i_{R_q}} f_R) B_{RR}^T & \cdot \\ \cdot & & \vdots & & 1 \\ \cdot & & 1 & \vdots & \end{bmatrix}_{(\hat{v}_R, \hat{i}_R)}$$

(178)

It is clear that $\hat{\mathfrak{X}}_R$ depends only on (v_R, i_R) and that

$$|\det \hat{\mathfrak{X}}_R(\hat{v}_R, \hat{i}_R)| = |\det \mathfrak{X}_R(v_R, i_R)| . \quad (179)$$

Now if $(\hat{v}_R, \hat{i}_R) \in \hat{\Lambda}_R$, then $(v_R, i_R) \in \Lambda_R$, because

$$\hat{\Lambda}_R = \{ (\hat{v}_R, \hat{i}_R) | (v_R, i_R) \in \Lambda_R, i_g = g^{-1} v_g, v_r = r i_r \} .$$

(180)

By the strong local solvability assumption, we have

$$|\det \mathfrak{X}_R(v_R, i_R)| > 0, \quad \text{for all } (v_R, i_R) \in \Lambda_R .$$

This and (179) imply

$$|\det \hat{\mathfrak{X}}_R(\hat{v}_R, \hat{i}_R)| > 0, \quad \text{for all } (\hat{v}_R, \hat{i}_R) \in \hat{\pi}_R(\hat{\Sigma}) .$$

It follows from *Corollary 7* that $\hat{\mathfrak{N}}$ is locally solvable. By *Proposition 11*, we have $\hat{\Lambda} \pitchfork \hat{K}$. Therefore, $\hat{\Sigma}$ is an $(n_c + n_l)$-dimensional C^2 submanifold.

(3) The resistor constitutive relations $\hat{\Lambda}_R$ for $\hat{\mathfrak{N}}$ is described by (180) where $\hat{v}_R = (v_R, v_g, v_r)$, $\hat{i}_R = (i_R, i_g, i_r)$. Therefore, the function $\hat{W}_R$ corresponding to W_R is given by

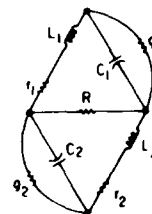$$\hat{W}_R(\hat{v}_R, \hat{i}_R) = W_R(v_R, i_R) + v_g^T g^{-1} v_g + i_r^T r i_r .$$

Fig. 12. Perturbation of network of Fig. 10(a).

It follows from condition (ii) that there is a compact set $\Omega_R \subset \Lambda_R$ such that (136) holds. For any $\alpha > 0$, let

$$\Omega_{gr} \triangleq \left\{ (v_g, v_r, i_g, i_r) \,\middle|\, \begin{matrix} i_g = g^{-1} v_g, v_r = r i_r \\ \|(v_g, v_r, i_g, i_r)\| \le \alpha \end{matrix} \right\} .$$

Then the set $\hat{\Omega}_R \triangleq \Omega_R \times \Omega_{gr}$ has the property that

$$\hat{W}_R > 0 \text{ on } \hat{\Lambda}_R - \hat{\Omega}_R$$

because g and r are diagonal matrices with positive elements. Finally, to show that the fundamental topological hypothesis is satisfied, let R_g, L, r_γ, and g_L represent the branches of the resistors in $\hat{\mathfrak{N}}$, inductors in $\hat{\mathfrak{N}}$, the added resistors r_k's and the added resistors g_k's. Then $\hat{\mathfrak{N}}(R) = R_g \cup r_\gamma \cup g_L$ is a tree for $\hat{\mathfrak{N}}$ which consists only of resistors. Also $\hat{\mathfrak{N}}(CL) = \hat{\mathfrak{N}} \cup L$ is a tree for $\hat{\mathfrak{N}}$ which contains all capacitors and inductors. It follows from *Lemma 2* and condition (iii) that $\hat{\mathfrak{N}}$ is eventually strictly passive. □

Example 16: Consider the circuit of Example 15. Since the circuit is linear, all the conditions of *Proposition 16* are satisfied. The perturbed circuit is shown in Fig. 12. It follows from *Proposition 16* that this perturbed circuit has a compact set of attraction. In fact the linear subspace $H = 0$ in Fig. 10(b) degenerates into the origin and any closed ball centered at the origin serves as a compact set of attraction.

Remark: As we have seen $\hat{\mathfrak{X}}_R(\hat{v}_R, \hat{i}_R) = \mathfrak{X}_R(v_R, i_R)$ for $(\hat{v}_R, \hat{i}_R) \in \hat{\pi}_R(\hat{\Sigma})$. But $(\hat{v}_R, \hat{i}_R) \in \hat{\pi}_R(\hat{\Sigma})$ does not necessarily imply $(v_R, i_R) \in \pi_R(\Sigma)$ even though $(v_R, i_R) \in \Lambda_R$. Recalling *Corollary 7* and *Definition 5*, one sees why we needed the strong local solvability hypothesis.

We will next replace strong local solvability with another condition.

Proposition 17: Replace the "strong local solvability" hypothesis in *Proposition 16* with the following hypothesis:

(i)' π is a global diffeomorphism.

Then, under the same perturbation as in *Proposition 16*, the same conclusion holds.

Proof: The preceding proof for *Proposition 16* remains applicable except for the fact that $\hat{\Sigma}$ is an $(n_c + n_l)$-dimensional C^2 submanifold and that $\hat{\mathfrak{N}}$ is locally solvable. In order to prove this recall (145)–(149). Hypothesis (i)' implies that (v, i, q, ϕ) is expressible as a C^2 function of
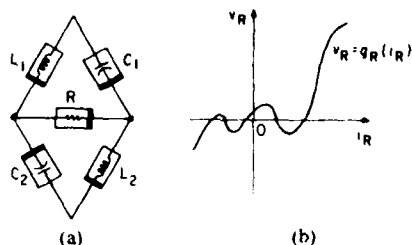
Fig. 13. A nonlinear network which becomes eventually strictly passive after perturbation. (a) The circuit diagram. (b) Resistor constitutive relation.

(q, ϕ);

$$(v, i, q, \phi) = \pi^{-1}(q, \phi).$$

Recall (150)–(158) and set $v'_L \triangleq v_L + v_r$, $i'_C \triangleq i_C - i_g$. Then (150), (151), (153), (154), and (156) are exactly the same as (145)–(149). Therefore,

$$(v_R, v_C, v'_L, i_R, i'_C, i_L, q, \phi) = \pi^{-1}(q, \phi). \quad (181)$$

It follows from (152), (155), (157), (158) and (181) that (v_r, v_g, i_r, i_g) is also a C^2 function of (q, ϕ). Therefore, all the variables of $\hat{\mathfrak{N}}$ are expressible as a C^2 function of (q, ϕ);

$$(\hat{v}, \hat{i}, q, \phi) = \hat{\pi}^{-1}(q, \phi).$$

If follows from the way $\hat{\pi}^{-1}$ was determined that $\hat{\pi}^{-1}$ is a global diffeomorphism and hence so is $\hat{\pi}$. Therefore, $\hat{\Sigma}$ is an $(n_C + n_L)$-dimensional C^2 submanifold. Since $\hat{\pi}$ is a global diffeomorphism, it is a local diffeomorphism. It follows from Theorem 1 that $\hat{\mathfrak{N}}$ is locally solvable. □

Example 17: Consider the network of Fig. 13(a) where the resistor is described by Fig. 13(b). The resistor is eventually strictly passive. It is easy to show that π is a global diffeomorphism. Therefore we can make the same perturbation as in Example 16 so that the network will have a compact set of attraction.

We will next relax the "strong local solvability" hypothesis and the global diffeomorphism assumption, while imposing a stronger condition on Λ_R to derive a different perturbation result. Recall that Λ_R is said to be *globally hybrid* [3] if

$$\Lambda_R = \{(v_R, i_R) | y = h(x)\}$$

where $y = (y_1, \cdots, y_{n_R})$, $x = (x_1, \cdots, x_{n_R})$ and if y_k is the current (resp., voltage) of the kth resistor then x_k is the voltage (resp., current) of the kth resistor. If y_k is the current (resp., voltage), then that particular resistor is called *voltage controlled* (resp., *current controlled*). The following result says that most practical networks can be perturbed in such a manner that the resulting network are locally solvable and have compact sets of attraction.

Theorem 2: Given a nonlinear network $\mathfrak{N}$ assume the following:

(i) Λ_R is closed, globally hybrid and eventually strictly passive.

(ii) Λ_C (resp. Λ_L) is locally charge (resp. flux) controlled and $\|(q, \phi)\| \to \infty$ implies $\|(v_C, i_L)\| \to \infty$ on $\Lambda_C \times \Lambda_L$.

(iii) $\Lambda \cap K \neq \varnothing$.

Perturb $\mathfrak{N}$ in the following manner:

(a) Let $\mathfrak{T}$ be a proper tree containing a maximum number of voltage controlled resistors and a minimum number

of current controlled resistors and let $\mathfrak{L}$ be its associated cotree. Insert a small linear capacitor in parallel with each voltage controlled resistor in $\mathfrak{T}$ and insert a small linear inductor in series with each current controlled resistor in $\mathfrak{L}$. Call the resulting network $\hat{\mathfrak{N}}$.

(b) Insert a large linear resistor g_k in parallel with each capacitor of $\hat{\mathfrak{N}}$ and insert a small linear resistor r_k in series with each inductor of $\hat{\mathfrak{N}}$. Call the resulting network $\bar{\mathfrak{N}}$. Then the following hold:

(1) $\bar{\Lambda} \cap \bar{K} \neq \varnothing$ and $\bar{\Sigma} = \bar{\Lambda} \cap \bar{K}$ is an $(n_C + n_L + k)$-dimensional C^2 submanifold where k is the number of reactive elements added, $\bar{\Lambda}$ and $\bar{K}$ are the resistor constitutive relations and the Kirchhoff space of $\bar{\mathfrak{N}}$, respectively.

(2) $\bar{\mathfrak{N}}$ is locally solvable.

(3) $\bar{\mathfrak{N}}$ is eventually strictly passive. Consequently $\bar{\mathfrak{N}}$ has a compact set of attraction.

Proof: (1) It is clear that one can prove $\bar{\Lambda} \cap \bar{K} \neq \varnothing$ in a similar manner to the proof of Proposition 16. We will prove $\bar{\Lambda} \pitchfork \bar{K}$ later.

(2), (3) We first claim that $\hat{\mathfrak{N}}$ is strongly locally solvable. To this end partition (v, i) of $\hat{\mathfrak{N}}$ as in the proof of Proposition 16. Since Λ_R is assumed to be globally hybrid, it can be represented as follows:

$$i_{V_\pi} - f_{V_\pi}(v_{V_\pi}, v_{V_\ell}, i_{I_\pi}, i_{I_\ell}) = 0$$
$$i_{V_\ell} - f_{V_\ell}(v_{V_\pi}, v_{V_\ell}, i_{I_\pi}, i_{I_\ell}) = 0$$
$$v_{I_\pi} - f_{I_\pi}(v_{V_\pi}, v_{V_\ell}, i_{I_\pi}, i_{I_\ell}) = 0$$
$$v_{I_\ell} - f_{I_\ell}(v_{V_\pi}, v_{V_\ell}, i_{I_\pi}, i_{I_\ell}) = 0$$

where V and I denote voltage controlled and current controlled resistors, respectively. We write these equations as

$$f_R(v_R, i_R) = 0.$$

It follows from (120) that for $\hat{\mathfrak{N}}$ we have

$$\mathfrak{K}_R(\hat{v}_R, \hat{i}_R) = \left[D_{(v_{V_\pi}, v_{V_\ell})} f_R \ \vdots \ D_{(i_{I_\ell}, i_{I_\pi})} f_R \right]_{(\hat{v}_R, \hat{i}_R)}$$

$$\begin{array}{cccc} v_{I_\pi} & v_{I_\ell} & i_V & i_{V_\pi} \end{array}$$

$$= \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

Therefore, $\mathfrak{K}_R(\hat{v}_R, \hat{i}_R)$ is a constant nonsingular matrix and $\hat{\mathfrak{N}}$ is strongly locally solvable. Clearly, $\hat{\Lambda}_R = \Lambda_R$ because no resistors are added in (a). This implies that $\hat{\Lambda}_R$ is eventually strictly passive. Since $\hat{\mathfrak{N}}$ satisfies the hypothesis of Proposition 16, by taking procedure (b), we obtain $\bar{\mathfrak{N}}$ which is locally solvable, $\bar{\Lambda} \pitchfork \bar{K}$ and eventually strictly passive. □

Example 18: Consider the network of Fig. 14(a), where R_1 and R_2 are as in Fig. 2(b). Other elements are linear. By a similar reasoning to that of Example 2, one can show that $\Lambda \pitchfork K$. Pick the proper tree $\mathfrak{T} = \{C_1, C_2, C_3, R_2\}$. Then applying procedure (a) of Theorem 2, we obtain $\hat{\mathfrak{N}}$ which is strongly locally solvable (Fig. 14(b)). The network $\hat{\mathfrak{N}}$ of Fig. 14(b) does not satisfy the fundamental topological hypothesis, however, because there is a capacitor-only cut set. Insert large linear resistors, $g_1, g_2, g_3,$ and g_4 according
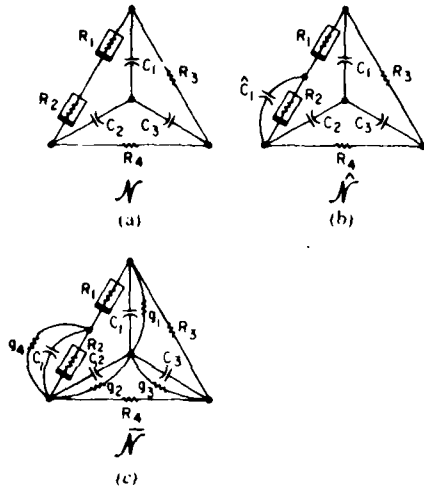
Fig. 14. A nonlinear network which becomes locally solvable and eventually strictly passive after perturbations. (a) Original network $\mathcal{N}$. (b) Perturbed network $\hat{\mathcal{N}}$. (c) Perturbed network $\bar{\mathcal{N}}$.

to procedure (b) and obtain $\bar{\mathcal{N}}$ (Fig. 14(c)). *Theorem 2* says that $\bar{\mathcal{N}}$ has a compact set of attraction.

Remark: The elements added in *Theorem 2* can be thought of as parasitic elements of $\mathcal{N}$. Therefore *Theorem 2* formally justifies the fact that in most networks of practical interest, voltage and current waveforms eventually approach a fixed compact set.

APPENDIX

Proof of Proposition 3: We will first prove the following:

Lemma A. Suppose that Λ_C (resp. Λ_L) is locally voltage (resp. current) controlled. Then $\Lambda \bar{\cap} K$ if and only if $\Lambda_R \bar{\cap} \pi_R'(K)$, where π_R' is defined by (20).

Proof: If Λ_C (resp. Λ_L) is locally voltage (resp. current) controlled, then v_C (resp. i_L) serves as a local coordinate system for Λ_C (resp. Λ_L). This implies that $(Dv_C)_{p_L}$ (resp. $(Di_L)_{p_L}$) in (42) is nonsingular. By elementary operations, one can show that (41) holds if and only if

$$\text{rank}\left[\begin{bmatrix} Dv_{R_e} \\ Dv_{R_q} \\ Di_{R_e} \\ Di_{R_q} \end{bmatrix}_{p_R} : \begin{matrix} -B_{RR} & -B_{RC} & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & B_{RR}^T & B_{LR}^T \end{matrix}\right] = 2n_R. \tag{A.1}$$

Next, observe that

$$T_{(v_R, i_R)}\pi_R'(K) = \text{Im}\begin{matrix} v_{R_q} & v_C & i_{R_e} & i_L \\ \begin{bmatrix} -B_{RR} & -B_{RC} & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & B_{RR}^T & B_{LR}^T \end{bmatrix}\end{matrix} \tag{A.2}$$

$$T_{(v_R, i_R)}\Lambda_R = \text{Im}\begin{bmatrix} Dv_{R_e} \\ Dv_{R_q} \\ Di_{R_e} \\ Di_{R_q} \end{bmatrix}_{p_R}. \tag{A.3}$$

Equations (A.2) and (A.3) imply that $\Lambda_R \bar{\cap} \pi_R'(K)$ if and only if (A.1) holds. $\square$

Knowing that $\Lambda \bar{\cap} K$ is equivalent to $\Lambda_R \bar{\cap} \pi_R'(K)$, one sees that *Proposition 3* can be proved in a similar manner to that of *Theorem 3* in [3] which is the same as the proof of (ii-a) of *Theorem 2* of [3]. Proof of (ii-a) of *Theorem 2* uses *Lemmas 1, 2,* and *4* of [3]. It is easy to show that *Lemma 1* is true for C^2 submanifolds. *Lemma 2* has nothing to do with differentiability. Therefore we need to only show that *Lemma 4* is true in the C^2 category. We state this in the following:

Lemma B: Let A be an $n \times n$ matrix such that $\|A - 1\|$ is sufficiently small. Then there are neighborhoods U_1 and U_2 of the origin of $\mathbf{R}^n$ with $\bar{U}_1 \subset U_2$ and there is a C^2 function $G: \mathbf{R}^n \to \mathbf{R}^n$ such that

(i) $G = A$ on U_1,

(ii) $G = i_d$ off U_2, where i_d is the identity map of $\mathbf{R}^n$.

(iii) G is arbitrarily close to i_d in the strong C^2 topology.

Proof: Let $\mathfrak{N}^2(i_d; \epsilon(\cdot))$ be a sufficiently small neighborhood of i_d in $C^2(\mathbf{R}^n, \mathbf{R}^n)$ with respect to the strong C^2 topology. Since $\epsilon(x) > 0$ for all $x \in \mathbf{R}^n$, there are numbers $\epsilon > 0$ and $\delta > 0$ such that $\epsilon(x) \geq \epsilon$ for all x with $\|x\| < \delta$. Let δ_0 satisfy $0 < \delta_0 < \delta$. Then there is a C^2 function (bump function [6]) $\mu: \mathbf{R}^n \to \mathbf{R}$ such that

(i) $\mu(x) = \begin{cases} 1, & \text{if } \|x\| < \delta_0 \\ 0, & \text{if } \|x\| \geq \delta \end{cases}$ \hspace{1cm} (A.4)

(ii) There is a $k > 0$ such that

$$\|(D\mu)_x\| < k, \quad \|(D^2\mu)_x\| < k \tag{A.5}$$

for all $x \in \mathbf{R}^n$. Now, choose A close enough to i_d so that

$$\|A - 1\| < \frac{\epsilon}{(1 + \delta)(1 + 2k)} \tag{A.6}$$

and define

$$G(x) \triangleq \mu(x)Ax + (1 - \mu(x))x.$$

We will show that $G \in \mathfrak{N}^2(i_d; \epsilon(\cdot))$. Since $\mu(x) \equiv 0$ for $\|x\| \geq \delta$, we need to check the C^2 size of $G - i_d$ only for $\|x\| < \delta$. Since $G(x) - x = \mu(x)(Ax - x)$, we have, using (A.4)-(A.6), that

$\|G(x) - x\| + \|(DG)_x - 1\| + \|(D^2G)_x\|$

$\leq \mu(x)\|Ax - x\| + \|(D\mu)_x\|\|Ax - x\|$

$+ \mu(x)\|A - 1\| + \|(D^2\mu)_x\|\|Ax - x\| + 2\|(D\mu)_x\|\|A - 1\|$

$\leq \|A - 1\|(\|x\| + k\|x\| + 1 + k\|x\| + 2k)$

$\leq \|A - 1\|(1 + \delta)(1 + 2k) < \epsilon.$

Take $U_1 \triangleq \{x \in \mathbf{R}^n | \|x\| < \delta\}$ and $U_2 \triangleq \{x \in \mathbf{R}^n | \|x\| < \delta_0\}$. Then all the properties are satisfied. $\square$

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