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A THEORY OF LAMINATED COMPOSITE PLATES AND RODS. (U)

MAY 81 A E GREEN; P M NAGHDI

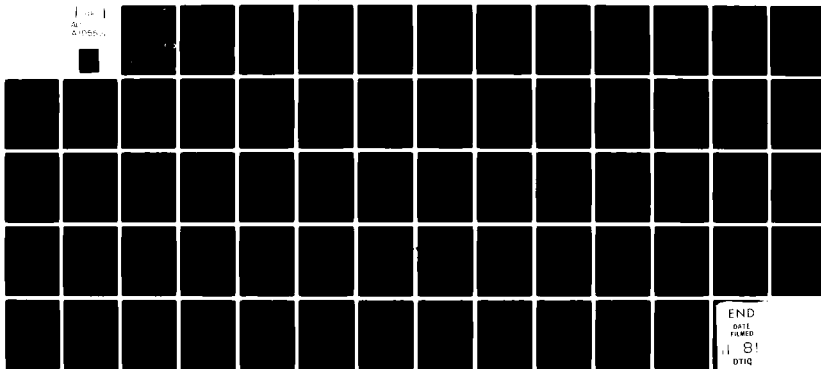
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20. Abstract (continued)

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# A Theory of Laminated Composite Plates and Rods

by

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Abstract. This paper is concerned with small deformations of a class of laminated layered composite plates and a class of laminated layered composite rods utilizing a direct formulation of thermomechanical theories of shells and rods. The paper is arranged in two parts, namely Part A (for laminated composite plates) and Part B (for laminated composite rods), and can be read independently of each other. In each part, after providing the main ingredients of a direct formulation, fairly general linear theories are developed which include both dynamical and thermal effects. Some applications to sandwich plates and layered rods are discussed in detail.

## 1. Introduction

Among the many types of composite materials which have been studied, layered media have received much attention. One group of such media consists of a large number of alternating plane parallel layers of two homogeneous and usually isotropic, elastic materials. Three main approaches have been used to construct theories to describe their mechanical behavior. One, termed the effective modulus theory, consists in replacing the composite by a homogeneous anisotropic medium whose material constants are determined in terms of the geometry and material properties of the constituents of the composite. The second, called the effective stiffness theory, yields more detailed information about the composite than the effective modulus theory; and is expounded, for example, by Sun, Achenbach and Herrmann (1968) with extensions by Drumheller and Bedford (1973). This theory also has some connection with microstructure theories as shown by Herrmann and Achenbach (1968) and Achenbach (1975). The third, discussed by Stern and Bedford (1972) is based on a multi-continuum theory for closely coupled mixtures.

Another class of layered media consists of multilayered shells and plates. The earliest theory used for these seems to have been of the effective modulus type for elastic composites, which gives useful, but limited information about the behavior of the composites. A different approach, used by Malcolm and Glockner (1976) for a three layered isothermal, elastostatic sandwich shell and by Epstein and Glockner (1977) for an  $n$  layered elastostatic shell involves a single surface and a single director, or  $n$  directors, respectively, one associated with each layer of the sandwich, together with the use of variational procedures. Another type of theory given by Epstein and Glockner (1979) models the elastic shell by a surface and  $n$  directors associated with the surface,

and again employs variational methods. Here there is considerable difficulty in identifying the elastic coefficients of the sandwich shell or plate in order to render this theory useful for specific problems.

The effective modulus or classical theory for laminates consisting of linear elastic material, has been extended to a higher order theory by Lo, Christensen and Wu (1977,1978) and has been discussed also by Christensen (1979). This theory utilizes one expansion for displacements throughout the laminates, whatever the number of laminates, together with a variational procedure, and has been derived for statical problems under isothermal conditions. Displacements and stresses are continuous at the interfaces and the theory yields more detailed information than the classical theory. A different higher order theory given by Pagano (1978), again is limited to statical problems under isothermal conditions. Pagano (1978) assumes stress distributions for each laminate which satisfy appropriate surface conditions at interfaces, and derives the displacement fields and equations of equilibrium with the help of a variational procedure.

With the use of an expansion procedure, Hegemier and Bache (1973) have discussed special types of one-dimensional motions in a plate with isotropic layers. Similarly, Fourier integral methods were used earlier by Buffer (1971) for infinite plates with isotropic layers.

In the present paper a different approach from those noted above is used to obtain a linear theory of elastic sandwich plates, one which is available for dynamical problems and which allows for thermal effects. The method used can also be applied to sandwich type shells and to materials which are not necessarily elastic. The theory of plates and shells for single phase media has a long history — see, e.g., Naghdi (1972). In recent years considerable attention has been given to direct two-dimensional theories from a thermodynamical point of view where temperature effects

along the major directions of the plate or shell are considered. The full thermomechanical theory for plates or shells in which temperature effects across the thickness, as well as along the major surfaces, are incorporated, has been given by Green and Naghdi (1979a). In particular, the linear thermoelastic theory for a plate is studied in detail (Green and Naghdi 1979a) and all the thermoelastic coefficients for an isotropic plate were found in terms of the three-dimensional coefficients. This theory, extended here to orthotropic plates, is immediately available for each individual member of a sandwich type plate (or shell), together with appropriate conditions to ensure continuity in displacements and surface forces at interfaces, or appropriate specified values for surface forces or displacements if other conditions prevail at interfaces. The final form of the theory has some similarities to that given by Pagano (1978) when the results are specialized to the isothermal static case. It does not, however, yield immediate values for the actual stress distribution in each laminate, but is concerned with force resultants and higher order couple resultants. These, if desired, can be used to obtain approximate values for stresses by a procedure similar to that of Pagano (1978).

In a similar way, the complete direct formulation of a thermoelastic theory of a rod with rectangular cross-section, which has been developed by Green et al. (1974) based on earlier work on the subject (Green and Laws 1966, 1973), is employed in Part B (sections 10-14) to study the behavior of laminated composite beams in which each layer of the composite is a plane rod (or a beam) of rectangular cross-section with different orthotropic properties. Although our main developments for composite rods are limited to a linear isothermal theory, the same approach together with the use of recent results for nonisothermal rods (Green and Naghdi 1979b) can be used to account for thermal effects and a more general formulation of composites

in the presence of finite deformation. A different approach from that given here for rods has been used by Dökmeci (1973) who considered a nonlinear isothermal theory of elastic, composite beams based on an expansion for displacements in each layer together with a variational approach for obtaining equations of motion. Dökmeci (1973) obtains his constitutive equations for force resultants from the energy equation in the three-dimensional theory but this portion of his work suggests some doubt about the values of some of his constitutive coefficients especially those for flexure.

Specifically, the contents of the paper are arranged in two parts and can be read independently of each other: One part (Part A) is concerned with the linear theory of laminated layered composite plates and the other (Part B) is devoted to the linear theory of laminated layered composite rods. In Part A (sections 2-9), first a concise summary of the basic equations of the linear theory of orthotropic elastic plates, constructed by a direct approach via the linearized theory of a Cosserat surface with a single director, is given and this is followed (section 3) by a corresponding summary of a restricted linear theory of plates by direct approach. Next, the developments in sections 2 and 3 are used to formulate a linear theory for a laminated plate containing  $N$  laminates each consisting of a plate of homogeneous orthotropic material of constant density and thickness and each at constant temperature along with appropriate continuity conditions for the tractions and the displacements at the interfaces.

The results in sections 4-6 are specialized to composites with three laminates and are applied to a number of cases, which include the thermo-static problems of the torsion of a three layer laminated rectangular plate in the context of thermo-statics, the study of harmonic waves and the case



of periodic force resultant distribution in a stratified medium consisting of a large number of alternating parallel layers of two homogeneous orthotropic materials.

In Part B (sections 10-14), first a concise summary of the basic equations of the linear theory of orthotropic elastic rods, constructed by a direct approach via the linearized theory of a Cosserat rod, is given. These are then applied (sections 12-14) to some simple static solutions for composite rods and harmonic wave propagation along the composite rod.

## Part A. Laminated Layered Composite Plates

### 2. Basic equations of the linear theory of orthotropic elastic plate

Consider a plate of constant thickness  $h$ , constant density  $\rho^*$  and at constant temperature  $\bar{\theta}$ , bounded by the planes  $z = \pm \frac{1}{2}h$ , where  $x_i = (x, y, z)$  are rectangular Cartesian coordinates, with unit vectors  $\underline{e}_i$  along the  $x_i$ -axes. The plate has homogeneous orthotropic symmetry with respect to the  $z$ -axis and two axes orthogonal to the  $z$ -axis. The basic equations for the deformation of such a plate have been given in a number of previous papers and, in particular, in the monograph by Naghdi (1972) which also contains values of the mechanical constitutive coefficients for a linear isotropic, elastic plate (Naghdi 1972, Ch. E). The full thermodynamical coefficients for an isotropic, elastic plate have been given by Green and Naghdi (1979a). The main equations and constitutive coefficients are summarized below in Cartesian tensor notation with Greek indices taking the values 1, 2. Partial differentiation with respect to  $x_\alpha$  is denoted by  $(\ )_{,\alpha}$  and a superposed dot denotes partial differentiation with respect to time.

We recall that a Cosserat surface  $C$  comprises a material surface  $S$  and a director assigned to every point of the material surface. In the reference configuration, let the material surface be a plane  $S_R$  which we identify with the middle plane  $z = 0$  of the plate described in the preceding paragraph. Let  $\underline{u}$  and  $\underline{\delta}$  denote, respectively, the infinitesimal displacement vector and the infinitesimal director displacement of  $S_R$  (or the plane  $z = 0$ ). Then, referred to the basis  $\underline{e}_i$ ,  $\underline{u}$  and  $\underline{\delta}$  may be expressed as

$$\underline{u} = u_{\alpha} \underline{e}_{\alpha} + u_3 \underline{e}_3, \quad \underline{\delta} = \delta_{\alpha} \underline{e}_{\alpha} + \delta_3 \underline{e}_3 \quad (2.1)$$

and the strain measures are

$$\begin{aligned} e_{\alpha\beta} &= \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \quad , \quad \rho_{3\alpha} = \delta_{3,\alpha} \quad , \\ \rho_{\alpha\beta} &= \delta_{\alpha,\beta} \quad , \quad \gamma_{\alpha} = \delta_{\alpha} + u_{3,\alpha} \quad . \end{aligned} \quad (2.2)$$

We denote the temperature change in the plane  $S_R$  by  $\theta$  and the temperature change along the thickness of the plate by  $\phi$ .

The contact force vector  $\underline{N}$ , the contact director force vector  $\underline{M}$ , the internal force director  $\underline{m}$  and the entropy fluxes  $k, \ell$  are given by

$$\begin{aligned} \underline{N} &= (N_{\alpha\beta} e_{\beta} + V_{\alpha} e_3) v_{\alpha} \quad , \quad \underline{M} = (M_{\alpha\beta} e_{\beta} + M_{\alpha 3} e_3) v_{\alpha} \quad , \\ \underline{m} &= V_{\alpha} e_{\alpha} + V_3 e_3 \quad , \quad k = p_{\alpha} v_{\alpha} \quad , \quad \ell = p_{\alpha}^1 v_{\alpha} \end{aligned} \quad (2.3)$$

where  $v_{\alpha} = v_{\alpha} e_{\alpha}$  is the outward unit normal to a curve in the  $(x,y)$  plane in a specified direction. The force vectors  $\underline{u}^t, \underline{\ell}^t$  on the major surfaces  $z = \pm \frac{1}{2}h$ , respectively, in terms of their components are given by

$$\underline{u}^t = u_{\alpha}^t e_{\alpha} + u_3^t e_3 \quad , \quad \underline{\ell}^t = \ell_{\alpha}^t e_{\alpha} + \ell_3^t e_3 \quad (2.4)$$

and  $\underline{u}^k, \underline{\ell}^k$  are the entropy flux vectors, respectively, at these surfaces. Then the field equations for forces and entropy fluxes, in which body forces and entropy sources are zero, are

$$\begin{aligned} N_{\alpha\beta,\alpha} + u_{\alpha}^t e_{\beta} + \ell_{\alpha}^t e_{\beta} &= \rho \ddot{u}_{\beta} \quad , \\ M_{\alpha 3,\alpha} - V_3 + \frac{1}{2} h (u_3^t - \ell_3^t) &= \rho \alpha \ddot{\delta}_3 \quad , \\ -p_{\alpha,\alpha} - u^k - \ell^k &= \rho \dot{\eta} \quad , \end{aligned} \quad (2.5)$$

$$M_{\alpha\beta,\alpha} - V_{\beta} + \frac{1}{2} h(u t_{\beta} - l t_{\beta}) = \rho \ddot{\alpha}_{\beta} ,$$

$$V_{\alpha,\alpha} + u t_3 + l t_3 = \rho \ddot{u}_3 , \quad (2.6)$$

$$-p_{\alpha,\alpha}^1 + \rho \xi_1 - \frac{1}{2} h(u k^* - l k^*) = \rho \dot{\eta}_1 ,$$

where  $\eta, \eta_1$  are entropy densities and  $\xi_1$  is an internal rate of production of entropy.

The desired thermoelastic constitutive equations for an orthotropic plate can be derived from a Helmholtz free energy function  $\psi$  of the form

$$\begin{aligned} 2\rho\psi = & A_{\lambda\mu}^{\alpha\beta} e_{\alpha\beta} e_{\lambda\mu} + 2A_{\lambda\mu} e_{\lambda\mu} \delta_3 + A(\delta_3)^2 + E_{\lambda\mu} \rho_{3\lambda} \rho_{3\mu} \\ & + D_{\lambda\mu}^{\alpha\beta} \rho_{\alpha\beta} \rho_{\lambda\mu} + D_{\lambda\mu} \gamma_{\lambda} \gamma_{\mu} \\ & + 2B\delta_3 \theta + 2B_{\lambda\mu} e_{\lambda\mu} \theta + D\theta^2 \\ & + 2F_{\lambda\mu} \rho_{\lambda\mu} \phi + F\phi^2 , \end{aligned} \quad (2.7)$$

where for compactness we have designated the usual Cartesian tensor constitutive coefficients  $A_{\alpha\beta\lambda\mu}, D_{\alpha\beta\lambda\mu}$  as  $A_{\lambda\mu}^{\alpha\beta}, D_{\lambda\mu}^{\alpha\beta}$  defined by

$$\begin{aligned} A_{\lambda\mu}^{\alpha\beta} &= \delta^{\alpha\xi} \delta^{\beta\eta} A_{\xi\eta\lambda\mu} , \quad D_{\lambda\mu}^{\alpha\beta} = \delta^{\alpha\xi} \delta^{\beta\eta} D_{\xi\eta\lambda\mu} , \\ \delta^{11} &= \delta^{22} = 1 , \quad \delta^{12} = \delta^{21} = 0 . \end{aligned} \quad (2.8)$$

Also, the Cartesian tensor components of the thermoelastic coefficients satisfy the symmetries

$$\begin{aligned} A_{\alpha\beta\lambda\mu} &= A_{\beta\alpha\lambda\mu} = A_{\alpha\beta\mu\lambda} = A_{\lambda\mu\alpha\beta} , \quad A_{\lambda\mu} = A_{\mu\lambda} , \\ D_{\alpha\beta\lambda\mu} &= D_{\lambda\mu\alpha\beta} , \quad D_{\lambda\mu} = D_{\mu\lambda} , \quad B_{\lambda\mu} = B_{\mu\lambda} . \end{aligned} \quad (2.9)$$

The corresponding constitutive equations are

$$\begin{aligned}
N_{\alpha\beta} &= A_{\alpha\beta}^{\lambda\mu} e_{\lambda\mu} + A_{\alpha\beta} \delta_3 + B_{\alpha\beta} \theta, \\
V_3 &= A_{\alpha\beta} e_{\alpha\beta} + A \delta_3 + B \theta, \quad M_{\alpha 3} = E_{\alpha\lambda} \rho_{3\lambda}, \\
\rho\eta &= -(B \delta_3 + B_{\alpha\beta} e_{\alpha\beta} + D \theta), \quad p_\alpha = -a_{\alpha\lambda} \theta_{,\lambda},
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
M_{\alpha\beta} &= D_{\alpha\beta}^{\lambda\mu} \rho_{\mu\lambda} + F_{\alpha\beta} \phi, \quad V_\alpha = D_{\alpha\lambda} \gamma_\lambda, \quad \rho \xi_1 = -b \phi, \\
\rho\eta_1 &= -(F_{\alpha\beta} \rho_{\alpha\beta} + F \phi), \quad p_\alpha^1 = -b_{\alpha\lambda} \phi_{,\lambda}.
\end{aligned} \tag{2.11}$$

The various coefficients are given below in terms of three-dimensional coefficients of linear orthotropic elastic materials for the case in which the coordinate axes may be at an angle with the principal directions of orthotropy in the plane.

In order to introduce the relevant three-dimensional constitutive coefficients we recall that the (three-dimensional) constitutive equations for the stress tensor  $t_{ij}$ , entropy  $\eta^*$ , heat flux vector  $q^*$  and entropy flux vector have the forms

$$\begin{aligned}
t_{ij} &= c_{ij}^{rs} e_{rs} - c_{ij} \theta^*, \quad \rho^* \eta^* = c_{ij} e_{ij} + \rho^* (c/\bar{\theta}) \theta^*, \\
q^* &= \bar{\theta} p^*, \quad p^* = p_i^* e_i, \quad p_i^* = -(d_{ij}/\bar{\theta}) \theta_{,j}^*,
\end{aligned} \tag{2.12}$$

where  $e_{ij}$  and  $\theta^*$  are, respectively, the strain components and temperature in the three-dimensional theory, and Latin indices have the values 1,2,3. Again, instead of the Cartesian tensor coefficients  $c_{ijrs}$ , for compactness we have used the notation  $c_{rs}^{ij}$  defined as

$$c_{rs}^{ij} = \delta^{im} \delta^{jn} c_{mrs} \quad ,$$

(2.13)

$$\delta^{11} = \delta^{22} = \delta^{33} = 1 \quad , \quad \delta^{ij} = 0 \text{ for } i \neq j \quad .$$

Also, the various coefficients in (2.12) and (2.13) satisfy the symmetries

$$c_{ijrs} = c_{jirs} = c_{ijsr} = c_{rsij} \quad , \quad c_{ij} = c_{ji} \quad , \quad d_{ij} = d_{ji} \quad . \quad (2.14)$$

In the special case when the principal axes of orthotropy coincide with the coordinate axes, the non-zero components, distinguished by a prefix o, are

$$o c_{11}^{11}, o c_{22}^{11}, o c_{33}^{11}, o c_{22}^{22}, o c_{22}^{33}, o c_{33}^{33}, o c_{12}^{12}, o c_{23}^{23}, o c_{13}^{13}, \quad (2.15)$$

$$o c_{11}^{11}, o c_{22}^{22}, o c_{33}^{33}, o d_{11}^{11}, o d_{22}^{22}, o d_{33}^{33},$$

together with the values given by (2.14). When the axes of orthotropy in the x,y plane are at an angle (say  $\chi$ ) to the coordinate axes, the non-zero coefficients are

$$c_{11}^{11}, c_{22}^{11}, c_{11}^{33}, c_{22}^{22}, c_{22}^{33}, c_{33}^{33}, c_{12}^{12}, c_{23}^{23}, c_{13}^{13}, \quad (2.16)$$

$$c_{12}^{11}, c_{22}^{12}, c_{12}^{33}, c_{23}^{13}, c_{11}^{11}, c_{22}^{22}, c_{33}^{33}, c_{12}^{12},$$

$$d_{11}^{11}, d_{22}^{22}, d_{33}^{33}, d_{12}^{12}, d_{21}^{21},$$

together with the values given by (2.14). The values of the coefficients (2.16) may be expressed in terms of the coefficients (2.15) by two-dimensional tensor transformations between

$$c_{\lambda\mu}^{\alpha\beta}, c_{\alpha\beta}^{33}, c_{\beta\beta}^{\alpha 3}, c_{\alpha\beta}^{\alpha\beta}, d_{\alpha\beta}^{\alpha\beta},$$

and

$$o c_{\lambda\mu}^{\alpha\beta}, o c_{\alpha\beta}^{33}, o c_{\beta\beta}^{\alpha 3}, o c_{\alpha\beta}^{\alpha\beta}, o d_{\alpha\beta}^{\alpha\beta},$$

(2.17)

respectively, together with

$$c_{33}^{33} = {}_0c_{33}^{33} , \quad c_{33} = {}_0c_{33} , \quad d_{33} = {}_0d_{33} . \quad (2.18)$$

In order to specify the constitutive coefficients in (2.10) and (2.11) in terms of the three-dimensional coefficients (2.16) it is simplest to use axes which coincide with the coordinate axes  $x, y$  and relate simple exact solutions of three-dimensional elasticity with solutions of the equations (2.5), (2.6), (2.10), (2.11). Coefficients with respect to general axes in the  $x, y$  plane may then be found by appropriate tensor transformations. The results are

$$\begin{aligned} A_{\lambda\mu}^{\alpha\beta} &= hc_{\lambda\mu}^{\alpha\beta} , \quad A_{\alpha\beta} = hc_{\alpha\beta}^{33} , \quad A = hc_{33}^{33} , \\ E_{\alpha\beta} &= \beta h^3 \delta_{\alpha\lambda} c_{\beta 3}^{\lambda 3} , \quad \rho = \rho^* h , \quad B_{\alpha\beta} = -hc_{\alpha\beta} , \quad B = -hc_{33} , \\ D &= -\rho^* ch/\bar{\theta} , \quad a_{\alpha\beta} = h d_{\alpha\beta}/\bar{\theta} , \\ \delta_{11} &= \delta_{22} = 1 , \quad \delta_{12} = \delta_{21} = 0 \end{aligned} \quad (2.19)$$

and

$$\begin{aligned} D_{\lambda\mu}^{\alpha\beta} &= \frac{1}{12} h^3 (c_{\lambda\mu}^{\alpha\beta} - \frac{c_{33}^{\alpha\beta} c_{\lambda\mu}^{33}}{c_{33}^{33}}) , \\ D_{\alpha\beta} &= \bar{\beta} h \delta_{\alpha\lambda} c_{\beta 3}^{\lambda 3} , \quad \alpha = \frac{1}{12} h^2 , \\ F_{\alpha\beta} &= -\frac{1}{12} h^3 (c_{\alpha\beta} - \frac{c_{\alpha\beta}^{33} c_{33}^{33}}{c_{33}^{33}}) , \\ F &= -\frac{1}{12} h^3 (\frac{\rho^* c}{\bar{\theta}} + \frac{c_{33}^{33} c_{33}^{33}}{c_{33}^{33}}) , \\ b &= \frac{h}{\bar{\theta}} d_{33} , \quad b_{\alpha\beta} = \frac{1}{12} \frac{h^3}{\bar{\theta}} d_{\alpha\beta} . \end{aligned} \quad (2.20)$$

An approximate comparison with the torsion problem for a plate, or an exact comparison with one of the flexural modes of wave propagation, leads to the

choice  $\bar{\beta} = \pi^2/12$  for  $\bar{\beta}$ . This is approximately 5/6, the value specified in previous papers. The value of  $\beta$  is much more dependent on the particular problem chosen as a basis of comparison. Here we choose  $\beta = \frac{7}{120}$  as in previous papers.

For one purpose below we require equations of the form (2.12) for a plate which has symmetry of structure at any point only with respect to directions normal to the middle surface, i.e., in the z-direction, but does not necessarily have symmetry with respect to the x and y directions. The non-zero coefficients in (2.12) are then the set given by (2.16), (2.13) and (2.14).



### 3. Summary of a restricted linear theory of plates

In addition to the theory summarized in section 2, in subsequent sections we wish to consider also the use of a system of equations which correspond to those of a classical (Poisson-Kirchhoff) linear theory for a plate having symmetry with respect to directions which are normal to the plate but which is not necessarily orthotropic. These equations may be obtained from the present equations by a limiting or approximate procedure. Alternatively, they may be derived from linearization of the restricted theory discussed by Naghdi (1972, Secs. 10,15), but extended here to account for full thermal effects and expressed in a slightly more general form. The plane faces of the plate are given by  $z = a$ ,  $z = a + h$  where  $a$  is constant. Again, we identify the material surface  $\mathcal{S}_R$  in the reference configuration with the plane  $z = 0$ , which is any suitable reference plane in the plate. In the restricted theory the director is coincident with the outward unit normal to the plane  $\mathcal{S}_R$  so that  $\underline{\delta} = \underline{\beta}$ ,  $\underline{\beta} = \beta_{\alpha} \underline{e}_{\alpha}$ ,  $\beta_{\alpha} = -u_{3,\alpha}$ , while the infinitesimal displacement  $\underline{u}$  is still given by (2.1)<sub>1</sub>. The relevant field equations in the restricted linear theory are

$$\begin{aligned}
 N_{\alpha\beta,\alpha} + u^t_{\beta} + \ell^t_{\beta} &= \rho \ddot{u}_{\beta} \quad , \quad u^t_3 - \ell^t_3 = 0 \quad , \\
 -p_{\alpha,\alpha} - u^k_{\alpha} - \ell^k_{\alpha} &= \rho \dot{\eta} \quad , \\
 M_{\alpha\beta,\alpha} - V_{\beta} + (a+h)u^t_{\beta} + a \ell^t_{\beta} &= 0 \quad , \\
 V_{\alpha,\alpha} + u^t_3 + \ell^t_3 &= \rho \ddot{u}_3 \quad , \\
 -p^1_{\alpha,\alpha} + \rho \xi_1 - (a+h)u^k_{\alpha} - a \ell^k_{\alpha} &= \rho \dot{\eta}_1 \quad ,
 \end{aligned}
 \tag{3.1}$$

where for a plate which has symmetry only with respect to the  $z$ -direction

$$\begin{aligned}
N_{\alpha\beta} &= \bar{A}_{\alpha\beta}^{\lambda\mu} e_{\lambda\mu} + \bar{A}'_{\alpha\beta}{}^{\lambda\mu} \rho_{\lambda\mu} + \bar{B}_{\alpha\beta} \theta + \bar{B}'_{\alpha\beta} \phi, \\
\rho\eta &= -(\bar{B}_{\alpha\beta} e_{\alpha\beta} + \bar{B}'_{\alpha\beta} \rho_{\alpha\beta} + \bar{D}\theta + \bar{D}'\phi), \\
p_\alpha &= -\bar{a}_{\alpha\lambda} \theta_{,\lambda} - \bar{a}'_{\alpha\lambda} \phi_{,\lambda}
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
M_{\alpha\beta} &= \bar{D}_{\alpha\beta}^{\lambda\mu} \rho_{\lambda\mu} + \bar{A}_{\alpha\beta}{}^{\lambda\mu} e_{\lambda\mu} + \bar{F}_{\alpha\beta} \phi + \bar{F}'_{\alpha\beta} \theta, \\
\rho\eta_1 &= -(\bar{F}_{\alpha\beta} \rho_{\alpha\beta} + \bar{B}'_{\alpha\beta} e_{\alpha\beta} + \bar{F}\phi + \bar{D}'\theta), \\
p_\alpha^1 &= -\bar{b}_{\alpha\lambda} \phi_{,\lambda} - \bar{b}'_{\alpha\lambda} \theta_{,\lambda}, \quad \rho\xi_1 = -\bar{b}\phi, \\
\rho_{\alpha\beta} &= -u_{3,\alpha\beta}.
\end{aligned} \tag{3.3}$$

Also  $V_\alpha$  is not now determined by a constitutive equation. The various coefficients may be specified in terms of the coefficients (2.14) by studying simple problems which have exact solutions in three dimensions. For later convenience we assume here that the coefficients (2.14) are functions of  $z$ . Thus

$$\rho = \int_a^{a+h} \rho^* dz, \quad \bar{A}_{\lambda\mu}^{\alpha\beta} = \int_a^{a+h} C_{\lambda\mu}^{\alpha\beta} dz,$$

$$\bar{A}_{\lambda\mu}^{\alpha\beta} = \int_a^{a+h} C_{\lambda\mu}^{\alpha\beta} z dz,$$

$$\bar{B}_{\alpha\beta} = - \int_a^{a+h} C_{\alpha\beta} dz, \quad \bar{B}'_{\alpha\beta} = \bar{F}'_{\alpha\beta} = - \int_a^{a+h} C_{\alpha\beta} z dz,$$

$$\bar{D} = - \int_a^{a+h} C dz, \quad \bar{D}' = - \int_a^{a+h} C z dz,$$

(3.4a)

$$\bar{a}_{\alpha\beta} = \frac{1}{\theta} \int_a^{a+h} d_{\alpha\beta} dz, \quad \bar{a}'_{\alpha\beta} = \bar{b}'_{\alpha\beta} = \frac{1}{\theta} \int_a^{a+h} d_{\alpha\beta} z dz,$$

$$\bar{D}_{\lambda\mu}^{\alpha\beta} = \int_a^{a+h} C_{\lambda\mu}^{\alpha\beta} z^2 dz, \quad \bar{F}_{\alpha\beta} = - \int_a^{a+h} C_{\alpha\beta} z^2 dz,$$

$$\bar{F} = - \int_a^{a+h} C z^2 dz, \quad \bar{b}_{\alpha\beta} = \frac{1}{\theta} \int_a^{a+h} d_{\alpha\beta} z^2 dz,$$

$$\bar{b} = \frac{1}{\theta} \int_a^{a+h} d_{33} dz,$$

where

$$C_{\lambda\mu}^{\alpha\beta} = c_{\lambda\mu}^{\alpha\beta} - \frac{c_{33}^{\alpha\beta} c_{\lambda\mu}^{33}}{c_{33}},$$

$$C_{\alpha\beta} = c_{\alpha\beta} - \frac{c_{\alpha\beta}^{33} c_{33}^{33}}{c_{33}},$$

(3.4b)

$$C = \rho^* \frac{c}{\theta} + \frac{c_{33}^{33} c_{33}^{33}}{c_{33}}.$$

#### 4. Laminated plate

We suppose that the plate contains  $N$  laminates each consisting of a plate of homogeneous orthotropic material of constant density and thickness and each at constant temperature  $\bar{\theta}$ . A typical laminate, the  $n^{\text{th}}$ , has constant density  $\rho_n^*$  and thickness  $h_n$  and lies between the planes  $z = H_n \pm \frac{1}{2} h_n$  where  $H_n$  is constant, and each laminate is glued to its neighbours. Hence

$$H_{n+1} - H_n = \frac{1}{2}(h_{n+1} + h_n) \quad (n = 1, 2, \dots, N-1) \quad (4.1)$$

The principal directions of orthotropy of the  $n^{\text{th}}$  laminate are at angles  $\epsilon_n$  to the  $x$  and  $y$  axes; without loss of generality one value of  $\epsilon_n$  ( $n = 1, 2, \dots, N$ ) can be taken to be zero. The thermoelastic coefficients for the  $n^{\text{th}}$  laminate are denoted by the symbols used in (2.12) with an additional prefix  $n$ ; thus  ${}^n c_{ij}^{\gamma\delta}$ ,  ${}^n c_{ij}$ ,  ${}^n d_{ij}$ ,  ${}^n c$  and the relevant coefficients are specified by (2.13), (2.14) and (2.16). These coefficients may be expressed in terms of the principal values (2.15) and the angle  $\epsilon_n$  by the usual tensor transformations.

Each laminate is assumed to be governed by plate equations of the type given in section 2 with constitutive coefficients specified by  ${}^n A_{\lambda\mu}^{\alpha\beta}, \dots, {}^n F, \dots, {}^n b_{\alpha\beta}$ . These, in turn, are given in terms of the coefficients in (2.16) by formulae of the type (2.19) and (2.20), each coefficient having the prefix  $n$  as in  ${}^n c_{\lambda\mu}^{\alpha\beta}$ . The displacement of the middle plane  $z = H_n$  of the  $n^{\text{th}}$  laminate, together with its associated director displacement, are denoted by

$$u_n^n = u_{\alpha\sim\alpha}^n e_{\alpha\sim\alpha} + u_{3\sim3}^n e_{3\sim3} \quad , \quad \delta_n^n = \delta_{\alpha\sim\alpha}^n e_{\alpha\sim\alpha} + \delta_{3\sim3}^n e_{3\sim3} \quad (4.2)$$

The two temperatures associated with the laminate are denoted by  $\theta_n, \phi_n$ . Since displacement and temperature are continuous at each glued face of the laminate, it follows that

$$\bar{u}_{n+1} - \frac{1}{2} h_{n+1} \delta_{n+1} = \bar{u}_n + \frac{1}{2} h_n \delta_n, \quad (n = 1, 2, \dots, N-1) \quad (4.3)$$

$$\bar{\theta}_{n+1} - \frac{1}{2} h_{n+1} \phi_{n+1} = \bar{\theta}_n + \frac{1}{2} h_n \phi_n$$

The forces acting on the faces  $z = H_n \pm \frac{1}{2} h_n$  of the  $n^{\text{th}}$  laminate are

$$\bar{t}_n = \bar{u}^n_{\alpha} e_{\alpha} + \bar{u}^n_3 e_3, \quad \bar{\ell}_n = \bar{\ell}^n_{\alpha} e_{\alpha} + \bar{\ell}^n_3 e_3 \quad (4.4)$$

and the corresponding entropy fluxes are  $\bar{u}^{*n}, \bar{\ell}^{*n}$ . Because forces at the boundary of two laminates are equal and opposite and entropy fluxes are continuous, we have

$$\bar{t}_n + \bar{\ell}_n^{n+1} = 0, \quad \bar{u}^{*n} + \bar{\ell}^{*n+1} = 0 \quad (n = 1, 2, \dots, N-1) \quad (4.5)$$

Forces, entropy fluxes and temperatures at the outer boundaries  $z = H_1 - \frac{1}{2} h_1$ ,  $z = H_N + \frac{1}{2} h_N$  of the plate are respectively

$$\bar{\ell}_1^1, \bar{\ell}_1^{*1}, \bar{\theta}_1 - \frac{1}{2} h_1 \phi_1 \quad \text{and} \quad \bar{u}_N^N, \bar{u}_N^{*N}, \bar{\theta}_N + \frac{1}{2} h_N \phi_N \quad (4.6)$$

Although we have considered only surface conditions for laminates attached continuously to its neighbors, the above conditions may easily be modified to allow for other interfacial conditions. Force and director force vectors and tensors, entropy fluxes, vectors and densities, etc., for the  $n^{\text{th}}$  laminate are denoted by the symbols used in section 2 with an added index  $n$ . Thus, e.g., we have  $\bar{N}^n, \bar{M}^n, \bar{N}_{\alpha\beta}^n, \dots, \bar{\eta}^n, \bar{\eta}_1^n, \dots, \bar{p}_{\alpha}^{1n}$ . The equations of motion and entropy balances for the  $n^{\text{th}}$  laminate are of the same form as (2.5) and (2.6) with the added index  $n$ , so that

$$N_{\alpha\beta,\alpha}^n + u_{\beta}^n + \ell t_{\beta}^n = \rho_n \ddot{u}_{\beta}^n, \quad (4.7)$$

$$M_{\alpha 3,\alpha}^n - V_3^n + \frac{1}{2} h_n (u_3^n - \ell t_3^n) = \rho_n \alpha_n \ddot{\delta}_3^n,$$

$$-p_{\alpha,\alpha}^n - u^k{}^{*n} - \ell k^k{}^{*n} = \rho_n \dot{\eta}^n,$$

$$M_{\alpha\beta,\alpha}^n - V_{\beta}^n + \frac{1}{2} h_n (u_{\beta}^n - \ell t_{\beta}^n) = \rho_n \alpha_n \ddot{\delta}_{\beta}^n,$$

$$V_{\alpha,\alpha}^n + u_3^n + \ell t_3^n = \rho_n \ddot{u}_3^n, \quad (4.8)$$

$$-p_{\alpha,\alpha}^{1n} + \rho_n \xi_1^n - \frac{1}{2} h_n (u^k{}^{*n} - \ell k^k{}^{*n}) = \rho_n \dot{\eta}_1^n$$

for  $n=1,2,\dots,N$ . The constitutive equations are

$$N_{\alpha\beta}^n = n_{A\alpha\beta}^{\lambda\mu} e_{\lambda\mu}^n + n_{A\alpha\beta}^n \delta_3^n + n_{B\alpha\beta}^n \theta_n,$$

$$V_3^n = n_{A\alpha\beta}^n e_{\alpha\beta}^n + n_A^n \delta_3^n + n_B^n \theta_n, \quad n_{M\alpha 3}^n = n_{E\alpha\lambda}^n \rho_{3\lambda}^n, \quad (4.9)$$

$$\rho_n \eta^n = - (n_B^n \delta_3^n + n_{B\alpha\beta}^n e_{\alpha\beta}^n + n_D^n \theta_n), \quad p_{\alpha}^n = - n_{a\alpha\lambda}^n \theta_{n,\lambda},$$

$$M_{\alpha\beta}^n = n_{D\alpha\beta}^{\lambda\mu} \rho_{\mu\lambda}^n + n_{F\alpha\beta}^n \phi_n, \quad V_{\alpha}^n = n_{D\alpha\lambda}^n \gamma_{\lambda}^n,$$

$$\rho_n \xi_1^n = - n_b^n \phi_n, \quad \rho_n \eta_1^n = - (n_{F\alpha\beta}^n \rho_{\alpha\beta}^n + n_F^n \phi_n), \quad (4.10)$$

$$p_{\alpha}^{1n} = - n_{b\alpha\beta}^n \phi_{n,\lambda}.$$

Equations (4.3) give  $N-1$  relations between the  $2N$  variables  $\underline{u}_n, \delta_n$  ( $n=1,\dots,N$ ) so that some restriction is placed on the freedom of choice of boundary conditions at the edges of the plate. To determine possible choices of boundary conditions we consider the rate of work of external forces  $\underline{N}^n$  and external director forces  $\underline{M}^n$  which is

$$R = \sum_{n=1}^N (\underline{N}^n \cdot \dot{\underline{u}}_n + \underline{M}^n \cdot \dot{\underline{\delta}}_n). \quad (4.11)$$

Now, from (4.2) we have

$$\begin{aligned} u_2 &= \frac{1}{2}(h_2 \delta_2 + h_1 \delta_1) + u_1, \\ u_n &= u_1 + \frac{1}{2}(h_n \delta_n + h_1 \delta_1) + \sum_{r=2}^{n-1} h_r \delta_r \quad (n=3, \dots, N), \end{aligned} \quad (4.12)$$

so that (4.11) may be written in the form

$$\begin{aligned} R &= \dot{u}_1 \cdot \sum_{n=1}^N N^n + \dot{\delta}_1 \cdot (M^1 + \frac{1}{2} h_1 \sum_{n=2}^N N^n) \\ &\quad + \dot{\delta}_N \cdot (M^N + \frac{1}{2} h_N N^N) \\ &\quad + \sum_{n=2}^{N-1} \dot{\delta}_n \cdot (M^n + \frac{1}{2} h_n N^n + h_n \sum_{r=n+1}^N N^r). \end{aligned} \quad (4.13)$$

The expression for  $R$  indicates that boundary values may be chosen for  $\dot{u}_1, \dot{\delta}_1, \dots, \dot{\delta}_N$  or their coefficients in (4.13).

Similarly, in view of (4.3), some restriction is placed on the freedom of choice temperature or entropy flux boundary conditions at the edges of the plate. The total heat flux at an edge

$$\begin{aligned} h &= \sum_{n=1}^N (k^n \theta_n + \ell^n \phi_n) \\ &= \theta_1 \sum_{n=1}^N k^n + \phi_1 (\ell^1 + \frac{1}{2} h_1 \sum_{n=2}^N k^n) + \phi_N (\ell^N + \frac{1}{2} h_N k^N) \\ &\quad + \sum_{n=2}^{N-1} \phi_n (\ell^n + \frac{1}{2} h_n k^n + h_n \sum_{r=n+1}^N k^r). \end{aligned} \quad (4.14)$$

Boundary values may be chosen for  $\theta_1, \phi_1, \dots, \phi_N$  or their coefficients in (4.14). Alternatively,  $N+1$  boundary conditions of a suitable type may be prescribed at each boundary.

The above theory of laminates provides values for  $N_{\alpha\beta}^n, M_{\alpha\beta}^n, V_{\alpha}^n, V_3^n$  but not

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\* As is customary, through appropriate definitions of stress resultants (see Naghdi 1972, Secs. 11-12), the quantities  $N_{\alpha\beta}^n, M_{\alpha\beta}^n, V_{\alpha}^n$  may be interpreted as membrane force resultants, bending moments and transverse shear resultants, respectively.

for stresses. [In fact, any plate (or shell) theory, whether constructed by a direct approach or derived from integration of the three-dimensional equations across the thickness of plate (or shell), can only involve forces or resultants and not the components of the stress tensor.] Information about the latter, if desired, may be obtained by introduction of an additional assumption. In particular, in the context of the present paper, some estimation of the three-dimensional stress distribution across each laminate may be obtained by a procedure similar to that used by Pagano (1978). Approximate expressions for the in-plane stress components  $t_{\alpha\beta}^n$  for the  $n^{\text{th}}$  laminate which are consistent with the definitions of  $N_{\alpha\beta}^n, M_{\alpha\beta}^n$  as stress resultants are

$$t_{\alpha\beta}^n = \frac{1}{h_n} N_{\alpha\beta}^n + \frac{12z}{h_n^3} M_{\alpha\beta}^n . \quad (4.15)$$

With the help of the three-dimensional equations of motion for the  $n^{\text{th}}$  laminate together with the surface conditions (4.4) and equations (4.7) and (4.8), we may obtain the following expression for  $t_{\beta 3}^n$  and  $t_{33}^n$ , namely



$$t_{\beta 3}^n = t_{3\beta}^n = \frac{1}{2}(u t_{\beta}^n - \ell t_{\beta}^n) + \frac{z}{h_n}(u t_{\beta}^n + \ell t_{\beta}^n) + \frac{6}{h_n^3}(z^2 - \frac{1}{4}h_n^2)[-v_{\beta}^n + \frac{1}{2}h_n(u t_{\beta}^n - \ell t_{\beta}^n)] , \quad (4.15)$$

$$\begin{aligned} t_{33}^n &= z t_{\beta 3, \beta}^n - \frac{1}{h_n^2}(4z^3 - \frac{1}{2}zh_n^2)(u t_{\beta, \beta}^n - \ell t_{\beta, \beta}^n) \\ &+ \frac{8z}{h_n^3}(z^2 - \frac{1}{4}h_n^2)v_{\beta, \beta}^n - \frac{1}{2h_n}(3z^2 - \frac{1}{4}h_n^2)(u t_{\beta, \beta}^n + \ell t_{\beta, \beta}^n) \\ &- \frac{6}{h_n^3}(z^2 - \frac{1}{4}h_n^2)(v_3^n - M_{\alpha 3, \alpha}^n) \\ &+ \frac{1}{h_n^2}(3z^2 - \frac{1}{4}h_n^2)(u t_3^n - \ell t_3^n) \\ &+ \frac{z}{h_n}(u t_3^n + \ell t_3^n) . \end{aligned} \quad (4.17)$$

We note from (4.16) and (4.17) that

$$\int_{-\frac{1}{2}h_n}^{\frac{1}{2}h_n} t_{\beta 3}^n dz = v_{\beta}^n , \quad \int_{-\frac{1}{2}h_n}^{\frac{1}{2}h_n} (t_{33}^n - z t_{\beta 3, \beta}^n) dz = v_3^n - M_{\beta 3, \beta}^n , \quad (4.18)$$

in conformity with the values of  $v_{\beta}^n, v_3^n, M_{\beta 3}^n$  but (4.16) does not fit in with the value for  $M_{\beta 3}^n$ . This is partly due to the nature of approximation in the special assumption (4.15).

## 5. Effective modulus theory

We summarize here an effective modulus theory similar to the classical theory for sandwich plates of the type described at the beginning of section 4. The theory is dynamical and includes thermal effects. The composite plate is regarded as a single plate with symmetry with respect to directions normal to the plate, and its motion is governed by equations (3.1) to (3.3). The coefficients in these equations are specified in terms of the constant coefficients of the separate laminates by using the formulae (3.4).

With the help of the values (2.19) and (2.20) for each laminate, the coefficients in (3.1) to (3.3) have the values listed below:

$$\overline{A}_{\lambda\mu}^{\alpha\beta} = \sum_{n=1}^N (n_{A\lambda\mu}^{\alpha\beta} - \frac{n_{A\alpha\beta} n_{A\lambda\mu}}{n_A}) , \quad \rho = \sum_{n=1}^N \rho_n^* h_n ,$$

$$\overline{A}_{\lambda\mu}^{\alpha\beta} = \sum_{n=1}^N H_n (n_{A\lambda\mu}^{\alpha\beta} - \frac{n_{A\alpha\beta} n_{A\lambda\mu}}{n_A}) ,$$

$$\overline{B}_{\alpha\beta} = \sum_{n=1}^N (n_{B\alpha\beta} - \frac{n_{A\alpha\beta} n_B}{n_A}) ,$$

$$\overline{B}'_{\alpha\beta} = \overline{F}'_{\alpha\beta} = \sum_{n=1}^N H_n (n_{B\alpha\beta} - \frac{n_{A\alpha\beta} n_B}{n_A}) ,$$

$$\overline{D} = \sum_{n=1}^N (n_D - \frac{n_B n_B}{n_A}) , \quad \overline{D}' = \sum_{n=1}^N H_n (n_D - \frac{n_B n_B}{n_A}) ,$$

(5.1)

$$\overline{a}_{\alpha\beta} = \sum_{n=1}^N n_{a\alpha\beta} , \quad \overline{a}'_{\alpha\beta} = \overline{b}'_{\alpha\beta} = \sum_{n=1}^N H_n n_{a\alpha\beta} ,$$

$$\overline{D}_{\lambda\mu}^{\alpha\beta} = \sum_{n=1}^N \{ n_{D\lambda\mu}^{\alpha\beta} + H_n^2 (n_{A\lambda\mu}^{\alpha\beta} - \frac{n_{A\alpha\beta} n_{A\lambda\mu}}{n_A}) \} ,$$

$$\overline{F}_{\alpha\beta} = \sum_{n=1}^N \{ n_{F\alpha\beta} + H_n^2 (n_{B\alpha\beta} - \frac{n_{A\alpha\beta} n_B}{n_A}) \} ,$$

$$\overline{F} = \sum_{n=1}^N \{ n_F + H_n^2 (n_D - \frac{n_B n_B}{n_A}) \} ,$$

$$\overline{b}_{\alpha\beta} = \sum_{n=1}^N (n_{b\alpha\beta} + H_n^2 n_{a\alpha\beta}) , \quad \overline{b} = \sum_{n=1}^N n_b .$$

## 6. Simple static problems for a laminated plate

We consider first the simple homogeneous extension of the plate, using the theory of section 4. Recalling (4.2) and the conditions (4.3), we suppose that the components of the displacement and the director displacement vectors and temperatures of each laminate are given by

$$\begin{aligned} u_1^n &= Lx + My, \quad u_2^n = Mx + Ny, \quad \delta_\alpha^n = 0, \quad \theta_n = \theta, \quad \phi_n = 0, \\ u_3^{n+1} - u_3^n &= \frac{1}{2} h_{n+1} \delta_3^{n+1} + \frac{1}{2} h_n \delta_3^n \quad (n = 1, 2, \dots, N-1), \end{aligned} \quad (6.1)$$

where  $L, M, N, \theta, u_3^n, \delta_3^n$  are constants. Then,

$$\begin{aligned} e_{11}^n &= e_{11} = L, \quad e_{12}^n = e_{12} = M, \quad e_{22}^n = e_{22} = N, \\ \rho_{\alpha\beta}^n &= 0, \quad \rho_{3\alpha}^n = 0, \quad \gamma_\alpha^n = 0. \end{aligned} \quad (6.2)$$

With the help of (4.9) and (4.10), it can be seen that equations (4.5), (4.7) and (4.8) are satisfied provided

$$u_\beta^n = 0, \quad \ell_\beta^n = 0, \quad u_3^n = 0, \quad \ell_3^n = 0, \quad u^{k*n} = 0, \quad \ell^{k*n} = 0 \quad (6.3)$$

for  $n = 1, 2, \dots, N$  and

$$V_3^n = 0, \quad {}^n_A \delta_3^n = - {}^n_A e_{\alpha\beta}^n - {}^n_B \theta_n. \quad (6.4)$$

By (6.4)<sub>2</sub>, the kinematical quantities  $\delta_3^n$  are determined in terms of the constant strain components  $e_{\alpha\beta}^n = e_{\alpha\beta}$  and constant temperature  $\theta_n = \theta$ . Then,  $u_3^1$  may be an arbitrary constant and  $u_3^n$  is given by (6.1) for  $n = 2, \dots, N$ . From (6.4), (4.9) and (4.10) we have

$$\begin{aligned}
N_{\alpha\beta}^n &= (n_{A\alpha\beta}^{\lambda\mu} - \frac{n_A^{\lambda\mu} n_{A\alpha\beta}}{n_A}) e_{\lambda\mu} + (n_{B\alpha\beta} - \frac{n_A^{\lambda\mu} n_B}{n_A}) \theta , \\
\rho_n \eta^n &= - (n_{B\alpha\beta} - \frac{n_A^{\lambda\mu} n_B}{n_A}) e_{\alpha\beta} - (n_D - \frac{n_B n_B}{n_A}) \theta , \quad p_\alpha^n = 0 , \\
M_{\alpha\beta}^n &= 0 , \quad n_{V_\alpha} = 0 , \quad \eta_1^n = 0 , \quad p_\alpha^{1n} = 0 .
\end{aligned} \tag{6.5}$$

Making use of (5.1), we calculate the expressions for the forces  $N_{\alpha\beta}$ , director forces  $M_{\alpha\beta}$  with respect to the reference surface  $S_R$  (or  $z = 0$ ) in section 4, and entropy densities for the whole plate, in the form

$$\begin{aligned}
N_{\alpha\beta} &= \sum_{n=1}^N N_{\alpha\beta}^n = \bar{A}_{\alpha\beta}^{\lambda\mu} e_{\lambda\mu} + \bar{B}_{\alpha\beta} \theta , \\
\rho \eta &= \sum_{n=1}^N \rho_n \eta^n = - \bar{B}_{\alpha\beta} e_{\alpha\beta} - \bar{D} \theta , \\
M_{\alpha\beta} &= \sum_{n=1}^N H_n N_{\alpha\beta}^n = \bar{A}'_{\alpha\beta}{}^{\lambda\mu} e_{\lambda\mu} + \bar{F}'_{\alpha\beta} \theta , \\
\rho \eta_1 &= \sum_{n=1}^N H_n \rho_n \eta^n = - \bar{B}'_{\alpha\beta} e_{\alpha\beta} - \bar{D}' \theta .
\end{aligned} \tag{6.6}$$

Comparison of (6.6) with the constitutive relations (3.2) and (3.3) shows that for homogeneous extension of the composite plate the overall moduli are the same as those used in the effective modulus theory of section 5.

Next, we consider flexure of the plate with the reference surface  $S_R$  ( $z = 0$ ) unstretched and the entire plate remaining unchanged in temperature. Again recalling the continuity condition (4.2) and the relations (4.1) we take displacement components and temperatures in the forms

$$\delta_1^n = Px + Qy, \quad \delta_2^n = Qx + Ry, \quad u_\alpha^n = H_n \delta_\alpha^n, \quad (6.7)$$

$$u_3^n = -\frac{1}{2} Px^2 - Qxy - \frac{1}{2} Ry^2 + K^n, \quad \phi_n = 0, \quad \theta_n = 0,$$

where  $P, Q, R, K^n, \phi$  are constants. Also  $\delta_3^n$  are constants subject to the conditions

$$K^{n+1} - K^n = \frac{1}{2} h_{n+1} \delta_3^{n+1} + \frac{1}{2} h_n \delta_3^n \quad (n = 1, 2, \dots, N-1). \quad (6.8)$$

Then,

$$\rho_{11}^n = \rho_{11} = P, \quad \rho_{12}^n = \rho_{21}^n = \rho_{12} = Q, \quad \rho_{22}^n = \rho_{22} = R, \quad \rho_{3\alpha}^n = 0, \quad (6.9)$$

$$e_{\alpha\beta}^n = H_n \rho_{\alpha\beta}, \quad \gamma_i^n = 0.$$

Equations (4.7) and (4.8) are satisfied provided (6.3) and (6.4) hold, where (6.4)<sub>2</sub> determine  $\delta_3^n$ . The constants  $K^n$  can then be found from (6.8) with  $K^1$  being chosen arbitrarily. From (6.9), (6.4), (4.9) and (4.10) we have

$$N_{\alpha\beta}^n = H_n \left( n_{A\alpha\beta}^{\lambda\mu} - \frac{n_{A\alpha\beta}^{\lambda\mu} n_{A\lambda\mu}}{n_A} \right) \rho_{\lambda\mu}, \quad \rho_n^n = -H_n \left( n_{B\alpha\beta}^{\lambda\mu} - \frac{n_{A\alpha\beta}^{\lambda\mu} n_B}{n_A} \right) \rho_{\alpha\beta}, \quad (6.10)$$

$$M_{\alpha\beta}^n = n_{D\alpha\beta}^{\lambda\mu} \rho_{\lambda\mu}, \quad \rho_n^n = -n_{F\alpha\beta} \rho_{\alpha\beta},$$

From (6.10) and (5.1) we then obtain

$$N_{\alpha\beta} = \sum_{n=1}^N N_{\alpha\beta}^n = \bar{A}_{\alpha\beta}^{\lambda\mu} \rho_{\lambda\mu} \quad , \quad \rho\eta = \sum_{n=1}^N \rho_n \eta^n = -\bar{F}'_{\alpha\beta} \rho_{\alpha\beta} \quad , \quad (6.11)$$

$$M_{\alpha\beta} = \sum_{n=1}^N (M_{\alpha\beta}^n + H_n N_{\alpha\beta}^n) = \bar{D}_{\alpha\beta}^{\lambda\mu} \rho_{\lambda\mu} \quad , \quad \rho\eta_1 = \sum_{n=1}^N \rho_n (\eta_1 + H_n \eta^n) = -\bar{F}_{\alpha\beta} \rho_{\alpha\beta} \quad ,$$

where  $N_{\alpha\beta}$  and  $M_{\alpha\beta}$  are computed with respect to the reference surface  $S_R$  (or  $z=0$ ) in section 4. Comparison of (6.11) with the constitutive relations (3.2) and (3.3) for pure homogeneous bending, in which  $e_{\alpha\beta}=0$ ,  $\phi=0$ ,  $\theta=0$  shows that the overall bending moduli are the same as those used in the effective modulus theory of section 5.

In the foregoing examples the overall moduli for simple extension and flexure of the composite plate are the same as those found from the effective modulus theory, but this is not generally true for more complex problems. For example, if the plate is subjected to a constant temperature difference between its major surfaces, then the overall moduli differ from those for the effective modulus theory. Moreover, for more complicated problems it is difficult to complete the analysis based on the theory of section 4 without specifying a particular value for the number  $N$  of laminates. On the other hand, it is no more difficult to use the effective modulus theory of section 5 for  $N$  laminates than for 1 or 2 laminates. For many purposes this theory may be adequate but sometimes a more detailed theory, such as that in section 4, is required. For example, studies of simple harmonic wave propagation by the effective modulus theory fail to indicate any dispersion effects.

The theory of section 4 can be used as a basis for a numerical discussion of thermo-mechanical response in a composite once  $N$  and the composition of the laminates are given. In the next two sections, we limit our attention to further analysis for one particular composite consisting of 3 laminates of orthotropic material whose directions of orthotropy coincide with the  $x, y$  axes, the two outer laminates having the same thermo-elastic moduli.

### 7. Composite with three laminates

In the theory discussed in section 4, we now restrict the axes of orthotropy of each laminate to coincide with the  $x, y$  axes so that non-zero constitutive coefficients for each laminate are given by (2.15), with an added prefix  $n$ , and by (2.13) and (2.14). The composite consists of three laminates in which the outer ones have the same mass density, thickness and elastic moduli. We choose the reference plane  $S_R$  (or  $z = 0$ ) to be midway between the faces of the composite plate. Then,

$$\begin{aligned} h_3 &= h_1, \quad H_2 = 0, \quad H_3 = -H_1 = \frac{1}{2}(h_1 + h_2), \\ \rho_3^* &= \rho_1^*, \quad {}^3A_{\lambda\mu}^{\alpha\beta} = {}^1A_{\lambda\mu}^{\alpha\beta}, \quad \text{etc.} \end{aligned} \tag{7.1}$$

Evidently, for 3-layer composite, the equation of motion (4.7)-(4.8) and the constitutive equations (4.9)-(4.10) separate into two groups: One group represents the extensional motion (or the stretching) and the other characterizes the flexural motion (or the bending). In order to record the relevant equations in an economical manner, we introduce the quantities  $u_1^+$ ,  $u_1^-$ , etc., defined as follows:



$$\begin{aligned}
u_i^+ &= u_i^1 + u_i^3, \quad u_i^- = u_i^1 - u_i^3, \quad \delta_i^+ = \delta_i^1 + \delta_i^3, \quad \delta_i^- = \delta_i^1 - \delta_i^3, \\
e_{\alpha\beta}^+ &= \frac{1}{2}(u_{\alpha,\beta}^+ + u_{\beta,\alpha}^+), \quad e_{\alpha\beta}^- = \frac{1}{2}(u_{\alpha,\beta}^- + u_{\beta,\alpha}^-), \\
\rho_{3\alpha}^+ &= \delta_{3,\alpha}^+, \quad \rho_{3\alpha}^- = \delta_{3,\alpha}^-, \\
\rho_{\alpha\beta}^+ &= \delta_{\alpha,\beta}^+, \quad \rho_{\alpha\beta}^- = \delta_{\alpha,\beta}^-, \quad \gamma_\alpha^+ = \delta_\alpha^+ + u_{3,\alpha}^+, \quad \gamma_\alpha^- = \delta_\alpha^- + u_{3,\alpha}^-, \\
\theta^+ &= \theta_1 + \theta_2, \quad \theta^- = \theta_1 - \theta_2, \quad \phi^+ = \phi_1 + \phi_2, \quad \phi^- = \phi_1 - \phi_2, \\
N_{\alpha\beta}^+ &= N_{\alpha\beta}^1 + N_{\alpha\beta}^3, \quad N_{\alpha\beta}^- = N_{\alpha\beta}^1 - N_{\alpha\beta}^3, \\
V_i^+ &= V_i^1 + V_i^3, \quad V_i^- = V_i^1 - V_i^3, \\
M_{\alpha\beta}^+ &= M_{\alpha\beta}^1 + M_{\alpha\beta}^3, \quad M_{\alpha\beta}^- = M_{\alpha\beta}^1 - M_{\alpha\beta}^3, \\
M_{\alpha 3}^+ &= M_{\alpha 3}^1 + M_{\alpha 3}^3, \quad M_{\alpha 3}^- = M_{\alpha 3}^1 - M_{\alpha 3}^3, \\
\eta^+ &= \eta^1 + \eta^3, \quad \eta^- = \eta^1 - \eta^3, \quad \eta_1^+ = \eta_1^1 + \eta_1^3, \quad \eta_1^- = \eta_1^1 - \eta_1^3, \\
p_\alpha^+ &= p_\alpha^1 + p_\alpha^3, \quad p_\alpha^- = p_\alpha^1 - p_\alpha^3, \quad p_\alpha^{1+} = p_\alpha^{11} + p_\alpha^{13}, \quad p_\alpha^{1-} = p_\alpha^{11} - p_\alpha^{13}, \\
\xi_1^+ &= \xi_1^1 + \xi_1^3, \quad \xi_1^- = \xi_1^1 - \xi_1^3.
\end{aligned} \tag{7.2}$$

Then, with reference to the 3-layer composite, the first group of equations appropriate for extensional motions are given by

$$\begin{aligned}
N_{\alpha\beta,\alpha}^2 + u t_{\beta}^2 - u t_{\beta}^1 &= \rho_2 \ddot{u}_{\beta}^2, \\
M_{\alpha 3,\alpha}^2 - V_3^2 + \frac{1}{2} h_2 (u t_3^2 + u t_3^1) &= \rho_2 \alpha_2 \ddot{\delta}_3^2, \\
-p_{\alpha,\alpha}^2 - u k^{*2} + u k^{*1} &= \rho_2 \dot{\eta}^2,
\end{aligned} \tag{7.3}$$

$$\begin{aligned}
N_{11}^2 &= {}^2A_{11}^{11} e_{11}^2 + {}^2A_{22}^{11} e_{22}^2 + {}^2A_{11} \delta_3^2 + {}^2B_{11} \theta_2, \\
N_{22}^2 &= {}^2A_{22}^{11} e_{11}^2 + {}^2A_{22}^{22} e_{22}^2 + {}^2A_{22} \delta_3^2 + {}^2B_{22} \theta_2, \\
N_{12}^2 &= N_{21}^2 = 2 {}^2A_{12}^{12} e_{12}^2, \\
V_3^2 &= {}^2A_{11} e_{11}^2 + {}^2A_{22} e_{22}^2 + {}^2A \delta_3^2 + {}^2B \theta_2, \\
M_{13}^2 &= {}^2E_{11} \rho_{31}^2, \quad M_{23}^2 = {}^2E_{22} \rho_{32}^2, \\
\rho_2 \eta^2 &= - {}^2B \delta_3^2 - {}^2B_{11} e_{11}^2 - {}^2B_{22} e_{22}^2 - {}^2D \theta_2, \\
p_1^2 &= - {}^2a_{11} \theta_{2,1}, \quad p_2^2 = - {}^2a_{22} \theta_{2,2},
\end{aligned} \tag{7.4}$$

and by

$$\begin{aligned}
N_{\alpha\beta,\alpha}^+ + u t_{\beta}^1 - u t_{\beta}^2 + u t_{\beta}^3 + \ell t_{\beta}^1 &= \rho_1 \ddot{u}_{\beta}^+, \\
M_{\alpha 3,\alpha}^+ - V_3^+ + \frac{1}{2} h_1 (u t_3^1 + u t_3^2 + u t_3^3 - \ell t_3^1) &= \rho_1 \alpha_1 \ddot{\delta}_3^+, \\
M_{\alpha\beta,\alpha}^- - V_{\beta}^- + \frac{1}{2} h_1 (u t_{\beta}^1 - u t_{\beta}^2 - u t_{\beta}^3 - \ell t_{\beta}^1) &= \rho_1 \alpha_1 \ddot{\delta}_{\beta}^-, \\
V_{\alpha,\alpha}^- + u t_3^1 + u t_3^2 + \ell t_3^1 - u t_3^3 &= \rho_1 \ddot{u}_3^-, \\
-p_{\alpha,\alpha}^+ - u k^{*1} + u k^{*2} - u k^{*3} - \ell k^{*1} &= \rho_1 \dot{\eta}^+, \\
-p_{\alpha,\alpha}^{1-} + \rho_1 \xi_1^- - \frac{1}{2} h_1 (u k^{*1} - u k^{*2} - u k^{*3} - \ell k^{*1}) &= \rho_1 \dot{\eta}_1^-,
\end{aligned} \tag{7.5}$$

$$\begin{aligned}
N_{11}^+ &= {}^1A_{11}^{11}e_{11}^+ + {}^1A_{22}^{11}e_{22}^+ + {}^1A_{11}\delta_3^+ + {}^1B_{11}\theta^+ , \\
N_{22}^+ &= {}^1A_{22}^{11}e_{11}^+ + {}^1A_{22}^{22}e_{22}^+ + {}^1A_{22}\delta_3^+ + {}^1B_{22}\theta^+ , \\
N_{12}^+ &= 2 {}^1A_{12}^{12}e_{12}^+ , \\
V_3^+ &= {}^1A_{11}e_{11}^+ + {}^1A_{22}e_{22}^+ + {}^1A\delta_3^+ + {}^1B\theta^+ , \\
M_{13}^+ &= {}^1E_{11}\rho_{31}^+ , \quad M_{23}^+ = {}^1E_{22}\rho_{32}^+ , \\
M_{11}^- &= {}^1D_{11}^{11}\rho_{11}^- + {}^1D_{22}^{11}\rho_{22}^- + {}^1F_{11}\phi^- , \\
M_{22}^- &= {}^1D_{22}^{11}\rho_{11}^- + {}^1D_{22}^{22}\rho_{22}^- + {}^2F_{22}\phi^- , \\
M_{12}^- &= M_{21}^- = {}^1D_{12}^{12}(\rho_{12}^- + \rho_{21}^-) , \\
V_1^- &= {}^1D_{11}\gamma_1^- , \quad V_2^- = {}^1D_{22}\gamma_2^- , \\
\rho_1\eta^+ &= -{}^1B_{11}e_{11}^+ - {}^1B_{22}e_{22}^+ - {}^1B\delta_3^+ - {}^1D\theta^+ , \\
\rho_1\eta_1^- &= -{}^1F_{11}\rho_{11}^- - {}^1F_{22}\rho_{22}^- - {}^1F\phi^- , \\
p_1^+ &= -{}^1a_{11}\theta_{,1}^+ , \quad p_2^+ = -{}^1a_{22}\theta_{,2}^+ , \\
p_1^{1-} &= -{}^1b_{11}\phi_{,1}^- , \quad p_2^{1-} = -{}^1b_{22}\phi_{,2}^- , \quad \rho_1\xi_1^- = -{}^1b\phi^- ,
\end{aligned} \tag{7.6}$$

where

$$u_\alpha^+ = 2u_\alpha^2 - \frac{1}{2} h_1 \delta_\alpha^- , \quad u_3^- = -h_2 \delta_3^2 - \frac{1}{2} h_1 \delta_3^+ , \quad \theta^+ = 2\theta_2 - \frac{1}{2} h_1 \phi^- . \tag{7.7}$$

The second group of equations appropriate for flexural motions of the 3-layer composite plate is

$$M_{\alpha\beta,\alpha}^2 - V_\beta^2 + \frac{1}{2} h_2 (u t_\beta^2 + u t_\beta^1) = \rho_2 \alpha_2 \ddot{\delta}_\beta^2 ,$$

$$V_{\alpha,\alpha}^2 + u t_3^2 - u t_3^1 = \rho_2 \ddot{u}_3^2 , \quad (7.8)$$

$$-p_{\alpha,\alpha}^{12} + \rho_2 \xi_1^2 - \frac{1}{2} h_2 (u k^{*2} + u k^{*1}) = \rho_2 \dot{\eta}_1^2 ,$$

$$M_{11}^2 = {}^2D_{11}^{11} \rho_{11}^2 + {}^2D_{22}^{11} \rho_{22}^2 + {}^2F_{11} \phi_2 ,$$

$$M_{22}^2 = {}^2D_{22}^{11} \rho_{11}^2 + {}^2D_{22}^{22} \rho_{22}^2 + {}^2F_{22} \phi_2 ,$$

$$M_{12}^2 = M_{21}^2 = {}^2D_{12}^{12} (\rho_{12}^2 + \rho_{21}^2) , \quad V_1^2 = {}^2D_{11} \gamma_1^2 , \quad V_2^2 = {}^2D_{22} \gamma_2^2 , \quad (7.9)$$

$$\rho_2 \eta_1^2 = - {}^2F_{11} \rho_{11}^2 - {}^2F_{22} \rho_{22}^2 - {}^2F \phi_2 ,$$

$$p_1^{12} = - {}^2b_{11} \phi_{2,1} , \quad p_2^{12} = - {}^2b_{22} \phi_{2,2} , \quad \rho_2 \xi_1^2 = - {}^2b \phi_2 ,$$

and by

$$N_{\alpha\beta,\alpha}^- + u t_\beta^1 + u t_\beta^2 - u t_\beta^3 + \ell t_\beta^1 = \rho_1 \ddot{u}_\beta^- ,$$

$$M_{\alpha 3,\alpha}^- - V_3^- + \frac{1}{2} h_1 (u t_3^1 - u t_3^2 - u t_3^3 - \ell t_3^1) = \rho_1 \alpha_1 \ddot{\delta}_3^- ,$$

$$M_{\alpha\beta,\alpha}^+ - V_\beta^+ + \frac{1}{2} h_1 (u t_\beta^1 + u t_\beta^2 + u t_\beta^3 - \ell t_\beta^1) = \rho_1 \alpha_1 \ddot{\delta}_\beta^+ ,$$

$$V_{\alpha,\alpha}^+ + u t_3^1 - u t_3^2 + u t_3^3 + \ell t_3^1 = \rho_1 \ddot{u}_3^+ , \quad (7.10)$$

$$-p_{\alpha,\alpha}^- - u k^{*1} - u k^{*2} + u k^{*3} - \ell k^{*1} = \rho_1 \dot{\eta}^- ,$$

$$-p_{\alpha,\alpha}^{1+} + \rho_1 \xi_1^+ - \frac{1}{2} h_1 (u k^{*1} + u k^{*2} + u k^{*3} - \ell k^{*1}) = \rho_1 \dot{\eta}_1^+ ,$$

$$\begin{aligned}
N_{11}^- &= {}^1A_{11}^{11} e_{11}^- + {}^1A_{22}^{11} e_{22}^- + {}^1A_{11} \delta_3^- + {}^1B_{11} \theta^- , \\
N_{22}^- &= {}^1A_{22}^{11} e_{11}^- + {}^1A_{22}^{22} e_{22}^- + {}^1A_{22} \delta_3^- + {}^1B_{22} \theta^- , \\
N_{12}^- &= 2 {}^1A_{12}^{12} e_{12}^- , \quad V_3^- = {}^1A_{11} e_{11}^- + {}^1A_{22} e_{22}^- + {}^1A \delta_3^- + {}^1B \theta^- , \\
M_{13}^- &= {}^1E_{11} \rho_{13}^- , \quad M_{23}^- = {}^1E_{22} \rho_{23}^- , \\
M_{11}^+ &= {}^1D_{11}^{11} \rho_{11}^+ + {}^1D_{22}^{11} \rho_{22}^+ + {}^1F_{11} \phi^+ , \\
M_{22}^+ &= {}^1D_{22}^{11} \rho_{11}^+ + {}^1D_{22}^{22} \rho_{22}^+ + {}^1F_{22} \phi^+ , \\
M_{12}^+ &= M_{21}^+ = {}^1D_{12}^{12} (\rho_{12}^+ + \rho_{21}^+) , \\
V_1^+ &= {}^1D_{11} \gamma_1^+ , \quad V_2^+ = {}^1D_{22} \gamma_2^+ , \\
\rho_1 \eta^- &= - {}^1B_{11} e_{11}^- - {}^1B_{22} e_{22}^- - {}^1B \delta_3^- - {}^1D \theta^- , \\
\rho_1 \eta_1^+ &= - {}^1F_{11} \rho_{11}^+ - {}^1F_{22} \rho_{22}^+ - {}^1F \phi^+ , \\
p_1^- &= - {}^1a_{11} \theta^- ,_1 , \quad p_2^- = - {}^1a_{22} \theta^- ,_2 , \\
p_1^{1+} &= - {}^1b_{11} \phi^+ ,_1 , \quad p_2^{1+} = - {}^1b_{22} \phi^+ ,_2 , \\
\rho_1 \xi_1^+ &= - {}^1b \phi^+ ,
\end{aligned} \tag{7.11}$$

where

$$u_\alpha^- = -h_2 \delta_\alpha^2 - \frac{1}{2} h_1 \delta_\alpha^+ , \quad u_3^+ = 2u_3^2 - \frac{1}{2} h_1 \delta_3^- , \quad \theta^- = -h_2 \phi_2 - \frac{1}{2} h_1 \phi^+ . \tag{7.12}$$

In the remainder of this section we consider an example in the context of thermostatics, namely the torsion of a three layer laminated rectangular plate. Let the plate be bounded by the planes  $x = \pm \frac{1}{2} a$ ,  $y = \pm \frac{1}{2} \ell$ ,  $z = \pm(h_1 + \frac{1}{2} h_2)$  and let the plate be subjected to the action of couples about the  $y$  axis over the ends  $y = \pm \frac{1}{2} \ell$ . For this problem,  $\phi_1, \phi_2, \phi_3$ ,

$\theta_1, \theta_2, \theta_3$  are all zero and we choose the following forms for the components of displacements and director displacements which are compatible with (7.7) and (7.12):

$$\begin{aligned}
u_1^1 &= H_1 \tau y, \quad u_1^3 = -H_1 \tau y, \quad u_1^2 = 0, \quad \delta_1^1 = \delta_1^2 = \delta_1^3 = \tau y, \\
u_2^1 &= -\frac{1}{2} \tau [h_2 A(x) + h_1 B(x)], \quad u_2^3 = \frac{1}{2} \tau [h_2 A(x) + h_1 B(x)], \quad u_2^2 = 0, \\
\delta_2^1 &= \delta_2^3 = \tau B(x), \quad \delta_2^2 = \tau A(x), \\
u_3^1 &= u_3^2 = u_3^3 = -\tau xy, \quad \delta_3^1 = \delta_3^2 = \delta_3^3 = 0,
\end{aligned} \tag{7.13}$$

where  $\tau$  is the twist per unit length and  $A, B$  are functions of  $x$  only. It follows that  $u_\alpha^+, u_\alpha^-, \delta_\alpha^+, \delta_\alpha^-$  are all zero and relevant equations are (7.8) to (7.12) for flexural motions. The components of displacements and measures of deformations are

$$\begin{aligned}
u_1^- &= 2H_1 \tau y, \quad u_2^- = -\tau [h_2 A(x) + h_1 B(x)], \quad u_3^- = -2\tau xy, \\
\delta_1^+ &= 2\tau y, \quad \delta_2^+ = 2\tau B(x), \quad \delta_3^+ = 0, \\
\rho_{12}^2 + \rho_{21}^2 &= \tau \left( \frac{dA}{dx} + 1 \right), \quad \gamma_2^2 = \tau(A-x), \quad \gamma_2^+ = 2\tau(B-x), \\
e_{12}^- &= -\frac{1}{2} \tau [h_1 \left( \frac{dB}{dx} + 1 \right) + h_2 \left( \frac{dA}{dx} + 1 \right)], \quad \rho_{12}^+ + \rho_{21}^+ = 2\tau \left( \frac{dB}{dx} + 1 \right),
\end{aligned} \tag{7.14}$$

the remaining components of strain being zero. It follows from (7.4), (7.6), (7.9) and (7.11) that the components of various forces and director forces are given by

$$M_{12}^2 = M_{21}^2 = 2 D_{12}^{12} \tau \left( \frac{dA}{dx} + 1 \right) , \quad V_2^2 = 2 D_{22} \tau (A-x) .$$

$$N_{12}^- = N_{21}^- = - {}^1 A_{12}^{12} \tau [h_1 \left( \frac{dB}{dx} + 1 \right) + h_2 \left( \frac{dA}{dx} + 1 \right)] , \quad (7.15)$$

$$M_{12}^+ = M_{21}^+ = 2 {}^1 D_{12}^{12} \tau \left( \frac{dB}{dx} + 1 \right) , \quad V_2^+ = 2 {}^1 D_{22} \tau (B-x) ,$$

the remaining components being zero. Given that the forces  $u_i^3, u_i^1$  on the faces of the plate are zero, the equations of equilibrium (7.3), (7.5), (7.8) and (7.10) are satisfied provided

$$\begin{aligned} \frac{dM_{12}^2}{dx} - V_2^2 + \frac{1}{2} h_2 (u_2^2 + u_2^1) &= 0 , \quad \frac{dN_{12}^-}{dx} + u_2^1 + u_2^2 = 0 , \\ \frac{dM_{12}^+}{dx} - V_2^+ + \frac{1}{2} h_1 (u_2^1 + u_2^2) &= 0 , \quad u_2^1 = u_2^2 , \end{aligned} \quad (7.16)$$

$$u_1^1 = u_1^2 = u_3^1 = u_3^2 = 0 .$$

Substituting (7.15) into (7.16) and eliminating  $u_2^1 + u_2^2$ , we obtain the following differential equations for A and B:

$$\begin{aligned} b \frac{d^2 A}{dx^2} + \frac{d^2 B}{dx^2} - d(A-x) &= 0 , \\ c \frac{d^2 B}{dx^2} + \frac{d^2 A}{dx^2} - e(B-x) &= 0 , \end{aligned} \quad (7.17)$$

where the coefficients b,c,d,e are given by

$$b = \frac{2 D_{12}^{12} + \frac{1}{2} h_2^2 A_{12}^{12}}{\frac{1}{2} h_1 h_2 A_{12}^{12}}, \quad c = \frac{1 D_{12}^{12} + \frac{1}{4} h_1^2 A_{12}^{12}}{\frac{1}{4} h_1 h_2 A_{12}^{12}}, \quad (7.18)$$

$$d = \frac{2 D_{22}^{22}}{h_1 h_2 A_{12}^{12}}, \quad e = \frac{4 D_{22}^{12}}{h_1 h_2 A_{12}^{12}}.$$

We need to supplement the differential equations (7.17) with suitable boundary conditions. To this end, we first note that in the torsion problem under discussion there are no applied forces at the edges  $x = \pm a$ . Then, in view of (4.13) for the special case  $N = 3$  and recalling the nonvanishing expressions in (7.15), the appropriate edge conditions are

$$M_{12}^+ - \frac{1}{2} h_1 N_{12}^- = 0, \quad M_{12}^2 - \frac{1}{2} h_2 N_{12}^- = 0 \quad \text{on } x = \pm a. \quad (7.19)$$

The above conditions are satisfied if

$$\frac{dA}{dx} + 1 = 0, \quad \frac{dB}{dx} + 1 = 0 \quad \text{on } x = \pm a. \quad (7.20)$$

From (7.17) and (7.20), it follows that

$$A = x + L \sinh nx + M \sinh mx, \quad (7.21)$$

$$B = x - \frac{bn^2 - d}{n^2} L \sinh nx - \frac{bm^2 - d}{m^2} M \sinh mx,$$

where

$$Ld(m^2 - n^2) \cosh na + 2n[(b+1)m^2 - d] = 0, \quad (7.22)$$

$$Md(m^2 - n^2) \cosh ma - 2m[(b+1)n^2 - d] = 0,$$

and  $n, m$  are the positive roots of the equation

$$(bc-1)\lambda^4 - (dc+be)\lambda^2 + de = 0. \quad (7.23)$$

The couple at the edge of the plate, measured per unit length, is



given by

$$\begin{aligned}
 & (xe_1 + ye_2) \times \tilde{N}^2 + \tilde{e}_3 \times \tilde{M}^2 \\
 & + (xe_1 + ye_2 + H_1 \tilde{e}_3) \times \tilde{N}^1 + \tilde{e}_3 \times \tilde{M}^1 \\
 & + (xe_1 + ye_2 + H_3 \tilde{e}_3) \times \tilde{N}^3 + \tilde{e}_3 \times \tilde{M}^3 .
 \end{aligned} \tag{7.24}$$

Over any edge  $y = \text{constant}$ , with the help of (7.15), (7.24) becomes

$$\tilde{e}_1 y (V_2^2 + V_2^+) + \tilde{e}_2 [M_{21}^2 + M_{21}^+ + H_1 N_{21}^- - x(V_2^2 + V_2^+)] . \tag{7.25}$$

This yields a resultant couple  $\tau G$  over  $y = \text{constant}$  about the  $y$ -axis given by

$$\tau G = \int_{-a}^a \{M_{21}^2 + M_{21}^+ + H_1 N_{21}^- - x(V_2^2 + V_2^+)\} dx . \tag{7.26}$$

It can be seen from (7.16) that

$$V_2^2 + V_2^+ = \frac{d}{dx} (M_{21}^2 + M_{21}^+ + H_1 N_{21}^-) , \tag{7.27}$$

so that with the help of this result and the edge conditions (7.19), (7.26) becomes

$$\tau G = 2 \int_{-a}^a \{M_{21}^2 + M_{21}^+ + H_1 N_{21}^-\} dx . \tag{7.28}$$

With the help of (7.15), (7.18), (7.21) and (7.22), it follows from (7.28)

that the torsional rigidity of the laminated plate is

$$\begin{aligned}
 G = 4h_1 h_2 \frac{1}{A_{12}^{12}} & \left[ (b+c+2)a + \frac{\{d - (b+1)m^2\} \{(1-bc)n^2 + d(c+1)\} \tanh na}{nd(m^2 - n^2)} \right. \\
 & \left. - \frac{\{d - (b+1)n^2\} \{(1-bc)m^2 + d(c+1)\} \tanh ma}{md(m^2 - n^2)} \right] .
 \end{aligned} \tag{7.29}$$

It can be shown that when  $h_1 \rightarrow 0$  (7.29) reduces to the known value

$$G \rightarrow \frac{2a h_2^3 c_{12}^{212}}{3} \left(1 - \frac{\tanh ma}{ma}\right), \quad m = \frac{\pi}{h_2} \left(\frac{c_{23}^{223}}{c_{12}^{212}}\right) \quad \text{as } h_1 \rightarrow 0 \quad (7.30)$$

for a plate of thickness  $h_2$  and moduli  $c_{12}^{212}, c_{23}^{223}$  computed from Cosserat plate theory. Similarly, when  $h_2 \rightarrow 0$ , (7.29) reduces to the expected value

$$G \rightarrow \frac{16a h_1^3 c_{12}^{112}}{3} \left(1 - \frac{\tanh ma}{ma}\right), \quad m = \frac{\pi}{2h_1} \left(\frac{c_{23}^{123}}{c_{12}^{112}}\right)^{\frac{1}{2}} \quad (7.31)$$

for a plate of thickness  $2h_1$  and moduli  $c_{12}^{112}, c_{23}^{123}$ .

When the width  $2a$  of the plate is large compared with both thicknesses  $h_1, h_2$  of the laminates, then  $na$  and  $ma$  will be large and the torsional rigidity (7.29) reduces approximately to

$$G = 4h_1 h_2 a (b+c+2) A_{12}^{112}. \quad (7.32)$$

Also, with the help of (7.18) and (5.1), the result (7.32) becomes

$$G = 8a \bar{D}_{12}^{12}. \quad (7.33)$$

This is the same value which would be obtained by using the effective modulus theory of section 5. However, if  $2a$  is not large compared with both  $h_1$  or  $h_2$  the effective modulus theory is not sufficient for the discussion of torsion. This is in contrast to the situation for pure tension and flexure of the plate, where the effective modulus theory gives satisfactory values for overall moduli.

### 8. Harmonic waves in 3-layered plate

We consider here wave propagation in a 3-layered composite plate of the type discussed in section 7 utilizing the two groups of equations (7.3) to (7.7) and (7.8) to (7.12). We limit our attention to the propagation of harmonic waves, under isothermal conditions, along the x-direction of a composite plate whose major faces are free of forces and couples. We assume that all variables contain a factor  $\exp i(\xi x - \omega t)$ , where  $\xi, \omega$  are constants, so that this factor can be removed throughout the calculations that follow. The group of equations (7.3) to (7.7) then separate into two further groups, the first being concerned with symmetric SH waves and the second with symmetric plane strain waves. For SH waves we have non-zero values for  $u_2^2, \delta_2^-, u_2^+$  and

$$\begin{aligned} N_{12}^2 &= i\xi({}^2A_{12}^{12})u_2^2, \quad N_{12}^+ = i\xi({}^1A_{12}^{12})u_2^+, \\ M_{12}^- &= i\xi({}^1D_{12}^{12})\delta_2^-, \quad V_2^- = {}^1D_{22}\delta_2^-, \quad u_2^+ = 2u_2^2 - \frac{1}{2}h_1\delta_2^-, \\ i\xi N_{12}^2 + u_2^2 t_2^2 - u_2^1 t_2^1 &= -\rho_2 \omega^2 u_2^2, \\ i\xi N_{12}^+ + u_2^1 t_2^1 - u_2^2 t_2^2 &= -\rho_1 \omega^2 u_2^+, \\ i\xi M_{12}^- - V_2^- + \frac{1}{2}h_1(u_2^1 t_2^1 - u_2^2 t_2^2) &= -\rho_1 \alpha_1 \omega^2 \delta_2^-. \end{aligned} \tag{8.1}$$

A non-zero solution of equations (8.1) leads to the dispersion relation

$$a\xi^4 + (b - c\omega^2)\xi^2 - d\omega^2 + e\omega^4 = 0, \tag{8.2}$$

where

$$\begin{aligned}
a &= h_1^2 ({}^1c_{12}^{12}) (h_2^2 c_{12}^{12} + \frac{1}{2} h_1 {}^1c_{12}^{12}) , \\
b &= \frac{\pi^2}{4} ({}^1c_{23}^{23}) (h_2^2 c_{12}^{12} + 2h_1 {}^1c_{12}^{12}) , \\
c &= \rho_1^* h_1^2 h_2^2 c_{12}^{12} + (\rho_1^* h_1^3 + \rho_2^* h_2^2 h_1^2) {}^1c_{12}^{12} , \\
d &= \frac{\pi^2}{4} ({}^1c_{23}^{23}) (\rho_2^* h_2 + 2\rho_1^* h_1) , \\
e &= \frac{1}{2} \rho_1^* h_1^2 (2\rho_2^* h_2 + \rho_1^* h_1) ,
\end{aligned} \tag{8.3}$$

and where use has been made of the expressions of the form (2.19)-(2.20).

The dispersion relation (8.2) yields two expressions for  $\xi^2$  as a function of  $\omega^2$ , corresponding to two different modes of propagation of SH waves. In one of these

$$\xi^2 + \frac{d}{b} \omega^2 = \frac{\rho_2^* h_2 + 2\rho_1^* h_1}{h_2^2 c_{12}^{12} + 2h_1 {}^1c_{12}^{12}} \omega^2 , \tag{8.4}$$

as  $\omega^2 \rightarrow 0$ . This agrees with the values obtained for SH waves from the effective modulus theory of section 5. [Recall that there is no dispersion relation in the effective modulus theory.] In the case of isotropic layers the limiting value (8.4) as  $\omega \rightarrow 0$  agrees with the full three-dimensional theory by Lee and Chang (1979).

Symmetric plane strain wave propagation on the basis of the theory characterized by (7.3) to (7.7) yields equations for  $u_1^2, u_1^+, u_3^-, \delta_3^2, \delta_1^-, \delta_3^+$ . A non-zero solution of these equations gives a quartic dispersion equation for  $\xi^2$  in terms of  $\omega^2$ . We omit the details but note that one root of this quartic is such that

$$\frac{\xi^2}{\omega^2} + \frac{\rho_2^* h_2 + 2\rho_1^* h_1}{h_2 \{ {}^2c_{11}^{11} - \frac{({}^2c_{33}^{11})^2}{2c_{33}^{33}} \} + 2h_1 \{ {}^1c_{11}^{11} - \frac{({}^1c_{33}^{11})^2}{c_{33}^{33}} \}} \tag{8.5}$$

as  $\omega \rightarrow 0$ . This agrees with the value predicted for symmetric plane strain

waves from the effective modulus theory of section 5 which has no dispersion relation. In the case of isotropic plates the limiting value (8.5) as  $\omega \rightarrow 0$  agrees with that found from the three-dimensional theory by Lee and Chang (1979)..

In a similar way, the group of equations (7.8) to (7.12) separate into two further groups, one for antisymmetric SH waves and the other for antisymmetric plane strain waves. In the first group we have nonzero values for  $\delta_2^2, \delta_2^+, u_2^-$  and in the second group  $\delta_1^2, u_3^2, \delta_1^+, \delta_3^-, u_1^-, u_3^+$  are nonzero. For antisymmetric SH waves we find a quadratic dispersion relation for  $\xi^2$  in terms of  $\omega^2$ . There is no wave of this type to be found from the effective modulus theory of section 5. Also, in the case of antisymmetric plane strain waves there is a quartic dispersion equation for  $\xi^2$  in terms of  $\omega^2$ . One root of this quartic is such that

$$\frac{\xi^4}{\omega^2} \rightarrow \frac{\rho_2^* h_2 + 2\rho_1^* h_1}{\frac{1}{12} h_2^3 \left\{ c_{11}^{11} - \frac{(c_{33}^{11})^2}{c_{33}} \right\} + \left( \frac{2}{3} h_1^3 + h_1^2 h_2 + \frac{1}{2} h_1 h_2^2 \right) \left\{ c_{11}^{11} - \frac{(c_{33}^{11})^2}{c_{33}} \right\}} \quad (8.6)$$

as  $\omega \rightarrow 0$ . This agrees with the value predicted for antisymmetric plane strain waves from the effective modulus theory of section 5 which has no dispersion relation. In the case of isotropic plates the limiting value (8.6) as  $\omega \rightarrow 0$  agrees with that found from the three-dimensional theory by Lee and Chang (1979).

### 9. Large number of laminates

The theory of section 4 and section 7 may also be used to discuss periodic stress distribution in a stratified medium consisting of a large number of alternating, parallel layers of two homogeneous orthotropic materials whose directions of orthotropy coincide. We consider only the case in which deformation and temperature fields are identical in every other layer so that

$$h_{n+2} = h_n, \quad u_{n+2} = u_n, \quad \delta_{n+2} = \delta_n, \quad \theta_{n+2} = \theta_n, \quad \phi_{n+2} = \phi_n. \quad (9.1)$$

In view of the continuity conditions (4.3), it follows that

$$\begin{aligned} u_{n+1} &= u_n, \quad h_{n+1} \delta_{n+1} = -h_n \delta_n, \\ \theta_{n+1} &= \theta_n, \quad h_{n+1} \phi_{n+1} = -h_n \phi_n. \end{aligned} \quad (9.2)$$

We may now select three typical laminates, say 1,2,3, and use the theory of section 7 in which

$$\begin{aligned} u_i^- &= 0, \quad \delta_i^- = 0, \quad \theta^- = 0, \quad \phi^- = 0, \\ u_i^+ &= 2u_i^2, \quad h_1 \delta_i^+ = -2h_2 \delta_i^2, \quad \theta^+ = 2\theta_2, \quad h_1 \phi^+ = -2h_2 \phi_2, \\ u_t^3 &= u_t^1, \quad \ell_t^3 = \ell_t^1 = -u_t^2, \\ u^k{}^3 &= u^k{}^1, \quad \ell^k{}^3 = \ell^k{}^1 = -u^k{}^2. \end{aligned} \quad (9.3)$$

The two groups of equations (7.3) to (7.7) and (7.8) to (7.12) simplify considerably. We restrict attention here only to the group (7.3) to (7.7) which determine extensional deformations parallel to the laminates. Under isothermal conditions, (7.3) and (7.5) reduce to

$$\begin{aligned}
N_{\alpha\beta,\alpha}^2 + u t_{\beta}^2 - u t_{\beta}^1 &= \rho_2 \ddot{u}^2, \\
M_{\alpha 3,\alpha}^2 - V_3^2 + \frac{1}{2} h_2 (u t_3^2 + u t_3^1) &= \rho_2 \alpha_2 \ddot{\delta}_3^2, \\
N_{\alpha\beta,\alpha}^+ + 2(u t_{\beta}^1 - u t_{\beta}^2) &= 2\rho_1 \ddot{u}_{\beta}^2, \\
M_{\alpha 3,\alpha}^+ - V_3^+ + h_1 (u t_3^1 + u t_3^2) &= -2(\rho_1 \alpha_1 h_2 / h_1) \ddot{\delta}_3^2
\end{aligned} \tag{9.4}$$

and this leads to

$$\begin{aligned}
(2N_{\alpha\beta}^2 + N_{\alpha\beta}^+),_{\alpha} &= 2(\rho_1 + \rho_2) \ddot{u}_{\beta}^2, \\
(2h_1 M_{\alpha 3}^2 - h_2 M_{\alpha 3}^+),_{\alpha} - (2h_1 V_3^2 - h_2 V_3^+) &= 2[h_1 \rho_2 \alpha_2 + h_2 \rho_1 \alpha_1 / h_1] \ddot{\delta}_3^2.
\end{aligned} \tag{9.5}$$

Also, from (7.4) and (7.6) we obtain

$$\begin{aligned}
2N_{11}^2 + N_{11}^+ &= 2({}^2A_{11}^{11} + {}^1A_{11}^{11})e_{11}^2 + 2({}^2A_{22}^{11} + {}^1A_{22}^{11})e_{22}^2 + 2({}^2A_{11}^{12} - \frac{h_2}{h_1} {}^1A_{11}^{12})\delta_3^2, \\
2N_{22}^2 + N_{22}^+ &= 2({}^2A_{22}^{11} + {}^1A_{22}^{11})e_{11}^2 + 2({}^2A_{22}^{22} + {}^1A_{22}^{22})e_{22}^2 + 2({}^2A_{22}^{12} - \frac{h_2}{h_1} {}^1A_{22}^{12})\delta_3^2, \\
2N_{12}^2 + N_{12}^+ &= 2N_{21}^2 + N_{21}^+ = 4({}^2A_{12}^{12} + {}^1A_{12}^{12})e_{12}^2, \\
2h_1 M_{13}^2 - h_2 M_{13}^+ &= 2(h_1 {}^2E_{11} + \frac{h_2}{h_1} {}^1E_{11})\rho_{31}^2, \\
2h_1 M_{23}^2 - h_2 M_{23}^+ &= 2(h_1 {}^2E_{22} + \frac{h_2}{h_1} {}^1E_{22})\rho_{32}^2, \\
2h_1 V_3^2 - h_2 V_3^+ &= 2(h_1 {}^2A_{11} - h_2 {}^1A_{11})e_{11}^2 \\
&\quad + 2(h_1 {}^2A_{22} - h_2 {}^1A_{22})e_{22}^2 + 2(h_1 {}^2A_{12} + \frac{h_2}{h_1} {}^1A_{12})\delta_3^2.
\end{aligned} \tag{9.6}$$

It is now a simple matter to discuss harmonic wave propagation in the x-direction. There is a symmetrical SH wave for which  $u_2^2$  is non-zero and

$$\frac{\xi^2}{\omega^2} = \frac{\rho_1 + \rho_2}{2A_{12}^{12} + 1A_{12}^{12}} = \frac{\rho_1^* h_1 + \rho_2^* h_2}{h_2 2c_{12}^{12} + h_1 1c_{12}^{12}} \quad (9.7)$$

When the layers are isotropic, this reduces to the result given by Sun, Achenbach and Herrmann (1968). In addition, equations (9.4) to (9.6) yield longitudinal waves for which  $u_1^2, \delta_3^2$ .

$$\begin{aligned} & [(2E_{11} + \frac{h_2^2}{h_1^2} 1E_{11})\xi^2 - (\rho_2\alpha_2 + \frac{h_2^2\rho_1\alpha_1}{h_1^2})\omega^2 + 2A + \frac{h_2^2}{h_1^2} 1A] \\ & \times [(2A_{11}^{11} + 1A_{11}^{11})\xi^2 - (\rho_1 + \rho_2)\omega^2] - (2A_{11} - \frac{h_2}{h_1} 1A_{11})^2 \xi^2 = 0 \quad , \end{aligned}$$

or

$$\begin{aligned} & [\frac{1}{12} (h_2 2c_{13}^{13} + h_1 1c_{13}^{13})h_2^2 \xi^2 - \frac{1}{12} (\rho_2^* h_2 + \rho_1^* h_1)h_2^2 \omega^2 \\ & + \frac{h_2}{h_1} (h_1 2c_{33}^{33} + h_2 1c_{33}^{33})] \times [(h_2 2c_{11}^{11} + h_1 1c_{11}^{11})\xi^2 - (\rho_2^* h_2 + \rho_1^* h_1)\omega^2] \\ & - (2c_{33}^{11} - 1c_{33}^{11})^2 h_2^2 \xi^2 = 0 \quad . \end{aligned} \quad (9.8)$$

For the stratified medium considered here, we use the value  $\beta = 1/12$  in the expressions for  $2E_{11}, 1E_{11}$  in (2.19). Again, when the layers are isotropic, (9.8) reduces to the result given by Sun, Achenbach and Herrmann (1968).



## Part B. Laminated Layered Composite Rods

### 10. Basic equations of the linear theory of orthotropic elastic rods.

Consider a straight rod of constant density  $\rho^*$  and at constant temperature  $\bar{\theta}$ , bounded by the surface

$$F(x,y) = 0, \quad (10.1)$$

where  $x_i = (x,y,z)$  are rectangular Cartesian coordinates. The rod has orthotropic symmetry with respect to the  $x_i$ -axes and has geometric symmetry with respect to the  $x$  and  $y$ -axes. The basic equations of the linear theory for thermoelastic deformation of such a rod have been given in a number of previous papers. In particular, we refer to the recent paper by Green and Naghdi (1979b), which contains explicit values for the constitutive coefficients. The main equations and constitutive coefficients for isothermal deformation of an elastic orthotropic rod are summarized below in Cartesian tensor notation with Greek indices taking the values 1,2 and Latin indicates the values 1,2,3.

We recall that a Cosserat curve  $\mathcal{R}$  comprises of a material curve  $\mathcal{L}$  and a set of two directors assigned to every point of the material curve; the two directors are regarded as modelling the shape and the deformation of the cross-section of the rod. In the reference configuration of a straight rod, let the material curve be a straight line  $\mathcal{L}_R$  which we identify with the line of centroid  $x=y=0$  of the rod. Let  $\underline{u}$  and  $\underline{\delta}_\alpha$  denote, respectively, the infinitesimal displacement vector and the infinitesimal director displacement of  $\mathcal{L}_R$  (or the line  $x=y=0$ ). Then, referred to the unit vectors  $\underline{e}_i$  along the  $x_i$  coordinate axes,  $\underline{u}$  and  $\underline{\delta}$  may be expressed as\*

\*The overbars on  $\underline{\delta}_\alpha$  and  $\underline{\delta}_{\alpha i}$  in Green and Naghdi (1979b) are omitted here for convenience.

$$\underline{u} = u_i \underline{e}_i , \quad \underline{\delta}_\alpha = \delta_{\alpha i} \underline{e}_i , \quad (10.2)$$

and the relevant strain measures are

$$\begin{aligned} \gamma_{\alpha\beta} &= \delta_{\alpha\beta} + \delta_{\beta\alpha} , \quad \gamma_{\alpha 3} = \delta_{\alpha 3} + \partial u_\alpha / \partial z , \quad \gamma_{33} = 2\partial u_3 / \partial z , \\ \kappa_{\alpha\beta} &= \partial \delta_{\alpha\beta} / \partial z , \quad \kappa_{\alpha 3} = \partial \delta_{\alpha 3} / \partial z . \end{aligned} \quad (10.3)$$

The contact force vector  $\underline{n}$ , the contact director forces  $\underline{p}_\alpha$ , the intrinsic director forces  $\underline{\pi}_\alpha$  and the external fields  $\underline{f}, \underline{l}_\alpha$ , referred to the basis  $\underline{e}_i$ , can be expressed as

$$\begin{aligned} \underline{n} &= n_i \underline{e}_i , \quad \underline{p}_\alpha = p_{\alpha i} \underline{e}_i , \quad \underline{\pi}_\alpha = \pi_{\alpha i} \underline{e}_i , \\ \underline{f} &= f_i \underline{e}_i , \quad \underline{l}_\alpha = l_{\alpha i} \underline{e}_i . \end{aligned} \quad (10.4)$$

The basic equations of the linear theory under discussion separate into four groups, two for flexure, one for extension and one for torsion. These are summarized below.

#### Flexure F1

$$\begin{aligned} \pi_{23} = n_2 = k_5 \gamma_{23} , \quad m_1 = p_{23} = k_{15} \kappa_{23} , \\ \partial n_2 / \partial z + \rho f_2 = \rho \partial^2 u_2 / \partial t^2 , \quad \partial m_1 / \partial z - n_2 + \rho l_{23} = \rho \alpha_2 \partial^2 \delta_{23} / \partial t^2 . \end{aligned} \quad (10.5)$$

#### Flexure F2

$$\begin{aligned} \pi_{13} = n_1 = k_6 \gamma_{13} , \quad m_2 = -p_{13} = -k_{16} \kappa_{13} , \\ \partial n_1 / \partial z + \rho f_1 = \rho \partial^2 u_1 / \partial t^2 , \quad \partial m_2 / \partial z + n_1 - \rho l_{13} = -\rho \alpha_1 \partial^2 \delta_{13} / \partial t^2 . \end{aligned} \quad (10.6)$$

### Extension E

$$\pi_{11} = 2\bar{k}_1\gamma_{11} + k_7\gamma_{22} + k_8\gamma_{33} \quad , \quad \pi_{22} = k_7\gamma_{11} + 2\bar{k}_2\gamma_{22} + k_9\gamma_{33} \quad ,$$

$$n_3 = k_8\gamma_{11} + k_9\gamma_{22} + 2k_3\gamma_{33} \quad ,$$

$$p_{11} = k_{10}\kappa_{11} + \frac{1}{2} k_{17}\kappa_{22} \quad , \quad p_{22} = \frac{1}{2} k_{17}\kappa_{11} + k_{11}\kappa_{22} \quad , \quad (10.7)$$

$$\partial n_3 / \partial z + \rho f_3 = \rho \partial^2 u_3 / \partial t^2 \quad ,$$

$$\partial p_{11} / \partial z + \rho \ell_{11} - \pi_{11} = \rho \alpha_1 \partial^2 \delta_{11} / \partial t^2 \quad , \quad \partial p_{22} / \partial z + \rho \ell_{22} - \pi_{22} = \rho \alpha_2 \partial^2 \delta_{22} / \partial t^2 \quad .$$

### Torsion T

$$\pi_{12} = \pi_{21} = \frac{1}{2} k_4 (\gamma_{12} + \gamma_{21}) \quad , \quad p_{12} - p_{21} = m_3 \quad ,$$

$$p_{12} = k_{12}\kappa_{12} + \frac{1}{2} k_{14}\kappa_{21} \quad , \quad p_{21} = k_{13}\kappa_{21} + \frac{1}{2} k_{14}\kappa_{12} \quad , \quad (10.8)$$

$$\partial p_{12} / \partial z - \pi_{12} + \rho \ell_{12} = \rho \alpha_1 \partial^2 \delta_{12} / \partial t^2 \quad , \quad \partial p_{21} / \partial z - \pi_{21} + \rho \ell_{21} = \rho \alpha_2 \partial^2 \delta_{21} / \partial t^2 \quad .$$

We also record here the value of the constitutive coefficients for a rod of rectangular cross-section bounded by the planes  $x = \pm \frac{1}{2} a$ ,  $y = \pm \frac{1}{2} h$ . In terms of the elastic constants  $s_{rs}^{ij}$  (or  $c_{rs}^{ij}$ ) for an orthotropic material in the three-dimensional theory, the relevant constitutive coefficients are<sup>§</sup>

$$\rho = ah\rho^* \quad , \quad \rho\alpha_1 = ha^3\rho^*/12 \quad , \quad \rho\alpha_2 = ah^3\rho^*/12 \quad , \quad (10.9)$$

$$\bar{k}_1 = \frac{1}{4} ah c_{11}^{11} \quad , \quad \bar{k}_2 = \frac{1}{4} ah c_{22}^{22} \quad , \quad k_3 = \frac{1}{4} ah c_{33}^{33} \quad ,$$

$$k_7 = \frac{1}{2} ah c_{22}^{11} \quad , \quad k_8 = \frac{1}{2} ah c_{33}^{11} \quad , \quad k_9 = \frac{1}{2} ah c_{33}^{22} \quad , \quad (10.10)$$

$$k_{10} = \frac{1}{12} ha^3 c_{13}^{13} \quad , \quad k_{11} = \frac{1}{12} ah^3 c_{23}^{23} \quad , \quad k_{17} = 0 \quad ,$$

These coefficients are special cases of those given by Green and Naghdi (1979b) but revised values are given here for  $k_{12}$ ,  $k_{14}$  and  $k_4$ . We also note that in the paper of Green, Naghdi and Wenner (1974, p. 501), Eq. (8.55) should read  $\hat{k}_{12} - \frac{1}{2}\hat{k}_{14} = \mu\pi R^4/4[1 + \frac{2}{3}(R')^2]$ . In addition, we now set  $\hat{k}_{14} = 0$ .

$$k_{14} = 0, \quad k_{12} = k_{13} = \frac{1}{2} \mathcal{D}, \quad k_4 = \gamma a h c_{12}^{12}/8 \quad (10.11)$$

and

$$k_{15} = a h^3 / (12 s_{33}^{33}), \quad k_{16} = h a^3 / (12 s_{33}^{33}). \quad (10.12)$$

Previously  $\gamma$  was taken to be 8. For vibration problems we set  $\gamma = 5.135$ , the first zero of the Bessel function  $J_2(\beta)$ . The coefficient  $\mathcal{D}$  is the classical torsional rigidity. For a rod in which  $a/h \geq (c_{13}^{13}/c_{23}^{23})^{1/2}$  this is given approximately by

$$\mathcal{D} \approx \frac{1}{3} a h^3 c_{13}^{13} \left[ 1 - \frac{192 h}{\pi^5 a} \left( \frac{c_{13}^{13}}{c_{23}^{23}} \right)^{1/2} \tanh \left[ \frac{\pi a}{2 h} \left( \frac{c_{23}^{23}}{c_{13}^{13}} \right)^{1/2} \right] \right]. \quad (10.13)$$

The coefficients  $k_5, k_6$  are given by expressions of the form (10.26) in Green and Naghdi (1979b). For a rectangular section these are calculated to be \*

$$\begin{aligned} \frac{h a^3}{12 k_6} &= \frac{1}{2} a^2 s_{13}^{13} + h^2 s_{33}^{22} \left\{ \frac{1}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^2 n^2 \cosh \left[ \frac{n \pi a}{h} \left( s_{13}^{13} / s_{23}^{23} \right)^{1/2} \right]} \right\} \\ &= \frac{1}{4} a^2 s_{13}^{13} \left( \frac{s_{33}^{22}}{s_{23}^{23}} + 2 \right) - \frac{4 h a s_{33}^{22}}{\pi^2} \left( \frac{s_{13}^{13}}{s_{23}^{23}} \right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2 \sinh \left[ \frac{(2n+1) \pi h}{2 a} \left( s_{23}^{23} / s_{13}^{13} \right)^{1/2} \right]}, \end{aligned} \quad (10.14)$$

$$\begin{aligned} \frac{a h^3}{12 k_5} &= \frac{1}{2} h^2 s_{23}^{23} + a^2 s_{33}^{11} \left\{ \frac{1}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^2 n^2 \cosh \left[ \frac{n \pi h}{a} \left( s_{23}^{23} / s_{13}^{13} \right)^{1/2} \right]} \right\} \\ &= \frac{1}{4} h^2 s_{23}^{23} \left( \frac{s_{33}^{11}}{s_{13}^{13}} + 2 \right) - \frac{4 h a s_{33}^{11}}{\pi^2} \left( \frac{s_{23}^{23}}{s_{13}^{13}} \right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2 \sinh \left[ \frac{(2n+1) \pi a}{2 h} \left( s_{13}^{13} / s_{23}^{23} \right)^{1/2} \right]}, \end{aligned}$$

Let the external force vectors applied to the plane surfaces  $x = \frac{1}{2} a$ ,  $x = -\frac{1}{2} a$  and  $y = \frac{1}{2} h$ ,  $y = -\frac{1}{2} h$  of the rod be denoted, respectively, by  $\underline{t}_+$ ,  $\underline{t}_-$  and by  $\underline{u}_+$ ,  $\underline{u}_-$  and write

\* The expression for the coefficients  $k_5$  and  $k_6$  are recorded in two different forms for convenience of computation depending on the magnitude of the quantity  $(a/h)(s_{13}^{13}/s_{23}^{23})^{1/2}$ .

$$+_{\sim} t = +_{i\sim i} t e_i , \quad -_{\sim} t = -_{i\sim i} t e_i , \quad (10.15)$$

$$u_{\sim}^t = u_{i\sim i}^t e_i , \quad l_{\sim}^t = l_{i\sim i}^t e_i .$$

Then, in the absence of the effect of body forces in the rod, we have

$$\begin{aligned} \rho \tilde{f} &= \int_{-\frac{1}{2}a}^{\frac{1}{2}a} (u_{\sim}^t + l_{\sim}^t) dx + \int_{-\frac{1}{2}h}^{\frac{1}{2}h} (+_{\sim} t + -_{\sim} t) dy , \\ \rho \tilde{l}_1 &= \int_{-\frac{1}{2}a}^{\frac{1}{2}a} x (u_{\sim}^t + l_{\sim}^t) dx + \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \frac{1}{2} a (+_{\sim} t - -_{\sim} t) dy , \\ \rho \tilde{l}_2 &= \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \frac{1}{2} h (u_{\sim}^t - l_{\sim}^t) dx + \int_{-\frac{1}{2}h}^{\frac{1}{2}h} y (+_{\sim} t + -_{\sim} t) dy . \end{aligned} \quad (10.16)$$

# 11. Composite rods

We suppose that a rod of rectangular cross-section contains  $N$  rod-like laminated layers, each consisting of a rod of rectangular cross-section and of homogeneous orthotropic elastic material at the same constant temperature. A typical cylindrical layer, the  $n^{\text{th}}$ , has constant density  $\rho_n^*$  and is bounded by the plane  $x = \pm \frac{1}{2}a$ ,  $y = H_n \pm \frac{1}{2}h_n$ ,  $z = \pm \frac{1}{2}l$ , where  $a, H_n, h_n, l$  are constants. Each cylindrical layer is fixed to its neighbors at its major surfaces  $y = H_n \pm \frac{1}{2}h_n$  so that

$$H_{n+1} - H_n = \frac{1}{2}(h_{n+1} + h_n) \quad , \quad n = 1, 2, \dots, N-1 \quad . \quad (11.1)$$

The mechanical behavior of each cylindrical layer is assumed to be governed by differential equations for rods in (10.5) to (10.8) but with an additional index  $n$  attached to each symbol. Thus, for the  $n^{\text{th}}$  layer, we write  $n_1^n, n_{23}^n, \dots, k_1^n, \dots, k_{16}^n, \rho^n, \rho^{*n}, \alpha_1^n, \alpha_2^n, \dots, u_{\sim}^n, u_i^n$ , etc. Since displacements and force vectors are continuous at each interface it follows that

$$\begin{aligned} \delta_{\sim 1}^{n+1} &= \delta_{\sim 1}^n \quad , \quad u_{\sim}^{n+1} - \frac{1}{2} h_{n+1} \delta_{\sim 2}^{n+1} = u_{\sim}^n + \frac{1}{2} h_n \delta_{\sim 2}^n \quad , \\ u_{\sim}^n + \ell_{\sim}^n &= 0 \quad (n = 1, 2, \dots, N-1) \quad . \end{aligned} \quad (11.2)$$

Forces applied respectively to the outer surfaces  $y = H_1 - \frac{1}{2}h_1$  and  $y = H_N + \frac{1}{2}h_N$  of the composite are

$$\ell_{\sim}^1 \quad , \quad u_{\sim}^N \quad , \quad (11.3)$$

while forces applied at the surfaces  $x = \frac{1}{2}a$  and  $x = -\frac{1}{2}a$  are, respectively, given by

$$t_{\sim}^n \quad \text{and} \quad -t_{\sim}^n \quad , \quad H_n - \frac{1}{2}h_n \leq y \leq H_n + \frac{1}{2}h_n \quad (n = 1, 2, \dots, N) \quad . \quad (11.4)$$

The relevant equations of motion and constitutive equations for each cylindrical layer are of the forms  $F_1, F_2, E$  and  $T$  in (10.5) to (10.9) with an index  $n$  added to each variable.

## 12. Simple static solutions for composite rods

We consider first simple homogeneous extension of the rod along its axis with zero applied forces on its major surfaces  $x = \pm \frac{1}{2} a$ ,  $y = H_1 - \frac{1}{2} h_1$ ,  $y = H_N + \frac{1}{2} h_N$  so that

$$\begin{aligned} \tilde{t}^n_+ &= 0, \quad \tilde{t}^n_- = 0 \quad (n=1,2,\dots,N), \\ \tilde{t}^1_l &= 0, \quad \tilde{t}^N_u = 0. \end{aligned} \quad (12.1)$$

Recalling (10.2) and the conditions (11.2) we choose the components of the displacement and director displacement vectors in the forms

$$\begin{aligned} u_3^n &= Lz, \quad \delta_{11}^n = M, \quad \delta_{22}^n = K^n, \\ u_2^{n+1} - u_2^n &= \frac{1}{2} h_{n+1} K^{n+1} + \frac{1}{2} h_n K^n \quad (n=1,2,\dots,N-1), \end{aligned} \quad (12.2)$$

where  $L, M, K^n, u_2^n$  are constants. The remaining components of displacements and director displacements are zero. It follows from equations (10.7) for the extensional case that  $n_3^n, \pi_{11}^n, \pi_{22}^n$  are constants and we choose the constants  $K^n$  such that  $\pi_{22}^n = 0$  for  $n=1,2,\dots,N$ . The equations of motion in the form recorded in (10.7) are then satisfied if

$$\rho^n f_3^n = 0, \quad \rho^n \ell_{22}^n = 0, \quad \rho^n \ell_{11}^n - \pi_{11}^n = 0, \quad (12.3)$$

where

$$\begin{aligned} \rho^n f_3^n &= \int_{-\frac{1}{2}a}^{\frac{1}{2}a} (u_3^n + \ell_3^n) dx, \\ \rho^n \ell_{11}^n &= \int_{-\frac{1}{2}a}^{\frac{1}{2}a} (\pi_{11}^n + \ell_{11}^n) x dx, \\ \rho^n \ell_{22}^n &= \frac{1}{2} h_n \int_{-\frac{1}{2}a}^{\frac{1}{2}a} (u_2^n - \ell_2^n) dx. \end{aligned} \quad (12.4)$$

In view of (11.2)<sub>3</sub>, (12.1)<sub>3,4</sub>, (12.3)<sub>1,2</sub> and (12.4)<sub>1,3</sub>, we have

$$\int_{-\frac{1}{2}a}^{\frac{1}{2}a} u_3^n dx = 0, \quad \int_{-\frac{1}{2}a}^{\frac{1}{2}a} u_2^n dx = 0 \quad \text{for } n=1,2,\dots,N. \quad (12.5)$$

Since  $\pi_{22}^n = 0$ , it follows that

$$2\bar{k}_2^n K^n = -k_7^n M - k_9^n L \quad (12.6)$$

and hence

$$\begin{aligned} \pi_{11}^n &= [4\bar{k}_1^n - \frac{(k_7^n)^2}{\bar{k}_2^n}]M + [2k_8^n - \frac{k_7^n k_9^n}{\bar{k}_2^n}]L, \\ n_3^n &= [2k_8^n - \frac{k_7^n k_9^n}{\bar{k}_2^n}]M + [4k_3^n - \frac{(k_9^n)^2}{\bar{k}_2^n}]L. \end{aligned} \quad (12.7)$$

With the help of (11.2)<sub>3</sub> and (12.1)<sub>3,4</sub>, it follows from (12.4)<sub>2</sub> and (12.3)<sub>3</sub> that

$$\sum_{n=1}^N \rho_n \pi_{11}^n = 0 \quad \text{or} \quad \sum_{n=1}^N \pi_{11}^n = 0. \quad (12.8)$$

This equation determines M in terms of L. Hence, from (12.7) we obtain

$$\begin{aligned} \frac{n_3}{L} &= \sum_{n=1}^N n_3^n \\ &= \sum_{n=1}^N [4k_3^n - \frac{(k_9^n)^2}{\bar{k}_2^n}] - \frac{\sum_{n=1}^N \{2k_8^n - \frac{k_7^n k_9^n}{\bar{k}_2^n}\}^2}{\sum_{n=1}^N \{4\bar{k}_1^n - \frac{(k_7^n)^2}{\bar{k}_2^n}\}}. \end{aligned} \quad (12.9)$$

Since  $n_3$  is the resultant force along the rod and L is the extension, this gives the effective modulus for extension in terms of the moduli of the individual layers of the composite.

Next, we consider flexure of a composite rod about the x-direction by couples over the ends of the rod. The solution of this problem requires equations of the form F1 in (10.5), but because of the conditions (11.2) at the interfaces between the layers of the composite, the extensional



equations E must also be used. We again assume that the applied forces on the major surfaces of the composite rod are zero so that the conditions (12.1) hold for the present problem. In addition, making use of (10.2) and (11.2), we assume the following values for the nonzero components of displacement and director displacement vectors:

$$\begin{aligned} u_2^n &= -\frac{1}{2} Pz^2 + \bar{P}z + S^n, \quad \delta_{23}^n = Pz - \bar{P}, \quad \delta_{11}^n = M, \\ u_3^1 &= Lz, \quad u_3^{n+1} - u_3^n = (H_{n+1} - H_n)(Pz - \bar{P}) \quad (n = 1, 2, \dots, N-1), \\ \delta_{22}^n &= K^n, \quad S^{n+1} - S^n = \frac{1}{2} h_{n+1} K^{n+1} + \frac{1}{2} h_n K^n \quad (n = 1, 2, \dots, N-1), \end{aligned} \quad (12.10)$$

where  $P, \bar{P}, S^n, L, M, K^n$  are constants. It follows from equations of the form (10.5) for flexural case F1 that

$$m_1^n = k_{15}^n P, \quad n_2^n = 0, \quad \ell_{23}^n = 0, \quad f_2^n = 0. \quad (12.11)$$

Also, as in the extension problem just discussed, we choose the constants  $K^n$  such that  $\pi_{22}^n = 0$  for  $n = 1, 2, \dots, N$ . Then

$$\begin{aligned} \pi_{11}^n &= [4\bar{k}_1^n - \frac{(k_7^n)^2}{\bar{k}_2^n}]M + [2k_8^n - \frac{k_7^n k_9^n}{\bar{k}_2^n}][L + P(H_n - H_1)], \\ n_3^n &= [2k_8^n - \frac{k_7^n k_9^n}{\bar{k}_2^n}]M + [4k_3^n - \frac{(k_9^n)^2}{\bar{k}_2^n}][L + P(H_n - H_1)] \end{aligned} \quad (12.12)$$

and equations (12.3) must again be satisfied. It follows that (12.8)<sub>2</sub> still holds; and, in addition, we require that there is no resultant force on the end sections of the rod so that

$$\sum_{n=1}^N \pi_{11}^n = 0, \quad \sum_{n=1}^N n_3^n = 0. \quad (12.13)$$

These equations express  $L$  and  $M$  in terms of  $P$ . As before, the conditions (12.3)<sub>1,2</sub> yield the equations (12.5) since zero tractions are specified

on the surfaces  $x = \pm \frac{1}{2} a$ ,  $y = H_1 - \frac{1}{2} h_1$ ,  $y = H_N + \frac{1}{2} h_N$ . Moreover, because of (12.5) the conditions (12.11)<sub>3,4</sub> are satisfied.

The resultant couple on any section of the composite rod about the x-axis is

$$m_1 = \sum_{n=1}^N m_1^n + H_n n_3^n = DP, \quad (12.14)$$

where  $P$  is the curvature of the line  $x=y=0$  introduced in (12.10) and  $D$  is the flexural rigidity of the composite rod. The latter may be expressed in terms of the flexural rigidities of each constituent layer of the composite rod and also other elastic coefficients. We illustrate these results by considering a 3-layer composite rod in which the outer layers 1 and 3 of the rod are identical and flexure is about the middle line of the layer 2. Then,

$$\begin{aligned} h_3 &= h_1, \quad H_2 = 0, \quad H_3 = -H_1 = \frac{1}{2}(h_1 + h_2), \\ \rho^{*3} &= \rho^{*1}, \quad \bar{k}_1^3 = \bar{k}_1^1, \quad \text{etc.} \end{aligned} \quad (12.15)$$

and with the help of (12.12)<sub>2</sub> the moment-curvature relation (12.14) simplifies to

$$\begin{aligned} m_1 &= 2m_1^1 + m_1^2 - H_1(n_3^3 - n_3^1) \\ &= \{2k_{15}^1 + k_{15}^2 + 2H_1^2(4k_{13}^1 - \frac{(k_9^1)^2}{\bar{k}_2^1})\}P. \end{aligned} \quad (12.16)$$

Flexure about the y-axis is simpler than that about the x-axis. When flexure is about the y-axis, the only nonzero components of displacement and director displacements are

$$u_1^n = -\frac{1}{2} Qz^2 + \bar{Q}z, \quad \delta_{13}^n = Qz - \bar{Q}, \quad (12.17)$$

so that from appropriate equations of the forms (10.6) we obtain

$$m_2^n = -k_{16}^n Q, \quad n_1^n = 0, \quad f_1^n = 0, \quad \ell_{13}^n = 0. \quad (12.18)$$

In view of the surface conditions on the composite rod and the continuity conditions (11.2)<sub>3</sub>, the last two equations in (12.18) lead to

$$\int_{-\frac{1}{2}a}^{\frac{1}{2}a} u_1^n dx = 0, \quad \int_{-\frac{1}{2}a}^{\frac{1}{2}a} x u_3^n dx = 0. \quad (12.19)$$

The resultant couple about the y-axis is

$$m_2 = \sum_{n=1}^N m_2^n = -Q \sum_{n=1}^N k_{16}^n. \quad (12.20)$$

Finally we deal with the torsion of the rod which is free from applied tractions over its major surfaces. In this case, we have nonzero components of displacement and director displacements of the form

$$\delta_{12}^n = -\delta_{21}^n = \beta z, \quad \delta_{13}^n = N, \quad u_1^n = Fz - (H_n - H_1)\beta z, \quad (12.21)$$

where  $\beta, N, F$  are constants. These satisfy the continuity conditions (11.2) and, from equations of the form recorded in (10.6) and (10.8), we have

$$\begin{aligned} m_2^n &= 0, \quad n_1^n = k_6^n [N + F - (H_n - H_1)\beta], \\ \rho_1^n f_1^n &= 0, \quad n_1^n - \rho^n \ell_{13}^n = 0, \\ \pi_{12}^n &= \pi_{21}^n = 0, \quad m_3^n = (2k_{12}^n - k_{14}^n)\beta = \mathcal{D}^n \beta, \\ \ell_{12}^n &= \ell_{21}^n = 0, \end{aligned} \quad (12.22)$$

where

$$\rho^n f_1^n = \int_{-\frac{1}{2}a}^{\frac{1}{2}a} (u_1^n + l_1^n) dx, \quad \rho^n l_{12}^n = \int_{-\frac{1}{2}a}^{\frac{1}{2}a} x(u_2^n + l_2^n) dx, \quad (12.23)$$

$$\rho^n l_{21}^n = \frac{1}{2} h_n \int_{-\frac{1}{2}a}^{\frac{1}{2}a} (u_1^n - l_1^n) dx, \quad \rho^n l_{13}^n = \int_{-\frac{1}{2}a}^{\frac{1}{2}a} x(u_3^n + l_3^n) dx.$$

In view of the conditions (11.2)<sub>3</sub>, (12.1)<sub>3,4</sub>, (12.22)<sub>3,7,8</sub>, from (12.23)<sub>1,2,3</sub> we have

$$\int_{-\frac{1}{2}a}^{\frac{1}{2}a} u_1^n dx = 0, \quad \int_{-\frac{1}{2}a}^{\frac{1}{2}a} u_2^n x dx = 0. \quad (12.24)$$

Also by (12.1)<sub>3,4</sub> and (11.2)<sub>3</sub> we conclude from (12.23)<sub>4</sub> and (12.22) that

$$\sum_{n=1}^N \rho^n l_{13}^n = 0 \quad \text{or} \quad \sum_{n=1}^N n_1^n = 0, \quad (12.25)$$

so that there is no resultant force over any section of the composite rod. The resultant couple acting on the composite rod about the z-axis is then given by

$$m_3 = \sum_{n=1}^N (m_3^n - H_n n_1^n) \quad (12.26)$$

and the constant  $N + F$  is determined in terms of  $\beta$  by (12.25)<sub>2</sub>.

When the composite rod consists of 3 layers in which the outer layers 1 and 3 of the rod are identical and flexure is about the line of symmetry of the middle layer, then the conditions (12.15) apply and the constant  $F$  is chosen so that  $u_1^2 = 0$ . Using also (12.25)<sub>2</sub>, we obtain

$$F + H_1 \beta = 0, \quad N = 0, \quad n_1^3 = -n_1^1 = k_6^1 H_1 \beta, \quad n_1^2 = 0. \quad (12.27)$$

The torsional rigidity of the composite rod can be obtained from (12.22), (12.26) and (12.27) and is given by

$$m_3/\beta = 2 \mathcal{D}^1 + \mathcal{D}^2 + 2k_6^1 H_1^2. \quad (12.28)$$

It is difficult to discuss more complex problems analytically without specifying the number  $N$  of the composite layers, but we leave the discussion of static problems at this point.

### 13. A constrained theory of rods

We consider here application of a constrained theory of rods which has a simpler structure in comparison with the more general theory summarized in section 10. A derivation of this constrained theory is contained in a paper of Green and Laws (1973). For a linear elastic rod which has symmetries of the type discussed in section 10, the equations again separate into four groups. The flexure groups are the same as  $F_{1,2}$  in (10.5) and (10.6) but the extension and torsion groups  $E$  and  $T$  are replaced by the simpler groups  $E_c, T_c$ . Thus

#### Extension $E_c$

$$n_3 = k\gamma_{33} \quad , \quad \gamma_{33} = 2\partial u_3/\partial z \quad , \quad (13.1)$$

$$\partial n_3/\partial z + \rho f_3 = \rho \partial^2 u_3/\partial t^2 \quad .$$

#### Torsion $T_c$

$$\delta_{12} = -\delta_{21} \quad , \quad m_3 = G\delta_{12} \quad ,$$

(13.2)

$$\partial m_3/\partial z + \rho(\ell_{12} - \ell_{21}) = \rho(\alpha_1 + \alpha_2)\partial^2 \delta_{12}/\partial t^2 \quad .$$

A further constraint is often applied to the flexure part of the problem. Thus, in (10.5) and (10.6),  $\gamma_{\alpha 3} \rightarrow 0$  while  $k_5$  and  $k_6 \rightarrow \infty$ , so that the resultant force components  $n_1, n_2$  are indeterminate, i.e., not given by a constitutive equation. Then, neglecting also the rotatory inertia terms, (10.5) and (10.6) are replaced by

#### Flexure $F_{c1}$

$$m_1 = -k_{15}\partial^2 u_2/\partial z^2 \quad ,$$

(13.3)

$$\partial n_2/\partial z + \rho f_2 = \rho \partial^2 u_2/\partial t^2 \quad , \quad \partial m_1/\partial z - n_2 + \rho \ell_{23} = 0 \quad ,$$

Flexure F<sub>C</sub> 2

$$m_2 = k_{16} \partial^2 u_1 / \partial z^2, \quad (13.4)$$

$$\partial n_1 / \partial z + \rho f_1 = \rho \partial^2 u_1 / \partial t^2, \quad \partial m_2 / \partial z + n_1 - \rho l_{13} = 0.$$

The constitutive coefficients  $k_{15}, k_{16}$  for a homogeneous rod of rectangular section are the same as those given in section 10, but the coefficient  $k$  has the value

$$k = 2[k_3 - \frac{\bar{k}_1 k_9^2 + \bar{k}_2 k_8^2 - k_7 k_8 k_9}{4\bar{k}_1 \bar{k}_2 - \bar{k}_7}] = \frac{ah}{2s_{33}}. \quad (13.5)$$

Also  $\mathcal{Q}$  is equal to the torsional rigidity  $\mathcal{D}$  given in (10.13).

For a composite rod of the type discussed in section 12, the theory of the present section may be used if we compute the coefficients  $k_{15}, k_{16}, \mathcal{Q}$  and  $k$  by comparing simple solutions with the corresponding four simple solutions for extension, flexure and torsion discussed in section 12.

#### 14. Harmonic wave propagation along the rod

The propagation of simple harmonic waves along a composite rod of  $N$  constituent layers may be discussed on the basis of the theory of section 11. In general the waves can be categorized into two groups, one described by equations of the type E and F1 and the other by equations of the type T and F2. To make the analysis definite, it is necessary to specify the number  $N$  of the composite layers and we do not discuss this further here. There is, however, one subgroup of waves, purely of the type F2, which can be discussed for the general case of  $N$  constituent layers and we consider this briefly. We assume that all variables contain a factor  $\exp i(\xi z - \omega t)$  where  $\xi, \omega$  are constants, and this factor is removed throughout. Then, recalling the continuity conditions (11.2), we choose

$$\delta_{13}^n = \delta, \quad u_1^n = u \quad (n=1, \dots, N) \quad (14.1)$$

and take all the remaining components of displacements and director displacements to be zero. Then, from (10.6) and section 11 we have

$$\begin{aligned} (k_6^n \xi^2 - \rho^n \omega^2)u - i k_6^n \xi \delta &= \rho^n f_1^n, \\ (k_{16}^n \xi^2 - \rho^n \alpha_1^n \omega^2 + k_6^n)\delta + i k_6^n \xi u &= \rho^n l_{13}^n. \end{aligned} \quad (14.2)$$

Assuming that the major surfaces of the composite rod of rectangular cross-section are free from applied forces, by (10.16) the quantities  $f_1^n$  and  $l_{13}^n$  are given by

$$\rho^n f_1^n = \int_{-\frac{1}{2}a}^{\frac{1}{2}a} (u t_1^n + l t_1^n) dx, \quad \rho^n l_{13}^n = \int_{-\frac{1}{2}a}^{\frac{1}{2}a} (u t_3^n + l t_3^n) x dx. \quad (14.3)$$

In view of (11.2)<sub>3</sub>, the surface conditions (12.1)<sub>3,4</sub> and (14.3) we conclude that



$$\sum_{n=1}^N \rho^n f_1^n = 0, \quad \sum_{n=1}^N \rho^n \lambda_{13}^n = 0. \quad (14.4)$$

With the help of (14.4) we may eliminate  $u, \delta$  from (14.2) to obtain the dispersion relation

$$\bar{\alpha}_1 \bar{\rho}^2 \omega^4 - \bar{\rho} \omega^2 (\bar{k}_6 + \bar{k}_{16} \xi^2 + \bar{\alpha}_1 \bar{k}_6 \xi^2) + \bar{k}_6 \bar{k}_{16} \xi^4 = 0, \quad (14.5)$$

where

$$\bar{k}_6 = \sum_{n=1}^N k_6^n, \quad \bar{k}_{16} = \sum_{n=1}^N k_{16}^n, \quad \bar{\rho} = \sum_{n=1}^N \rho^n, \quad \bar{\rho} \bar{\alpha}_1 = \sum_{n=1}^N \rho^n \alpha_1^n. \quad (14.6)$$

When  $\xi \rightarrow 0$ , one value of  $\omega^2 \rightarrow \bar{k}_{16} \xi^4 / \bar{\rho}$  and this is the single value which would be obtained by using the  $F_c^2$  flexure equations (13.3) for the same problem.

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