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COMPACT COVARIANCE OPERATORS[†]

by

Charles R. Baker and Ian W. McKeague

ABSTRACT

Let B be a real separable Banach space and $R: B^* \rightarrow B$ a covariance operator. All representations of R in the form $\sum e_n \otimes e_n$, $\{e_n, n \geq 1\} \subset B$, are characterized. Necessary and sufficient conditions for R to be compact are obtained, including a generalization of Mercer's theorem. An application to characteristic functions is given.

KEY WORDS AND PHRASES: Covariance operators, compact operators, probability in Banach spaces.

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1. Introduction

The study of covariance operators is a major component in the theory of probability measures on Banach spaces [10], [2], [14]. The covariance operator of a strong second-order measure is always compact [2]; however, the covariance operator of a weak second-order measure need not be compact. In this paper we first characterize series representations of covariance operators, and then give a set of necessary and sufficient conditions for a covariance operator to be compact. The classical Mercer's theorem [14] can be obtained as an immediate corollary. These results are then applied to extend a result of Prohorov and Sazanov [6] on relative compactness of probability measures from Hilbert space to Banach space.



2. Definitions and Notation

B is a real separable Banach space with norm $||\cdot||$ and topological dual B^* . A linear operator $R: B^* \rightarrow B$ is a covariance operator if R is symmetric and non-negative: $\langle Ru, v \rangle = \langle u, Rv \rangle$ and $\langle Ru, u \rangle \geq 0$, for all u, v in B^* . A probability measure μ on the Borel σ -field of B is said to be weak second-order if $\int_B \langle x, u \rangle^2 d\mu(x) < \infty$, for all u in B^* ; μ is strong second-order if $\int_B ||x||^2 d\mu(x) < \infty$. Every weak second order measure μ has a mean element m in B and a covariance operator $R: B^* \rightarrow B$ [9], [10], defined by

$$\langle m, v \rangle = \int_B \langle x, v \rangle d\mu(x)$$

$$\langle Ru, v \rangle = \int_B \langle x - m, u \rangle \langle x - m, v \rangle d\mu(x),$$

for all u, v in B^* . Strong second-order measures have compact covariances; the strong second order property is not necessary in order that μ have compact covariance.

For a covariance operator $R: B^* \rightarrow B$ it is well known [8], [1], that there exists a separable Hilbert space $H \subset B$ such that the natural injection

$j: H \rightarrow B$ is continuous and $R = jj^*$. H is the RKHS of R and is the completion of range (R) with respect to the inner product $\langle \cdot, \cdot \rangle_H$ defined by $\langle Ru, Rv \rangle_H = \langle Ru, v \rangle$.

I_H will denote the identity on H . For u, v in B^* , z in B (resp. in H), $(u \otimes v)(z) = \langle v, z \rangle u$ (resp., $\langle v, z \rangle_H u$). If T is any map $r(T) \equiv \text{range}(T)$. τ_C is the linear topology on B^* determined by a neighborhood base at zero of the form $V_{C, \epsilon}(0) = \{f \in B^*: \sup_{x \in C} \langle f, x \rangle^2 < \epsilon\}$ for all $\epsilon > 0$ and all compact sets $C \subset B$ (τ_C is the topology of uniform convergence on compact sets). For a given covariance operator $R: B^* \rightarrow B$, q_R is the real-valued quadratic functional on B^* defined by $q_R f = \langle Rf, f \rangle$. The notation $R = \sum_n e_n \otimes e_n$ for $\{e_n, n \geq 1\} \subset B$ means that the sequence $(\sum_1^N e_n \otimes e_n)$ converges to R in the strong operator topology: $\sum_1^N \langle e_n, f \rangle e_n \rightarrow Rf$ in the norm topology of B , for all f in B^* . $I_H = \sum_n e_n \otimes e_n$ has a similar interpretation. If $\{u_n, n \geq 1\}$ is any orthonormal basis for H , then $R = \sum j u_n \otimes j u_n$, [9]. K_R will denote the unit ball in H .

If μ is a probability measure on the Borel σ -field of B , its characteristic functional $\hat{\mu}$ is defined as

$$\hat{\mu}(x) = \int_B e^{i\langle x, y \rangle} d\mu(y), \text{ for } x \text{ in } B^*.$$

3. Representation of Covariance Operators.

In this section, R is an arbitrary covariance operator.

Theorem 1. $R = \sum_n e_n \otimes e_n$ if and only if $e_n = jv_n$, $v_n \in H$ for $n \geq 1$, and

$$I_H = \sum_n v_n \otimes v_n.$$

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Proof. It suffices to show that the stated conditions are necessary for $R = \sum_n e_n \otimes e_n$. Suppose $R = \sum_n e_n \otimes e_n$, and fix e_k . Let $P_k = e_k \otimes e_k$. To show $e_k \in \text{range}(j)$, let (as in [3]) $D: r(j^*) \rightarrow B$ be defined by $Dj^*f = P_k f$. Then $\|Dj^*f\|^2 = \|P_k f\|^2 = \|e_k\|^2 \langle e_k, f \rangle^2 \leq \|e_k\|^2 \sum_n \langle f, e_n \rangle^2 = \|e_k\|^2 \langle Rf, f \rangle = \|e_k\|^2 \|j^*f\|_H^2$. Thus D can be extended to a continuous linear map from $\overline{r(j^*)} = H$ into B . From its definition, $Dj^* = P_k$, so $P_k = jD^*$ and thus $e_k \in \text{range}(j)$.

To see that $I_H = \sum v_n \otimes v_n$, where $jv_n = e_n$, $n \geq 1$, define $Q_N = \sum_1^N v_n \otimes v_n$. $Q_N = Q_N^*$ and $Q_N \geq 0$, so $Q_N^{1/2}$ exists. $\|Q_N^{1/2} j^*f\|_H^2 = \sum_1^N \langle f, e_n \rangle^2 + \|j^*f\|_H^2$, so that $\|Q_N^{1/2}\| \leq 1$ and $\|Q_N^{1/2} x\|_H \rightarrow \|x\|_H$ for all x in $r(j^*)$. Thus,

$$\begin{aligned} \|\sum_1^N (v_n \otimes v_n) j^*f - j^*f\|_H^2 &= \|Q_N j^*f - j^*f\|_H^2 \\ &\leq \|Q_N^{1/2} j^*f\|_H^2 + \|j^*f\|_H^2, \end{aligned}$$

which converges to zero as $N \rightarrow \infty$ for any fixed f in B^* . Thus, $\sum v_n \otimes v_n = I_H$ on $r(j^*)$, and the result follows by $\overline{r(j^*)} = H$. $\square$

Remark. Suppose E is a locally convex topological vector space, $R: E' \rightarrow E$ is a covariance operator, and $R = jj^*$, where $j: H \rightarrow E$ is the injection and H is the RKHS of R . R will have such a representation, for example, if E is separable and quasi-complete [8]. In this case, it is easily shown that Theorem 1 holds without modification.

The representation $I_H = \sum v_n \otimes v_n$ does not require that $\{v_n, n \geq 1\}$ be a CONS in H ; however, sufficient conditions for $\{v_n, n \geq 1\}$ to be a CONS in H can be given.

Proposition 1. Suppose $I_H = \sum v_n \otimes v_n$; the following are equivalent:

- (1) $\|v_k\|_H = 1$
- (2) $v_k \notin \overline{\text{sp}\{v_n, n \neq k\}}$
- (3) $v_k \perp \overline{\text{sp}\{v_n, n \neq k\}}$.

If any of the above conditions hold for all $k \geq 1$, then $\{v_n, n \geq 1\}$ is a CONS for H .

4. Compact Covariance Operators.

Theorem 2. Suppose $R = \sum e_n \otimes e_n$, $\{e_n, n \geq 1\} \subset B$. Let $\{v_n, n \geq 1\} \subset H$ be such that $e_n = jv_n$, $n \geq 1$. The following are equivalent:

- (1) R is compact;
- (2) j is compact;
- (3) $j[K_R]$ is compact in B ;
- (4) the series $\sum v_n \otimes jv_n$ converges uniformly in H on bounded subsets of B^* ;
- (5) $(\sum_{n=1}^N e_n \otimes e_n)$ converges to R uniformly in B on bounded subsets of B^* ;
- (6) q_R is w^* -continuous on bounded subsets of B^* ;
- (7) q_R is τ_C -continuous.

Proof. (1) $\Rightarrow$ (2). Suppose $f_\alpha \rightarrow f$ in the w^* topology of B^* , where $\|f_\alpha\| \leq k$ for all α . Then $\|j^*f_\alpha - j^*f\|_H^2 = \langle R(f_\alpha - f), (f_\alpha - f) \rangle \leq 2k \|R(f_\alpha - f)\|_B$; since R is compact, $j^*f_\alpha \rightarrow j^*f$ in H [4, p. 486] and thus j is compact.

(2) $\Rightarrow$ (3). j compact implies $j[K_R]$ is relatively compact in B . Since K_R is weakly compact in H and j is weakly continuous, $j[K_R]$ is weakly compact in B , and thus closed.

(3) $\Rightarrow$ (2) by definition.

(2) $\Rightarrow$ (4). By Theorem 1, $\sum v_n \otimes v_n = I_H$. Set $Q_N = \sum_{n=1}^N v_n \otimes v_n$. If $A \subset B^*$ is bounded, then $j^*[A]$ is compact; by Dini's theorem $\|Q_N^*x\|_H \uparrow \|x\|_H$ uniformly on $j^*[A]$. Hence $\|(Q_N - I)j^*x\|_H^2 \leq \|j^*x\|_H^2 - \|Q_N^*j^*x\|_H^2 \rightarrow 0$ uniformly on A .

(4) $\Rightarrow$ (5), since j is continuous.

(5) $\Rightarrow$ (1), since R is the uniform limit of compact operators.

(2) $\Leftrightarrow$ (6). Follows from the fact that j is compact if and only if $j^*f_\alpha \rightarrow 0$ in the norm topology of H for all bounded generalized sequences (f_α) in B^* which are w^* convergent to zero [4, p. 486], and $q_R(f_\alpha) = \|j^*f_\alpha\|_H^2$.

(1) $\Rightarrow$ (7). Suppose R is compact. Writing $C = j[K_R]$, C is compact in B .
 $q_R(f) = \langle Rf, f \rangle = \|j*f\|_H^2 = \sup_{x \in K} \langle j*f, x \rangle_H^2 = \sup_{x \in C} \langle f, x \rangle^2$. Thus q_R is τ_C -continuous at zero. τ_C -continuity of q_R follows from $q_R(f_\alpha) = q_R(f_\alpha - f) - q_R(f) + 2\langle Rf, f_\alpha \rangle$.

(7) $\Rightarrow$ (1). Suppose q_R is τ_C -continuous. Using (6), R is compact if q_R is w^* continuous at 0 on bounded subsets of B^* . But B is separable so that the w^* topology on bounded subsets of B^* is metrizable and it suffices to consider sequences. Suppose $f_n \xrightarrow{w^*} 0$ and $\|f_n\| \leq k$. Let L be an arbitrary compact subset of B . Since $\{f_n\}$ is bounded in B^* the f_n are equicontinuous and uniformly bounded as continuous functions on L . Thus, by the Arzela-Ascoli Theorem [4, p. 266] $\{f_n\}$ is relatively compact as a subset of $C^{\mathbb{R}}(L)$. Thus since $f_n \xrightarrow{w^*} 0$, f_n converges to 0 uniformly on L . Therefore $f_n \xrightarrow{\tau_C} 0$ and $q_R(f_n) \rightarrow 0$. This completes the proof of Theorem 2. $\square$

Remarks. (1) Suppose $r: [0,1] \times [0,1] \rightarrow \mathbb{R}$ is continuous, symmetric and positive definite. For fixed $t \in [0,1]$, let $\pi_t(x) = x_t$ for x in $C[0,1]$; $\|\pi_t\| = 1$. A compact covariance operator $R: C^*[0,1] \rightarrow C[0,1]$ is defined by $[R\mu](t) = \int_0^1 r(t,s) d\mu(s)$ for any μ in $C^*[0,1]$ (by Arzela-Ascoli Theorem). Thus for $s, t \in [0,1]$, $\langle R\pi_t, \pi_s \rangle = r(t,s)$. The integral operator in $L_2[0,1]$, corresponding to the kernel r , has continuous orthonormal eigenvectors $\{y_n, n \geq 1\}$ and associated non-zero eigenvalues $\{\lambda_n, n \geq 1\}$; it is well known that $\{\lambda_n^{1/2} y_n, n \geq 1\}$ is a CONS in the RKHS H of R . Thus, from Theorem 2, $\sum_{n=1}^N \lambda_n y_n(t) y_n(s)$ converges uniformly to $r(t,s)$ for all t, s in $[0,1]$. This is the classical Mercer's Theorem [7, pp. 245-246].

(2) The fact that the unit ball of H is compact in B when R is compact was proved by Kuelbs [5] under the assumption that R is the covariance of a strong second-order measure.

5. Characteristic Functionals

Let Λ denote a family of probability measures on B (separable Banach) and $\hat{\Lambda}$ the corresponding family of characteristic functionals.

Theorem 3. Let B be a separable Banach space. Then the following are equivalent:

- a) There exists a topology τ on B^* such that for each family Λ of probability measures on B , $\hat{\Lambda}$ is equicontinuous in this topology if and only if Λ is relatively compact in the topology of weak convergence
- b) B is finite dimensional.

Proof. As in the Hilbert space case (see [6, Lemma 2]) τ_c is the weakest topology on B^* such that relative compactness of $\Lambda \Rightarrow$ equicontinuity of $\hat{\Lambda}$. Suppose that (a) holds. Then $\tau_c \subset \tau$ and τ_c equicontinuity of $\hat{\Lambda}$ implies relative compactness of Λ . Now let $R: B^* \rightarrow B$ be any compact covariance operator. Let $\{e_n\}$ be a CONS in the RKHS of R . Define μ_k to be the zero mean Gaussian measure on B with covariance operator $\sum_{n=1}^k e_n \otimes e_n$. Then $\{\hat{\mu}_k\}$ is τ_c equicontinuous by Theorem 2 and $\{\mu_k\}$ is relatively compact. Therefore R is the covariance of a Gaussian probability measure on B and by [9, Theorem 11] B is finite dimensional. □

Theorem 3 extends a result of Prohorov and Sazonov [6] who proved it for Hilbert spaces.

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