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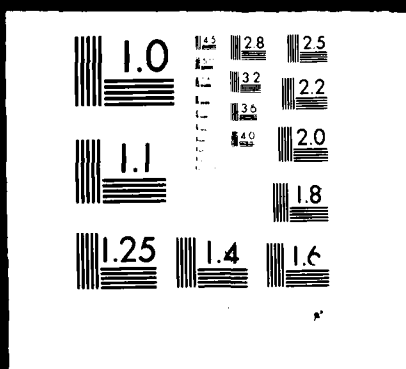

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VARIABLE METRIC RELAXATION METHODS, PART I: A CONCEPTUAL ALGORITHM

by

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1. Introduction

The most successful methods in classical optimization all use some information about second derivatives. First order methods, like the steepest descent, converge at a linear rate of convergence which is related to the condition number of the Hessian at the optimum, and thus convergence can be very slow if the Hessian is poorly conditioned. This poor behavior can be corrected by applying the first order method in a space obtained by a linear transformation of the variables; classical analysis clearly states that the best linear transformation (unique up to an equivalence class) is the square root of the Hessian at the optimum. This linear transformation leads to a transformed function whose Hessian at the optimum is a unit matrix, and thus superlinear, or quadratic, convergence ensues. Translated in terms of the original variable, this leads to a conceptual Newton method, which can be approximated in an implementable way, by Newton or quasi-Newton methods; both of these procedures, using explicitly, or implicitly, second order information, tend to approximate, iteratively, the Hessian, or the inverse Hessian, at the optimum, and, in doing so, preserve, to some extent, the superlinear or quadratic convergence rate of the conceptual method.

The extension of these ideas to the problem of minimizing a nondifferentiable convex function, or, somewhat equivalently, the problem of solving a system of linear inequalities (which includes the general linear programming problem), runs in the difficulty that the concept of Hessian at the optimum does not exist.

The ellipsoid method was introduced by Shor [32] as an attempt to reproduce, as well as possible, the behavior of classical quasi-Newton methods; Yudin and Nemirovski [36,37] showed that the ellipsoid method converges, on any convex function, at a rate which depends only upon the dimension of the space, but not on the specific function (the rate being approximately $1 - (1/2)n^{-2}$), and furthermore that nothing much better can be expected from any algorithm which uses only information given by an oracle, which speaks only the function value and one sub-gradient at every consultation (and thus superlinear convergence is ruled out). Khacian [24,25] showed that in the case of a system of linear inequalities the ellipsoid method will find a solution in polynomial time (using the meaning of that word given in the theory of computational complexity, i.e., polynomial in the length of the input data, and not in the size of the problem). The method has been studied further, and improved, by, among others, Akgul [2], Aspvall and Stone [3], Bland, Goldfarb and Todd [6], Gacs and Lovasz [11], Goldfarb and Todd [17], Grotschel, Lovasz and Schrijver [18] and in [16].

In this paper, we will study the conceptual method which the ellipsoid method implements in an approximate way, while this last statement will be justified in a sequel; all of this will be done only for the problem of solving a system of linear inequalities.

In section 2 we describe the first order method which is used, the maximal distance relaxation method of Agmon [1] and Motzkin and

Schoeberg [29], while also reproducing the theory of its convergence which involves critically the condition number μ^* (whose definition is mostly algebraic); the convergence theory says that convergence is linear, at a rate given by $(1-\mu^{*2})^{1/2}$, which may be very slow. By applying the algorithm in a transformed space, and translating it in terms of the original variables, one defines a variable metric, maximal ellipsoidal distance, relaxation method; the convergence rate is given by the equivalent of μ^* in this transformed space, and thus it can be expected that if the linear transformantion is well chosen, convergence will be improved.

In section 3, two other "condition numbers," which have a purely geometrical definition, are described: ν, where $\sin^{-1} \nu$ measures the angles of the feasible set, and σ, the asphericity of the feasible set. Various relationships between μ^*, ν and σ are given, and their behavior under perturbations of the feasible set and under linear transformations is investigated. This requires a rather detailed study of the behavior of the face lattice (and related lattices) of the feasible set, when perturbations and linear transformations are introduced. A concept of nondegeneracy, which slightly differs from the usual definitions is defined, and it is shown that all but a finite number of perturbations of the feasible set are nondegenerate. The section ends by showing that μ^* may be bounded below by the inverse of the asphericity of a compact and full dimensional polyhedron, which is obtained from the feasible set by perturbation and compactification; this will permit the geometrical results of section 6 to always be applicable.

In section 4, a termination routine is given, which allows the usually infinitely convergent relaxation method to terminate; it is a classical projection method (in the transformed space) whose convergence is not impaired by the type of degeneracy defined in section 3. As it is essentially a projected inverse "Hessian" method, it permits the variable metric information to be used at the level of the termination routine, which might be a useful feature.

In section 5, under the assumption that the data is integer, the issue of the representation of all computed numbers by the ratio of polynomial space integers is treated, so as to be ignored later. A priori estimates of the various quantities used in the method are given, so as to make the whole algorithm implementable.

In section 6, the key geometrical result, due to John [23], is discussed and proved: for any compact, convex set with an interior, there always exists an affine transform of this set, whose asphericity is at most the dimension of the space; it is also shown that the linear map defined through the largest ellipsoid inscribed in the set (or the smallest ellipsoid circumscribed around it) have that property. Thus the ellipsoid matrix corresponding to the largest ellipsoid inscribed in a polyhedron (or in a convex set) may be viewed as a natural extension of the Hessian; a characterization is given, which shows that the inverse "Hessian" is a positive linear combination of symmetric rank one matrices build upon the normals to the facets of the polyhedron. It also reduces to the usual concept of an inverse when the set is an ellipsoid, or a parallelotope.

In section 7, we thus show that every system of linear inequalities may be solved in polynomial time and space by the variable metric, maximal ellipsoidal distance, relaxation method, using an integer (scaled) inverse "Hessian" (where the integers are polynomial space). The length of the input data is denoted by L, and will never be used, but often referred to.

2. Relaxation methods for systems of linear inequalities

Let $Ax \leq b$ be a system of linear inequalities, where $A \in R^{m,n}$ (or sometimes $Z^{m,n}$, where Z is the set of integers), $x \in R^m$ and $b \in R^m$ (or sometimes Z^m).

This system may also be written as

$$(a^i,x) \leq b_i\,, \qquad i \in M \qquad \text{(LI)},$$

where $a^i = A^t e^i$, $b_i = e^{it} b$ (e^i is the ith column of the identity matrix of dimension m, $M = \{1, 2, \ldots, m\}$, $(\cdot,\cdot)$ is the scalar product and t means transpose).

It will be assumed that no row of A is identically zero.

The solution set of (LI) is a polyhedron P:

$$P = \{x \in R^n: Ax \leq b\}$$
$$= \{x \in R^n: ((a^i,x) - b_i)/\|a^i\| \leq 0,\ i \in M\}\,,$$

where $\|\ \|$ is the Euclidean norm.

The problem is to find a point satisfying all the inequalities, or to decide that no such point exists. The algorithm used is the maximal distance relaxation method of Agmon [1], with an n step termination routine added to it.

Algorithm 1: the maximal distance relaxation method.

1. $x = 0$, $q = 0$;
2. select any $i \in I(x)$, where

$$I(x) = \{i \in M: (a^{it}x - b_i)/\|a^i\| = \max_{j \in M} ((a^{jt}x - b_j)/\|a^j\|)\}$$

3.

3.1 if $(a^i,x)-b_i \le 0$, stop: P is feasible and $x \in P$;

? other termination criteria?

4. $x_+ = x - ((a^i,x)-b_i)\ a^i/\|a^i\|^2$.

5. $q \leftarrow q+1$, $x \leftarrow x_+$, go to 2.

A notation indicating the iteration count (x^q, i^q) will be used only when unavoidable. In step 4, a relaxation parameter may be introduced [29,13,14], but it does not affect the theory given here, even though, in practice, it seems to significantly improve the convergence of the algorithm.

The ? other termination criteria? will be specified later, but they consist of:

3.2. if $(a^i,x)-b_i > 0$ is small, P is feasible and go to a (n step) termination routine to find $x' \in P$.

3.3. if q is large, stop: P is empty.

If we define

$$f(x) = \max_{i \in M} \{((a^i,x)-b_i)/\|a^i\|\},$$

then $a^{i^q}/\|a^{i^q}\| \in \partial f(x^q)$, the subdifferential of f at x^q. If $f(x)$ is positive, then it is the maximal distance from x to any hyperplane representing a violated constraint; if $f(x)$ is negative, then $-f(x)$ is the radius of the largest sphere centered at x and contained in P.

The theory of convergence of algorithm 1 will use the properties of f but also of the function g:

$$g(x) = \begin{cases} \min\{\|x-x'\|: x' \in P\} = \sup\{r \geq 0: (x+rS) \cap P = \phi\}, & \text{if } x \notin P \\ -\sup\{r \geq 0: x+rS \subset P\}, & \text{if } x \in P \end{cases}$$

where $S = \{x \in R^n: \|x\| \leq 1\}$ is the unit ball.

The condition number $\mu^*(P)$ is defined by $\mu^*(P) = \text{Inf}\{f(x)/g(x): x \notin P\}$; it is well known that $\mu^*(P) \in (0,1]$ (see Agmon [1], Hoffman [21], Todd [33], and [13,14]). It should be emphasized that $\mu^*(P)$ is not a function of P, as a geometrical object, but depends on the representation of P by a specific system of inequalities.

Lemma 2.1

The functions f and g are convex, and

$$f(x) = g(x), \qquad \text{if } x \in P$$

$$\mu^*(P)\, g(x) \leq f(x) \leq g(x), \qquad \text{if } x \notin P,$$

where both bounds are reached.

Proof.

The convexity of f follows as f is a maximum of linear functions. That of g follows from geometrical results given in Hadwiger [20, pp. 149-150], which imply that the array of outer $(w \geq 0)$ and inner $(w < 0)$ parallel sets of P

$$w \to Y_w = [x \in R^n : g(x) \leq w\}$$

is a concave array. This means that $Y = \{(x,w) \in R^{n+1} : x \in Y_w\}$ is a convex set; but as $Y = \{(x,w) \in R^{n+1} : g(x) \leq w\}$, it follows that g is convex.

The facts that $f \leq g$ and that both bounds are tight are easy to show [14]. QED

Theorem 2.2

The maximal distance relaxation method applied to the system of inequalities (LI), assumed to be consistent, generates a sequence of iterates which converges finitely or infinitely to a point x^* which solves (LI). Furthermore:

$$\|x^q - x^*\| \leq 2g(x^o)\ \theta^q$$

$$g(x^q) \leq \theta g(x^{q-1}) \leq \theta^q\ g(x^o)$$

$$f(x^q) \leq (\mu^*(P))^{-1}\ \theta^q\ f(x^o)$$

where $\theta = (1-\mu^*(P))^2)^{1/2}$.

Proof: See Agmon [1], or [13, 14].

A rough sketch of the key part of the proof goes as follows: $(x + g(x)S) \cap P = \{\tilde{x}\}$, where $\tilde{x}$ is the closest point to x in P; now x, x_+ and $\tilde{x}$ define a triangle which is obtuse at x_+, and thus

$$\|x_+ - \tilde{x}\|^2 \leq \|x-\tilde{x}\|^2 - \|x-x_+\|^2 = g^2(x) - f^2(x) \leq \theta^2 g^2(x) .$$

Hence,

$$(x + \theta g(x)S) \cap P \supset (x_+ + \|x_+ - \tilde{x}\|S) \cap P \supset \{\tilde{x}\} ,$$

and $g(x_+) \leq \theta g(x)$. QED

The method has the property that the sequence of iterates is Fejér-monotone [29], i.e.,

$$\|x^{q+1} - x'\| < \|x^q - x'\| , \qquad \text{for all } x' \in P .$$

Theorem 2.2 is valid if a relaxation parameter $\rho \in (0,2)$ is introduced in step 4 of algorithm 1, provided that θ is defined by

$$\theta = \sqrt{1 - \rho(2-\rho)\ (\mu^*(P))^2}$$

[13,14]; the introduction of a relaxation parameter greater than one seems, in practice, to significantly improve the convergence of the algorithm, but it still may be excrutiatingly slow, or non-polynomial [15,33].

It appears sensible to expect that a well chosen linear transformation of the space may improve the rate of convergence of algorithm 1.

Let $x = Ty$, where T is a nonsingular linear transformation; then (LI) becomes

$$(a^i, Ty) \le b_i, \qquad i \in M ,$$

or

$$(T^t a^i, y) \le b_i , \qquad i \in M .$$

The solution set of this system of linear inequalities is $T^{-1}P$. The maximal distance relaxation method applied to this transformed problem is given below.

Algorithm 1'.

1. $y = 0$, $q = 0$.
2. select any $i \in I(y,T)$, where

$$I(y,T) = \{i \in M: ((T^t a^i, y) - b_i)/\|T^t a^i\| = \max_{j \in M} (((T^t a^j, y) - b_j)/\|T^t a^j\|)\} .$$

3.

3.1. If $(T^t a^i, y) - b_i \le 0$, stop:
$T^{-1}P$ is feasible, and $y \in T^{-1}P$ (also P is feasible, and $x = Ty \in P$).
? other termination criteria?

4. $y_+ = y - ((T^t a^i, y) - b^i)\, T^t a^i/\|T^t a^i\|^2$
5. $q \leftarrow q+1$, $y \leftarrow y_+$, go to 2.

The convergence of algorithm 1' is given by a theorem analogous to Theorem 2.2, but which uses the functions:

1. $$f(y,T) = \max_{i \in M} \frac{(T^t a^i, y) - b_i}{\|T^t a^i\|} ;$$

clearly $(T^t a^{i^q})/\|T^t a^{i^q}\| \in \partial f(y^q,T)$.

2.

$$g(y,T) = \begin{cases} \min\{\|y-y'\|: y' \in T^{-1}P\} = \sup\{r \geq 0: (y+rS) \cap T^{-1}P \neq \phi\}, & \text{if } y \notin T^{-1}P \\ -\sup\{r \geq 0: (y+rS) \subset T^{-1}P\}, & \text{if } y \in T^{-1}P. \end{cases}$$

If one defines

$$\mu^*(T^{-1}P) = \text{Inf}\{f(y,T)/g(y,T): y \notin T^{-1}P\} ,$$

then Lemma 2.1 and Theorem 2.2 apply. One should note that $f(y,T)$ is not equal to $f(Ty)$ because of the normalizations used in defining $f(x)$ and $f(y,T)$.

Algorithm 1' may be expressed in terms of the original variable x, and leads to a (fixed) variable metric, maximal ellipsoidal distance, relaxation method.

Algorithm 2

Define $H = TT^t$

1. $x = 0$, $q = 0$.

2. select any $i \in I(x,T)$, where

$$I(x,T) = \{i \in M: ((a^i,x) - b_i)/(a^{it}Ha^i)^{1/2}$$

$$= \max_{j \in M} (((a^j,x) - b_j)/(a^{jt}Ha^j)^{1/2}\}$$

3.

3.1.

If $(a^i,x) - b_i \leq 0$, stop: P is feasible and $x \in P$;

?other termination criteria?

4. $x_+ = x = ((a^i,x) - b_i)\, Ha^i/(a^{it}Ha^i)$.

5. $q \leftarrow q+1$, $x \leftarrow x_+$, go to 2.

One could describe algorithm 2 as an ellipsoid method, with a fixed ellipsoid.

It is clear that, if, in steps 2 of algorithms 1' and 2, ties are broken in the same fashion, then the sequences generated satisfy

$$x^q = Ty^q , \quad \text{for all } q;$$

and thus Lemma 2.1 and Theorem 2.2 can be adapted to give a convergence theory for algorithm 2.

Let $E = TS = \{x \in R^n: x^tH^{-1}x \leq 1\}$ be "the" ellipsoid, and E^d be its dual $E^d = T^{-t}S = \{x \in R^n: x^tHx \leq 1\}$; the corresponding ellipsoidal norms are given by

$$\|x\|_E \doteq \text{Inf}\{r \geq 0: x \in rE\} = \|T^{-1}x\| = (x^t H^{-1} x)^{1/2} \quad ,$$

$$\|x\|_{E^d} = \text{Inf}\{r > 0: x \in rE^d\} = \|T^t x\| = (x^t H x)^{1/2} \; .$$

One thus defines the functions:

$$1. \quad \phi(x,T) = \underset{i \in M}{\text{Max}} \frac{(a^i,x)-b_i}{(a^{it}Ha^i)^{1/2}} = \underset{i \in M}{\text{Max}} \frac{(a^i x)-b^i}{\|a^i\|_{E^d}}$$

$$\gamma(x,T) = \begin{cases} \text{Min}\{\|x-x'\|_E: x' \in P\} = \text{Sup}\{r > 0: (x+rE) \cap P \neq \phi\} & \text{if } x \notin P \\ -\text{Sup}\{r > 0: (x+rE) \subset P\} & \text{if } x \in P \; . \end{cases}$$

The direction used at step q is given by Ha^{i^q}, where

$$a^{i^q}/(a^{i^q t}Ha^{i^q})^{1/2} \in \partial\phi(x^q,T),$$

and thus it is a subgradient of ϕ multiplied by the positive definite symmetric matrix H; algorithm 2 is thus quite similar to a Newton method, with a fixed variable metric, and where H plays the role of the inverse Hessian.

The hyperplane selected at step q is, in the terminology of the ellipsoid method (see Goldfarb and Todd [17], and Bland, Goldfarb and Todd [6]), one giving the "deepest" cut; it is also a (violated) hyperplane most distant from x in the metric $\| \; \|_E$. The next iterate x_+ is thus the projection, in the metric $\| \; \|_E$, of x

on this hyperplane. It is also true that the sequence of iterates satisfies an extension of the Fejér-monotonicity:

$$\|x^{q+1}-x'\|_E < \|x^q-x'\|_E , \qquad \text{for all } x' \in P .$$

Lemma 2.3

Let $\phi(x,T)$, $\gamma(x,T)$, $f(y,T)$, $g(y,T)$, $f(x)$ and $g(x)$ be defined as above; then:

1. $\phi(x,T) = f(T^{-1}x, T)$,

   $\gamma(x,T) = g(T^{-1}x, T)$.

2. ϕ and γ are convex functions of x.
3. $\phi(x,T) = \gamma(x,T)$, if $x \in P$,

   $\mu^*(T^{-1}P)\ \gamma(x,T) \leq \phi(x,T) \leq \gamma(x,T)$, if $x \notin P$

   where both bounds are reached.
4. $\mu^*(T^{-1}P) = \operatorname{Inf}\{\phi(x,T)/\gamma(x,T) : x \notin P\}$
5.

$$\Lambda^{-1/2}(H)|f(x)| \leq |\phi(x,T)| \leq \lambda^{-1/2}(H)|f(x)|$$
$$\Lambda^{-1/2}(H)|g(x)| \leq |\gamma(x,T)| \leq \lambda^{-1/2}(H)|g(x)|$$
$$\text{for all } x \in R^n$$

(where λ and Λ mean the smallest and largest eigenvalues).

Proof

The fact that $\phi(x,T) = f(T^{-1}x,T)$ follows directly from the definitions of $\phi(x,T)$ and $f(y,T)$; $\gamma(x,T) = g(T^{-1}x,T)$ is also clear as

$$T^{-1}(x+rE) = y + rS , \quad \text{if} \quad Ty = x .$$

Thus 2,3 and 4 are rewritings of Lemma 2.1.

And 5 follows from:

$$\lambda^{1/2}(H^{-1}) \|x\| \leq \|x\|_E \leq \Lambda^{1/2}(H^{-1}) \|x\| ,$$

$$\lambda^{1/2}(H) \|x\| \leq \|x\|_{Ed} \leq \Lambda^{1/2}(H) \|x\| . \qquad \text{QED}$$

Theorem 2.4

If algorithm 2 is applied to the system of inequalities (LI), assumed to be consistent, then it generates a sequence of iterates which converge finitely or infinitely to a point x^* which solves (LI). Furthermore,

$$\|x^q - x^*\|_E \leq 2\theta^q(T) \gamma(x^o,T)$$

$$\gamma(x^q,T) \leq \theta(T) \gamma(x^{q-1},T) \leq \theta^q(T) \gamma(x^o,T)$$

$$\phi(x^q,T) \leq (\mu^*(T^{-1}P))^{-1} \theta^q(T) \phi(x^o,T)$$

$$f(x^q) \leq \Lambda^{1/2}(H) \lambda^{-1/2}(H) (\mu^*(T^{-1}P))^{-1} \theta^q(T) f(x^o) ,$$

where

$$\theta(T) = (1 - (\mu^*(T^{-1}P))^2)^{1/2} .$$

Proof

The sequences generated by algorithms 1' and 2 satisfy $x^q = Ty^q$ (if ties are broken in the same fashion); and, thus, this theorem simply translates Theorem 2.2, using Lemma 2.3. QED

The relationship between algorithms 1' and 2 clearly means that all the results about finite convergence of the maximal distance relaxation method (see Motzkin and Schoenberg [29], Eaves, [9], Todd [33], and [13,14]) extend to algorithm 2.

The proofs given in this paper do not extend to other implementations of the relaxation method, like the maximal residual relaxation method; it is not clear whether the results would extend, or not.

In the maximal residual relaxation method, the only change from algorithms 1 (or 1' and 2) is in the selection of a violated constraint (step 2):

select $i \in \tilde{I}(x)$, where

$$\tilde{I}(x) = \{i \in M: (a^i,x)-b_i = f(\tilde{x})\} \quad \text{and} \quad f(\tilde{x}) = \max_{i \in M} ((a^i,x)-b_i).$$

The convergence theory [13,14] is based upon a condition member $\tilde{\mu}(P)$ defined by

$$\tilde{\mu}(P) = \inf_{x \notin P} \frac{1}{g(x)} \min_{i \in \tilde{I}(x)} \frac{\tilde{f}(x)}{\|a^i\|} ,$$

which can be related to $\mu^*(P)$ by

$$\tilde{\mu}(P) \geq \mu^*(P) \max_{i \in M} \|a^i\| / \min_{i \in M} \|a^i\| .$$

Thus, the nice behavior of μ^* under linear transformations does not extend to $\tilde{\mu}$, because the norms of the rows of A do not behave sensibly under linear transformations.

A minor annoyance where one studies polynomiality is that one should worry about the fact that numbers should be represented in a space polynomial in the length of the input. The function f clearly takes irrational values at points n which are rational. If the data (A,b) is integer, then this can be ignored if one simply uses the function $\hat{f}(x)$ in step 2 of algorithm 1 (or a function $\hat{\phi}(x,T)$ in algorithm 2, if T is integer):

$$\hat{f}(x) = \underset{i \in M}{\mathrm{Max}} \left(((a^i,x) - b^i) / \lfloor \| a^i \| \rfloor \right) .$$

Let also $\hat{I}(x) = \{ i \in M: ((a^i,x)-b_i)/\lfloor \| a^i \| \rfloor = \hat{f}(x) \}$, and

$$\hat{\mu}(P) = \underset{x \notin P}{\mathrm{Inf}} \frac{1}{g(x)} \underset{i \in \hat{I}(x)}{\mathrm{Min}} \frac{\hat{f}(x) \lfloor \| a^i \| \rfloor}{\| a^i \|} .$$

It is easy to see that, if a^i is integer, then

$$\| a^i \| \leq \lfloor \| a^i \| \rfloor \sqrt{2} \quad \text{and} \quad \hat{\mu}(P) \geq \mu^*(P)/\sqrt{2} ;$$

also

$$|f(x)| \leq |\hat{f}(x)| \leq |f(x)| \sqrt{2} , \qquad \text{for all } x \in R^n .$$

Thus, if in step 2 one selects any $i \in \hat{I}(x)$ (which can be done using integer arithmetic, if x is rational) the convergence theory is not affected in any significant manner.

The convergence theory is easy to rewrite, and gives

$$\hat{f}(x^q) \leq \sqrt{2}\ \hat{\theta}^q(\hat{\mu}(P))^{-1}\ \hat{f}(x^o) ,$$

with

$$\hat{\theta} = (1 - (\hat{\mu}(P))^2)^{1/2} .$$

The issue of polynomial space will be neglected throughout this paper, but it is clear that by using $\hat{f}(x)$ (or $\hat{f}(y,T)$ or $\hat{\phi}(x,T)$ if T is integer) it can be taken care of quite easily.

3. Behavior of condition numbers and face lattices under perturbation

Throughout the remainder of this paper, the following assumption will sometimes be made, mostly for notational convenience, as it does not imply any loss in generality.

Assumption 3.1:

> Rank $A = n$, or, equivalently, the null space of A reduces to the origin.

This means that if P is not empty then it does not contain any proper subspace. This assumption is not restrictive because it is always true within the subspace $R(A^t)$, the range of A^t, and the sequence x^q remains in $x^o + R(A^t)$; the same holds in algorithms 1' or 2 if $R(HA^t) = R(A^t)$.

If Assumption 3.1 holds then any nonempty polyhedron has vertices, and is the sum of a bounded polyhedron and a pointed cone; if Assumption 3.1 does not hold then this statement is true within $R(A^t)$.

Define the family of perturbed polyhedra

$$P_w = \{x \in R^n: (a^i,x)-b_i \leq \|a^i\|w , \quad i \in M\}$$

$$= \{x \in R^n: f(x) \leq w\} ;$$

and the epigraph of f:

$$Q = \{(x,w) \in R^{n+1}: f(x) \leq w\}$$

$$= \{(x,w) \in R^{n+1}: x \in P_w\} \quad .$$

Clearly $P = P_0$, and it should be noticed that the definition of P_w and Q includes a normalization of the inequalities. One needs to define other perturbed sets if other normalizations are used,

$$\tilde{Q} = \{(x,w) \in R^{n+1}: \tilde{f}(x) \leq w\}$$

$$\hat{Q} = \{(x,w) \in R^{n+1}: \hat{f}(x) \leq w\}$$

$$Q(T) = \{(x,w) \in R^{n+1}: \phi(x,T) \leq w\}$$

$$\hat{Q}(T) = \{(x,w) \in R^{n+1}: \hat{\phi}(x,T) \leq w\}$$

and the corresponding $\tilde{P}_w$, $\hat{P}_w$, $P_w(T)$ and $\hat{P}_w(T)$; clearly $P = \tilde{P}_0 = \hat{P}_0 = \hat{P}_0 = P_0(T) = \hat{P}_0(T)$.

In what follows we will study condition numbers for systems of linear inequalities, and how they vary under perturbations. In order to do so, the behavior of the face lattice (and related lattices) under perturbations mst be described.

The lattice, under the ordering induced by set inclusion, of faces of P will be denoted by $\mathcal{f}(P)$, where $\mathcal{f}(P)$ includes the empty set, unless P is a cone. To the face lattice, one may associate the following lattices:

1. indices $\mathscr{i}(P)$

$$F \to I(F) = \{i \in M: (a^i,x)-b_i = 0, \quad \text{for all} \quad x \in F\}$$

$$= \{i \in M: (a^i,x)-b_i = 0, \quad \text{for some} \quad x \in \text{ri}F\}$$

where ri means relative interior.

2. tangent cones $\mathscr{t}(P)$

$$F \to C_P(F) = \{x \in R^n: (a^i,x) \leq 0, \quad \text{for all} \quad i \in I(F)\}$$

3. normal cones $\mathscr{n}(P)$

$$F \to N_P(F) = [C_P(F)]^P$$

$$= \{y \in R^n: (x,y) \leq 0, \quad \text{for all} \quad x \in C_P(F)\}$$

$$= \{\sum_{i \in I(F)} \lambda_i a^i: \ \lambda_i \geq 0, i \in I(F)\}$$

4. subdifferentials $\mathscr{d}(P)$

$$F \to \partial f(F) = \mathscr{h}\{a^i/\|a^i\|: i \in I(F)\}$$

$$= \partial f(x) \qquad \text{for some (or any)} \quad x \in \text{ri}F$$

and $\mathscr{h}$ denotes convex hull.

Clearly $\mathscr{t}(P)$ is isomorphic to $\mathscr{f}(P)$, while $\mathscr{i}(P)$, $\mathscr{n}(P)$ and $\mathscr{d}(P)$ are isomorphic to one another and antiisomorphic, or dual, to $\mathscr{f}(P)$ and $\mathscr{t}(P)$.

It will be necessary to describe the behavior of these lattices under various normalizations of the defining inequalities, and under linear transformations.

The index lattice is constant under affine transformations and normalizations (by constant, it is meant that the lattices are isomorphic, and that the objects composing them are identical).

The lattices $\mathcal{F}$, and $\mathcal{N}$ are constant under normalization, while affine transformations induce isomorphisms.

Normalizations and affine transformations induce isomorphisms of $\mathcal{D}$.

Difficulties occur at the level of the face lattice of the subdifferentials $\partial f(F)$ (the elements of $\mathcal{D}(P)$), where affine transformations induce isomorphisms, but normalizations completely change the lattice structure. Some key proofs will need to operate at the level of the face lattices of the subdifferential.

The following lemma will be used repeatedly, and is but a rewriting of the characterization of the projection map on a convex set (see [13]).

<u>Lemma 3.2</u>

For every polyhedron P in R^n, both

$$\{\mathrm{ri}F + N_P(F) : F \in \mathcal{F}(P)\} ,$$

and

$$\{F + \mathrm{ri}N_P(F) : F \in \mathcal{F}(P)\}$$

are partitions of R^n. Equivalently, every $x \in R^n$ may be written uniquely as $x = y+z$, where y and z belong to dual elements of $\mathcal{F}(P)$ and $\mathcal{N}(P)$; y is the projection of x on P, while z is the outer normal of the halfspace which is the most distant from x, among the family of halfspaces containing P. If P is a cone, then y and z are orthogonal.

Proof

Given $x \in R^n$, let y be the projection of x on P and $z = x-y$.

For the first partition, associate to x the smallest face of P which contains y; for the second partition, if $z = 0$ associate P to x, while if $z \neq 0$ associate to x the face of P which is the set of maximizers of (z,x') for $x' \in P$. QED

The rate of convergence of the relaxation method is related, critically, to the condition number $\mu^*(P)$. Its definition was mostly algebraic, but now more geometric interpretations of $\mu^*(P)$ will be given, as well as bounds in terms of purely geometric concepts.

Definition 3.3

The asphericities of P, where P is assumed bounded and full dimensional:

$$\sigma(P) = \mathrm{Inf}\{\sigma \geq 0: x + rS \subset P \subset x + \sigma rS\}$$

and

$$\sigma'(P) = \mathrm{Inf}\{\sigma \geq 0: x^* + r^*S \subset P \subset x^* + \sigma r^*S, \ x^* \in P^*\}$$

where

$$r^* = \text{Sup}\{r \geq 0: x+rS \subset P\}$$

$$P^* = \{x \in R^n: x+r^*S \subset P\}$$

$$= \{x \in R^n: f(x) = \max_{x' \in R^n} f(x')\} .$$

The first definition is more natural in geometry, while the second one is more natural within the context of linear inequalities. It is clear that $\sigma(P) \leq \sigma'(P)$, and that if Assumption 3.1 does not hold, then σ and σ' could be defined within $R(A^t)$.

<u>Definition 3.4</u>: The condition number ν [13,14]:

1. For a cone (say Cp(F)), the following are equivalent definitions of $\nu(Cp(F))$:

$$\nu(C_p(F))$$

$$= \text{Sup}\{\sin \alpha: \{x \in R^n: \frac{(x,e)}{\|x\| \ \|e\|} \geq \cos\alpha\} \subset C_p(F)\}$$

$$= \text{Inf}\{\sin \alpha: \{x \in R^n: \frac{(x,e')}{\|x\| \ \|e'\|} \geq \sin \alpha\} \supset N_p(F)\}$$

$$= \text{Min}\{\|g\|: g \in \partial f(F)\}$$

$$= \text{Sup}\{r \geq 0: (rS) \cap \partial f(F) = \phi\}$$

$$= -\text{Inf}\{f'(x;d): \|d\| = 1\}$$

$$= \text{Min}\{(\lambda^t \Gamma(F) \lambda)^{1/2}/\|\lambda\|_1: \lambda \geq 0\} ,$$

where $f'(x;\cdot)$ is the directional derivative of $f(x)$, $\Gamma(F)$ is the Grammian associated to the vectors $a^i/\|a^i\|$, $i \in I(F)$, and $\|\lambda\|_1 = \sum_{i\in I(F)} |\lambda_i|$ is the L_1 norm.

Some properties of ν will be used repeatedly (for proofs, see [13,14]):

i. $\nu(C_p(F)) > 0$ if and only if $\dim C_p(F) = n$, $\nu(C_p(F)) \leq 1$.

ii. $C_1 \subset C_2$, C_1 and C_2 are convex cones, implies $\nu(C_1) \leq \nu(C_2)$.

iii. $\nu(C_p(F))$ is a lattice monotone function on $\mathcal{F}(P)$, i.e.,
$F' \subset F$, F' and $F \in \mathcal{F}(P)$ implies $\nu(C_p(F')) \leq \nu(C_p(F))$.

iv. $\nu(C_p(F))$ is a function of $C_p(F)$, or $N_p(F)$, or $\partial f(F)$ as geometrical objects.

v. $\sin^{-1} \nu(C_p(F))$ could be called the angle of the cone $C_p(F)$

vi. let e, e', g, d and λ be the vectors where the various infima and suprema are attained, then

$$g = -\nu(C_p(F))e/\|e\| ,$$

$$d = e/\|e\| = -e'/\|e'\| ,$$

$$g = \Big(\sum_{i\in I(F)} \lambda_i a^i/\|a^i\|\Big)\Big/ \sum_{i\in I(F)} \lambda_i .$$

2. $\qquad \nu(P) = \mathrm{Min}\{\nu(C_p(F)) : F \in \mathcal{F}(P),\ F \neq \phi\}$:

the same notation is used as in 1., and this is consistent because ν is lattice monotone (and thus if $P = C$, a convex cone, then $\nu(C) = \nu(C_C(L'))$, where L', the lineality space of C, is the minimal element of $\mathcal{F}(C)$).

Definition 3.5: the condition numbers μ and μ^* [1,13,14]:

1. $$\mu(C_P(F)) = \inf_{\substack{x \in N_P(F) \\ x \neq 0}} \max_{i \in I(F)} \frac{(a^i,x)}{\|a^i\| \, \|x\|} \; ;$$

this definition is due to Agmon [1], who showed that $\mu(C_P(F) \in (0,1]$, and this under no assumptions whatsoever on P (except $P \neq \phi$). It should be said that μ is not a function of $C_P(F)$ as a geometrical object, but depends on the actual representation of $C_P(F)$ by an index set $I(F)$; also $\mu(C_P(F))$ is not lattice monotone on $\mathcal{F}(P)$.

2. $$\mu^*(C_P(F)) = \mathrm{Min}\{\mu(C_P(F')) : F' \in \mathcal{F}(P),\ F' \supset F\} \; ;$$

this is, obviously, lattice monotone.

3. $$\mu^*(P) = \mathrm{Min}\ \{\mu^*(C_P(F)) : F \in \mathcal{F}(P),\ F \neq \phi\} \; ;$$

$$= \mathrm{Min}\{\mu(C_P(F)) : F \in \mathcal{F}(P),\ F \neq \phi\} \; ;$$

the same notation has been used as in §2, as it will be shown later that the two definitions coincide.

The following lemma gives a geometric characterization of μ^*, which shows that it depends on the lattice of subdifferentials $\partial(P)$.

Lemma 3.6

Let $(a^i,x) - b_i \leq 0$, $i \in M$, be a consistent system of linear inequalities, $f(x) = \mathrm{Max}_{i \in M}(((a^i,x) - b_i)/\|a^i\|)$, and $P = \{x \in R^n : f(x) \leq 0\}$, then

$$\mu^*(C_p(F)) = \operatorname{Sup}\{r > 0: (rS) \cap N_p(F) \subset \mathcal{H}(\partial f(F) \cup \{0\})\} .$$

Proof:

Let r^* be the value of the supremum, where $r^* \leq 1$, as $\partial f(F) \subset S$.

Denote by $h(y;K) = \operatorname{Sup}\{y^t x: x \in K\}$ the support function of a set K; and by $h_1(y)$ the support function of $\mathcal{H}(\partial f(F) \cup \{0\})$, i.e.,

$$h_1(y) = \operatorname{Max}\left(\operatorname*{Max}_{i \in I(F)} \frac{(a^i, y)}{\|a^i\|}, 0\right) .$$

Now

$$\begin{aligned} & h(y; (rS) \cap N_p(F)) \\ &= \operatorname{Inf}\{h(y_1; rS) + h(y_2; N_p(F)): y_1 + y_2 = y\} , \end{aligned}$$

(see Rockafellar [30], p. 146)

$$= \operatorname{Inf}\{r\|y - y_2\|: y_2 \in C_p(F)\}$$

$$\left(\text{as } h(y_1; rS) = r\|y_1\|;\ h(y_2; N_p(F)) = \begin{cases} 0 & \text{if } y_2 \in C_p(F) \\ +\infty & \text{if } y_2 \notin C_p(F) \end{cases}\right)$$

$$= rd(y; C_p(F)) ,$$

where $d(\cdot;\cdot)$ represents the distance between a point and a set.

Using the fact that $K_1 \subset K_2$ (K_1 and K_2 compact and convex) if and only if $h(y; K_1) \leq h(y; K_2)$ for all y, one gets

$$\begin{aligned} r^* &= \operatorname{Sup}\{r > 0: rd(y; C_p(F)) \leq h_1(y), \quad \text{for all } y \in R^n\} \\ &= \operatorname{Inf}\{h_1(y)/d(y; C_p(F)): y \notin C_p(F)\} , \end{aligned}$$

using the fact that h_1 and d are zero on $C_p(F)$, and positive elsewhere.

Using Lemma 3.2, every y may be written as $y = y_1+y_2$, with $y_1 \in N_P(F)$, $y_2 \in C_P(F)$ and $(y_1,y_2) = 0$; to every y, one may associate, uniquely, a face F' of P such that $y_2 \in \text{ri}C(F')_P$.

Clearly

$$(a^i,y_2) = 0 , \qquad \text{for } i \in I(F')$$

and

$$(a^i,y_2) < 0 , \qquad \text{for } i \in I(F),\ i \notin I(F')$$

Thus

$$r^* = \underset{\substack{F' \supset F \\ F \in \mathcal{F}(F)}}{\text{Min}} \ \text{Inf}\{(h_1(y_1+y_2))/\|y_1\|: y_1 \in N_P(F'),\ y_1 \neq 0 ,\ y_2 \in \text{ri}C_P(F')\} ;$$

but, if $y_1 \neq 0$, $y_1 \in N_P(F')$, one has

$$\text{Inf}\{(h_1(y_1+y_2))/\|y_1\|: y_2 \in r_iC_P(F')\}$$

$$= \underset{\substack{y_2 \in \text{ri}C_P(F') \\ \|y_2\|=1}}{\text{Inf}} \ \underset{\varepsilon > 0}{\text{Inf}} \ \underset{i \in I(F)}{\text{Max}} \left\{ \frac{(a^i,y_1)}{\|a^i\|\ \|y_1\|} + \varepsilon \frac{(a^i,y_2)}{\|a^i\|\|y_1\|} \right\}$$

$$= \text{Max}\left\{\frac{(a^i,y_1)}{\|a^i\|\|y_1\|}: i \in I(F')\right\} .$$

Hence,

$$r^* = \text{Min}\{\mu(C_P(F)): F' \supset F,\ F' \in \mathcal{F}(P)\}$$

$$= \mu^*(C_P(F)) .$$

QED

Lemma 3.7

The definitions of $\mu^*(P)$ given in §2 and 3 are identical.

Proof:

Let $\mu^*(P) = \text{Inf}\{f(x)/g(x): x \notin P\}$ and let $\mu^{**} = \text{Min}\{\mu(C_p(F)): F \in \mathcal{F}(P), F \neq \phi\}$. Then it has been shown [13,14] that $\mu^*(P) \geq \mu^{**}$; an alternative proof of this could be given by taking the dual of the sets involved in the definition of $\mu^*(C_p(F))$ given as Lemma 3.6. Now, choose F, x and e such that

$$\mu(C_p(F)) = \mu^{**}, \quad x \in \text{ri}F, \quad e \in N_p(F), \quad \|e\| = 1$$

and

$$\mu^{**} = \underset{i \in I(F)}{\text{Max}} \frac{(a^i,e)}{\|a^i\|} ;$$

such an e exists because the function $\underset{i \in I(F)}{\text{Max}}[(a^i,y)/\|a^i\|]$ is continuous on the compact set $N_p(F) \cap \{y \in R^n: \|y\| = 1\}$, see Agmon [1].

But for ε small enough $(\varepsilon > 0)$, $f(x+\varepsilon e) = \varepsilon\mu^{**}$, $g(x+\varepsilon e) = \varepsilon$ (as x is the projection of $x+\varepsilon e$ on P); and thus $\mu^*(P) \leq \mu^{**}$.

QED

Definition 3.8

A representation of a polyhedron P by a system of inequalities $(a^i,x) - b_i)/\|a^i\| \leq 0$, $i \in M$, where $P = \{x \in R^n: f(x) \leq 0\}$ is nondegenerate if every face F of P is nondegenerate; a face F is nondegenerate if the affine hull of $\partial f(F)$ does not contain the origin. Equivalent statements of the nondegeneracy of F are:

(i) $\sum_{i \in I(F)} \lambda_i a^i / \| a^i \| = 0, \quad \sum_{i \in I(F)} \lambda_i = 1$, does not have a solution ;

(ii) $(a^i,x)/\| a^i \| = 1$ for all $i \in I(F)$ has a solution .

This definition of degeneracy is implied by the usual definition (every set $\{a^i : i \in I(F)\}$ is linearly independent).

If P has an empty interior, then every face of P is degenerate.

The definition of degeneracy depends upon the normalization chosen, and thus it is quite possible for $P = \{x: f(x) \leq 0\}$ to be nondegenerate, while $P = \{x: \tilde{f}(x) \leq 0\}$ or $P = \{x: \phi(x,T) \leq 0\}$ would be degenerate. It is invariant under linear transformations, but not under a linear transformation followed by a renormalization.

Now, using Lemma 3.6 or [13,14], it follows that $\mu^*(P) \geq \nu(P)$, while a simple extension of a proof of [14] (to the degeneracy as defined here, rather than using the usual definition) gives $\mu^*(P) = \nu(P)$ if P is nondegenerate. It also follows immediately from Lemma 3.11 (but that is a very roundabout proof).

We shall now describe a classification of the faces of the subdifferentials $\partial f(F)$, where F is a face of P; this will lead to alternate definitions of μ^* and ν, which will permit a study of the behavior of μ^* and ν under perturbations of P.

Let $\partial f(F) = \mathcal{h}\{a^i/\|a^i\|: i \in I(F)\}$ be the subdifferential of f at F; denote its face lattice by $\mathcal{F}(\partial f(F))$ and define $\mathcal{J}(\partial f(F))$ to be the associated index lattice (which to D, a face of $\partial f(F)$, associates $\mathcal{J}(D) = \{i \in I(F): a^i/\|a^i\| \in D\}$), and which is clearly isomorphic to $\mathcal{F}(\partial f(F))$. Notice that $I(F) = \mathcal{J}(\partial f(F))$.

<u>Definition 3.9</u>

Let D be a face of $\partial f(F)$, $f \in \mathcal{F}(P)$, and $\mathcal{J}(D)$ be the corresponding index set; then D is an <u>outside face</u> (resp. <u>inside face</u>, resp. <u>side face</u>) of $\partial f(F)$ if, for $\varepsilon = 1$ (resp. $\varepsilon = -1$, resp. $\varepsilon = 0$), there exists a solution to:

$$(a^i,x)/\|a^i\| = \varepsilon , \qquad i \in \mathcal{J}(D) ,$$

$$(a^i,x)/\|a^i\| < \varepsilon , \qquad i \in I(F),\ i \notin \mathcal{J}(D) .$$

This definition means that there exists a halfspace whose intersection with $\partial f(F)$ is D, and such that this halfspace does not contain the origin (resp. contains the origin in its interior, resp. contains the origin on its boundary); with the proviso that $\partial f(F)$ is always defined as a side face (with $x = 0$).

Every face of $\partial f(F)$ satisfies at least one of the three definitions. Also $\partial f(F)$ is an outside face, if and only if it is an inside face, if and only if F is nondegenerate; the side faces of $\partial f(F)$ are essentially the faces of $Np(F)$. The polyhedron P has dimension n if and only if every, or any, $\partial f(F)$ has an inside face (see Lemma 3.11).

The definition of outside, and inside, faces is "invariant" under linear transformations, but not under normalizations, while side faces are "invariant" for both.

Similar definitions may be given for the subdifferentials of $\tilde{f}$, $\hat{f}$, ϕ, $\hat{\phi}$.

<u>Lemma 3.10</u>

Let $\partial f(F)$ be the subdifferential of f at a face F of P; then any face D of $\partial f(F)$ which is both an inside and outside face is also a side face. If F is nondegenerate, then D is inside if and only if it is outside. A face F is nondegenerate if and only if every face D of $\partial f(F)$ is both inside and outside.

<u>Proof:</u>

Let D be a face of $\partial f(F)$, which is both outside and inside, and x_1 and x_2 be the corresponding values of x given by the definition of outside and inside faces; then x_1+x_2 satisfies the definition of a side face, and hence D is a side face.

Now, assume that F is nondegenerate, and thus there exists a y such that

$$(a^i,y)/\|a^i\| = 1 , \qquad \text{for all } i \in I(F);$$

if D is an outside face, and n_1 is given by the definition of an outside face, then

$$(a^i, x_1-2y)/\|a^i\| = -1 , \qquad i \in J(D) ,$$
$$(a^i, x_1-2y)/\|a^i\| < -1 , \qquad i \in I(F),\ i \notin J(D)$$

and thus D is an inside face. The converse statement follows similarly.

If F is nondegenerate, then:

i) if D is an outside face, it is also inside, and side

ii) if D is an inside face, it is also outside, and side

iii) if D is a side face, and x_3 is given by the definition of a side face, then x_3+y satisfies the definition of an outside face, and x_3-y satisfies the definition of an inside face;

as every face is either inside, outside or side, the lemma is proved.

QED

Lemma 3.11

Let $\partial f(F)$ be the subdifferential of f at a face F of P, then

$$\mu^*(C_p(F)) = \mathrm{Min}\{d(0;D)\colon \ D \text{ is an outside face of } \partial f(F)\},$$

while if P has an interior, then

$$\nu(C_p(F)) = \mathrm{Min}\{d(0;D)\colon \ D \text{ is an inside face of } \partial f(F)\},$$

where $d(0;D)$ is the distance between the origin and the set D.

Before proving Lemma 3.11, two technical lemmas will be given.

Lemma 3.12

Let D be a compact polyhedron, such that its affine hull does not contain the origin, then

$$d(0;D) = \mathrm{Sup}\{r \geq 0: (rS) \cap \mathcal{C}(D) \subset \mathcal{H}(D \cup \{0\})\} ,$$

where $\mathcal{C}$ means cone hull.

<u>Proof</u>:

By hypothesis, there exists a y such that $(y,x) = 1$, for all $x \in D$. Also

$$\mathcal{C}(D) = \bigcup_{\lambda \geq 0} \lambda D \quad \text{and} \quad \mathcal{H}(D \cup \{0\}) = \bigcup_{\lambda \in [0,1]} \lambda D ,$$

and thus

$$D = \mathcal{C}(D) \cap \{x: (y,x) = 1\} .$$

$$\mathcal{H}(D \cup \{0\}) = \mathcal{C}(D) \cap \{x: (y,x) \leq 1\} .$$

Now

$$\begin{aligned} d(0;D) &= \mathrm{Min}\{r \geq 0: (rS) \cap D \neq \phi\} \\ &= \mathrm{Min}\{r \geq 0: (rS) \cap \mathcal{C}(D) \cap \{x : y^t x = 1\} \neq \phi\} \\ &= \mathrm{Min}\{r \geq 0: (rS) \cap \mathcal{C}(D) \cap \{x : y^t x \geq 1\} \neq \phi\} \\ &= \mathrm{Sup}\{r \geq 0: (rS) \cap \mathcal{C}(D) \subset \{x : y^t x < 1\}\} \\ &= \mathrm{Sup}\{r \geq 0: (rS) \cap \mathcal{C}(D) \subset \{x : y^t x \leq 1\}\} \\ &= \mathrm{Sup}\{r \geq 0: (rS) \cap \mathcal{C}(D) \subset \mathcal{H}\{D \cup \{0\}\}\} . \end{aligned}$$

QED

<u>Lemma 3.13</u>

Let $\partial f(F)$ be as before, then every $x \in \mathcal{H}(\partial f(F) \cup \{0\})$, $x \neq 0$, belongs to one and only one of the sets $\mathcal{H}((\mathrm{ri}D) \cup \{0\})$, where D is any outside face of $\partial f(F)$.

Proof:

Let D be any outside face of $\partial f(F)$, and y be such that:

$$(y,x) = 1 , \qquad x \in D$$

$$(y,x) < 1 , \qquad x \in \partial f(F),\ x \notin D$$

Every face D' of D is an outside face: let y' satisfy

$$D' = \{x \in D: (y',x) = \max_{x' \in D} (y',x')\} ,$$

the $(y+\varepsilon y')/(1+\varepsilon(y',x'))$ where $x' \in D'$, and ε is positive and small enough, shows that D' satisfies the definition of an outside face.

If $x \in D$, then $(y,\lambda x) = \lambda$, and thus $\lambda x \notin D$ if $\lambda \neq 1$. Also, if $\lambda > 1$, then $(y,\lambda x) > 1$ and $\lambda x \notin \partial f(F)$.

Now, let $x \in \mathcal{A}(\partial f(F) \cup \{0\})$, $x \neq 0$; define $x' = \rho' x$ where $\rho' = \max\{\rho \geq 1: \rho x \in \partial f(F)\}$. Clearly $\{\rho x': \rho \geq 1\}$ and $\partial f(F)$ have disjoint relative interiors, and thus they can be (strictly) separated; i.e., there exists a $z \in R^n$ such that:

$$(z,\rho x') > (z,x') \geq (z,x'') , \quad \text{for all } \rho > 1,\ x'' \in \partial f(F) .$$

Hence $(z,x') > 0$; but as $x' \in \partial f(F)$, $x' = \sum_{i \in I(F)} \lambda_i a^i/\|a^i\|$, $\lambda_i \geq 0$, $i \in I(F)$ and $\sum_{i \in I(F)} \lambda_i = 1$, it follows that:

$$\sum_{i \in I(F)} \lambda_i[(z,x') - (z,a^i)/\|a^i\|] = 0 .$$

Using this, and $\lambda_i \geq 0$, $(z,x') \geq (z,a^i)/\|a^i\|$, leads to $\lambda_i[(z,x') - (z,a^i)/\|a^i\|] = 0$, for all $i \in I(F)$. Hence there exists a face D' of $\partial f(F)$, which is an outside face, such that $J(D') = \{i \in I(F): (z,a^i)/\|a^i\| = (z,x')\}$, and furthermore $\lambda_i = 0$ if $i \in I(F)$, $i \notin J(D')$, implying that $x' = \sum_{i\in J(D')} \lambda_i \, a^i/\|a^i\|$, and $x' \in D'$.

Now let D'' be the smallest face of D' which contains x'; one has $x' \in riD''$, D'' is an outside face, and $x \in ((riD'') \cup \{0\})$. It is also clear that D'' is unique. QED

Proof of Lemma 3.11

One has

$$\text{Min}\{d(0;D): D \text{ is an outside face of } \partial f(F)\}$$

$$= \underset{D \text{ outside}}{\text{Min}} \text{Sup}\{r \geq 0: (rS) \cap c(D) \subset \mathcal{K}(D \cup \{0\})\}, \quad \text{(by Lemma 3.12)},$$

$$= \text{Sup}\{r \geq 0: (rS) \cap N_p(F) \subset \mathcal{K}(\partial f(F) \cup \{0\})\}, \quad \text{(by Lemma 3.13)},$$

$$= \mu^*(C_p(F)) \quad \text{(by Lemma 3.6)}.$$

Now, it is clear that if $\dim P < n$, then there are no inside faces and $\nu(Cp(F)) = 0$ for all $F \in \mathcal{F}(P)$.

Using the characterization of $\nu(Cp(F))$ as $d(0; \partial f(F))$, and letting g to be the element of minimum norm in $\partial f(F)$, it follows that (if $\dim P = n$):

$$(g, x-g) \geq 0 , \qquad \text{for all } x \in \partial f(F) ,$$

and $d(0; \partial f(F)) = \|g\| \neq 0$; now if we let

$$D^* = \{x \in \partial f(F): (g,x-g) = 0\} ,$$

it is clear (with $-g$) that D^* is an inside face of $\partial f(F)$. But, as $D \subset \{x: (g, x-g) \geq 0\}$ it is clear that $\|g\| = d(0; D^*) \leq d(0;D)$ for every face D of $\partial f(F)$. Hence the lemma. QED

Note that the last part of the proof implies that if $\dim P = n$, then every $\partial f(F)$, $F \in \mathcal{F}(P)$, has at least one inside face.

<u>Theorem 3.14</u>

Let $(a^i,x) \leq b_i$, $i \in M$, be a consistent system of linear inequalities, with solution set P, then

$$\mu^*(P) \geq \nu(P) \geq (\sigma(P))^{-1} \geq (\sigma'(P))^{-1} ,$$

where $\sigma(P)$ and $\sigma'(P)$ are taken as $+\infty$ if P is unbounded, or has no interior; furthermore if P is nondegenerate, then $\mu^*(P) = \nu(P)$.

<u>Proof:</u>

Lemma 3.11 implies that, for every $F \in \mathcal{F}(P)$, one has

$$\begin{aligned} \mu^*(C_P(F)) &= \text{Min}\{d(0;D): D \text{ is an outside face of } \partial f(F)\} \\ &\geq \text{Min}\{d(0;D): D \text{ is a face of } \partial f(F)\} \\ &= \nu(C_P(F)); \quad \text{hence } \mu^*(P) > \nu(P) . \end{aligned}$$

If F is nondegenerate, then Lemma 3.10 and 3.11 imply that $\mu^*(C_p(F)) = \nu(C_p(F))$, and thus if P is nondegenerate then $\mu^*(P) = \nu(P)$.

From the definition of $\sigma'(P)$ (or of $\sigma(P)$), there exists a sphere of some radius $r > 0$ contained in P, such that a concentric sphere of radius σr contains P (where σ is either $\sigma(P)$ or $\sigma'(P)$); thus every tangent cone $C_p(x)$, where $x \in$ boundary P, contains a spherical cone of angle $\sin^{-1}(r/\sigma r)$, and thus $\nu(C_p(x)) \geq \sin[\sin^{-1}(r/\sigma r)] = \sigma^{-1}$, whence $\nu(C_p(F)) \geq \sigma^{-1}$, and

$$\nu(P) \geq (\sigma(P))^{-1} \geq (\sigma'(P))^{-1} . \qquad \text{QED}$$

In all that precedes, P should be viewed as a representative of the class

$$P_w = \{x \in R^n: ((a^i,x) - b_i)/\|a^i\| \leq w, \quad i \in M\} ,$$

and thus every definition, or result, which relates to P also extends, verbatim, to every P_w. The behavior of the various lattices, and condition numbers, as w varies will now be described. The key to this somewhat exciting study and munificent enterprise is given by the relationship between the faces of P_w, and the faces of $Q = \{(x,w) \in R^{n+1}: f(x) \leq w\}$.

Definition 3.15; the vertex level set W

Define $W = \{w_1, w_2, \ldots, w_K\}$, where $0 < w_1 < w_2 < \cdots < w_K$, as the set of the positive w coordinates of the vertices of the set Q.

A similar definition of W can be given if Assumption 3.1 does not hold. Analogous concepts can also be defined for $\tilde{Q}$, $\hat{Q}$, etc. The level $w = w_0 = 0$ may or may not contain vertices of Q, but it always does if P is not full dimensional (under Assumption 3.1). The symbol w_{-1} will be used, at times, for the first negative vertex level.

Definition 3.16

A face F_Q of Q is called horizontal if it is entirely contained in a (horizontal) hyperplane, w = constant; if we let

$$I(F_Q) = \{i \in M: ((a^i,x)-b_i)/\|a_i\| = w, \quad \text{for all } (x,w) \in F_Q\}$$

where $F_Q \in \mathcal{F}(Q)$, then F_Q is horizontal if and only if

$$(a^i,x)/\|a^i\| = 1 , \qquad i \in I(F_Q) ,$$

does not have a solution.

Every horizontal face of Q (which has positive w coordinates) is contained in one of the horizontal halfspaces

$$w = w_k , \qquad k = 1, \ldots, K .$$

For a given w, the faces of P_w are related to the faces of Q; let F_Q be any face of Q such that its relative interior intersects the hyperplane w, and let F be the corresponding face of P_w (where $I(F_Q) = I(F)$), then there are two (disjoint) cases:

i. F_Q is horizontal, then $\dim F = \dim F_Q$, and F is degenerate.

ii. F_Q is not horizontal, then $\dim F = \dim F_Q - 1$, and F is not degenerate.

This is clear, as the degeneracy of F is given by the same condition as the horizontality of F_Q.

<u>Lemma 3.17</u>

On each open interval between consecutive vertex levels (say (w_k, w_{k+1})), the lattices $\mathcal{i}(P_w)$, $\mathcal{n}(P_w)$, $\mathcal{l}(P_w)$, $\mathcal{d}(P_w)$ are constant, while the lattices $\mathcal{f}(P_w)$ are isomorphic, and $\mu^*(P_w) = \nu(P_w)$ is constant.

The polyhedron P_w is degenerate if and only if w is a vertex level.

<u>Proof</u>:

Take $w \in (w_k, w_{k+1})$ (say), then every face F of P_w is generated by a nonhorizontal face of F_Q of Q, such that $I(F) = I(F_Q)$. The projection of F_Q on the w axis contains at least the closed interval $[w_k, w_{k+1}]$, and thus the projection of $\operatorname{ri} F_Q$ contains at least (w_k, w_{k+1}).

Thus $\mathcal{I}(P_w) = \{I(F_Q): F_Q \in \mathcal{F}(Q)$, and F_Q intersects the horizontal hyperplane w' for some, or any, $w' \in (w_k, w_{k+1})\}$; and the constancy of $\mathcal{I}(P_{w'})$ on $w' \in (w_k, w_{k+1})$ follows, as well as the statements for the other lattices, and for the condition numbers. QED

Lemma 3.18

Let w_k be an arbitrary vertex level of Q, then for any $w \in (w_k, w_{k+1})$ (resp. (w_{k-1}, w_k), assuming that P_{w_k} has dimension n) the index lattice of P_w is precisely the set of index sets corresponding to the outside (resp. inside) faces of the subdifferentials $\partial f(F)$, for all $F \in \mathcal{F}(P_{w_k})$. The index lattice of P_w is also precisely the set of index sets corresponding to the faces of $\partial f(F)$, for all $F \in \mathcal{F}(P_w)$.

Proof:

Let F be an arbitrary face of P_w, with $w \in (w_k, w_{k+1})$, and $I(F)$ be the corresponding index set; as F is nondegenerate, there exists a nonhorizontal face F_Q of Q such that $I(F_Q) = I(F)$. If F_Q extends below the level w_k, then the intersection of F_Q with the horizontal hyperplane w_k gives a face F' of P_{w_k}, such that $I(F') = I(F_Q) = I(F)$, and F' is nondegererate; and $\partial f(F')$ is an outside face (of itself) such that $\mathcal{J}(\partial f(F')) = I(F)$.

If F_Q stops at w_k, let F' be defined as before, and choose $x' \in \text{ri}F'$; clearly $(x', w_k) \in F_Q$ and $I(x') = I(F') \supset I(F_Q) = I(F)$. Let $x \in \text{ri}F$, then:

$$((a^i,x')-b_i)/\|a^i\| = w_k\ , \qquad \text{for all } i \in I(x')\ ,$$

$$((a^i,x)-b_i)/\|a^i\| = w\ , \qquad \text{for all } i \in I(F) = I(x)\ ,$$

$$((a^i,x)-b_i)/\|a^i\| < w\ , \qquad \text{for all } i \in I(x'),\ i \notin I(x)\ ;$$

hence

$$(a^i,x-x')/\|a^i\| = w-w_k > 0\ , \qquad \text{for all } i \in I(x)$$

$$(a^i,x-x')/\|a^i\| < w-w_k\ , \qquad \text{for all } i \in I(x'),\ i \notin I(x)\ ,$$

and the consideration of $(x-x')/(w-w_k)$ shows that $I(F)$ is the index set of an outside face of $\partial f(F')$, where F' is a face of P_{w_k}.

For the converse, let $J = J(D)$ be the index set of an outside face D of a subdifferential $\partial f(F')$, where F' is a face of P_{w_k}. Let $x' \in \text{ri}F'$, and y be such that

$$(a^i,y)/\|a^i\| = 1\ , \qquad i \in J\ ,$$

$$(a^i,y)/\|a^i\| < 1\ , \qquad i \in I(F'),\ i \notin J\ ;$$

for ε small enough (and positive) $I(x+\varepsilon y) = J$ and $x+\varepsilon y \in P_{w_k+\varepsilon}$. Now take F to be the unique face of $P_{w_k+\varepsilon}$ such that $x+\varepsilon y \in \text{ri}F$, then $I(F) = I(x+\varepsilon y) = J$. And Lemma 3.17 thus implies that there exists a face $F_w \in \mathcal{F}(P_w)$, for all $w \in (w_k, w_{k+1})$ such that $I(F_w) = J$.

The proof for inside faces is similar.

The last statement of the lemma says that if F is a nondegenerate face of P_w, and D is a face of $\partial f(F)$, then D is a side face of $\partial f(F)$, and, thus, there exists a face F' of P_w such that $D = \partial f(F')$. QED

Theorem 3.19

Let $(a^i,x) \leq b_i$, $i \in M$, be a system of linear inequalities, and

$$P_w = \{x \in R^n: \ ((a^i,x) - b_i)/\|a^i\| \leq w, \quad i \in M\} ,$$

be the family of perturbed polyhedra; then the condition numbers $\mu^*(P_w)$ and $\nu(P_w)$ have the following properties:

1. they are nondecreasing functions of w;
2. they are constant and equal between consecutive vertex levels;
3. at every vertex level w_k, ν is left continuous (or left constant) while μ^* is right continuous (or right constant) and $\mu^*(P_{w_k}) \geq \nu(P_{w_k})$.

The asphericities $\sigma'(P_w)$ and $\sigma(P_w)$ are decreasing functions of w and

$$\mu^*(P_w) \geq \nu(P_w) > (\sigma(P_w))^{-1} > (\sigma'(P_w))^{-1} .$$

Proof:

Property 2 was given earlier, as Lemma 3.17, while property 1 follows from 2 and 3. So one only needs to show property 3; Lemma 3.11 shows that

$$\mu^*(P_w) = \text{Min}\{d(0;D)\text{: } D \text{ is an outside face of some } \partial f(F),$$
$$F \in \mathcal{F}(P_w)\} \quad ,$$

and thus Lemma 3.18 gives

$$\mu^*(P_{w_k}) = \mu^*(P_w) \ , \qquad \text{for all } w \in [w_k, w_{k+1}) \ .$$

Similarly $\nu(P_{w_k}) = \nu(P_w)$ for all $w \in (w_{k-1}, w_k]$; also Theorem 3.14 shows that $\mu^*(P_{w_k}) \geq \nu(P_{w_k})$ and thus property 3 is proved.

Now, for the asphericities, we will prove that $\sigma'(P_w)$ is a decreasing function of w (a similar proof works for $\sigma(P_w)$).

By definition of $\sigma'(P_w)$, there exists an $x^* \in P^* = \{x\text{: } f(x) = \text{Min}_y f(y)\}$ and $r^* = -f(x^*)$ such that

$$x^* + (r^*+w)S \subset P_w \subset x + (r^*+w)\ \sigma'(P_w)S \ ;$$

note that $x^* + (r^*+w)S$, for $x^* \in P^*$, are all the largest spheres contained in P_w (and this for any w).

Now, using the definition of $\mu^*(P_w)$ given in §2, one has, for any $v > w$,

$$P_v \subset P_w + (\mu^*(P_w))^{-1}\ (v-w)S \quad ,$$

and

$$\begin{aligned} x^* + (r^*+v)S \subset P_v &\subset P_w + (\mu^*(P_w))^{-1}\ (v-w)S \\ &\subset x^* + [(r^*+w)\ \sigma'(P_w) + (\mu^*(P_w))^{-1}\ (v-w)]S \\ &\subset x^* + (r^*+v)\ \sigma'(P_w)S \qquad \text{(using Theorem 3.14)} \ ; \end{aligned}$$

and thus $\sigma'(P_v) \leq \sigma'(P_w)$. QED

This theorem essentially means that condition numbers, and asphericities, improve as the perturbation w increases; this critically depends upon the assumption that the inequalities have been normalized before being perturbed, and it is not true if this assumption is not made. The normalization assumption also implies that every n dimensional face of Q extends up to $w = +\infty$, and that every singleton $\{i\}$, $i \in M$, belongs to $\mathcal{E}(Q)$ (if one assumes that no rows of A are positively proportional).

Theorem 3.19 will not be of much use unless a compactification scheme is introduced, which makes the asphericities finite.

Compactification scheme 3.20

Define by $c(w)$ (and $c = c(0)$) a positive number such that the cube $\{x: \|x\|_\infty \leq c(w)\}$ contains a point in the relative interior of every face of P_w. Then let $P_w^c = \{x \in P_w: \|x\|_\infty \leq c(w)\}$; one clearly has the fact that

$$\mu^*(P_w) \geq \nu(P_w) \geq \nu(P_w^c) \geq (\sigma(P_w^c))^{-1} \geq (\sigma'(P_w^c))^{-1} ,$$

where now $\sigma'(P_w^c) \in [1, +\infty)$.

This leads to the main result of this section, which says that, for any consistent system of linear inequalities with solution set P, the condition number $\mu^*(P)$, which measures the convergence rate of the maximal distance method, is greater than the asphericity of a full dimensional and bounded set which is given by a perturbation cum compactification of the set P.

Theorem 3.21

Let $(a^i,x) \leq b_i$, $i \in M$, be any consistent system of linear inequalities, with solution set P, and let

$$P_w = \{x \in R^n\colon ((a^i,x)-b_i)/\|a^i\| \leq w, \quad i \in M\} \ ,$$

where $w \in (0,w_1)$, and

$$P_w^c = P_w \cap \{x \in R^n\colon \|x\|_\infty \leq c(w)\} \ ,$$

which is always bounded and full dimensional, then

$$\mu^*(P) = \nu(P_w) \geq \nu(P_w^c) \geq (\sigma(P_w^c))^{-1} \geq (\sigma'(P_w^c))^{-1} \ .$$

4. Finite termination

The relaxation methods given by Algorithms 1 and 2 are not, in general, finitely convergent procedures. It has been shown by Motzkin and Schoenberg [29], Eaves [9] and in [13,14] that, if $\dim P = n$, then some values of the relaxation parameter (including the value of 2) lead to finite convergence; no bound on the number of iterations has been proved, and thus those results do not permit a discussion of polynomiality.

It will thus be necessary to stop Algorithms 1 and 2 after a finite number of steps, and either decide that P is empty, or go to a termination routine which will identify a point x in P.

If w_1 is the first positive vertex level of Q, then, if there exists an x such that $f(x) < w_1$, there also exists an x' such that $f(x') \leq 0$, and thus P is feasible; this is a geometric interpretation of results given in Chernikov [35] and Khacian [24,25]. Similar statements with $\tilde{f}$ and $\tilde{w}_1$ and the other related functions are of course valid.

It should be apparent that it is possible to design an n step termination procedure, which is a descent method, starting from a point x such that $f(x) \leq w_1$, which will follow faces of decreasing dimension; if the faces used decrease by one dimension at each iteration, then, after n iterations, it must be a vertex x', such that $f(x') < w_1$, and thus $f(x') \leq 0$.

Thus, we will now specify ?other termination criteria? for algorithm 1; something similar applies to Algorithms 1' and 2.

3.2. if $f(x) < w_1$, go to

termination routine

3.3. if $q > [\log(f(x^0)/w_1\mu^*(P)))]/\log(1 - (\mu^*(P))^2)^{-1/2}$, stop, P is infeasible.

The stop in step 3.3 is correct, because if q is as above, and if P were feasible, then by Theorem 2.2, $f(x^q) < w_1$. This part of the algorithm is not quite implementable unless lower bounds on w_1 and $\mu^*(P)$ can be given, and this requires the assumption that the data is integer (see §5).

If the switch to the termination routine is based upon an incorrect value of w_1, then the termination routine will end at a vertex of Q, which may not be a point of P; it is clear that the relaxation method (or the simplex method) could be restarted from that point.

In the study of the ellipsoid method, various methods have been suggested, in order to find an exact solution from an approximate one; some proposals involved solving m systems using Algorithm 1 (Aspvall and Stone [3], Khacian [25]), or used rationality arguments and continued fractions (Bland, Goldfarb and Todd [6]).

The termination procedure given here follows more closely the ones given by Goldfarb and Todd [17], and Akgul [2], and also is in the spirit of a proof given by Gacs and Lovasz [11].

It is essentially a projection method, or a rank deflating ellipsoid method [16], and is similar to the algorithms of Lemke [27], Rosen [31], Zoutendijk [39,40], Gill and Murray [12], Cline [7] and Bartels, Conn and Charalambous [4], but with special attention being paid to the kind of degeneracy which has been defined in Section 3.

In order to keep the notation a bit less cumbersome, the termination routine will be described using the function $\tilde{f}$, but it is easy to translate the routine in terms of the other functions f, $\hat{f}$, ϕ, $\hat{\phi}$, etc.

Termination routine 4.1

1. Set $K = K_0$, a positive definite symmetric matrix (possibly an identity, or the matrix H of Algorithm 2),

$$x = x^q \qquad \text{(where } \tilde{f}(x^q) < \tilde{w}_1\text{)}$$
$$z = 0\ , \qquad p = 0\ .$$

2. ? is $(a^i,z) = 1$ for all $i \in \tilde{I}(x)$, where $\tilde{I}(x) = \{i \in M: (a^i,x) - b_i = \tilde{f}(x)\}$?

   if yes, go to 4 (linesearch)

   if no, go to 3 (update)

   (an alternative test would be

   ? is $(a^i,z) > 1$ for all $i \in I(\tilde{x})$?)

3. Update

Select $i \in \tilde{I}(x)$ such that $(a^i,z) \neq 1$ (or $(a^i,z) < 1$ for the alternative test of step 2), always one has $Ka^i \neq 0$; set

$$z_+ = z + (1 - (a^i,z))\, Ka^i/(a^{it}Ka^i)$$

$$K_+ = K - \frac{Ka^i a^{it} K}{a^{it} K a^i}$$

set $z \leftarrow z_+$, $K \leftarrow K_+$, $p \leftarrow p+1$, and go to step 2.

4. linesearch

$$\delta = \text{Max}\{\delta' : \tilde{f}(x) - \tilde{f}(x-\delta' z) = \delta'\}$$

$$= \text{Max}\{(f(x) - (a^j,x) + b_j)/(1 - (a^j,z)) : j \in \{i \in M : (a^i,z) < 1\}\}$$

(note that $\delta > 0$),

if $\delta = +\infty$, $\text{Min}\{\tilde{f}(y) : y \in R^n\} = -\infty$, and $x' = x - \delta' z$ solves P for any $\tilde{\delta}' \geq f(x)$,

set $x_+ = x - \delta z$, if $f(x_+) \leq 0$, stop: $x_+ \in P$,

otherwise, set $x \leftarrow x_+$ and go to 2.

Theorem 4.2

If the assumption that $\tilde{f}(x_0) < \tilde{w}_1$ is correct, then the termination routine stops after at most n steps, with a point x which solves P; if the assumption is incorrect, then the routine may "fail" in step 3, with $Ka^i = 0$, and this implies that x belongs to a horizontal face of Q.

Proof:

Let $A_p = (a^{i^o}, a^{i^1}, \ldots, a^{i^{p-1}})^t$, then it has been shown in [16] that

$$K_p = K_o - K_o A_p^t (A_p K_o A_p^t)^{-1} A_p K_o ,$$

$A_p K_p = 0$, rank $A_p = p$, rank $K_p = n-p$, and also that $K_p a = 0$ if and only if $a = A_p^t \lambda$.

Also, by induction of p, one has $A_p z_p = 1_p$ (a p dimensional vector of ones) and $z_p = K_o A_p^t (A_p K_o A_p^t)^{-1} 1_p$; in fact z_p is the unique solution of $A_p z_p = 1_p$ which belongs to $R(K_o A_p^t)$. Also, by induction, $A_p x_p - b(p) = \tilde{f}(x_p) 1_p$ (where $b(p) = (b_o, \ldots, b_{i^{p-1}})^t$), and thus $\tilde{I}(x_p) \supset \{i^o, \ldots, i^{p-1}\}$; if one denotes by $F_Q(x^p)$ the face of Q such that $I(F^p(x_Q)) = \tilde{I}(x^p)$ (clearly the smallest face of Q containing $(x^p, \tilde{f}(x^p))$), then $\dim F_Q(x^p) \leq n-p$.

Every $a \in R^n$ may be decomposed uniquely as $a = A_p^t \lambda + a_N$, where $a_N \in N(A_p K_o)$ (where N means null space); hence $a^t z_p = \lambda^t 1_p$ and $K_p a = K_p a_N$ (note that $N(K_p) = R(A_p^t)$).

Thus failure will occur in step 3 if and only if $(a^i, z_p) \neq 1$ and $K_p a_i = 0$; this is true if and only if $a_p^i - A_p^t \lambda = 0$ and $1 - \lambda^t 1_p \neq 0$, which implies (by Definition 3.16 and a theorem of the alternative) that $F_Q(x^p)$ is horizonal. Note that if the first alternative for step 2 is used, then failure must occur at a horizontal face, while this need not be the case with the second alternative.

Now if $\tilde{f}(x^p) \in (0,\tilde{w}_1)$, as the linesearch stops the algorithm if $\tilde{f}(x^p) \leq 0$, it follows that in steps 2 and 3 $\tilde{f}(x^p) \in (0,\tilde{w}_1)$ and hence $F_Q(x^p)$ is a nonhorizontal face, and thus failure cannot occur in Step 3. After at most n steps $F_Q(x^n)$ has dimension zero, and thus is a vertex of Q, implying that $\tilde{f}(x^n) \leq 0$; thus the algorithm stops after p steps $(p \leq n)$ in a point x^p which belongs to P. If the algorithm takes the full n steps, then it terminates at a vertex of Q, and $K_{n-1} = 0$.

If $\tilde{f}(x^p)$ was not smaller than $\tilde{w}_1$, then failure may occur in step 3, for some $p < n$, but then $F_Q(x^p)$ is a horizontal face of Q. QED

If rank $A < n$, then the *termination routine stops after at most* rank A steps, and it is probably more natural to restrict K_o to satisfy $R(K_oA^t) = R(A^t)$ (so that $x^p \in x^o + R(A^t)$).

The termination routine may be amended so that when "failure" occurs at iteration p, one searches for, and finds, a vertex of Q after at most $n-p$ additional iterations. This is done by updating K_p (using the same formula) until $K_p a^i = 0$ for all $i \in \tilde{I}(x^p)$, and by then doing a horizontal line search in the direction on $-K_p a^j$ (where $j \in M$, $j \notin \tilde{I}(x^p)$); the line search is called horizontal because it is done in the null space of A_p, and thus the function $\tilde{f}$ remains constant.

If the matrix K_o is an identity then the termination routine is a projection method [4,5,7,12,27,31,39,40], while, if $K_o = H = TT^t$, it is a projection method within the space of the variable y ($x = Ty$), which uses the function $\tilde{f}(y,T)$. It seems reasonable to expect that using $K_o = H$, where H is the matrix used in algorithm 2, or H is a matrix generated by the ellipsoid method, may do some good.

It should be pointed out that the termination routine, while it decreases $\tilde{f}$ (or f, if one used f to guide the descent) does not necessarily decrease g; in fact, except in special cases (see Todd [33]), the direction z_p is not a subgradient of $\tilde{f}$ at x^p.

This means that it is not possible to guarantee convergence if one tried to switch too frequently between the relaxation method and the termination routine (if it fails). By not too frequently, we mean that the convergence theory of Theorem 2.2 indicates that if q is large enough so that $\theta^q \leq \mu^*(P)/\varepsilon$, with $\varepsilon > 1$, then it is guaranteed that, after q steps of the maximal distance relaxation method, one has $f(x^q) \leq g(x^q) \leq \varepsilon^{-1}\mu^*(P)\ g(x^q)$, and thus, after n additional steps of the termination routine (applied to $f(x)$), one has

$$g(x^{q+n}) \leq f(x^{q+n})/\mu^*(P) \leq f(x^q)/\mu^*(P) \leq \varepsilon^{-1}\ g(x^0)$$

The rate of convergence is badly affected (in theory), unless ε is taken to be of the order of $(\mu^*(P))^{-1}$, in which case it is reduced by about half.

Another possibility, after the termination routine finds a vertex of Q, would be to switch to the simplex method; note that if one updated $K_o A_p^t (A_p K_o A_p^t)^{-1}$ rather than K_p, then at step $n-1$ one would have computed A_{n-1}^{-1}, and thus the information needed for the simplex method would be available.

An interesting feature of the termination routine is that it is possible to implement it when the function $\tilde{f}$ and its subgradients are given by an oracle, or by the solution of subproblems (as is done when one uses Dantzig-Wolfe decomposition, or other similar schemes). Under such a description of the function $\tilde{f}$ one usually assumes that the sets M and $\tilde{I}(x)$ cannot be listed (or even, reasonably computed); but the condition (in step 2) $(a^i,z) \geq 1$ for all $i \in \tilde{I}(x)$ can be tested, and an a^j, such that $j \in \tilde{I}(x)$ and $(a^j,z) < 1$, can be generated, by taking a small ("null") step in the direction z. By a null step is meant any step δ such that if $j \in \tilde{I}(x-\delta z)$, then $f(x) - f(x-\delta z) = \delta(a^j,z)$; then clearly $j \in \tilde{I}(x)$, and $(a^j,z) < 1$. It is possible to implement, at the level of the master problem, a line search which will find such a δ, but it may take a large though finite (i.e., exponential in the <u>size</u> of the problem) number of steps; but usually the line search can be performed polynomially (i.e., in the <u>size</u> of the problem) at the subproblem level (within the oracle).

5. Integrality

The assumption that the data is integer is needed for two reasons:

1. in order to give computable bounds on w_1 and $\mu^*(P)$, so as to guarantee that the switch to the termination routine is not done prematurely;
2. to guarantee that all numbers needed in the computation can be represented in a space which is polynomial in the length of the input (L).

The sequence $\{x^q\}$ generated by algorithm 1 is not integer, unless (see Eaves [9]) each a^i has components 0, 1 and -1, at most two of them being nonzero, and one uses a relaxation parameter equal to 2.

In what follows, we will use

$$\bar{a} = \mathrm{Max}\{\|a^i\|_\infty : i \in M\} \quad \text{and} \quad \bar{b} = \|b\|_\infty ,$$

rather than L, the length of the input.

Lemma 5.1

If Algorithm 1 is polynomial time, then all numbers in the computation may be represented as a ratio of integers, which are polynomial space. The same holds for Algorithms 1' and 2 if T and H are integer matrices, and polynomial space.

Proof:

For Algorithm 1, let $\varepsilon_q = \Pi_{j=0}^{q-1} \|a^{i^j}\|^2$, then $x^q \varepsilon_q$ is an integer vector (assuming $x^o = 0$, or integer). Let $\delta_q = \|x^q \varepsilon_q\|_\infty$, then x^q is a ratio of an integer vector $x^q \varepsilon_q$ of size at most δ_q, and an integer scalar ε_q.

Now

$$x^{q+1} \varepsilon_{q+1} = (x^q \varepsilon_q) \|a^{i^q}\|^2 + ((a^{i^q}, x^q) - b^{i^q}) \varepsilon_q a^{i^q} ,$$

hence

$$\delta_{q+1} \leq 2n \delta_q \bar{a}^2 + \varepsilon_q \bar{b}\bar{a} ,$$

(using $\|a\|^2 \leq n\|a\|_\infty^2$, and

$$(a,x) \|a\|_\infty \leq \|a\| \|x\| \|a\|_\infty \leq n\|x\|_\infty \|a\|_\infty^2) ;$$

also $\varepsilon_q \leq (n\bar{a}^2)^q$.

Proceeding by induction on q, one is led to (assuming $\delta_o = 0$):

$$\delta_q \leq (2^q - 1) (n\bar{a}^2)^{q-1} \bar{b}\bar{a} \leq (2n\bar{a}^2)^q \bar{b} ;$$

and thus

$$\log_2 \delta_q \leq q \log_2 2n\bar{a}^2 + \log_2 \bar{b} ,$$

$$\log_2 \varepsilon_q \leq q \log_2 n \bar{a}^2 ,$$

and the first part of the lemma is satisfied for the relaxation phase of Algorithm 1.

The termination routine lends itself to the same type of analysis, provided K_o is integer and polynomial space, and one does not use $f(x)$, or $f(y,T)$ or $\phi(x,T)$, which introduce irrational numbers, but $\hat{f}$, $\tilde{f}$, $\hat{\phi}$, etc. The representation of K_p and z_p make it clear that both may be represented by ratios of integers which are polynomial space (using the same reasoning as above). The line search can also be performed using integer arithmetic, and then δ and x^p may also be represented as ratios of polynomial space integers.

For Algorithm 1', the results extend if one inteprets $\bar{a}$ as the largest entry, in absolute value, of AT; for Algorithm 2 the relationship $x = Ty$ shows that the lemma is also correct. QED

The function $f(x) = \mathrm{Max}_{i \in M}(((a^i,x)-b_i)/\|a^i\|)$ may be used in the relaxation phase of the algorithm as the set $I(x)$ may be computed by using integer arithmetic:

$$I(x) = \{i \in M: ((a^i,x)-b_i)^2/\|a^i\|^2 = f^2(x),\ (a^i,x)-b_i > 0\} .$$

<u>The estimates 5.2</u>

1.

$$\begin{aligned} w_1 &> 2^{-(n+1)/2}\, \mathrm{Min}\{\prod_{i \in I} \|a^i\|^{-1}: I \subset M,\ |I| = n+1\} \\ &> (2n\bar{a}^2)^{-(n+1)/2} \end{aligned}$$

Proof:

Every nonzero vertex level w solves

$$a^{it}x - b_i = \|a^i\|w \qquad I \subset M,\ |I| = n+1 ,$$

hence

$$w = \begin{vmatrix} a^{it} & -b_i \\ \vdots & \vdots \end{vmatrix} \Bigg/ \begin{vmatrix} a^{it} & \|a^i\| \\ \vdots & \vdots \end{vmatrix} \quad \text{by Cramer's rule;}$$

using the fact that the numerator is nonzero and integer, and Hadamard's inequality for the denumerator, one gets

$$w \geq 1/\prod_{i \in I} (\|a^i\|^2 + \|a^i\|^2)^{1/2} . \qquad \text{QED}$$

The same proof, using an orthogonal transformation, shows that, if Assumption 3.1 does not hold, the same result holds with rank A replacing n. Also the same estimate is valid for w_{-1}, the first negative vertex level.

2.

$$\tilde{w}_1 \geq \operatorname{Min}\{\prod_{i \in I} (1 + \|a^i\|^2)^{-1/2} : I \subset M,\ |I| = n+1\}$$
$$> (1 + n\bar{a}^2)^{-(n+1)/2} .$$

3.

$$\hat{w}_1 \geq (2na^{-2})^{-(n+1)/2} \quad .$$

4. If A is totally unimodular, then

$$w_1 \geq \frac{1}{(n+1)n^{1/2}} \quad , \quad \hat{w}_1 \geq \frac{1}{(n+1)n^{1/2}} \quad , \quad \tilde{w}_1 \geq \frac{1}{n+1} \quad ,$$

(this is done by expanding the bottom determinant with respect to its last column, and noticing that all $n+1$ minors are -1, 0 or 1).

If one had written a totally unimodular linear program as a system of linear inequalities (by using primal and dual constraints, and a reverse weak duality constraint), the resulting system is not totally unimodular (because of the weak duality row), but estimates of w_1 are still polynomial in n, and in the size of the numbers involved in the right hand side and in the objective.

5. $w_1(T)$ which is identical for $f(y,T)$ and $\phi(x,T)$:

$$w_1(T) \geq w_1 \det^{1/2} H/(\Lambda(H))^{(n+1)/2}.$$

<u>Proof</u>:

Solving $a^{it}\,Ty - b_i = w\|T^t a^i\|$, $i \in I \subset M$, $|I| = n+1$, one gets

$$w = \begin{vmatrix} a^{it}T & -b_i \\ \vdots & \vdots \end{vmatrix} \Bigg/ \begin{vmatrix} a^{it}T & \|T^t a^i\| \\ \vdots & \vdots \end{vmatrix}$$

$$\geq (\det T)/\prod_{i \in I} (2\|T^t a^i\|^2)^{1/2}$$

(using the fact that $\det(\bar{A}T,b) = \det T \cdot \det(\bar{A},b)$)

$$\geq (\det T)/((2\,\Lambda(H))^{(n+1)/2} \prod_{i \in I} \|a^i\|) \,.$$ QED

6. $\mu^*(P) = \nu(P_w)$ (where $w = 0$ if P is full dimensional, while if not $w \in (0,w_1]$):

$$\mu^*(P) = \nu(P_w) \geq n^{-1/2} e^{-1/2} \mathrm{Min}\{\prod_{i \in I} \|a^i\|^{-1}: I \subset M, |I| \leq n\}$$

$$\geq e^{-1/2} n^{-(n+1)/2} \bar{a}^{-n} \,.$$

Proof:

Using the last definition of ν (definition 3.4),

$$\nu = \mathrm{Min}\{(\lambda^t \Gamma \lambda)^{1/2}/\|\lambda\|_1 : \lambda \geq 0\}$$

for some nonsingular Grammian build on the vectors $a^i/\|a^i\|$ belonging to an index set I, $|I| \leq n$; hence

$$\nu \geq \mathrm{Min}\{(\lambda^t \Gamma \lambda)^{1/2}/(n^{1/2}\|\lambda\|)\}$$
$$\geq n^{-1/2} \lambda^{1/2}(\Gamma) \quad .$$

(where this last λ means the smallest eigenvalue).

Now (see [16])

$$\lambda(\Gamma) \geq (\det \Gamma)/(\mathrm{Tr}\, \Gamma/(n-1))^{n-1} \,,$$

and $\mathrm{Tr}\,\Gamma = |I| \leq n$, while

$$\det \Gamma = \det(A_I A_I^t) / \prod_{i \in I} \|a^i\|^2 \quad ,$$

(where A_I is a matrix whose rows are a^{it}; $i \in I$; it is also integer); thus

$$\det \Gamma > \prod_{i \in I} \|a^i\|^{-2}$$

Hence

$$\lambda(\Gamma) > \prod_{i \in I} \|a^i\|^{-2} / (n/(n-1))^{n-1}$$

$$> e^{-1} \prod_{i \in I} \|a^i\|^{-2}$$

and thus

$$\nu > e^{-1/2}\, n^{-1/2}\, \mathrm{Min}\{ \prod_{i \in I} \|a^i\|^{-1} : I \subset M,\ |I| \leq n\}$$

$$\geq e^{-1/2}\, n^{-1/2} (n^{1/2}\, \bar{a})^{-n} \; . \qquad \text{QED}$$

For totally unimodular problems, $\|a^i\| \leq n^{1/2}$, and the estimate becomes

$$\mu^*(P) = \nu(P_w) > e^{-1/2}\, n^{-(n+1)/2} \; .$$

7. The compactification scheme constant $c(w)$, where $w \in [0, w_1]$,

$$c(w) > n(n^{1/2}\, \bar{a})^{n-1} \, (\bar{b} + n^{1/2}\, \bar{a} w) \; .$$

Proof:

The constant $c(w)$ was defined so that $P_w \cap \{x: \|x\|_\infty \le c(w)\}$ contains at least one point in the relative interior of every face of P_w. As every face of P_w contains at least one vertex (under Assumption 3.1), one may take any $c(w) > \mathrm{Max}\{\|x\|_\infty: x \text{ is a vertex of } P_w\}$.

Every vertex of P_w is the solution of

$$(a^i,x) - b^i = \|a^i\|w \ , \qquad i \in I \subset M, \ |I| = n \ ,$$

where w is a constant. Using Cramer's rule, one gets (where x_k is the k^{th} coordinate of x)

$$x_k = \begin{vmatrix} a_1^i & \cdots & a_{k-1}^i & a_{k+1}^i & \cdots & b_i + \|a^i\|w \\ \vdots & & \vdots & \vdots & & \vdots \end{vmatrix} \Big/ \begin{vmatrix} a^i \\ \vdots \end{vmatrix}$$

$$|x_k| \le \frac{\left| \sum_{i \in I} [(|b_i| + \|a^i\|w) \prod_{j \ne i} \|a^j\|] \right|}{1}$$

$$\le n\bar{b}(n^{1/2}\,\bar{a})^{n-1} + nw(n^{1/2}\,\bar{a})^n \ ,$$

and thus

$$c(w) > n(n^{1/2}\,\bar{a})^{n-1}\,(\bar{b} + n^{1/2}\,\bar{a}w) \ . \qquad \text{QED}$$

If A is totally unimodular, then

$$c(w) > n(\bar{b} + n^{1/2}\,w) \ .$$

8. $f(0) \leq \bar{b}$.

All constants computed are exponential in the length of the input, and thus their logarithms are polynomial (and hence they are representable in polynomial space); the constants w_1, $c(w)$ are polynomial in n, and $\bar{b}$, if one assumes total unimodularity of the matrix A.

But, the constant that really matters $(\mu^*(P))$, as it determines the rate of convergence (or the polynomality) of the algorithm, is clearly exponential, and it seems hard, though not impossible, to find better estimates except on very special classes of problems (total unimodularity is not special enough).

It should be noted that m, the number of constraints, does not appear anywhere.

<u>Theorem 5.2</u>

Let $Ax \leq b$, where A and b are integers, be a system of linear inequalities, then the maximal distance relaxation method (Algorithm 1) will decide that the system is infeasible or find an exact solution to it in polynomial time (and polynomial space) if $(\mu^*(P))^{-1}$ is a polynomial of n, the dimension of the space.

The variable metric, maximal ellipsoidal distance, relaxation method (Algorithm 2) with ellipsoid matrix $H = TT^t$, will do the same if $\mu^*(T^{-1}P)$ is a polynomial of n, if $\log(\Lambda(H)/\lambda(H))$ is polynomial in the length of the input, and, for polynomial space, if T (or H) are matrices of polynomial space integers.

<u>Proof</u>:

Using the fact that $\ln(1-x^2)^{-1/2} \geq x^2/2$, the convergence Theorem 2.2 and the analysis of the termination routine 4.1, it is clear that Algorithm 1 will take at most

$$n + (2/(\mu^*(P))^2) \ln(f(0)/(w_1\ \mu^*(P))) .$$

iterations to solve the problem. Because of the estimates 5.2, the logarithm is polynomial in the length of the input, and the algorithm is polynomial time if $(\mu^*(P))^{-1} \leq n^k$ for some nonnegative k.

The issue of polynomial space is easily settled by using Lemma 5.1, and, for instance, by using the function $\hat{f}$.

For Algorithm 2, Theorem 2.4 indicates that the number of iterations will not exceed

$$n + (2/(\mu^*(T^{-1}P))^2) \ln(f(0)\ \Lambda^{1/2}(H)/(w_1\ \mu^*(T^{-1}P)\ \lambda^{1/2}(H))) ;$$

and hence the theorem follows similarly. QED

This theorem shows that except in very special cases, Algorithm 1 is exponential, and this may mean exceedingly bad, in practice. It works well on some classes of problems, if $\mu^*(P)$ is not too small; it has been conjectured that $(\mu^*(P))^{-1}$ is a polynomial of n for assignment problems, but this does not seem to extend to general totally unimodular problems.

Algorithm 2 shows a potential for improving over Algorithm 1; in fact, we will show in Sections 6 and 7 that there always exists a linear transformation such that $\mu^*(T^{-1}P) \geq 1/n$, and, in a sequel to this paper, that the ellipsoid method [2,3,6,11,16,17,18,24,25,32,36, 37] is, in fact, an algorithmic procedure (polynomial) to identify such at T.

6. Ellipsoids

A well known result due to John [23] says that the set of linear transformations $(P) = \{T: \sigma(T^{-1}P) \leq n,\ T$ nonsingular$\}$ is not empty, if P is full dimensional, compact and convex (P need not be a polyhedron). We will denote by

$$E_* = e_* + T_*S \qquad (\text{resp. } E^* = e^* + T^*S)$$

the largest volume (resp. smallest volume) ellipsoid contained in (resp. containing) P. Both ellipsoids are unique; see John [23], Dantzer-Laugwitz-Lenz [8], Zaguskin [38] and Grunbaum [19].

The fact that $T^* \in \mathcal{E}(P)$ has been proved by John [23], while $T_* \in \mathcal{E}(P)$ is alluded to by Grunbaum [19, p. 241].

Both these results may be interpreted in terms of the affine excentricity (or affine asphericity, or aellipsoidality) of a convex set P.

Definition 6.1

The affine excentricities, where P is compact, convex and full dimensional,

$$\begin{aligned} \tau(P) &= \mathrm{Inf}\{\sigma(T^{-1}P):\ T \text{ is nonsingular}\} \\ &= \mathrm{Inf}\{\tau \geq 0:\ x + E \subset P \subset x + \tau E\} \end{aligned}$$

where E stands for any ellipsoid centered at the origin; note that the second definition makes sense even if

$$1 \leq \dim P \leq n-1 \ .$$

$$\tau''(P) = \text{Inf}\{\tau > 0: P \subset e_* + \tau T_* S\} \ ,$$

this definition is sometimes easier to use or to characterize; one has $\tau''(P) = \sigma'(T_*^{-1}P)$.

Both affine excentricities are invariant under affine transformations, while $\tau(P)$ is the least asphericity of any member of the class of affine transforms of P; also $\tau(P) \leq \tau''(P)$.

Thus, it is true for any compact, convex and full dimensional set P, that $1 \leq \tau(P) \leq \tau''(P) \leq n$, and furthermore $\tau(P) = 1$, or $\tau''(P) = 1$, if and only if P is an ellipsoid, while $\tau(P) = \tau''(P) = n$ if and only if P is a simplex. If P is centrally symmetric then $1 \leq \tau(P) \leq \tau''(P) \leq \sqrt{n}$, and if $\tau(P) = \sqrt{n}$, or $\tau''(P) = \sqrt{n}$, and P centrally symmetric if and only if P is a parallelotope.

One somewhat interesting fact to notice is that the function $\phi(x,T*)$ always has a unique minimum at $x = e*$, and $\phi(e*, T*) = -1$ (under the assumption that P is bounded and full dimensional); this follows from the definition of ϕ, and the unicity of $E*$.

The ellipsoid principle (which we shall call primal), with shallow cuts, as given in Yudin and Nemirovskii [36,37], and Todd [34], and a dual ellipsoid principle, can be used to prove the fact that (P) is not empty, in a way which is somewhat constructive; by this, we mean that the ellilpsoid method (primal) and a dual ellipsoid method actually construct approximations to T^* and T_* (this statement will be proved in a sequel to this paper).

The primal ellipsoid principle 6.2 [34,36,37]

Let E be the ellipsoid

$$\{x \in R^n: (x-x^*)^t H^{-1}(x-x^*) \leq 1\} ,$$

and $V = \{x \in R^n: (a,x) \leq b\}$ be a halfspace, and define

$$\omega = ((a,x^*)-b)/(a^t Ha)^{1/2} ,$$

then the smallest volume ellipsoid containing $E \cap V$ is denoted by E_+, and:

1. if $\omega \leq -1/n$, $E_+ = E$;
2. if $\omega > -1/n$, Vol E_+ < Vol E.

The formula giving E_+ is well known; it is also easily computable.

If one used two halfspaces, containing x^*, and symmetric with respect to x_*, then n may be replaced by $\sqrt{n}$ (this is called the symmetric range ellipsoid principle, see Todd [34]).

The dual ellipsoid principle 6.3

Let $E_d = \{ x \in R^n: (x-x_*)^t G(x-x_*) \leq 1\}$ be an ellipsoid, x_*+v a point in R^n, and

$$\omega' = \text{Sup}\{\omega'' \geq 0: x_*+\omega''v \in E_d\}$$
$$= (v^t Gv)^{-1/2} ;$$

then if one defines E_{d+} to be the largest volume ellipsoid contained in $\mathcal{H}(E_d \cup \{x_*+v\})$, one has

1. if $\omega' \geq 1/n$, $E_{d+} = E_d$;
2. if $\omega' < 1/n$, $\text{Vol } E_{d+} > \text{Vol } E_d$,

and

$$E_{d+} = \{x \in R^n: (x-x_{*+})^t \, G_+(x-x_{*+}) \leq 1\}$$

where

$$G_+ = \frac{1}{\alpha'} \left[G - \frac{\beta'}{1+\beta'} \frac{Gvv^tG}{v^tGv} \right] ,$$

$$x_{*+} = x_* + \delta'(\omega')^{-1} \, v ,$$

$$\alpha' = \frac{(n-1)(1+\omega')}{(n+1)(1-\omega')} , \qquad \beta' = \frac{1-\omega'^2}{n^2-1} \frac{1}{\omega'^2} - 1 ,$$

$$\delta' = \frac{\omega'-1}{\omega'(n+1)} - \frac{n-1}{n+1} ,$$

$$\text{Vol}^2 \, E_{d+} = \frac{(n-1)^{n-1}}{(n+1)^{n+1}} \, \frac{(1+\omega')^{n+1}}{(1-\omega')^{n-1}} \, \frac{1}{\omega'^2} \, \text{Vol}^2 \, E_d \, .$$

No proof shall be given here. It is interesting to note that if one took the duals (or polars, see [30, p. 125]) of the sets involved in the primal ellipsoid principle (where $\omega < 0$, and $\omega' = -\omega$), with respect to the center of E, then one can prove a result similar to the dual ellipsoid principle; the only difference being that weaker values of α', β' and δ' ensue. The proof of this involves a bit of work, but is straightforward.

One could say that the dual of the primal ellipsoid principle is a weaker form of the dual ellipsoid principle. On the other hand, the dual of the symmetric range primal ellipsoid principle gives exactly the equivalent dual principle.

Theorem 6.4

Let $\mathscr{P}(P) = \{T: \sigma(T^{-1}P) \leq n,\ T \text{ nonsingular}\}$ where P is a full dimensional, compact, convex set, then $\mathscr{P}(P)$ is not empty. If $E_* = e_* + T_*S$ and $E^* = e^* + T^*S$ are respectively the largest ellipsoid inscribed in P, and the smallest ellipsoid circumscribed around P, then both T_* and T^* belong to $\mathscr{P}(P)$. If P is centrally symmetric, then this remains valid with $\sqrt{n}$ replacing n.

Proof:

We will show that $e_* + nT_*S \supset P$. If this is not true, then there exists a point $v \in R^n$ such that $v+e_* \in P$, and $v+e_* \notin e_* + nT_*S$; hence

$$\text{Sup}\{\omega'' \geq 0: \omega''v \in T_*S\} = \omega' < 1/n \quad .$$

Thus the dual ellipsoid principle 6.3 shows that there exists, and in fact constructs, an ellipsoid E_+ such that $\text{Vol } E_+ > \text{Vol } E_*$ and

$$E_+ \subset \mathscr{H}(E \cup \{e_*+v\}) \subset P \ ,$$

a contradiction.

A proof, dual to this one, using shallow cuts in the primal ellipsoid principle shows that $T^* \in \mathcal{E}(P)$ (this is John's theorem [23]).

The case of centrally symmetric sets follows, similarly, from the symmetric range ellipsoid principles. QED

As the ellipsoids E_* and E^* play a crucial role in the proofs of Section 7, we shall characterize E_* and E^* by the use of the Kuhn-Tucker or Fritz John conditions. For E_*, it is necessary to assume that P is given by a system of linear inequalities, while for E^* it is necessary to assume that P is given by the convex hull of a set of points. The study of E^* is given in John [23], and is probably the first application of the necessary optimality conditions of mathematical programming.

Both of these are optimization problems, with unknowns H (or $G = H^{-1}$), a positive definite symmetric matrix, and x, the center of the ellipsoid

$$E = \{y \in R^n : (y-x)^t H^{-1}(y-x) \leq 1\} .$$

One could regard H as a point in $R^{n,n}$, subject to symmetry constraints, but is is more natural to view H as an element of $p(R^n)$, the cone of positive semidefinite symmetric matrices, which we will assume to be defined within $s(R^n)$, the (linear) space of symmetric matrices. The interior of $p(R^n)$ is $p^0(R^n)$, the cone of positive definite symmetric matrices, while the extreme rays of

$p(R^n)$ are the symmetric positive semidefinite rank one matrices (i.e., of the form aa^t, with $a \in R^n$).

If $H, K \in s(R^n)$, the scalar product is $(H,K) = \text{Tr}\, HK$, where Tr means trace, and this induces the Froebinius norm $\text{Tr}^{1/2} H^2$. It is known that $p(R^n)$ is a self-dual convex cone (i.e., $\text{Tr}\, HK \geq 0$ for all $K \in p(R^n) \Rightarrow H \in p(R^n)$).

The equality $\text{Tr}\, VV^tK = \text{Tr}\, V^tKV$, valid if K is symmetric, and where V may be rectangular, will be used.

The volume of E is given by $(\det^{1/2} H)$ Vol S, and hence one shall maximize $\det H$, or $\ell n \det H$, and thus an expression of

$$\frac{d\, \ell n \det H}{dH}, \qquad \text{for } H \in p^0(R^n), \text{ is needed .}$$

It is known that $\ell n \det H$ is a concave function of H (see Fan [10]).

Lemma 6.5

Let $H \in p^0(R^n)$, then $(d\, \ell n \det H)/dH = H^{-1}$, and $\ell n \det H$ is strictly concave on $p^0(R^n)$.

Proof:

We shall compute, for ε small enough,

$$\ell n \det(H+\varepsilon K) - \ell n \det H \qquad (H \in p^0(R^n),\ K \in s(R^n)),$$

$$= \ell n \det H^{1/2}(I+\varepsilon H^{-1/2} KH^{-1/2})\, H^{1/2} - \ell n \det H$$

$$= \ell n \det(I+\varepsilon H^{-1/2} KH^{-1/2})$$

$$= \ell n \prod_{i-1}^{n} (1+\varepsilon\lambda_i) \;, \; (\lambda_i \text{ are the eigenvalues of } H^{-1/2} KH^{-1/2} \text{ or } H^{-1}K, \text{ all real numbers})$$

$$= \sum_{i=1}^{n} \ell n(1+\varepsilon\lambda_i)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{\varepsilon^j \lambda_i^j}{j!} \qquad (|\varepsilon\lambda_i| < 1)$$

$$= \sum_{j=1}^{\infty} (-1)^{j-1} \varepsilon^j \operatorname{Tr}(H^{-1/2} KH^{-1/2})^j \; .$$

Note that $\operatorname{Tr}(H^{-1/2} KH^{-1/2})^j = \operatorname{Tr}(H^{-1}K)^j$, and thus

$$\lim_{\varepsilon \to 0} [\ell n \det(H+\varepsilon K) - \ell n \det H] \, \varepsilon^{-1} = \operatorname{Tr} H^{-1}K \; ,$$

and thus

$$\frac{d \; \ell n \det H}{dH} = H^{-1} \; .$$

For the strict concavity, note that

$$\operatorname{Tr}(H^{-1/2} KH^{-1/2})^2 > 0 \; , \qquad \text{for all } K \neq 0 \; ,$$

and this means that the second derivative is negative definite. QED

It may be checked that for $\ln \det H$, the steepest ascent direction is H^{-1}, while the Newton direction is H.

Theorem 6.6

Let $P = \{x \in R^n: (a^i,x) \leq b_i,\ i \in M\}$ be a compact, full dimensional, polyhedron; then a necessary and sufficient condition for

$$E_* = e_* + T_*S = \{x \in R^n: (x-e_*)^t\, H_*^{-1}(x-e_*) \leq 1\} ,$$

where $H_* = T_*T_*^t$, to be the largest ellipsoid inscribed in P, is that there exist nonnegative multipliers λ_i, $i \in M$, such that:

$$H_*^{-1} = \sum_{i \in M} \lambda_i\, a^i a^{it} , \qquad (\in p^0(R^n)) ,$$

$$\sum_{i \in M} \lambda_i (b_i-(a^i,e_*))a^i = 0$$

$$\lambda_i[(a^{it}H_*a^i))^{1/2} - (b_i-(a^i,e_*))] = 0 , \qquad \text{for all } i \in M$$

$$(a^{it}H_*a^i)^{1/2} \leq b_i-(a^i,e_*) , \qquad \text{for all } i \in M .$$

Proof:

The largest ellipsoid inscribed in P is given by the solution of the following optimization problem:

$$\text{Max}\{\det H^{1/2}: x + H^{1/2}S \subset P,\ H \in p^0(R^n)\} .$$

But $x + H^{1/2}S \subset P$ if and only if

$$x + H^{1/2}S \subset \{y \in R^n: (a^i,y) \leq b_i\} , \quad \text{for all } i \in M$$

or

$$(a^{it}Ha^i)^{1/2} \leq b_i - (a^i,x) , \qquad \text{for all } i \in M$$

and

$$b_i - (a^i,x) > 0 , \qquad \text{for all } i \in M .$$

Thus the optimization problem becomes:

$$\text{Max} \qquad \ell n \det H$$

$$\text{subject to} \quad (a^i a^{it}, H) \leq (b_i - (a^i,x))^2 , \quad \text{for all } i \in M$$

with also

$$(a^i,x) - b_i < 0 , \qquad \text{for all } i \in M$$

$$H \in p^0(R^n) .$$

It is clear that the constraints $(a^i,x) < b_i, i \in M$ and $H \in p^0(R^n)$ have no impact on the optimality conditions. The objective is strictly concave in H and its gradient is H^{-1}.

The constraints

$$(a^i a^{it}, H) - (b_i - (a^i,x))^2 \leq 0 , \quad \text{for all } i \in M$$

are such that:

$(a^i a^{it}, H)$ is linear in H, and its gradient is $a^i a^{it}$

$-(b_i - a^{it}x)^2$ is concave in x, but also

$$\text{quasiconvex on } \{y \in R^n: (a^i, y) \leq b_i\} ;$$

they also satisfy the (extended) Slater constraint qualification.

Thus the Kuhn-Tucker conditions, that is, the theorem, are necessary and sufficient. QED

The condition $H_*^{-1} = \sum_{i \in M} \lambda_i a^i a^{it}$ (which is also the optimality condition for the largest ellipsoid with a given center) means that $G_* = H_*^{-1}$ is a positive linear combination of symmetric rank one matrices, which use the normals to the facets of P:

$$G_* = H_*^{-1} \in \text{Convex cone hull}\{a^i a^{it} : \{i\} \in (P)\} .$$

In the primal ellipsoid method [16], one always has $\alpha H_+^{-1} = H^{-1} + \gamma a a^t$, where $a \in \{a^i : i \in M\}$ and $\gamma > 0$; and thus the primal ellipsoid method may be described as updating one of the multipliers λ_i, at every iteration.

The simplex method may be described by a matrix $H^{-1} = \sum_{i \in N} a^i a^{it} = \tilde{A}^t \tilde{A}$, where $|N| = n$ and $\tilde{A}$ is square and full rank (see Gill and Murray [12]), and updates are performed by adding to H^{-1} a new $a^j a^{jt}$, $j \notin N$, and subtracting an old one $a^i a^{it}$, $i \in N$; expressed in terms of H or $\tilde{A}^{-1}$ this is the pivoting operation. So, in a sense, the simplex method always keeps all multipliers, but n, equal to zero, while the n nonzero multipliers are kept equal to one; and a pivoting is simply an exchange of two λ_i.

The optimality conditions may be expressed in many different ways; one interesting option, as it permits the interpretation of the multipliers in terms of sensitivity analysis is

$$\mathrm{Max}\{\tfrac{1}{2}\,\ell n \det H:\ (a^{it}Ha^{i})^{1/2} + a^{it}x \le b_i \ , \quad \text{for all} \quad i \in M\},$$

and it gives

$$H_*^{-1} = \sum_{i \in M} \mu_i \frac{a^{i}a^{it}}{(a^{it}H_*a^{i})^{1/2}} \ , \qquad \sum_{i \in M} \mu_i a^{i} = 0 \ ,$$

(note that $\lambda_i = \mu_i/(a^{it}H_*a^{i})^{1/2} = \mu_i/(b_i - a^{it}e^*)$): then (under standard technical assumptions) a perturbation of b_i to $b_i+\varepsilon$ (ε small) implies that the maximum volume is multiplied, approximately, by $e^{\varepsilon\mu_i}$.

The minimum volume ellipsoid containing a polyhedron $P = \{v^i: i \in I\}$ is

$$E^* = e^* + T^*S = \{x \in R^n:\ (x-e^*)^t\,(H^*)^{-1}\,(x-e^*) \le 1\} \ ,$$

where $H^* = T^*T^{*t} = (G^*)^{-1}$, and is characterized by

$$H^* = \sum_{i \in I} \nu_i (v^i - e^*)\,(v^i - e^*)^t$$

$$0 = \sum_{i \in I} \nu_i (v^i - e^*), \qquad \nu_i \ge 0,\ i \in I \ ,$$

$$(v^i - e^*)^t (H^*)^{-1} (v^i - e^*) \leq 1 \quad \text{with equality if } \nu_i > 0, \quad (i \in I) .$$

Note that H^* is given by a positive linear combination of symmetric rank-one matrices, constructed on the set of vertices of P.

In both cases (E_* and E^*), the complementarity conditions are hard to use, and it is possible to give explicit expressions for E_* and E^* only in a few particular cases.

If P is an ellipsoid, where $P = \{x \in R^n: x^t X x \leq w\}$, X positive definite symmetric, then $H_*^{-1} = (H^*)^{-1} = X/w$, and thus the ellipsoid matrix (the "variable" metric) is the inverse Hessian.

If $P = \{x \in R^n: |(a^i x) - b_i| \leq w, i = 1, \ldots, n\}$ where $A = (a^1, \ldots, a^n)^t$ is square and nonsingular, while w is positive, then P is a parallelotope, and if one lets $x = A^{-1}(y+b)$ then P transforms into a cube, $P' = \{y \in R^n: \|y\|_\infty \leq w\}$; the largest ellipsoid in P' is clearly the sphere $\{y \in R^n: y^t y \leq w^2\}$, which transforms back into the maximum volume ellipsoid in P:

$$E_* = \{x \in R^n: (x-x^*)^t A^t A(x-x^*) \leq w^2\} ,$$

where $x^* = e_* = e^* = A^{-1}b$, $H_* = w^2 A^{-1} A^{-t}$, and $H^* = nH_*$. If one did write the problem of finding the solution of $Ax = b$ as a quadratic programming program, i.e., $\text{Min}\|Ax-b\|^2$, and if one defines the level set

$$P' = \{x \in R^n: \|Ax-b\|^2 \leq w^2\} ,$$

then the largest ellipsoid inscribed in P" is exactly the same as the one inscribed in $P = \{x \in R^n: \|Ax-b\|_\infty \leq w\}$.

The last example of a set P for which E_* (and E^*) can be given somewhat explicitly is that of a simplex $P = \{x \in R^n: Ax \leq b\}$, where $A \in R^{n+1,n}$ has rank n; P has an interior if and only if $\pi^t A = 0$, $\pi^t b = 1$, $\pi > 0$ has a solution (π is unique). The linear transformation T_* (and also T^*) has the property that $T_*^{-1}P$ is a regular simplex; this follows because the largest ellipsoid inscribed in a regular simplex is a sphere, and $T^{-1}E_*$ is the largest ellipsoid inscribed in $T^{-1}P$, for any T.

Also, this shows that e_* is the center of gravity, or the centroid, of P.

After somewhat lengthy calculations, one gets

$$H_*^{-1} = n(n+1)\ A^t\ \mathrm{Diag}(\pi_i^2)A\ ,$$

where π was defined earlier, and $\mathrm{Diag}(\pi_i^2)$ is a $(n+1) \times (n+1)$ diagonal matrix, whose diagonal elements are π_i^2; also e_* solves the system

$$Ae_* + \mathrm{Diag}(1/\pi_i)\ w = b\ ,$$

where w is a variable (but its value is $w = 1/(n+1)$). If P were defined by its vertices, then let $V \in R^{n,n+1}$ be a matrix whose columns are the vertices of P, then

$$e_* = \frac{1}{n+1} V\ 1_{n+1}\ ,$$

$$H_* = \frac{1}{n(n+1)} V[I_{n+1} - \frac{1}{n+1} 1_{n+1} 1^t_{n+1}] V^t ,$$

where 1_{n+1} as a $n+1$ dimensional vector of ones, and I_{n+1} is a $n+1$ dimensional unit matrix.

All of this suggests rather compellingly that the matrix H_* (or H^*) plays the role of the inverse Hessian, in classical optimization.

The matrix TT^t which satisfies $\tau(P) = \sigma(T^{-1}P)$ might be a more satisfactory definition of the "inverse Hessian", but the Kuhn-Tucker conditions, which characterize it, do not seem terribly insightful.

7. The potential for polynomiality of the variable metric, maximal ellipsoidal distance, relaxation method

In this section, it will be shown that for every system of linear inequalities (assumed consistent), there exists an ellipsoid such that the variable metric, maximal ellipsoidal distance relaxation method (Algorithm 2) is polynomial, or, quite equivalently, that there exists an affine transformation such that the maximal distance relaxation method applied in this transformed space is polynomial. The concept of polynomality may be interpreted, in a somewhat more practical sense, by saying that the methods converge at a rate of at least $(1-n^{-2})^{1/2}$.

The class of ellipsoids which lead to the polynomiality of Algorithm 2 contain the largest ellipsoid inscribed in (and the smallest ellipsoid circumscribed amount) a perturbation cum compactification of the feasible set P (see compactification scheme 3.20), i.e., P_w^c where $w \in [0,w_1]$, with $w = 0$ acceptable if $\dim P = n$, while if $\dim P < n$, $(w)^{-1}$ should be no worse than exponential in the input data (that is $w = w_1/2$, or $w = w_1$, etc.).

All of this will be proved first for the case when $\dim P = n$, and then extended to the general case.

Lemma 7.1

Let $P = \{x \in R^n: Ax \leq b\}$ be the solution set of a system of linear inequalities, with P assumed to have a nonempty interior, and let $P^c = \{x \in R^n: \|x\|_\infty \leq c\} \cap P$ be a compactification of P, such that P^c contains a point of the relative interior of every face of P, then there exists a nonsingular linear map T such that

$$\mu^*(T^{-1}P) \geq \nu(T^{-1}P) \geq (\tau(P^c))^{-1} \geq n^{-1} ;$$

if T_* is the linear map given by the largest ellipsoid inscribed in P^c, then

$$\mu^*(T_*^{-1}P) \geq \nu(T_*^{-1}P) \geq (\tau''(P^c))^{-1} \geq n^{-1} .$$

If P is centrally symmetric, and if the compactification cube is centered at the center of symmetry of P, then everything is valid with $n^{-1/2}$ replacing n^{-1}.

If $A \in Z^{m,n}$ and $b \in Z^m$ are integer, then T_* and T may be "approximated" by polynomial space integer matrices $\tilde{T}_*$ and $\tilde{T}$ which satisfy

$$\mu^*(\tilde{T}^{-1}P) \geq \nu(\tilde{T}^{-1}P) \geq (\tau(P^c))^{-1} \frac{\alpha+1}{\alpha-1} \geq n^{-1} \frac{\alpha+1}{\alpha-1} ,$$

and

$$\mu^*(\tilde{T}_*^{-1}P) > \nu(\tilde{T}_*^{-1}P) \geq (\tau''(P^c))^{-1} \frac{\alpha+1}{\alpha-1} \geq n^{-1} \frac{\alpha+1}{\alpha-1} ,$$

where $\alpha > 1$, and such that $\log \alpha$ is polynomial in the length of the input.

Proof:

Using Theorems 3.14 and 3.21, one has that for any nonsingular transformation T:

$$\mu^*(T^{-1}P) \geq \nu(T^{-1}P) \geq (\sigma(T^{-1}P^c))^{-1} \geq (\sigma'(T^{-1}P^c))^{-1} .$$

If one takes in this the linear transformation T such that $\sigma(T^{-1}P^C) = \tau(P^C)$, or the linear transformation T_*, given by the largest ellipsoid inscribed in P, then Theorem 6.4 proves the first part of this theorem.

Now, assume that A and b are integers, so that the estimates of Section 5 hold. The work will be done on T, but clearly extends to T_*, or any map with similar properties.

We will assume, without restriction, that T is symmetric positive definite (if it is not, then $(TT^t)^{1/2} = H^{1/2}$ is, and defines the same ellipsoid). Now we have $e + TS \subset P^C \subset e + \tau TS$. Define T^{-1} as a, symmetric, perturbation of T, such that $\mathrm{Max}_{i,j} |t_{ij}-t'_{ij}| \leq \varepsilon$, where $\varepsilon > 0$ is arbitrary for now.

Then

$$\rho(T-T') \leq n \max_{i,j} |t_{ij}-t'_{ij}| \leq n\varepsilon ,$$

where ρ means the spectral norm. If $\rho_i(T)$ (resp. $\rho_i(T')$), $i = 1, \ldots, n$, are the ordered eigenvalues of T (resp. T'), then

$$|\rho_i(T) - \rho_i(T')| \leq \rho(T-T') \leq n\varepsilon ,$$

and thus T' is positive definite if $\rho_1(T) > n\varepsilon$ (where $\rho_1(T)$ is the smallest eigenvalue of T), as $\rho_1(T') > \rho_1(T) - n\varepsilon$.

In order to keep the notation consistent with the remainder of this paper, let

$$\lambda = \lambda(H) = (\rho_1(T))^2 , \qquad \Lambda = \Lambda(H) = (\rho_n(T))^2 ,$$

$$\lambda' = \lambda(H') = (\rho_1(T'))^2 , \quad \Lambda' = \Lambda(H') = (\rho_n(T'))^2 ,$$

where

$$H = T^2 \quad \text{and} \quad H' = T'^2 .$$

Now let

$$\delta = \text{Inf}\{\delta' > 0 : TS + \delta' S \supset T'S,\ T'S + \delta' S \supset TS\} ,$$

be the Hausdorff distance between the two ellipsoids TS and $T'S$; then

$$T'S \subset TS + \delta S \subset (1 + \delta\lambda^{-1/2})TS ,$$

and

$$TS \subset T'S + \delta S \subset (1 + \delta(\lambda')^{-1/2})T'S .$$

Hence

$$e + (1 + \delta\lambda^{-1/2})^{-1}\ T'S \subset P^c \subset e + \tau(1 + \delta(\lambda')^{-1/2})\ T'S ,$$

and thus

$$\sigma((T')^{-1} P^c) \leq \tau(1 + \delta\lambda^{-1/2}) (1 + \delta(\lambda')^{-1/2}) \quad ;$$

but, also (see [22]):

$$\delta \leq \rho(T-T') \leq n\varepsilon \ .$$

Now $\lambda^{1/2} \geq w_{-1}\tau^{-1}$ (where $-w_{-1}$ is the first negative vertex level, and w_{-1} can be bounded by the same estimates as w_1); this follows because a sphere of radius w_{-1} is contained in P^c, and thus also in $e + \tau TS$.

Choose $\varepsilon = w_{-1}/(\alpha n\tau)$, where $\alpha > 1$; then

$$(\lambda')^{1/2} \geq \lambda^{1/2} - n\varepsilon \geq w_{-1} \tau^{-1}(1-\alpha^{-1})$$

$$\delta \leq w_{-1}(\alpha\tau)^{-1} \ ,$$

and thus

$$\sigma((T')^{-1} P^c) \leq \tau \frac{\alpha+1}{\alpha-1} \ .$$

The ellipsoid $e + TS$ is contained in $\{x \in R^n: \|x\|_\infty \leq c\}$, and thus $\Lambda^{1/2} \leq n^{1/2}c/2$; but $\Lambda > (1/n)\mathrm{Tr}\ T^2 = (1/n) \sum_{i,j} t_{ij}^2$, and thus $|t_{ij}| \leq n^{1/2} \Lambda^{1/2} \leq nc/2$.

If one chooses $D = \lfloor \alpha n\tau/w_{-1} \rfloor$, $\tilde{t}_{ij} = \lfloor t_{ij}D \rfloor$ and $t'_{ij} = \tilde{t}_{ij}/D$, then

$$\max_{i,j} |t'_{ij}-t_{ij}| \leq D^{-1} \leq w_{-1}/(\alpha n\tau) \ ;$$

and thus

$$\sigma(\tilde{T}^{-1}P^c) = \sigma((T')^{-1}P^c) \leq \tau(P^c)\,\frac{\alpha+1}{\alpha-1}\,.$$

The matrix $\tilde{T}$ is an integer matrix, respresentable in polynomial space if α is representable in polynomial space (i.e., $\log \alpha$ is polynomial), as

$$|\tilde{t}_{ij}| \leq \frac{nc}{2} \left\lceil \frac{\alpha n \tau}{w_{-1}} \right\rceil .$$ QED

In the case where $Ax \leq b$ is totally unimodular, then $\mathrm{Sup}_{i,j}|\tilde{t}_{ij}|$ is approximately given by $n^{11/2}\,\bar{b}$ (if one takes $\alpha = 2$).

Lemma 7.1 and Theorem 5.2 lead immediately to Theorem 7.2, which says that any consistent system of linear inequalities may be solved in polynomial time (and space) by applying Algorithm 2, with some matrices H, to the perturbed system P_w (with w taken as $w_1/2$, say, if P has no interior, while $w = 0$ is satisfactory if P has an interior), with a termination routine appended to it. The matrix $H^t = TT$ should be chosen as any matrix such that $\sigma^{-1}(P_w^c)$ is polynomial in n, where P_w^c is a compactification (see 3.20) of P_w; the matrix H_* which defines the largest ellipsoid in P_w^c is a most sensible choice. We would like to point out that the proof of this is quite natural, and in fact does not require any of the work done in the later part of Section 3; but the resulting algorithm, which solves a perturbed problem, is somewhat of a mathematical artifact.

In Theorem 7.3 the perturbation is used at the level of the proof, which makes it a bit trickier, as it requires the whole of Section 3, but the algorithm is much crisper.

Theorem 7.2

Let $P = \{x \in R^n: (a^i, x) \leq b_i,\ i \in M\}$ be the solution set of a consistent system of linear inequalities, and let $w \in [0, w_1]$ (where $w = 0$ is acceptable if P has an interior, while, if not, then $\log(w)^{-1}$ and $\log(w_1 - w)^{-1}$ should be polynomial; $w = w_1/2$ will do in all cases), then algorithm 2, applied to the system

$$P_w = \{x \in R^n: (a^i, x) \leq b_i + \|a^i\| w,\ i \in M\},$$

using a matrix $H = TT^t$ which satisfies the fact that $\sigma(T^{-1}P_w^c)$ is polynomial in n (where P_w^c is a compactification of P_w), and switching to the termination routine 4.1, on the basis of the termination criterion $f(x^q) < w_1$, will converge in polynomial time to a solution of P.

A most sensible choice for H is $H_* = T_* T_*^t$ which gives the largest ellipsoid inscribed in P_w^c.

All of the computations can be done by using (polynomial space) integer arithmetic, provided that T is taken as an integer matrix (polynomial space), which is always possible.

<u>Proof:</u>

The function f associated to the system of linear inequalities

$$P_w = \{x \in R^n: ((a^i,x)-b_i)/\|a^i\| \leq w,\ i \in M\} \quad ,$$

is simply $f(x)-w$, and the corresponding first vertex level is w_1-w.

Now let T be such that $\tau = \sigma(T^{-1}P_w^c)$ is polynomial in n ($\tau \leq n$ is always possible), then, following Lemma 7.1, $\lambda(H) \geq w^2\ \tau^{-2}$ and $\Lambda(H) \leq nc^2$, where $H = TT^t$.

Theorem 5.2 implies that $f(x^q) < w_1$, or equivalently $f(x^q)-w < w_1-w$, will be reached after at most

$$2\tau^2\ \ln[(\Lambda^{1/2}(H)\ \tau(f(0)-w))/(\lambda^{1/2}(H)\ (w_1-w))] \ ,$$

iterations, after which at most n steps of the termination routine are required.

All of that is polynomial time, and if T is selected as a polynomial space integer matrix, which can be done by Lemma 7.1, then all computations can be performed in polynomial space (Lemma 5.1).

QED

<u>Theorem 7.3</u>

Let $P = \{x \in R^n: Ax \leq b\}$ be the solution of a consistent system of linear inequalities, and let $P_w^c = \{x \in P_w: \|x\|_\infty \leq c(w)\}$ (see compactification scheme 3.20), where $w \in (0,w_1)$, be a perturbation cum compactification of P, which does preserve some of the relative interior of every face of P_w, then there exists a nonsingular linear map T such that

$$\mu^*(T^{-1}P) \geq \nu(T^{-1}P_w) \geq (\tau(P_w^c))^{-1} \geq n^{-1} \;;$$

if T_* is the linear map given by the largest ellipsoid inscribed in P_w^c, then

$$\mu^*(T_*^{-1}P) \geq \nu(T_*^{-1}P_w) \geq (\tau''(P_w^c))^{-1} \geq n^{-1} \,.$$

If P_w is centrally symmetric, and if the compactification cube is centered at the center of symmetry of P_w, then everything is valid with $n^{-1/2}$ replacing n^{-1}.

If $A \in Z^{m,n}$ and $b \in Z^m$ are integer matrices, then T and T_* may be "approximated" by polynomial space integer matrices $\tilde{T}_*$. and $\tilde{T}$ which satisfy

$$\mu^*(\tilde{T}^{-1}P) \geq \nu(\tilde{T}^{-1}P_w) \geq (\tau(P_w^c))\,\frac{\alpha+1}{\alpha-1} \geq n^{-1}\,\frac{\alpha+1}{\alpha-1} \quad ,$$

and

$$\mu^*(\tilde{T}_*^{-1}P) \geq \nu(\tilde{T}_*^{-1}P_w) \geq (\tau''(P_w^c))\,\frac{\alpha+1}{\alpha-1} > n^{-1}\,\frac{\alpha+1}{\alpha-1} \quad ,$$

where $\alpha > 1$ is such that $\log \alpha$ is polynomial in the length of the input.

<u>Proof</u>:

Let T be a nonsingular linear map, and let $\tau = \sigma(T^{-1}P_w^c)$, or $\tau = \sigma'(T^{-1}P_w^c)$, then by Theorems 3.14 and 3.21 $\mu^*(T^{-1}P_w) \geq \nu(T^{-1}P_w) \geq \tau$, and thus, using Theorem 6.4, the present theorem is proved if $\mu^*(T^{-1}P) \geq \nu(T^{-1}P_w)$ can be shown.

The faces of $T^{-1}P$ are $T^{-1}F$, where F is a face of P, and the (face) index lattices are equal ($(P) = (T^{-1}P)$).

Let $T^{-1}F$ be an arbitrary face of $T^{-1}P$, D_T an arbitrary outside face of $\partial f(T^{-1}F,T)$, the subdifferential of $f(y,T)$ at $T^{-1}F$, and an arbitrary $z \in D_T$; Lemma 3.11 indicates that $\mu^*(T^{-1}P) > \nu$ if and only if for all such z one has $\|z\| \geq \nu$.

As z belongs to an outside face, Lemma 3.13 (see proof) implies that $\rho z \notin \partial f(T^{-1}F,T)$ for all $\rho > 1$; also $\|z\| > 0$.

But $z \in D_T \subset \partial f(T^{-1}F,T) \subset N_{T^{-1}P}(T^{-1}F)$, and thus, multiplying by T^{-t}, one has $T^{-t}z \in N_P(F)$ (it is not true that $T^{-1}z \in \partial f(F)$, because the lattice $\mathcal{A}$ is not isomorphic under linear transformations followed by normalization).

Using Lemma 3.13, there exists an outside face D of $\partial f(F)$, with index set $\mathcal{J}(D)$ (clearly $\mathcal{J}(D) \subset I(F)$), such that $T^{-t}z \in \lambda D$ ($\lambda > 0$); by Lemma 3.18, there always exist a face F_w of P_w such that $I(F_w) = \mathcal{J}(D)$ (note also that P_w is nondegenerate), and $T^{-t}z \in \lambda \partial f(F_w) \subset N_{P_w}(F_w)$.

Now, if F_w is a face of P_w, then $T^{-1}F_w$ is a face of $T^{-1}P_w$, where

$$T^{-1}P_w = \{y \in R^n\colon (a^{it}Ty-b_i - \|a^i\|w)/\|T^ta^i\| \leq 0,\ i \in M\}$$

is understood as being given by a renormalized system of linear inequalities.

Hence $z \in N_{T^{-1}P_w}(T^{-1}F_w)$, where

$$I(T^{-1}F_w) = I(F_w) = J(D) \subset I(F) = I(T^{-1}F) \ .$$

Now select any z' $(z' = \lambda' z, \ \lambda' > 0)$ which belongs to the subdifferential of $T^{-1}P_w$ at $T^{-1}F_w$, and thus

$$z' = \sum_{i \in J(D)} \lambda_i T^t a^i / \|T^t a^i\| \ ,$$

with $\sum_{i \in J(D)} \lambda_i = 1$, $\lambda_i \geq 0$, for all $i \in J(D)$; now Lemma 3.11 (see proof) implies that

$$\|z'\| > \nu(C_{T^{-1}P_w}(T^{-1}F_w)) \geq \nu \ .$$

But z' also belongs to $\partial f(T^{-1}F, \bar{F})$, as $J(D) \subset I(F) = I(T^{-1}F)$, and hence $\lambda' \leq 1$; whence $\|z\| = (\lambda')^{-1} \|z'\| \geq \nu$, and the first part of the theorem.

The remainder of the theorem follows from a proof similar to that given in Lemma 7.1. QED

The fact that a linear map T exists which satisfies $\mu^*(T^{-1}P) \geq (\tau(P_w^c))^{-1}$ indicates, somewhat misleadingly, that algorithm 2 may converge faster than linearly only if $\tau(P_w^c) = 1$, which can be true only if P_w is an ellipsoid; what happens is, in fact, one step convergence.

And thus Theorem 7.3 implies the following theorem, which is proved almost exactly as Theorem 7.2.

Theorem 7.4

Theorem 7.2 remains unchanged if Algorithm 2 is applied directly to the, unperturbed, system P; the only difference being that $\log(w_1-w)^{-1}$ need not be polynomial.

In fact $w = w_1$ could be used, always. It should also be clear that the best rate of convergence which Algorithm 2 may achieve, in theory, is $(1 - (\tau(P_w^c))^{-2})^{1/2}$, i.e., it depends upon the affine excentricity of a perturbation cum compactification of P.

If the system P were infeasible, then the solution of Min f(x) (where Min $f(x) = w_1$) identifies the, normalized, Cebysev solution of the infeasible system of linear inequalities; this is really an optimization problem which can be solved by subgradient optimization, which is a technique differs from the relaxation method only by the choice of the step size [32]. A variable metric subgradient optimization method, using, say, the maximum volume ellipsoid included in $P_{w_2}^c$ will also converge in polynomial time (under proper choices of the step size).

8. Conclusions.

It has been shown that variable metric, maximal ellipsoidal distance, relaxation algorithms may solve any system of linear inequalities in polynomial time, which, within the context of such methods, means fast or good (or better, or less bad), ellipsoid matrices which lead to polynomiality may be viewed as inverse Hessians, and are given, for instance, by the maximum volume ellipsoid inscribed in a perturbation cum compactification of the feasible set.

This method should probably be viewed as a conceptual algorithm, but an implementation of it is the ellipsoid method, and in fact it could be hoped that some of the insights gleaned from its study may lead to improvements of the ellipsoid method.

The method may be practical in problems where an educated guess of the matrix H may be accurate enough to be useful, we are thinking about linear programs derived from combinational problems, where a set of potential subgradients can be described a priori (maybe a simplex), and could approximate the set of subgradients at the optima.

Another issue of practical importance is the possibility of introducing a relaxation parameter in algorithms 1 and 2; this does not affect the theory given here, but, in practice, it has always led to significant improvements in the rate of convergence of the method, and we will conjecture that if one used the optimal variable metric, and the optimal relaxation parameter, then the rate of convergence would be of the order of $(1 - n^{-1})^{1/2}$.

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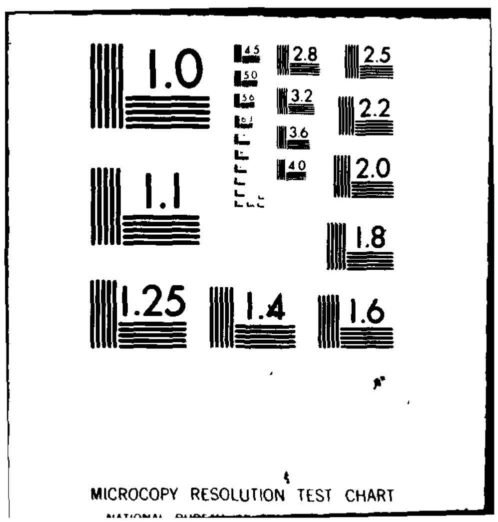
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A somewhat major annoyance is that the theory given here does not extend to other implementations of the relaxation method, like the maximal residual relaxation method.

The main difficulty is that the perturbed sets $\tilde{P}_w$ may behave poorly as w increases, and thus the proofs given here do not seem to extend to that case.

9. Acknowledgments

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A variable metric, maximal ellipsoidal distance, relaxation method for solving systems of linear inequalities is described and its rate of convergence analyzed. This algorithm, with a termination routine appended to it, is polynomial if the ellipsoid matrix approximates an inverse "Hessian," which may be defined through the largest ellipsoid contained in a perturbation cum compactification of the feasible set. This so called inverse Hessian is given by a positive linear combination of symmetric rank one matrices built on the vectors given by the facets of the perturbation cum compactification of the feasible set.

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