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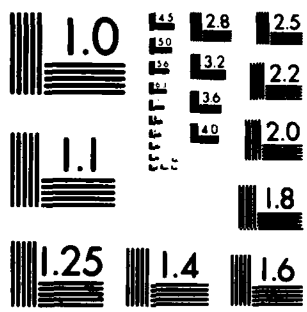

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LEVEL II

A CLOSED QUEUEING NETWORK MODEL
FOR MULTI-ECHELON REPAIRABLE
ITEM PROVISIONING

by

Donald Gross
Douglas R. Miller
Richard M. Soland

Serial T-446
30 June 1981

The George Washington University
School of Engineering and Applied Science
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1. INTRODUCTION

Multi-echelon inventory theory has been of interest over the last two decades, both for the theoretical problems it poses and for its realism in describing operating systems. After a relatively dormant period in the late 1960's and early 1970's, a resurgence of interest occurred. Most of the recent work in multi-echelon systems is keyed to a model called METRIC that was developed at the Rand Corporation for the U.S. Air Force. This model involves finding optimal spares levels at various locations in a two-echelon system, where lateral bases are supported by a single depot. More detail is provided in the sections that follow.

Of interest here is the study of the trade-off possible among spares levels *and* repair capacities, as well as a more realistic model than is presently available of the underlying stochastic process that describes components which randomly fail and require repair.

The general problem to be investigated is the determination of the optimal spares levels and repair capacities in a reparable item multi-echelon system in which a finite number of items is desired to be operational at any given time, and in which queueing may occur at the repair facilities when all channels--finite in number--are busy.

Before presenting the details of the model developed here, we first trace the historical development of inventory theory, with particular emphasis on multi-echelon efforts, and summarize the previous work in repairable item, multi-echelon inventory control.

2. HISTORICAL PERSPECTIVE

Inventory theory is said to have begun with the development by Ford Harris in 1915 of the Economic Order Quantity (EOQ) model [see HARRIS (1915)]. The same model was independently developed by R. H. Wilson at about the same time, and the model is sometimes referred to as the Wilson Lot Size Formula. This simple deterministic model still serves today as one of the cornerstones of applied inventory control.

In the 1950's and 1960's, interest in stochastic inventory control grew after the publication of the landmark paper by ARROW, HARRIS, and MARSCHAK (1951). A great deal of more "sophisticated" mathematical work then appeared, concerned mainly with proving that (s,S) types of control policies are optimal under a wide range of conditions [see, for example, ARROW, KARLIN, and SCARF (1958)]. Most of this work had to do with periodic review policies, that is, policies with the decision rule: "When it is time to review inventory, if the inventory position (on hand plus on order minus backorders) is below s, place an order to bring the inventory position level up to S." Only a few of the studies during that time were concerned with how actually to find optimal values of the three decision variables "s," "S," and "time between reviews."

In 1959, GALLIHER, MORSE, and SIMOND considered continuous review (s,S) policies, whose decision rule is: "Continuously monitor the inventory position. When it falls to a level s, place an order Q which will bring the inventory position level to S $(Q = S-s)$." These are also known as (r,Q) models [see HADLEY and WHITIN (1963)].

While there was interest in multi-echelon inventory models during the late 1950's and early 1960's [see, for example, CLARK (1958, 1960) and CLARK and SCARF (1960, 1962)], it was not until the 1970's that computers were able to handle the difficult task of solving problems of this magnitude. PINKUS (1971) extended the work of Clark and Scarf and designed a truly multi-echelon, multi-product periodic review model for consumable items, and showed that "real" solutions could be obtained.

The classic paper by FEENEY and SHERBROOKE (1966) appeared during this same period, and ultimately became the basis of the most popular multi-echelon reparable item model of today [see SHERBROOKE (1968)]. For reparable item control, a realistic model is a special case of the continuous review (s,S) policy, where $s = S-1$. This is also known as a one-for-one ordering policy, and is sometimes used in consumable item inventory control for items that are expensive, critically important, and infrequently demanded. It is a natural model for reparable item situations in that when an item fails, it is generally dispatched immediately to a repair facility and a spare, if available, is "plugged in." Repairing the item is analogous to ordering a new consumable item from an outside supplier with the repair time playing the same role as the replenishment leadtime.

The METRIC model of Sherbrooke "multi-echelonized" the basic (S-1,S) model of Feeney and Sherbrooke by allowing a certain fraction of the items to be repaired at the base and the remainder to be sent to a repair depot. The decision variables were the levels of spares (the S's) to be stocked in the field, i.e., at each of the bases, and at the depot. MUCKSTADT (1973) generalized METRIC to allow for a hierarchical, or indentured, parts structure; the resulting model was called MODMETRIC.

A key assumption of these METRIC models is commonly known as the *ample service assumption.* This means that repair capacity is infinite, i.e., that there is never any queueing of items waiting for a repair channel. This has the effect of causing successive replenishment leadtimes to be statistically independent and allows the invocation of a powerful theorem from queueing theory--Palm's theorem [see PALM (1938)]. Palm's theorem states that as long as there is ample service (Poisson or compound Poisson infinite calling population failure processes must be assumed as well), it is necessary to know only the mean turn-around time of failed items, and furthermore, that the steady-state probability distribution of the number of units in resupply is Poisson, with parameter equal to the mean number of failures during an average resupply time--in inventory jargon, the mean demand over the leadtime.

Other existing multi-echelon repairable item models based on this ample service assumption are ACCLOGTROM [see FORRY (1979)], SIMPLE SIMON [see KRUSE and KAPLAN (1973)], and TWOPT [see KAPLAN (1980)]. These models differ from each other in their respective "bells and whistles." For example, SIMPLE SIMON allows some old items to be discarded and replaced by purchases. ACCLOGTROM allows for the modeling of reliability networks; that is, components may be arranged in combinations of parallel, series, and k-out-of-n "circuits." Some of the models [METRIC, MODMETRIC, SESAME (see Kaplan, *op. cit.*), ACCLOGTROM] also consider finding optimal values of the decision variables, and their mathematical optimization techniques are somewhat dissimilar.

3. SUMMARY OF PREVIOUS WORK

The previous models that have the most direct bearing on our research are the ACCLOGTROM, METRIC/MODMETRIC, SESAME, SIMPLE SIMON, and TWOPT models. SIMPLE SIMON and TWOPT are stochastic process models only; that is, they give the steady state probabilities of the numbers of units in resupply. METRIC/MODMETRIC and ACCLOGTROM, in addition to modeling the stochastic process, have methodology for finding the optimal spares levels. SESAME basically concentrates on finding optimal spares levels and can use METRIC, TWOPT, ACCLOGTROM, or SIMPLE SIMON for modeling the stochastic process. There are three basic limitations to the preceding models:

- The stochastic process modeling is based on
  - (i) infinite source (calling population), and
  - (ii) ample service (infinite number of repair channels) assumptions.
- Because of the ample service assumption, the only decision variables in these models are spares levels, and thus no

trade-off considerations are possible among levels of spares and repair capacities.

Recent work by GRAVES and KEILSON (1979) considers a single echelon system allowing for a general birth-death stochastic process model, and introduces a new set of performance measures dealing with times for the system to recover after "failing" and times to "failure" when operating satisfactorily. While alluding to system design ramifications, no explicit optimization problem is formulated.

4. PROBLEM STATEMENT

The system we study here consists of a single base (or group of bases) with a single base (or field) repair facility and a single depot repair facility. The problem can be stated mathematically as

$$\underset{y,c_B,c_D}{\text{Minimize}} \quad Z = k_y y + k_B c_B + k_D c_D \tag{1}$$

$$\text{subject to} \quad \sum_{n=M}^{M+y} p_n \geq A , \tag{2}$$

where

p_n = steady-state probability that n units are operational,

M = number of components desired to be operating (operating population size),

A = minimum percentage of time all M components are to be operational (availability),

y = number of spare components to "stock",

c_B = base repair capacity in number of channels,

c_D = depot repair capacity in number of channels.

k_i = cost per unit (i=y,D,B) including annual operating costs and capi al investment amortization of a spare or a repair channel.

The variables y, c_B, and c_D are decision variables to be determined by an optimization algorithm. The steady-state probabilities, p_n, must be determined through a stochastic process analysis. We use closed network queueing theory for the latter and implicit enumeration for the former.

5. STOCHASTIC PROCESS MODEL

The stochastic process can be viewed as a network and is shown schematically in Figure 1.

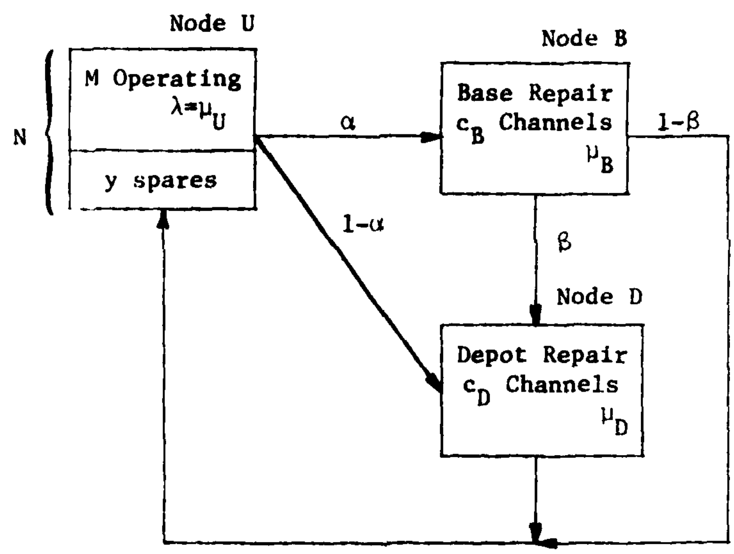

Figure 1.--Network for a two-echelon repairable item system.

The network has three nodes, which we refer to as U ("up" or operating), B (base repair facility), and D (depot repair facility). Additional parameters α and β are shown, where α is the fraction of failed items that are diagnosed as base repairable and sent directly to base repair ($1-\alpha$ is the fraction sent directly to depot repair). Of those that are sent to base repair, a further fraction β , after undergoing service, cannot be fixed and are sent to depot repair.

Holding times at all nodes are assumed to be independent exponentially distributed random variables. At node U, the holding time is the time to failure of a component, with the mean failure rate denoted by μ_U (often in queueing and reliability literature, this is denoted by λ). At nodes B and D, the holding times are repair times and the mean repair rates are denoted by μ_B and μ_D , respectively.

Since all holding times are exponential, we have a special case of a Jacksonian network [see JACKSON (1957, 1963)]; this special case is a closed queueing network where no items ever leave or enter the system, but circulate within the network only. Jackson (*op. cit.*) showed for general queueing networks with exponential holding times (which, because of his work, are often referred to as Jacksonian networks), and GORDON and NEWELL (1967) showed for the closed network case that the joint probability distribution of the number of customers at each node of the network is of product form. For closed networks, using the notation of BUZEN (1973), this means that for a k node network with a total of N units,

$$p(n_1,n_2,\ldots,n_k) = \frac{1}{G(N)} \prod_{i=1}^{k} \left[(x_i)^{n_i} / A_i(n_i)\right] , \tag{3}$$

where $p(n_1,n_2,\ldots,n_k)$ is the joint steady state probability that n_1 components are at node 1, n_2 at node 2, ..., n_k at node k . The x_i are the real positive solutions to the system of equations

$$\mu_j x_j = \sum_{i=1}^{k} \mu_i x_i p_{ij} , \quad j=1,2,\ldots,k , \tag{4}$$

and

$$A_i(n) = \begin{cases} n! & , \ n < c_i \\ c_i! c_i^{n-c_i} & , \ n \geq c_i \end{cases} \tag{5}$$

c_i = number of parallel channels at node i,

p_{ij} = Pr{unit goes to node j | service completed at node i} .

The x_i in equation (3) play the role of λ/μ in a standard M/M/c model, and thus the nodes *act as* independent M/M/c queues with $1/G(N)$ the normalizing constant (taking the place of the p_0's).

For our system, there are only three nodes $(i = U,B,D)$ and $c_U = M$, the desired number of components operating. The queue at this node represents the level of spares inventory. When servers at node U are idle, a spares backorder situation is in effect and the population is at a degraded strength (fewer than M components operating).

The matrix giving the p_{ij}'s for our problem is

$$P = \{p_{ij}\} = \begin{array}{c} \\ U \\ B \\ D \end{array} \begin{array}{c} \begin{array}{ccc} U & B & D \end{array} \\ \begin{pmatrix} 0 & \alpha & 1-\alpha \\ 1-\beta & 0 & \beta \\ 1 & 0 & 0 \end{pmatrix} \end{array} .$$

Using these p_{ij}'s in equation (4) yields

$$\begin{aligned} \mu_U x_U &= \mu_B x_B (1-\beta) + \mu_D x_D \\ \mu_B x_B &= \mu_U x_U \alpha \\ \mu_D x_D &= \mu_U x_U (1-\alpha) + \mu_B x_B \beta . \end{aligned} \tag{6}$$

The solution to the system of equations (6) is, arbitrarily setting $x_B = 1$ since one equation of the set (4) is always redundant,

$$x_U = \frac{\mu_B}{\mu_U \alpha}, \quad x_B = 1, \quad x_D = \frac{(1-\alpha+\alpha\beta)\mu_B}{\alpha\mu_D}.$$

Thus,

$$p(n_U,n_B,n_D) = \frac{1}{G(N)}\left(\frac{\mu_B}{\alpha\mu_U}\right)^{n_U} \frac{1}{A_U(n_U)} \cdot \frac{1}{A_B(n_B)} \cdot \left[\frac{(1-\alpha+\alpha\beta)\mu_F}{\alpha\mu_D}\right]^{n_D} \cdot \frac{1}{A_D(n_D)},$$

where the A's are given by equation (5), namely,

$$A(n) = \begin{cases} n! & , \ n < b \\ n!b^{n-b} & , \ n \geq b \end{cases},$$

where $b = M$, c_B, and c_D, respectively. Buzen's algorithm is used to calculate the constant $G(N)$; that is, the value of $G(N)$ so that

$$\sum_S p(n_U,n_B,n_D) = 1,$$

the sum being taken over the set S, which contains all triplets (n_U,n_B,n_D) such that $n_U + n_B + n_D = N$. Once the joint probabilities $p(n_U,n_B,n_D)$ are obtained, we can calculate the marginal probabilities p_{n_U}, which are required for the constraint(s), by

$$p_{n_U} = \sum_{S^1} p(n_U,n_B,n_D),$$

where S^1 now is the set of all pairs (n_B,n_D) such that $n_B + n_D = N-n_U$. This can be done efficiently using Buzen's algorithm; in fact, it results as a by-product when calculating $G(N)$.

The above probability distribution is a function of the decision variables y, c_B, and c_D. The distribution exhibits certain monotonicity properties in relation to these variables; this will play a crucial role in the optimization part of this study. Thus, before considering optimization, we verify monotonicity.

Let N_u represent the number of "up" machines for a steady-state network; the distribution of N_u depends on (y,c_B,c_D) and thus we represent the random variable as $N_u(y,c_B,c_D)$. We put the usual partial order on the decision space: $(y,c_B,c_D) \leq (y',c_B',c_D')$ if and only if $y < y'$, $c_B \leq c_B'$, and $c_D \leq c_D'$. Now consider the idea of stochastic ordering of random variables: $N \overset{st}{<} N'$ if and only if $P(N \geq n) < P(N' \geq n)$ for all n . We can now state the basic monotonicity property for the steady-state behavior of the system.

<u>Theorem</u>. If $(y,c_B,c_D) \leq (y',c_B',c_D')$, then $N_u(y,c_B,c_D) \overset{st}{\leq} N_u(y',c_B',c_D')$.

The transitivity of the inequalities implies that only three cases must be considered in proving this theorem: (i) $(y,c_B,c_D) \leq (y+1,c_B,c_D)$; (ii) $(y,c_B,c_D) \leq (y,c_B+1,c_D)$; and (iii) $(y,c_B,c_D) \leq (y,c_B,c_D+1)$.

First consider case (i): We must show that the steady-state number of "up" units increases stochastically when the total number of units is increased by one and the number of repair channels is unchanged. This is easily seen to be true by modelling the system with M+y+1 units as a preemptive priority network with M+y high priority customers and 1 low priority customer. The distribution of the total number of customers at each node will be identical to the nonpriority system with M+y+1 customers. The distribution of the number of high priority customers at each node will be identical to the nonpriority system with M+y customers. The one low priority customer will spend part of its time at node "U"; thus, $N_u(y+1,c_B,c_D) \overset{st}{=} N_u(y,c_B,c_D) + L_u$, where L_u equals the number of low priority customers at node "U" in steady state. Since $L_u \geq 0$ it follows that $N_u(y+1,c_B,c_D) \overset{st}{\geq} N_u(y,c_B,c_D)$.

Now consider case (ii): We must show that the steady-state number of "up" units increases stochastically when the number of base repair channels is increased by one and all other parameters are held fixed. This can be demonstrated by considering the form of the joint distribution of the number of items at each node, equation (3). Note that this is the conditional distribution of three independent random variables given that their sum equals the total number of items in the system. Let us denote these independent random variables as $Z_u(M)$, $Z_B(c_B)$ and $Z_D(c_D)$. Then, for example,

$$P(Z_B(c_B) = n) = \begin{cases} p_{OB} \dfrac{x_B^n}{n!} & n \leq c_B \\ p_{OB} \dfrac{x_B^n}{c_B! c_B^{n-c_B}} & c_B \leq n \leq M+y \end{cases} , \tag{7}$$

where p_{OB} is the appropriate normalization constant. By the product form, equation (3), it follows that

$$N_u(y, c_B, c_D) \overset{st}{=} Z_u(M) \mid Z_u(M) + Z_B(c_B) + Z_D(c_D) = M+y . \tag{8}$$

It can be shown by a straightforward algebraic analysis using equation (7) (see Appendix) that

$$Z_B(c_B + 1) \overset{st}{\leq} Z_B(c_B) .$$

This fact and equation (8) then imply that

$$N_u(y, c_B+1, c_D) \overset{st}{\geq} N_u(y, c_B, c_D) ,$$

the desired conclusion.

Case (iii) can be verified in exactly the same way as case (ii), completing the proof. The statement of the theorem gives the desired monotonicity property of the constraint function in equation (2). We can now proceed to the optimization.

6. OPTIMIZATION PROCEDURE

The optimization aspect of this study is a formidable one because an expression for the availability as a function of the decision variables (the spares level and repair capacities) does not exist in closed algebraic form. That is, the p_n's that appear in the constraint (2) are determined from the stochastic process model and can only be calculated numerically when the values of y, c_B, and c_D are specified.

The difficulty just indicated, and the fact that integer values are required for the decision variables, suggest the use of an implicit enumeration scheme for the optimization algorithm. One such scheme that has already been used when closed algebraic expressions were not available [see SOLAND (1973)] is that of LAWLER and BELL (1966). However, it requires that the objective and constraint functions each be expressible as the difference of two monotonic functions of the decision variables. Thus, use of this optimization scheme interacts with the stochastic process analysis in that the latter is charged with providing the required monotonicity properties of the model. We have shown in the stochastic process analysis that the monotonicity conditions hold (the higher y, c_B, c_D , the greater the availability). The cost is linear and therefore monotone. Thus we can use the Lawler-Bell (L-B) algorithm. Use of any other optimization algorithm would most likely place similar, or even more stringent, demands (e.g., convexity) on the stochastic process analysis.

An implicit enumeration approach also allows us to consider a much wider class of decision problems than has heretofore been treated in connection with multi-echelon inventory models. The optimization algorithms used in METRIC, MODMETRIC, and SESAME are each tailored to the specific form of the problem treated, i.e., a single specific constraint, either on service level or on budget, and are not easily generalized to other formulations. Through use of an implicit enumeration approach, however, we can treat a variety of objective and constraint functions and allow the use of multiple constraints. For example, we can impose the additional constraint of a lower limit on the average number of operating units:

$$\sum_{n_U=1}^{M-1} n_U p_{n_U} + M \sum_{n_U=M}^{M+y} p_{n_U} \geq B ,$$

or a constraint on the availability of a certain fraction of the population:

$$\sum_{n_U=.9M}^{M+y} p_{n_U} \geq A' .$$

In applying the algorithm it is necessary to have upper bounds for the decision variables. Certainly an upper bound for both c_B and c_D is $M+y$ (ample server case). To get an upper bound for y would require some knowledge of the particular application, for example, a budget limit or a manufacturing or supply limit.

The algorithm is based on representing the values of the decision variables in a single binary vector (a vector whose elements are either zero or one). Suppose we had a population of ten items and knew from budget considerations we could afford at most five spares. Then an upper bound of fifteen would be adequate and four bits (binary variables) would be adequate for describing each variable. Thus the L-B algorithm would work with a twelve bit vector, which might be arranged as follows:

$$(\underbrace{\_\ \_\ \_\ \_}_{y} \mid \underbrace{\_\ \_\ \_\ \_}_{c_B} \mid \underbrace{\_\ \_\ \_\ \_}_{c_D}) .$$

Hence in this case the vector $(0010 \mid 0010 \mid 0011)$, which has value $2^9 + 2^5 + 2^1 + 2^0 = 547$, represents the solution $y = 2$, $c_B = 2$, $c_D = 3$. The algorithm uses the binary vector whose value is 547 in determining which portions of the solution space to eliminate. For example, in the problem represented by (1) and (2), if the preceding vector cannot satisfy the constraints, no vector with value less than it can either, and hence all solutions represented by vectors of value less than 547 are eliminated from consideration.

It is not necessary to partition the vector into groups representing each decision variable; y, c_B, and c_D bits can be intermixed. For example, we could use the ordering

$$(y_4, c_{B4}, c_{D4}, y_3, c_{B3}, c_{D3}, y_2, c_{B2}, c_{D2}, y_1, c_{B1}, c_{D1})$$

where y_i, c_{Bi}, and c_{Di} represent the ith bit of y, c_B, and c_D , respectively. If this ordering were used, the vector with value 547 shown above would represent the solution $y = 0010 = 2$, $c_B = 0001 = 1$, $c_D = 1001 = 9$. Which ordering is most efficient to use depends on the problem and can only be determined with some experimentation. The reader is referred to Lawler and Bell (*op. cit.*) for a detailed description of the algorithm and further discussion.

7. SAMPLE RESULTS

The following problem was run on an IBM 4341 using Buzen's algorithm to calculate the normalizing constant needed to yield the joint probabilities $p(n_U, n_B, n_D)$, and using the L-B algorithm to find the optimal solution:

$$\text{Minimize} \quad Z = 20y + 8c_B + 10c_D$$

$$\text{subject to} \quad \sum_{n_U=M}^{M+y} p_{n_U} \geq .9 \qquad (A_1)$$

$$\sum_{n_U=.9M}^{M+y} p_{n_U} \geq .98 \qquad (A_2)$$

The parameters were set as follows:

$$\alpha = 0.5, \quad \beta = 0.5, \quad \mu_U = 1, \quad \mu_B = \mu_D = 5 .$$

The upper bound used on all variables was $2M$ and cases with $M = 5$, 10, 20, and 30 were solved. Both constraints were used, except for the case of $M = 30$, where only (A_1) was imposed. The results are given in Table 1. Four different orderings were used and the results for the best two are shown in Table 1, with ordering #1 being $(\ldots, y_i, c_{Bi}, c_{Di}, y_{i-1}, c_{Bi-1}, c_{Di-1}, \ldots)$ and ordering #2 being $(\ldots, y_i, y_{i-1}, \ldots, c_{Bi}, c_{Bi-1}, \ldots, c_{Di}, c_{Di-1}, \ldots)$. Given in the table are both the CPU running times in seconds and the number of times Buzen's algorithm was required (number of times the probabilities had to be calculated).

Both the L-B and Buzen algorithms appear to be quite efficient. The most demanding of the problems run was the case $M = 30$, both because it has the largest population size and because it imposes only one constraint, causing the L-B algorithm to enumerate more solutions than if both constraints had been invoked. Even so, the problem took only slightly over ten seconds to solve.

TABLE 1

SAMPLE RESULTS

M	c_B^*	c_D^*	y^*	Z^*	A_1	A_2	Ordering 1 CPU	Ordering 1 #	Ordering 2 CPU	Ordering 2 #
5	2	2	3	96	.938	.982	1.45	41	0.83	25
10	3	3	5	154	.926	.988	2.97	64	1.35	38
20	4	6	8	252	.907	.989	--	--	2.55	66
30	6	8	11	348	.904	--	--	--	10.11	137

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APPENDIX

PROOF THAT $Z_B(c_B + 1) \overset{st}{<} Z_B(c_B)$

Even though it is intuitively obvious that if one increases the number of servers, congestion decreases, so that it is certainly logical that $Z_B(c_B + 1)$ is stochastically smaller than $Z_B(c_B)$, it is not trivial to prove. We proceed as follows, using Equation (7).

Consider the ratio $P\{Z_B(c_B) = n\} / P\{Z_B(c_B) = n - 1\}$, which we shall call $R(c_B,n)$. From Equation (7),

$$R(c_B,n) = \begin{cases} \dfrac{x_B}{n} & 1 \leq n \leq c_B \\ \dfrac{x_B}{c_B} & c_B \leq n \leq M + y \end{cases} . \tag{A1}$$

It is clear that

$$R(c_B,n) \geq R(c_B + 1, n) \qquad n = 1,2,\ldots,M+y . \tag{A2}$$

This also implies that

$$\frac{P\{Z_B(c_B) = i\}}{P\{Z_B(c_B) = j\}} \leq \frac{P\{Z_B(c_B + 1) = i\}}{P\{Z_B(c_B + 1) = j\}} \qquad 0 \leq i \leq j \leq M+y \tag{A3}$$

since $\Pi_{n=i+1}^{j} R(c_B,n) \geq \Pi_{n=i+1}^{j} R(c_B + 1, n)$, and hence $1 / \Pi_{n=i+1}^{j} R(c_B,n) \leq 1 / \Pi_{n=i+1}^{j} R(c_B + 1, n)$.

From (A3) we can easily obtain

$$\frac{\sum_{i=0}^{j} P\{Z_B(c_B) = i\}}{P\{Z_B(c_B) = j\}} \leq \frac{\sum_{i=0}^{j} P\{Z_B(c_B + 1) = i\}}{P\{Z_B(c_B + 1) = j\}}$$

which implies, when taking reciprocals,

$$P\{Z_B(c_B) = j \mid Z_B(c_B) \leq j\} \geq P\{Z_B(c_B + 1) = j \mid Z_B(c_B + 1) \leq j\} ,$$

which in turn implies, by subtracting both sides from one,

$$P\{Z_B(c_B) \leq j-1 \mid Z_B(c_B) \leq j\} \leq P\{Z_B(c_B+1) \leq j-1 \mid Z_B(c_B+1) \leq j\} ,$$
$$1 \leq j \leq M+y .$$

Now

$$\begin{aligned} P\{Z_B(c_B) \leq j-1\} &= P\{Z_B(c_B) \leq j-1 \mid Z_B(c_B) \leq M+y\} \\ &= \prod_{i=j}^{M+y} P\{Z_B(c_B) \leq i-1 \mid Z_B(c_B) \leq i\} \\ &\leq \prod_{i=j}^{M+y} P\{Z_B(c_B+1) \leq i-1 \mid Z_B(c_B+1) \leq i\} \\ &= P\{Z_B(c_B+1) \leq j-1\} . \end{aligned}$$

Hence,

$$P\{Z_B(c_B) \geq j\} \geq P\{Z_B(c_B+1) \geq j\} . \qquad \text{Q.E.D.}$$

Ordering the ratios as in (A2) actually is a sufficient condition in general for stochastic ordering [see WHITT (1980)].

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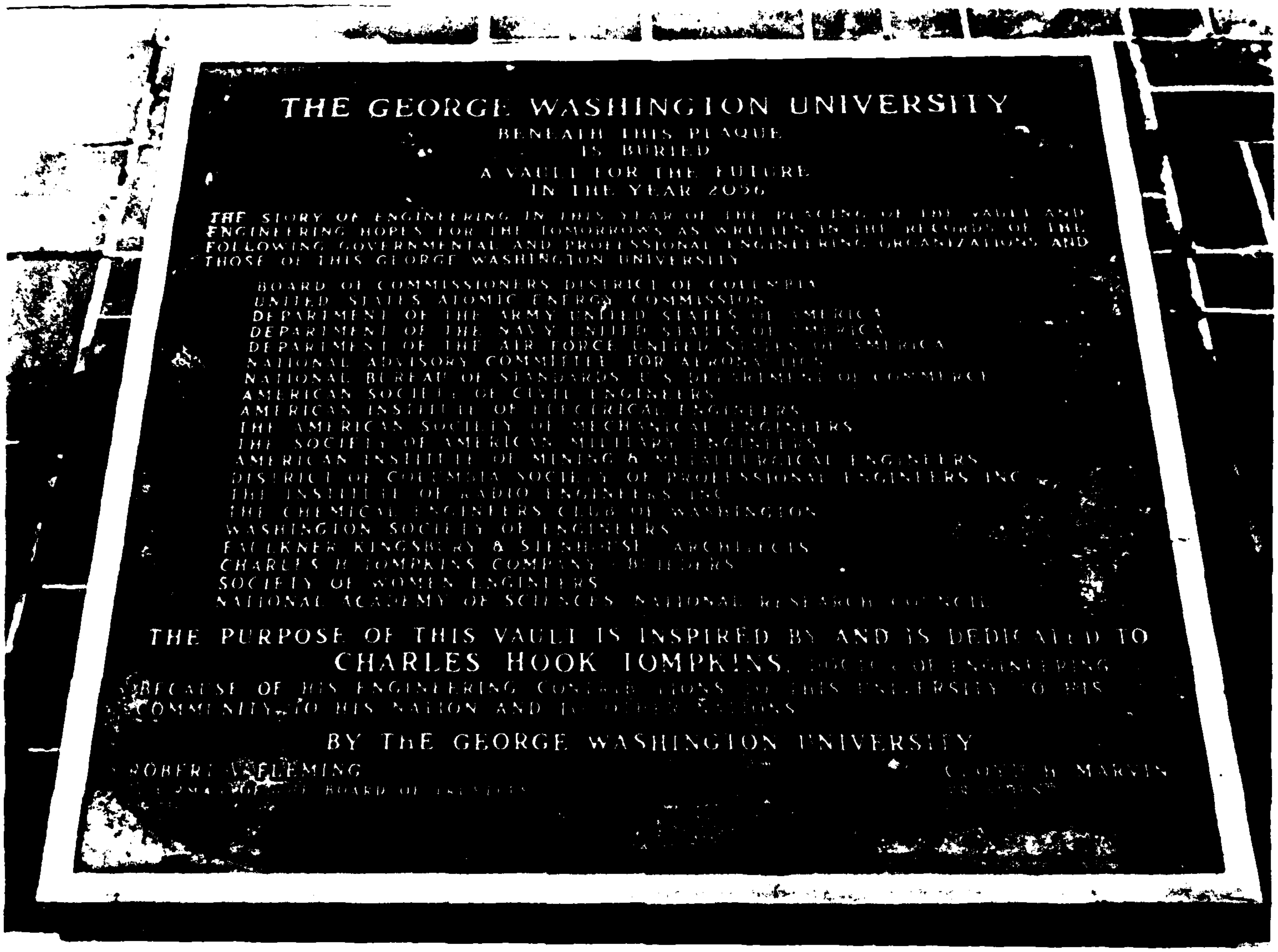

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