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ASYMPTOTIC NORMALITY OF AUTOREGRESSIVE PARAMETER ESTIMATES FOR MIXED TIME SERIES

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It is shown for mixed time series, a series generated by an autoregressive moving-average (ARMA) process or by an autoregressive process observed in additive white noise (AR+N), that estimates of the autoregressive (AR) parameters are asymptotically multivariate jointly normal with zero mean and finite covariance matrix. The structure of the asymptotic covariance matrix is evaluated for both types of mixed time series.

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I. INTRODUCTION

The problem addressed herein is the evaluation of asymptotic statistics for estimates of autoregressive (AR) parameters from observations of mixed time series of known order. The observed time series is considered to be mixed if the series is generated by an autoregressive moving-average (ARMA) process or by an autoregressive process observed in additive white noise (AR + N). It has been shown by Walker [1] and Pagano [2] that a time series formed by the addition of an autoregressive process of order (M) and white noise should be modeled as an ARMA process of order (M,M). In both cases the estimation of the AR parameters and the evaluation of estimate statistics are complicated by the mixed nature of the observed time series.

For an AR process of known order Mann and Wald [3] showed that the least-squares estimate of the process parameters coincides with their maximum likelihood estimate. It was also proven in [3] that the least-squares parameter estimate errors are jointly multivariate normal with zero mean and finite covariance matrix. The structure of the asymptotic covariance matrix was also evaluated. Walker [1] was the first to examine estimating the AR parameters for an AR + N process. In [1] the asymptotic efficiency and variance were evaluated for a first order AR process observed in the presence of additive noise. For the mixed ARMA process, Gersch [4] proved that asymptotically unbiased estimates of the AR parameters can be obtained by using the "higher order" Yule-Walker equations. The structure of the asymptotic covariance matrix was obtained, but asymptotic normality was not proven.

In this report, it is shown that the "higher order" Yule-Walker estimates of the AR parameters for both types of mixed processes are asymptotically jointly multivariate normal. The structure of the asymptotic covariance matrix is calculated for both types of process. The proof follows that of Kromer [5], in which a similar asymptotic normality result was proven for an AR process without noise.

These results extend the work of Gersch [4] in that the estimates are shown to be asymptotically jointly multivariate normal. The structure of the asymptotic covariance matrix, obtained herein, is shown to be equivalent to that obtained by Gersch. The demonstration of asymptotic normality for the AR parameters for an AR + N process is a new result. The evaluation of the structure of the asymptotic covariance matrix for the AR + N case extends the result Walker [1] obtained for a first order AR process.

---

1. AM Walker, Some consequences of superimposed error in time series analysis, Biometrika, vol 47, p 33–43, 1960
2. M Pagano, Estimation of models of autoregressive signal plus white noise, Ann Statist, vol 2, no 1, p 99-108, 1974
3. HB Mann and A Wald, On the statistical treatment of linear stochastic difference equations, Econometrica 11, p 173-200, 1943
4. W Gersch, Estimation of the autoregressive parameters of a mixed autoregressive moving average time series, IEEE Trans Automat Contr, vol AC-15, p 593-598, 1970
5. R Kromer, Asymptotic properties of the autoregressive spectral estimator, PhD Thesis, Dept of Statist, Stanford, CA, 1969

II. PARAMETER ESTIMATOR FOR MIXED SERIES

Assume that the observed discrete parameter time series $\{Y_t\}$ is a real, zero-mean, stationary, normal process. The series $\{Y_t\}$ is generated by a mixed autoregressive moving-average process of known order (M,M)

$$Y_t - a_1 Y_{t-1} - \cdots - a_M Y_{t-M} = \eta_t - b_1 \eta_{t-1} - \cdots - b_M \eta_{t-M} \tag{1}$$

where the sequence $\{\eta_t\}$ is assumed to be independent, identically distributed (i.i.d.) $N(0, \sigma_\eta^2)$. We evaluate the autocovariance function for this process by multiplying (1) through by Y_{t-k}, and, taking expectations term by term, we get

$$E[Y_{t-k} Y_t] - a_1 E[Y_{t-k}, Y_{t-1}] - \cdots - a_M E[Y_{t-k} Y_{t-M}]$$

$$= E[Y_{t-k} \eta_t] - b_1 E[Y_{t-k} \eta_{t-1}] - \cdots - b_M E[Y_{t-k} \eta_{t-M}] \tag{2}$$

Since Y_{t-k} depends only on inputs η_{t-j} for $t-j < t-k$, it follows that

$$E[Y_{t-k} \eta_{t-j}] = \begin{cases} 0 & k > j \\ R_{Y\eta} & k \leq j \end{cases}$$

From (2) we see that the range of j is 0 to M. Defining $R_y(k) = E[Y_t Y_{t-k}]$, we can write (2) as

$$R_Y(k) - a_1 R_Y(k-1) - \cdots - a_M R_Y(k-M) = 0 \qquad K = M+1, M+2, \ldots, 2M \tag{3}$$

Define the (M×M) matrix $\underline{\Gamma}_H$ by

$$\underline{\Gamma}_H = \begin{bmatrix} R_Y(M) \; R_Y(M-1) & \ldots R_Y(1) \\ R_Y(M+1) & \ldots R_Y(2) \\ & \\ R_Y(2M-1) & \ldots R_Y(M) \end{bmatrix}$$

and the (M×1) vectors

$$\underline{A}^T = [a_1, a_2, \ldots, a_M]$$

$$\underline{R}_H^T = [R_Y(M+1), R_Y(M+2), \ldots, R_Y(2M)] \; .$$

The relationships defined by (3) can be written as

$$\underline{\Gamma}_H \underline{A} = \underline{R}_H \tag{4}$$

These relationships are usually referred to as the "higher order" Yule-Walker equations.

Define the covariance estimator by

$$\hat{R}_Y(k) = \frac{1}{N} \sum_{t=1}^{N-|k|} Y_t \, Y_{t+|k|} \, .$$

If the true covariances in $\underline{\Gamma}_H$ and $\underline{R}_H$ are replaced with the corresponding covariance estimates, the resulting matrix and vector are denoted by $\hat{\underline{\Gamma}}_H$ and $\hat{\underline{R}}_H$ and the solution of

$$\hat{\underline{\Gamma}}_H \, \hat{\underline{A}} = \hat{\underline{R}}_H \tag{5}$$

provides consistent, but not efficient, estimates of the autoregressive parameters for a mixed autoregressive moving-average time series [6].

III. ASYMPTOTIC STATISTICAL PROPERTIES

In this section we prove that if the "higher order" Y-W equations are used to estimate the AR parameters of a mixed time series, the estimate errors are asymptotically jointly multivariable normal.

Define the following vectors:

$$\underline{R}_{2M}^T = [R_Y(1), R_Y(2), \ldots, R_Y(2M)]$$

$$\underline{G}_{2M}^T = [e^{i\lambda}, e^{i2\lambda}, \ldots e^{i2M\lambda}]$$

$$\underline{G}_{2M}^* = [e^{-i\lambda}, e^{-i2\lambda}, \ldots e^{-i2M\lambda}]$$

$$\underline{U}_{2M}^- = \underline{G}_{2M} \, \underline{G}_{2M}^*$$

$$\underline{U}_{2M}^+ = \underline{G}_{2M} \, \underline{G}_{2M}^T$$

$$\underline{U}_{2M} = \underline{U}_{2M}^- + \underline{U}_{2M}^+$$

$$\underline{\Phi}_M^T = [0, 0, \ldots, 0]$$

Assumption I: $\{Y_t\}$ is a normal, real, zero-mean stationary time series and

$$\sum_{k=-\infty}^{\infty} |k \, R_Y(k)| < \infty$$

6. E Parzen, Efficient estimation of stationary time series mixed schemes, Stanford U, Tech Report 16 on contract Nonr-225(80), 1971

Theorem 1: Let $\{Y_t\}$ satisfy Assumption I, then

$$N^{1/2}\left|\hat{R}_Y(1) - R_Y(1)\right|, \ldots, N^{1/2}\left|\hat{R}_Y(2M) - R_Y(2M)\right|$$

are asymptotically jointly multivariate normal with zero mean and covariance structure given by

$$\lim_{N\to\infty} \operatorname{cov}\left|N^{1/2}(\hat{R}_Y(k) - R_Y(k)), N^{1/2}(\hat{R}_Y(j) - R_Y(j))\right|$$

$$= 2\pi \int_{-\pi}^{\pi} \left|e^{i(k+j)\lambda} + e^{i(k-j)\lambda} \quad \phi_Y^2(\lambda)\, d\lambda\right| \quad (6)$$

The asymptotic covariance matrix structure is given by

$$\underline{\Psi} = 2\pi \int_{-\pi}^{\pi} \underline{U}_{2M}\, \phi_Y^2(\lambda)\, d\lambda\,. \quad (7)$$

<u>Proof</u>: This result follows directly from Theorem 5.2 of Brillinger [7].

<u>Lemma 1</u>: There exists a (M×2M) matrix $\underline{D}$ such that

$$\underline{D}\,(\hat{\underline{R}}_{2M} - \underline{R}_{2M}) = \hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H\, \underline{A} \quad (8)$$

where

$$\underline{D} = \begin{bmatrix} -a_M & \cdots & -a_1 & 1\; 0 & \cdots & 0 \\ 0\; -a_M & \cdots & & -a_1\; 1\; 0 \cdots & 0 \\ 0 & \cdots & 0 & -a_M \cdots -a_1 & 1 \end{bmatrix}$$

<u>Proof</u>: This result follows directly from the definition of the matrix $\underline{D}$ and the Y-W relationship

$$\underline{R}_H - \underline{\Gamma}_H\, \underline{A} = \underline{\Phi}.$$

■

<u>Lemma 2</u>: Let $\{Y_t\}$ satisfy Assumption I and assume $\{Y_t\}$ is an ARMA (M,M) series. The elements of the vector

$$N^{1/2}\,(\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H\, \underline{A})$$

7. DR Brillinger, Asymptotic properties of spectral estimates of second order, <u>Biometrica</u>, 56, 2, p 375-390, 1969

are asymptotically jointly multivariate normal with zero mean and covariance matrix structure

$$\lim_{N\to\infty} \operatorname{cov}\left| N^{1/2}(\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H \underline{A}), N^{1/2}(\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H \underline{A})^T \right| = \underline{D}\,\underline{\Psi}\,\underline{D}^T$$

$$= 2\pi \int_{-\pi}^{\pi} \underline{D}\,\underline{U}_{2M}\,\underline{D}^T\,\phi_Y^2(\lambda)\,d\lambda \tag{9}$$

Proof: From Lemma 1 we have that

$$(\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H \underline{A}) = \underline{D}\,(\hat{\underline{R}}_{2M} - \underline{R}_{2M})$$

Since $\underline{D}$ is a matrix of constants, we can conclude from Theorem 1 that the elements of the vector

$$N^{1/2}(\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H \underline{A})$$

are asymptotically jointly multivariate normal with zero mean and covariance matrix structure given by equation (9) ■

Lemma 3: Assume that $\{Y_t\}$ is an ARMA (M,M) series, then we have that

$$\hat{\underline{\Gamma}}_H^{-1} \xrightarrow[N\to\infty]{P} \underline{\Gamma}_H^{-1}$$

where the notation $\xrightarrow[N\to\infty]{P}$ indicates convergence as $N\to\infty$ is "in probability."

Proof: Given the previous definition of the (M×M) matrix $\underline{\Gamma}_H$ we have

$$\hat{\underline{\Gamma}}_H = \begin{bmatrix} \hat{R}_Y(M) & \hat{R}_Y(M-1) & \cdots & \hat{R}_Y(1) \\ \hat{R}_Y(M+1) & & \cdots & \hat{R}_Y(2) \\ & & & \\ \hat{R}_Y(2M) & & \cdots & \hat{R}_Y(M) \end{bmatrix}$$

thus, by Theorem 8B of Parzen [8] we can conclude

$$\hat{\underline{\Gamma}}_H \xrightarrow[N\to\infty]{P} \underline{\Gamma}_H\,.$$

8. E Parzen, An approach to time series analysis, Ann Math Stat, vol 32, p 951-989, 1961

Gersch [4] proves that the matrix Γ_H is nonsingular; thus, $\underline{\Gamma}_H^{-1}$ exists. Since the matrix inverse is a continuous function (from R^{M^2} to R^{M^2}) on a neighborhood containing $\underline{\Gamma}_H$, it follows that

$$\hat{\underline{\Gamma}}_H^{-1} \xrightarrow[N\to\infty]{P} \underline{\Gamma}_H^{-1} \tag{10}$$

■

We now state and prove the main result of this report. In the course of the following development we will use the notation

$$\mathrm{Cov}\left| N^{1/2}\,\hat{\underline{Q}},\, N^{1/2}\,\hat{\underline{Q}}^T \right| \xrightarrow[N\to\infty]{\mathcal{D}} \underline{\Theta}\ .$$

This expression should be interpreted as implying that the vector of random variables $N^{1/2}\,\hat{\underline{Q}}$ converses in probability, as $N\to\infty$, to a vector of random variables that are jointly multivariate normally distributed with zero mean and covariance matrix $\underline{\Theta}$.

Theorem 2: Let $\{Y_t\}$ satisfy Assumption I and assume $\{Y_t\}$ is an ARMA (M,M) series, then the AR parameter estimate errors

$$N^{1/2}(\hat{a}_1 - a_1),\ N^{1/2}(\hat{a}_2 - a_2), \ldots, N^{1/2}(\hat{a}_M - a_M)$$

are asymptotically jointly multivariate normal with zero mean and covariance matrix structure given by

$$\mathrm{cov}\left| N^{1/2}(\hat{\underline{A}} - \underline{A}),\ N^{1/2}(\hat{\underline{A}} - \underline{A})^T \right| \xrightarrow[N\to\infty]{\mathcal{D}}$$

$$2\pi \int_{-\pi}^{\pi} \underline{\Gamma}_H^{-1}\ \underline{D}\,\underline{U}_{2M}\,\underline{D}^T(\underline{\Gamma}_H^{-1})^T\ \phi_Y^2(\lambda)\, d\lambda \tag{11}$$

<u>Proof</u>: Define the (M×1) vector $\underline{Z}$ by

$$\underline{Z} = \begin{bmatrix} Z_{1,N} \\ Z_{2,N} \\ \vdots \\ Z_{M,N} \end{bmatrix} \triangleq N^{1/2}\,\hat{\underline{\Gamma}}_H^{-1}(\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H\,\underline{A}) - N^{1/2}\,\underline{\Gamma}_H^{-1}(\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H\,\underline{A})\ .$$

By the results of Lemma 2 and Lemma 3 we can conclude that

$$\underline{Z} = N^{1/2}(\hat{\underline{\Gamma}}_H^{-1} - \underline{\Gamma}_H^{-1})\ (\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H\,\underline{A}) \xrightarrow[N\to\infty]{P} \underline{\Phi}_M \tag{12}$$

Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space. For arbitrary $\epsilon > 0$ and $N > M$ define

$$\Lambda_{\epsilon,N} = \left| \omega \in \Omega: \ |Z_{i,N}| < \epsilon \ i = 1, 2, \ldots, M \right|$$

By (12) we have that for every $\eta \epsilon [0, 1]$ there exists a $N^*_{\epsilon,\eta}$ such that

$$P(\Lambda_{\epsilon,N}) \geq 1 - \eta \qquad N \geq N^*_{\epsilon,\eta} \quad .$$

But since $|Z_{i,N}| < \epsilon$, for $i = 1, 2, \ldots, M$, this implies the existence of the vector $\hat{\underline{\Gamma}}_H^{-1} (\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H \underline{A})$ for all $\omega \in \Lambda_{\epsilon,N}$. This result and (5) imply that we can write

$$(\hat{\underline{A}} - \underline{A}) = \hat{\underline{\Gamma}}_H^{-1} (\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H \underline{A})$$

for all $\omega \in \Lambda_{\epsilon,N}$. Substituting this result into the definition of the vector $\underline{Z}$ we get

$$\underline{Z} = N^{1/2} (\hat{\underline{A}} - \underline{A}) - N^{1/2} \underline{\Gamma}_H^{-1} (\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H \underline{A})$$

for all $\omega \in \Lambda_{\epsilon,N}$. Since the selection of ϵ and η is arbitrary we can conclude that

$$N^{1/2} (\hat{\underline{A}} - \underline{A}) - N^{1/2} \underline{\Gamma}_H^{-1} (\hat{\underline{R}}_H - \hat{\underline{\Gamma}}_H \underline{A}) \xrightarrow[N \to \infty]{P} \underline{\Phi}_M \tag{13}$$

Therefore, applying Lemma 2, the stated result follows.

■

We have proven that, when the higher order Yule-Walker equations are used to estimate the AR parameters of a mixed autoregressive moving-average process, the parameter estimation errors are asymptotically jointly multivariate normal with mean zero and covariance matrix structure given by

$$\text{cov} \left| N^{1/2} (\hat{\underline{A}} - \underline{A}), \ N^{1/2} (\hat{\underline{A}} - \underline{A})^T \right| \xrightarrow[N \to \infty]{\mathscr{D}}$$

$$2\pi \int_{-\pi}^{\pi} \underline{\Gamma}_H^{-1} \ \underline{D} \ \underline{U}_{2M} \ \underline{D}^T (\underline{\Gamma}_H^{-1})^T \ \phi_Y^2(\lambda) \ d\lambda$$

Let us now examine the form of a typical element of this covariance matrix, but first look at a typical element of $\underline{D} \ \underline{U}_{2M} \ \underline{D}^T$, call it $\xi_{k,j}$. We have

$$\xi_{k,j} = \sum_{n=1}^{2M} \sum_{m=1}^{2M} d_{kn} U_{nm} d_{jm}$$

$$= \sum_{n=1}^{2M} \sum_{m=1}^{2M} d_{kn} d_{jm} (e^{i(n-m)\lambda} + e^{i(n+m)\lambda})$$

$$= \sum_{n=1}^{2M} \sum_{m=1}^{2M} d_{kn} d_{jm} e^{i(n-m)\lambda} + \sum_{n=1}^{2M} \sum_{m=1}^{2M} d_{kn} d_{jm} e^{i(n+m)\lambda}$$

$$= \sum_{n=1}^{2M} d_{kn} e^{in\lambda} \sum_{m=1}^{2M} d_{jm} e^{-im\lambda} + \sum_{n=1}^{2M} d_{kn} e^{in\lambda} \sum_{m=1}^{2M} d_{jm} e^{im\lambda} .$$

Examining each term independently, using the definition of the matrix $\underline{D}$, and defining

$A(e^{-i\lambda}) = 1 - \sum_{j=1}^{M} a_j \exp^{(-ij\lambda)}$, we get

$$\sum_{n=1}^{2M} d_{kn} e^{in\lambda} = - \sum_{n=k}^{K+M-1} a_{M+k-n} e^{in\lambda} + e^{i(M+k)\lambda}$$

$$= A(e^{-i\lambda})\, e^{i(M+k)\lambda}$$

and

$$\sum_{m=1}^{2M} d_{jm} e^{-im\lambda} = - \sum_{m=j}^{j+M+1} a_{M+j-m} e^{-im\lambda} e^{-i(M+j)\lambda}$$

$$= A(e^{i\lambda})\, e^{-i(M+j)\lambda}$$

We can now express $\xi_{k,j}$ as

$$\xi_{k,j} = A(e^{-i\lambda})\; A(e^{i\lambda})\; e^{i(k-j)\lambda} + A(e^{-i\lambda})\; A(e^{-i\lambda})\; e^{i(k+j)\lambda}\; e^{i2M\lambda} .$$

Define the matrix $\underline{P}$ by

$$\underline{P} = 2\pi \int_{-\pi}^{\pi} \underline{D}\, \underline{U}_{2M}\, \underline{D}^T\, \phi_Y^2(\lambda)\, d\lambda.$$

a typical element of $\underline{P}$, call it $\rho_{k,j}$ is given by

$$\rho_{k,j} = 2\pi \int_{-\pi}^{\pi} \xi_{k,j} \; \phi_Y^2(\lambda)\, d\lambda$$

$$= 2\pi \int_{-\pi}^{\pi} A(e^{-i\lambda})\; A(e^{i\lambda})\; e^{i(k-j)\lambda}\; \phi_Y^2(\lambda)\, d\lambda$$

$$+ 2\pi \int_{-\pi}^{\pi} A(e^{-i\lambda})\; A(e^{-i\lambda})\; e^{i(k+j)\lambda}\; e^{i2M\lambda}\; \phi_Y^2(\lambda)\, d\lambda \tag{14}$$

At this point we state and prove the following Lemma.

Lemma 4: Let $\{Y_t\}$ satisfy Assumption I and assume $\{Y_t\}$ is generated by a mixed autoregressive moving-average process, then

$$2\pi \int_{-\pi}^{\pi} (A(e^{-i\lambda}))^2\; e^{i(k+j)\lambda}\; e^{i2M\lambda}\; \phi_Y^2(\lambda)\, d\lambda = 0$$

Proof: From the definition of the power spectral density for an ARMA process we can write

$$\phi_Y(\lambda) = \sigma_\eta^2 \frac{|B(e^{-i\lambda})|^2}{|A(e^{-i\lambda})|^2}$$

where $B(e^{-i\lambda}) = 1 - \sum_{\ell=1}^{M} b_\ell \exp(-i\ell\lambda)$ and $A(e^{-i\lambda}) = 1 - \sum_{n=1}^{M} a_n \exp(-in\lambda)$.

Let

$$\rho^*_{k,j} = 2\pi \int_{-\pi}^{\pi} (A(e^{-i\lambda}))^2\; e^{i(k+j)\lambda}\; e^{i2M\lambda}\; \phi_Y^2(k)\, d\lambda$$

then after the substitution for $\phi_Y(\lambda)$ given above we have

$$\rho^*_{k,j} = \frac{\sigma_\eta^4}{2\pi} \int_{-\pi}^{\pi} \frac{[B(e^{i\lambda})\, B(e^{-i\lambda})]^2}{(A(e^{i\lambda}))^2} \exp \left| i(k+j+2M)\lambda \right| d\lambda \ .$$

We can write

$$(1 - \sum_{n=1}^{M} a_n \exp(-in\lambda))^{-2} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} C_{n_1} C_{n_2} \exp \left| i(n_1+n_2)\lambda \right|$$

and with $b_o = 1$

$$[B(e^{i\lambda})\, B(e^{-i\lambda})]^2 = \sum_{\ell_1=0}^{M} \cdots \sum_{\ell_4=0}^{M} b_{\ell_1} b_{\ell_2} b_{\ell_3} b_{\ell_4} \exp \left| i(\ell_1-\ell_2+\ell_3-\ell_4)\lambda \right| .$$

Now, we have

$$\rho^*_{k,j} = \frac{\sigma_\eta^4}{2\pi} \int_{-\pi}^{\pi} \sum_{\ell_1=0}^{M} \cdot\cdot \sum_{\ell_4=0}^{M} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} b_{\ell_1} b_{\ell_2} b_{\ell_3} b_{\ell_4} C_{n_1} C_{n_2}$$

$$\cdot \exp \left| i[(k+j+2m) + (\ell_1-\ell_2+\ell_3-\ell_4) + (n_1 + n_2)]\, \lambda \right| d\lambda$$

$$= \frac{\sigma_\eta^4}{2\pi} \sum_{\ell_1=0}^{M} \cdots \sum_{\ell_4=0}^{M} b_{\ell_1} b_{\ell_2} b_{\ell_3} b_{\ell_4} \int_{-\pi}^{\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} C_{n_1} C_{n_2} \ ,$$

$$\cdot \exp \left| i\, [(n_1+n_2) + (k+j+2M+\ell_1-\ell_2+\ell_3-\ell_4)]\, \lambda \right| d\lambda$$

From equation (3.56) of Kromer [5] we have that

$$\int_{-\pi}^{\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} C_{n_1} C_{n_2} \exp \left| i\, (n_1+n_2+d)\, \lambda \right| d\lambda = 0 \text{ for } d > 0$$

thus it follows that $\rho^*_{k,j} = 0$ since

$$k + j + 2M + \ell_1 - \ell_2 + \ell_3 - \ell_4 > 0 \qquad \blacksquare$$

By the result of Lemma 4 we have that

$$\rho_{k,j} = 2\pi \int_{-\pi}^{\pi} A(e^{-i\lambda})\ A(e^{i\lambda})\ e^{i(k-j)\lambda}\ \phi_Y^2(\lambda)\, d\lambda \ . \tag{15}$$

We will now examine the form of this covariance term for two cases: (1) the observed process $\{Y_t\}$ is a mixed autoregressive moving-average ARMA process and (2) $\{Y_t\}$ is the sum of an autoregressive process plus white noise AR + N.

1. AUTOREGRESSIVE MOVING-AVERAGE CASE

We have previously stated that the power spectral density for the ARMA process is given by

$$\phi_Y(\lambda) = \frac{\sigma_\eta^2}{2\pi}\ \frac{B(e^{i\lambda})\, B(e^{-i\lambda})}{A(e^{i\lambda})\, A(e^{-i\lambda})} .$$

Substituting this expression into (15) we get

$$\rho_{k,j} = \sigma_\eta^2 \int_{-\pi}^{\pi} B(e^{i\lambda})\, B(e^{-i\lambda})\, e^{i(k-j)\lambda}\ \phi_Y(\lambda)\, d\lambda \ . \tag{16}$$

Define $\phi_v(\lambda)$ to be the power spectral density for the moving-average component of the process; ie,

$$\phi_v(\lambda) = \sigma_\eta^2\ B(e^{i\lambda})\, B(e^{-i\lambda}) \ .$$

Now we can rewrite (16) as

$$\rho_{k,j} = \int_{-\pi}^{\pi} \phi_v(\lambda)\, \phi_Y(\lambda)\, e^{i(k-j)\lambda}\, d\lambda$$

$$= \sum_{\ell=-M}^{M} r_v(\ell) R_y(\ell-k+j) \tag{17}$$

where, by definition, $r_v(\ell) = \sigma_\eta^2 \sum_{j=0}^{M-\ell} b_j b_{j+\ell}$ and $R_Y(\ell) = E\,[Y_t\, Y_{t-\ell}]$.

We can write (17) as

$$\rho_{k,j} = r_v(0)\, R_Y(k-j) + \sum_{\ell=1}^{M} r_v(\ell) \left| R_Y(\ell-k+j) + R_Y(-\ell-k+j) \right| .$$

Define the matrix $\underline{\Gamma}^\ell$ element by element as $\Gamma^\ell(k-j) = R_Y\,(\ell-k+j)$ and $\underline{P}$ is the matrix of elements $\rho_{k,j}$, then we have

$$\underline{P} = r_v(0)\, \underline{\Gamma}^0 + \sum_{\ell=1}^{M} r_v(\ell) \left| \underline{\Gamma}^\ell + (\underline{\Gamma}^\ell)^T \right| .$$

Thus, our final result follows from (11),

$$\mathrm{Cov} \left| N^{1/2} (\hat{\underline{A}} - \underline{A}),\ N^{1/2}\ (\hat{\underline{A}} - \underline{A})^T \right| \xrightarrow[N\to\infty]{\mathcal{D}} \underline{\Gamma}_H^{-1}\ \underline{P}\, (\underline{\Gamma}_H^{-1})^T ,$$

which is equivalent to the result obtained by Gersch [4]. Gersch did not prove asymptotic normality, but the forms of the asymptotic covariance matrices are equivalent.

2. AUTOREGRESSIVE PLUS NOISE CASE

It was shown by Walker [1] and Pagano [2] that when the observed time series is generated by an AR process of order M plus white noise an ARMA (M,M) model as given by (1) can be used to represent the series. If we let

$$Y_t = X_t + n_t$$

where X_t is an AR process of order M and n_t is i.i.d. $N(0, \sigma_n^2)$ then $\{Y_t\}$ can be represented by (1) and the general covariance relationship of (11) applies. A more specific relationship, that takes into account the properties of the AR + N series, is calculated below.

We can write the power spectral density for the AR + N process as

$$\phi_Y(\lambda) = \sigma_n^2 + \frac{\sigma_\epsilon^2}{A(e^{i\lambda})\, A(e^{-i\lambda})} = \sigma_n^2 + \phi_X(\lambda) .$$

Substituting this expression into (15), we get

$$\rho_{k,j} = \frac{\sigma_n^4}{2\pi} \int_{-\pi}^{\pi} A(e^{i\lambda})\, A(e^{-i\lambda})\, e^{i(k-j)\lambda}\, d\lambda$$

$$+ \frac{2\sigma_n^2 \sigma_\epsilon^2}{2\pi} \int_{-\pi}^{\pi} e^{i(k-j)\lambda}\, d\lambda \; + \; \sigma_\epsilon^2 \int_{-\pi}^{\pi} \phi_X(\lambda)\, e^{i(k-j)\lambda}\, d\lambda$$

$$= \frac{\sigma_n^4}{2\pi} \int_{-\pi}^{\pi} A(e^{i\lambda})\, A(e^{-i\lambda})\, e^{i(k-j)\lambda}\, d\lambda + 2\sigma_n^2 \sigma_\epsilon^2\, \delta(k-j) + \sigma_\epsilon^2\, R_X(k-j)$$

$$= \sigma_n^2\, \sigma_\epsilon^2\, \delta(k-j) + \sigma_\epsilon^2\, R_Y\,(k-j) + \frac{\sigma_n^4}{2\pi} \int_{-\pi}^{\pi} A(e^{i\lambda})\; A(e^{-i\lambda})\; e^{i(k-j)\lambda}\, d\lambda\ .$$

For the AR+N process we can express the matrix $\underline{\rho}$ as

$$\underline{P} = \sigma_n^2\; \sigma_2^2\; \underline{I} + \sigma_\epsilon^2\, \underline{\Gamma}^0 + \frac{\sigma_n^4}{2\pi} \int_{-\pi}^{\pi} A(e^{i\lambda})\;\; A(e^{-i\lambda})\;\; \underline{G}_M \underline{G}_M^{\,*}\;\; d\lambda$$

and the final result is

$$\text{cov}\left| N^{1/2}\,(\hat{\underline{A}} - \underline{A}),\, N^{1/2}\,(\hat{\underline{A}} - \underline{A})^T \right| \xrightarrow[N\to\infty]{\mathscr{D}} \underline{\Gamma}_H^{-1}\;\; \underline{P}(\underline{\Gamma}_H^{-1})^T.$$

IV. CONCLUSIONS

The estimation of the AR parameters for mixed time series has been studied for two types of mixed series–time series generated by an ARMA process and time series generated by an AR process observed in white noise. In both cases the estimates have been shown to be asymptotically jointly multivariate normal with zero mean and covariance matrix structure given by

$$\text{cov}\left| N^{1/2}\,(\hat{\underline{A}} - \underline{A}),\, N^{1/2}\,(\underline{A} - A) \right| \xrightarrow[N\to\infty]{\mathscr{D}} \underline{\Gamma}_H^{\;-1}\;\; \underline{P}(\underline{\Gamma}_H^{-1})^T$$

where the form of the matrix $\underline{P}$ is dependent on the type of mixed series assumed.

The results obtained relative to the asymptotic statistics associated with the AR+N time series are of considerable interest to researchers working in the area of AR spectral estimation of noisy signals.

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