AFOSR-TR- 81 -0842

LEVEL III

(12)

AD A108975

FINAL REPORT

PROGRAM FOR NONLINEAR STRUCTURAL ANALYSIS

Contract F49620-79-C-0057

Submitted to

Air Force Office of Scientific Research
Directorate of Aerospace Sciences
Washington, D.C.

August 31, 1981

BOEING AEROSPACE COMPANY
Engineering Technology
Seattle, Washington 98124

81 12 29 024

ABSTRACT

The development and evaluation of new theoretical and numerical approaches for strongly nonlinear finite element analysis are reported. The element technology uses interior nodes to create higher order in-plane displacement forms needed for nonlinear strain calculation. Several solution procedure types are discussed, based on an updated total Lagrangian formulation. Progress with this approach and current capability levels are discussed.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 81 -0842	2. GOVT ACCESSION NO. AD-A108 975	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Nonlinear Structural Analysis		5. TYPE OF REPORT & PERIOD COVERED JAN 79 - JUN 81 Final Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Dr. Robert E. Jones		8. CONTRACT OR GRANT NUMBER(s) F49620-79-C-0057
9. PERFORMING ORGANIZATION NAME AND ADDRESS Boeing Aerospace Company P. O. Box 3999 Seattle, Washington 98124		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 2307B1 61102F
11. CONTROLLING OFFICE NAME AND ADDRESS AFOSR/NA Bldg. 410 Bolling AFB, D.C. 20332		12. REPORT DATE Sept. 1981
		13. NUMBER OF PAGES 77
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Finite Element	Beams
Nonlinear Analysis	Computer Modeling
Euler Angles	Convergent Solutions
	Lagrangian Formulation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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1.0 INTRODUCTION

This report contains a summary of progress on AFOSR contract F49620-79-C-0057 on nonlinear finite element analysis and outlines suggested further research. Section 2.1 begins by describing the deficiencies in the state-of-the-art in nonlinear analysis as they relate to and have motivated the present research. Section 2.2 discusses methodology and progress in the current contract in detail. Section 2.3 presents results for several large deflection problems for beams, addressing features of the nonlinear behavior which illustrate both the advantages and the deficiencies of the developed methods. Section 2.4 suggests areas for future studies related to the work. Appendices A and B provide technical details on two areas of development which are briefly discussed in Sections 2.1 and 2.2, but for which the reader may desire clarification. Appendix C reproduces portions of the proposal (Reference 5) for the present contract, for convenient reference in the present context.

The goal of the present research has been the development and evaluation of improved displacement-method finite element approaches for the analysis of structural problems with geometrical nonlinearities. The initial work in this subject was done by Haftka, Mallett and Nachbar in reference 1. Further work by Jones (References 2, 3, 4) followed along the lines outlined in reference 1. Reference 3 concluded that an improved approach could be developed through new finite element formulations coupled with a stepwise nonlinear solution procedure. Reference 5 proposed and outlined the development of these new approaches, which have been pursued in the present AFOSR-sponsored research and are reported herein.

There are three technical areas associated with geometrically nonlinear finite element analysis which require investigation in order to achieve the desired research results. These are introduced briefly here in order to put into perspective more detailed discussions in the sections which follow. First, it is required to derive a new type of finite element which is numerically well behaved when the total nonlinear deformations are retained in analysis. This requirement addresses the role of the nonlinear contributions to the stresses as major controlling factors in the equilibrium state.

With conventional elements, retaining these stresses results in serious errors with consequent incorrectly convergent, or, frequently, divergent solution calculations. Secondly, it is required to develop geometrical representations of the deformations which avoid any cumulative inconsistency between the deformations and the displacements. Such errors cannot be eliminated by iteration. Finally, it is required to develop improved stepwise solution procedures, with residual load iteration, which are convergent for large load steps. The presence of large residual loads caused by the nonlinear deformations often causes failure of conventional solution procedures. These three requirements address proven deficiencies in the application of conventional finite element methods to strongly nonlinear analysis, as manifested by both unacceptable inaccuracies in problem solutions and serious difficulties in obtaining convergent stepwise solution processes.

2.0 TECHNICAL DISCUSSION

This section discusses the goals of the present research, progress on the current contract, and recommended areas for further research.

2.1 Goals and Approach of Current Research

The physical problems which have motivated the current research primarily involve those structures in which the nonlinearly induced stresses due to large rotations are critical in determining the correct equilibrium state of the structure. Examples of such problems include such difficult problems as the buckling of shells and the postbuckling action of panels, and also simple cases such as the stretching of cables and bending of beams. In all of these cases it is necessary to include a complete and accurate description of the nonlinear stress field within the structure in order to correctly address the equilibrium problem. In general, finite element analysis has encountered a great deal of difficulty in doing this. The difficulties arise because conventional element formulations retaining complete nonlinear stress calculations encounter several fundamental problems. One of these problems, and that which requires the development of new types of elements, is that the nonlinearly induced stresses include physically unrealistic components which become "locked" within the elements, unremovable except by major reduction of the displacement magnitudes. This causes excessive structural stiffness and results in inaccurate problem solutions. Another problem is that, due to including the nonlinear stresses, the solution procedure is required to deal with very large residual loads in the iterative portion of the calculation process. It is usually found that such large residual loads cause solution divergence or excessively slow convergence and very large computational costs. This difficulty requires the development of solution procedures which are improved in character and tailored specifically to problems having large nonlinear stresses.

An additional difficulty is that in problems of the types under discussion there are generally large rotations and corresponding deformations which are computed in a stepwise manner during the solution process. Total deformations are often determined by summing increments. This approach can cause cumulative errors in the solution which are not correctable through the residual load iteration process. There are several sources of such errors. The first and that most commonly encountered in conventional approaches is a cumulative error in the strains caused by the determination of the strain through stepwise incrementation. This error results from linearization of the strain increments, and is usually quite large. A second source of error occurs in cases having moderately large rotations in three dimensions. The representation of such three-dimensional rotations through incrementation of cartesian rotations will cause both incorrect total nonlinear strains and also a cumulative error in the orientation of the structure in space. The error will make the strains inconsistent with the true total displacement and rotation state, and hence is not recoverable through residual load iteration. This rotation problem must be solved in order to develop finite element procedures which are applicable to such problems as the combined lateral and torsional nonlinear deformation of beam structures. Another type of error occurs in total Lagrangian formulations, and results from an exchange of roles between the bending and membrane displacements when the rotations become large.

The above discussions point clearly to three primary technical goals for the current research. The first is the development of new element types which are formulated specifically for including complete nonlinear strains in finite element calculations. The beam and shell elements under development in this research accomplish this through the use of interior modes to incorporate higher order axial (beam) and membrane (plate, shell) displacement functions.

The second primary goal is to avoid unrecoverable cumulative error, that is, any type of error which is unrecognized, and hence uncorrectable, by the residual load iteration process. The cumulative errors due to incrementing element strains in a stepwise procedure have been avoided by computing the total nonlinear strains directly, rather than by incrementation. A total Lagrangian formulation with updating is used to accomplish this. The cumulative error in the structural rotation state due to summing cartesian rotation increments has been avoided through a rigorous three-dimensional rotation description. In this approach, the rotation state, both total and incremental, is represented by three sequenced angles, called Euler angles, which uniquely define the total rotation state for arbitrarily large motions. This appears to be a new approach in finite element formulations.

The third primary goal is technically distinct from the finite element research of the work, but was necessary in order to achieve numerical verifications of the element technology. It is the development of solution procedures which are rapidly convergent despite the numerical difficulties inherent in the solution process for strongly nonlinear problems. Several approaches have been investigated in this regard. The initial approach used an internally nonlinear stepwise solution procedure with iteration of the residual load state after each load step. The nonlinear stepwise capability was based on the static perturbation procedure (References 6, 7, 8). Additional means to convergence acceleration were found necessary in conjunction with the static perturbation method; these are discussed in Section 2.2. Ultimately, this approach was found inadequate (except for small load steps and moderately small displacements), and a method utilizing adaptive modification of the structural stiffness matrix (the BFGS method, reference 10) was successfully implemented in place of the static perturbation method.

The total set of technical goals for the present research include, in addition to the three primary goals described above, a number of important secondary items. These have been grouped into two categories. The first category relates to the matter of suitable finite element strain displacement equations. It is required that the basic rules governing finite element displacement states and strain-displacement relations be followed: the element can undergo rigid body displacements and constant strain states; large rigid body motions must not cause element deformations. These requirements are met by using cartesian-based elemental coordinate systems. Other requirements relate to the degree of approximation of the nonlinear portions of the strain-displacement equations and the accurate representation of geometry for shell elements. It is necessary to keep the strain-displacement equations simple and amenable to numerical processing as large displacements and rotations develop. In particular, it is required to avoid the complexity of a nonlinear shell strain displacement formulation of the intrinsic coordinate type. The use of the Marguerre type of strain formulation, together with element local coordinate system updating, has fulfilled these requirements. The second category relates to generality of solution procedures. It was originally intended that the solution procedures developed in this work be applicable to an extended set of of problem types including buckling (bifurcation, limit points and snap through), and also that the solution procedure be extendable to the case of dynamic response calculations where strong nonlinearities are present. The static perturbation approach appeared to have the potential to satisfy these requirements. The difficulties in achieving convergence of nonlinear stepwise solutions with the static perturbation method, and the adoption of the BFGS method for this purpose, have necessarily modified the original plan to develop a solution procedure with direct applicability to both dynamics and buckling problems.

2.2 Progress of Current Research

In the work accomplished to date the theoretical development of the finite element formulations for the new type of two-dimensional and three-dimensional beam elements has been completed and the Euler angle theory for both the beam elements and the plate and shell elements has been developed. By calculations, it has been demonstrated that the type of element formulation under development is completely successful in handling the large stresses inherent in the large rotation nonlinear state. A technically advanced set of solution procedure algorithms has been developed, refined and verified through numerical studies using a two-dimensional beam element code. The solution procedures appear to be rapidly convergent for large step sizes and strong nonlinearities. A number of technical difficulties, some expected and some unanticipated, have been encountered. The discussion which follows attempts to describe the chronology of the work accomplished and its present status, covering each task and difficulty in some detail.

The current research began with the development of the element technology for a curved beam element whose initial shape and subsequent deformation are constrained to take place in a single plane. This has been called the 2-D beam element (see Appendix C, Section 2.2 and Figure 1). The element has five nodes: nodes 1 and 5 are the end nodes; 2, 3, and 4 are internal. In its present form, the nodes are equally spaced along the length of the element. Nodes 1, 3 and 5 each have three freedoms, including the axial displacement, the lateral or bending displacement, and the bending rotation. Nodes 2 and 4 have only the axial displacement freedom. This unique nodal and freedom arrangement provides axial displacements which are quartic functions and bending displacements and rotations which are quadratic. By this means accurate stress and strain representations are obtained despite the rotations present in strongly nonlinear problems (see discussion in Appendix C, Section 2.2, Stability Elements). The development of this element encountered several technical difficulties related to its unusual nodal and freedom arrangement and in regard to geometrical updating. For example, in performing a rotational updating transformation, the calculation of the transformed values of the axial freedoms at nodes 2 or 4 requires including in the transformation the potentially large bending displacements at these nodes. Since the bending displacement is not an available freedom at nodes 2 and 4, its value must be generated by interpolation using the bending displacement values at nodes 1, 3 and 5. Related difficulties are encountered in transforming the loading on the beam element. The coupling together of the axial and bending displacements (or loads) in such updating transformations is the means by which any small angle approximations used in the nonlinear strain equations are removed by updating. This has implications regarding the performance of solution procedures, the calculation of residual loads, and the criteria controlling updating.

The next step in the research was the development of solution procedures appropriate to the implementation of the 2-D beam element. The solution procedure development was based on the static perturbation method of the second order (the static perturbation method is discussed in Appendix C, Section 2.2, Nonlinear Step Static Solution Procedure, and also in a mathematical derivation starting on page C30 of Appendix C. See also Appendix A, page A1, for a discussion of path parameters). In this formulation, the structural displacements are expressed as a second degree Taylor series in a path parameter. The path parameter is the Taylor series argument. In the initial development of this theory the path parameter was the load itself.

That is, the structural displacements were expressed as a second order Taylor series in the load. This approach is computationally simple and provides excellent results for many types of problems. It presumes that the displacements are well behaved functions of the load, and in particular that displacements cannot occur without corresponding changes in the value of the load. Hence this approach would be unsuitable for buckling problems, in which displacements can occur at constant or nearly constant load.

A proof-of-concept computer program was written to implement this procedure in conjunction with the 2-D beam element. The program was designed to have maximum adaptability to future extensions of element technology. In numerical work with this computer program, the surprising result was found that for certain problem types (large rotations with very small axial stresses) the second order static perturbation procedure often displayed poor convergence or even divergence unless the applied load steps were made small. In an attempt to improve this situation the static perturbation procedure was extended to include the third degree Taylor series terms. This extension led to a great deal of complexity, both in the area of theory development and also in coding work, and produced disappointing results. In numerical work it was found that the second order approach consistently performed better than the substantially more complex third order formulation. A considerable amount of study was done to explain this unexpected result and to understand more fully the behavior of the second and third order formulations.

It was determined that the second order procedure provides corrective displacements which primarily reduce errors in the axial force equilibrium state. The third order process, on the other hand, primarily provides corrective bending displacements in order to reduce errors in the bending (lateral) load equilibrium state. The static perturbation approaches accomplish these corrections through a type of residual load evaluation, which is made using <u>only</u> the start-of-step geometry and deformation description of the structure. Since this start-of-step state is approximate in its ability to forecast end-of-step residual loads, a corresponding approximation occurs in the static perturbation corrective displacement values. In contrast, corrective displacements computed by the stepwise iterative process, using rigorous, end-of-step evaluated residual loads, are able to respond exactly to the nonlinearity-induced errors which have occurred during a given load increment. In geometrically nonlinear analysis, axial equilibrium corrections are crucially important, because the axial forces combine with the rotations to produce potentially large bending equilibrium errors. Hence the second order static perturbation, despite its approximate nature, is beneficial to solution convergence. On the other hand, errors in bending displacement prediction can be crucially damaging to solution convergence, because of their potential to cause large rotations, nonlinear axial strains, and hence large axial force errors. Hence the approximations in the bending displacement corrections computed by the third order static perturbation process appear unacceptable. It was concluded for this reason that conventional static perturbation of the second order is superior to the third order procedure for geometrically nonlinear analysis of "thin" structures (beams, plates, shells).

At this point in the research it was felt worthwhile to extend the static perturbation approach to a more general type of formulation, in which the Taylor series path parameter is deformation-related (Appendix A discusses the path parameter in detail). A useful path parameter of this type is similar to the structural strain energy function, taking the form

$$S^2 = \Delta Q^T K \Delta Q$$

where S is the path parameter, K is the tangent stiffness matrix, and ΔQ is the incremental displacement of any load (or iteration) step. This type of path parameter has the advantage of applicability to buckling problems. It was felt that calculations using this particular type of path parameter might shed some light on the overall behavior of the static perturbation process in the types of problems under study. This particular extension again led to a great deal of complexity, in both theory and numerical approach. Numerical work with this approach was again disappointing, and showed that for problems in which the displacements are well behaved functions of the load, the energy-based path parameter formulation does not provide any advantages over the load-based path parameter approach. Only in cases in which the load-displacement relationship is poorly behaved, as occurs in buckling problems, would the generality of this approach be advantageous.

In order to obtain improved solution procedure performance, recourse was made to methods developed during previous experience (prior AFOSR contract, references 3, 4) in nonlinear analysis of beams and plates. In this work it had been concluded that the axial force equilibrium errors are primarily a result of bending displacements which take place without perfectly "matched" axial displacements. The axial force errors are usually very large. Together with the rotations they cause large error loads acting on the relatively flexible bending displacements. This, in turn, causes further bending displacement errors with even larger axial force errors. Thus, the errors in the axial forces have a tendency to magnify themselves and cause divergent calculations. In the previous work it was found that a successful method of accelerating convergence is to perform residual load iterations for the single purpose of "matching" the axial displacements to the bending displacements, thus removing the axial load equilibrium errors. This is implemented by a solution procedure employing "alternate-freedom" iterative corrections. In this procedure, the first incremental displacement of a load step includes all of the freedoms of the finite element model. The next increment is the first iteration. It only includes the axial/membrane freedoms, and thereby relaxes the axial/membrane force errors. The next increment is the second iteration; it includes all freedoms. The third iteration includes only the axial/membrane freedoms, and so on. It has been found that, despite element curvature and prior deformation, in each axial/membrane-freedom-only iterative correction the axial/membrane force equilibrium errors are significantly reduced. This prevents these errors from causing, in the subsequent iteration, large bending displacement errors. This type of solution procedure always accelerates convergence, and often achieves convergent solutions where other approaches encounter divergence. The alternate-freedom-iteration procedure was added to the static perturbation method to achieve a combined process having the benefits of both procedures. The extension was accomplished such that both the all-freedom and the axial/membrane freedom iterations are performed by the static perturbation procedure.

In calculations done with this method, it was found that the bending moment residual loads tended to remain excessive after the axial/membrane and transverse load residuals were reduced to relatively small values. Consequently, the alternate-freedom-iteration procedure was modified to include both the axial/membrane freedoms and the rotational freedoms. In this form the solution procedure appeared optimum.

The above-discussed solution procedure functioned well for problems with moderate displacements and rotations, but diverged when very large displacements (e.g., half the length for a simple end-loaded cantilever) were computed. To prevent the divergence, a procedure called a "line-search" was implemented. In this method, the amplitude of a computed displacement increment is scaled, or optimized, in such a way as to minimize the solution errors, as measured by residual load magnitudes, which correspond to the total displacements at the end of the increment. This avoids the use of a computed increment where that increment would increase, rather than decrease, the residual loads. The implementation involves evaluating a measure of the error, e.g., the root-sum-square of the residual loads, for several amplitudes of the computed increment, and interpolating on the amplitude to obtain a minimum error. The interpolation includes the axial/membrane/rotational iteration corrections. This approach performed well, particularly when employed with judicious updating of the structural stiffness matrix in order to assure that the "shape" of the computed increment is a good one. If the stiffness matrix is not updated, it sometimes occurs that the relative magnitudes of the incremental values of the structural freedoms, i.e., the "shape" of the increment, is sufficiently inaccurate that even a near zero amplitude of the computed increment will increase the error level. In this case the line-search fails, and an accurate problem solution is not obtained.

To avoid this difficulty, a method based on stepwise modification of the structural stiffness matrix can be used. Such an approach involves stepwise modifications of the stiffness matrix such that the matrix to be used for the <u>next</u> increment is the one which, had it been used for the <u>last</u> increment, would have produced an accurate incremental nonlinear response to the stepwise incremental loads. The procedure implemented was the BFGS method, described in reference 10. The BFGS method requires a line-search for each increment; the above-described line-search method appears suitable and was used, though it differs somewhat from the one discussed in reference 10. The use of the axial/membrane/rotational iterations, described earlier, was found to be still necessary for convergence, and this procedure was combined with the BFGS approach as follows. Each iterative increment was corrected by an axial/membrane/rotational iteration, and the "double increment" thus computed was subjected to a line-search. The axial/membrane/rotational correction was separately computed for each interpolation amplitude of the line-search, in order to account properly for the effects of nonlinearities. Without these corrections, the line-search always fails. The residual loads resulting after the increment are compared with those imposed at the start of the increment, resulting in the transformation matrix of the BFGS method.

The solution procedure described above was found to be uniformly convergent for both large and small displacements and for large load step sizes. In addition, the convergence was found to be rapid (typically 15 or less iterations). However, the convergence limit is to a nonzero error level which cannot be improved upon by further iterations. This minimum error level is a function of the amplitude of the displacements and also appears to be influenced by the transformations associated with geometrical updates of the element baseplanes. For the case of a simple cantilever beam, good accuracy is obtained when the end displacement is less than about half the length of the beam. This appears to be true whether one or more elements are used in the finite element model. The intentions of the current work are to handle much larger displacements than this apparent limitation.

A number of numerical experiments were carried out in attempts to understand and eliminate the minimum error level problem. These included: comparisons between results when the transverse shear and extensional deformations are represented by strain equations permitting large, as opposed to moderate, rotations; elimination (in deformation and residual loads calculations) of the quadratic-in-x component of the transverse shear strain; the use of double-precision arithmetic in the geometrical updating calculations; computing large deflection solutions with-and-without geometrical and stiffness matrix updates. The numerical experiments involving the large rotation strain equations improved but did not eliminate the minimum error level problem. The use of double precision geometrical update calculations resulted in somewhat modified solutions, indicating numerical sensitivity in this type of calculation. This suggests the probable need for extension to double precision arithmetic in all stress, deformation, and load computations. However, the minimum error level was not appreciably affected by this experiment.

Through omission of geometrical updating, it was found that the minimum error level could be made essentially zero. However, this is not a satisfactory solution to the problem, because it leads to a non-updated total Lagrangian approach which is subject to important limitations and errors. In particular, the desirable omission of certain normally negligible terms in the nonlinear strain equations is not admissible for a non-updated total Lagrangian approach. The geometrical updating used in the present approach is considered a valuable feature, not to be omitted as a solution to the minimum error problem. What has been gained through the geometrical updating experiment is: it has been verified that the element itself is capable of a "zero" error level in the large displacement state; it is clearly indicated that the geometrical updating introduces displacement and deformation forms which for some reason are not satisfactorily handled by the iterative solution procedure. It appears that the transverse shear strain becomes "locked" in the element, causing the nonzero minimum error level. Whether the fault lies in the element itself, or in the solution procedure, is not clear. It appears that the element should provide convergence to a correct shear strain and stress, since without geometrical updating it does so, and also since its nodal and freedom arrangements and its elemental coordinate system satisfy the required rigid motion and constant strain conditions. On the other hand, recent literature suggests that similar elements (though without the same internal nodes and freedoms) may have problems of a similar nature to those encountered in the present work (References 11, 12). Thus, the possibility of an element-level problem cannot yet be dismissed.

The total requirements for the solution procedure include a number of options in addition to those of the basic iteration process discussed above. Other needed features include: conditional geometrical updating of element local coordinate systems and displacement states; conditional updating of the stiffness matrix, with separate handling of the linearized portion of the matrix and the stress-dependent ("geometric stiffness") portion; conditional controls on solution continuance or termination; limits on the number of geometrical and stiffness updates and on the error measure which constitutes convergence.

The developed solution procedure contains user options for controlling all of these items. Table 1 gives a brief summary of the total set of solution procedure options. The static perturbation controls are covered in part (a), and the controls over the entire procedure (as currently coded) are given in part (b). One item which requires further discussion is the conditional updating of the stiffness matrix. The total stiffness matrix is the sum of the basic stiffness matrix and the geometric stiffness matrix. The updating of the basic matrix and the geometric matrix are not necessarily done at the same time, because the geometric matrix contains the stresses themselves. It is not satisfactory to update the geometric stiffness matrix when the stresses have relatively large errors. For this reason, the updating of the geometric stiffness matrix is only permitted when the error state of the solution is within certain bounds controlled by parameters within the code.

It was felt worthwhile to evaluate the new nonlinear element in comparison with conventional finite element approaches, in application to nonlinear analysis. In order for such a comparison to be valid, it must be made with the two element types having all features in common, i.e., solution procedure, element nodal and freedom formulation, all updating and transformations, etc., except for the single basic feature which distinguishes the new element: the use of higher order axial/membrane freedoms in combination with lower order bending freedoms. In order to accomplish the required comparison, the new element was provided with a special solution procedure option: a solution procedure constraint was developed which constrains the axial freedoms at nodes 2, 3 and 4 to values which are the interpolated values at these nodes of an element whose axial displacements are linear functions (a conventional axial displacement function). This constraint process is of the type called "multi-point constraint", and is implemented by a set of matrix transformations. It permits the element to be used in either its "nonlinear" mode of behavior, or in a conventional mode, with all other computational processes unchanged. As noted in Table 1, this option is effected by input of the value OPT2 = negative. Calculations made with this option converged very slowly to erroneous results, verifying that the conventional type of element is unsuited to the accurate computation of highly nonlinear problems. Section 2.3 discusses these results, which are in complete agreement with the earlier results of Haftka, Mallett and Nachbar (Reference 1). This capability was used in the static perturbation version of the code, and is not currently operative in the BFGS version.

Theory was developed for a beam element capable of bending and twisting in three-dimensions, called herein the 3-D beam element. This type of element is required for problems such as the nonlinear bending and torsional deformation and buckling of beam structures. A difficult technical problem arises at the outset of this type of derivation. It is recalled that one of the goals of the present research is to avoid unrecoverable cumulative error in the problem solution. A principle offender in this regard is the calculation of the strain itself. To avoid such errors, the deformation must be determined by direct calculation of the total strain, using the total displaced state of the structure, rather than by strain incrementation. To do this requires a precise definition of the rotation state. The three-dimensional beam element undergoes three components of rotation. These include the twist and the two bending rotations, all of which can be large for nonlinear problems. In the large rotation state the orientation of a rotated element or a node of an element in space cannot be represented by arbitrarily ordered cartesian components referred to a fixed coordinate system. Neither can the orientation be arrived at by summing small rotations referred to cartesian systems. The basic problem is that rotations are not vectors and therefore are not additive.

A correct large rotation state can be obtained through the use of sequenced angular rotations called Euler angles. Each Euler rotation takes place about an axis which has been subjected to all prior rotations in the sequence, which must take place in a specified order. This approach has been successfully used for the calculation of large motions of spacecraft as well as other types of large rotation dynamic problems (Reference 9). It is necessary to use this approach to develop the 3-D beam element. If the conventional small angle (cartesian) approach were used, the strains would be inconsistent with the rotation state of the structure, and it would be impossible to correct the equilibrium configuration through the residual load iteration process.

Strain displacement relations based on Euler angles were not found in the literature, and consequently a set of appropriate finite element deformation equations had to be developed. The derivation was carried out using a tensorial approach and convected coordinate systems. Appendix B describes the approach in some detail. This development presented a number of difficulties, including: the need to develop rational approximations associated with the relative importance of many different types of nonlinear terms in the strain displacement relations; the deformation-following beam cross-sectional axis system (Euler-angle-defined) does not maintain its "longitudinal" axis along the beam centerline axis, and it is necessary to define an additional Euler-angle-defined convected system which has this desirable property (see Appendix B, Figure B5); it is necessary to determine incremental cartesian bending and twisting angles as well as Euler increments, in order to maintain physical reality in interpretation of the deformations and the loads. In implementing the 3-D element together with a stepping solution procedure it is necessary to use a large number of transformations of geometrical types, in order to maintain and update the different convected coordinate systems and the two sets of Euler angle totals. One such transformation provides the needed relationship between the stepwise Euler angle rotation increments and the cartesian increments; the required transformation is called the π transformation (Reference 9). Other transformations are required to transform the stiffness matrix of the element from its derivation coordinate system to the coordinate systems of the solution process; that is, to transform the stiffness matrix from those definitions used in the strain displacement relations to those which are suitable for merging together adjacent elements and obtaining the problem solution in terms of meaningful cartesian quantities. Also, in the solution procedure the coordinate systems used for each element are convected, that is, they follow the elements throughout the deformation process in order to retain for each element a small deformation state. Hence, it is necessary to perform repeated transformations to accomplish the geometrical updating of various solution and geometry parameters. The total solution process for the 3-D beam element has been flow-charted to provide a methodology description suitable for computer coding. A computer code for this element has been about 75% completed.

2.3 Illustrative Numerical Results

This section presents numerical results for several beam bending problems. Results obtained with both the static perturbation method and the BFGS method are discussed. The purpose of the example problems is to illustrate the displacement magnitude capabilities, convergence characteristics, and limitations of the methods developed. Refer first to Figure 1. A simple two element beam structure is bent by an end load in the global Z direction. The loaded end of the beam is either completely free (Figure 1a) or constrained against X -direction displacement (Figure 1b). This problem was solved by the static perturbation approach.

Tables 2-6 present numerical data for the beam structure of Figures 1a and 1b. The tables include deflections, rotations, axial forces in the elements, and convergence data (numbers of iterations and percent error based on residual loads). The two values of axial force shown in the tables for node 5 are those computed for the two elements which connect to this mode.

Table 2 gives numerical results for the problem of Figure 1a. The load varies from 0 to 720 (pounds), while the end deflection varies from 0 to 2.70 (inches). A graph of the deflection versus the load would be very nearly a straight line, as the only nonlinearity in this problem results from the small foreshortening of the beam due to its deflection and rotation. The deflection of 2.70 in. is 93% of the theoretical value for this beam, a reasonable value for a two element model where each element has only a second degree displacement function capability in bending. The average rotation of element #2 at 720 lbs. is .189 radians. The average element rotation is the angle to which the "baseplane" of the element is updated in geometrical updating. The table shows the sequence of baseplane update angles for both elements. The deflection of 2.70 in. is 13.5% of the total beam length, an amplitude which is well into the potentially nonlinear range in structural analysis. However, since the right end of the beam is permitted to deflect freely, the beam is not stressed axially due to nonlinear strain buildup, and the behavior is essentially linear. The payoff of the new nonlinear element in this case is that it allows the axial stress to ignore the nonlinear strain effects, even though fully nonlinear strain calculation is done in the analysis. This is accomplished by the quartic axial displacement function.

The element axial stresses are small and essentially constant over each element. The jump in axial stress at node 5 balances a corresponding jump in the shear at this node. Nearly constant axial stress in an element is required by the equilibrium equations. The nonzero axial stress values are correct, and result from the inclination of the end of the beam with respect to the applied load. This is illustrated in Figure 2. The figure gives the equations of equilibrium which must be satisfied by the shear and axial forces at the end of the beam. It is seen that the inclination of the shear force requires the axial stress in the beam to be nonzero. The inclination of the shear force can be accounted for either in the element strain formation or in the solution procedure (by geometrical updating). To account for this in the strain formulation, it is necessary to retain nonlinear terms in the beam transverse shear strain, a type of nonlinearity not usually retained in geometrically nonlinear analysis. A simpler approach, and that used to compute the data under discussion here, is to use the geometrical updating to rotate the shear force. The result is that, at the 720 lb load, 707 lbs. is normal to the beam (the shear, S) and 135 lbs is directed along the beam axis, producing the axial stress shown in Table 2. These values are determined by the inclination of the updated baseplane which is seen in the table to be .189 radians. The total load in the global Z direction remains 720 lbs. It is noted that at larger displacements the element stresses do not necessarily follow the simple line of explanation given. For such problems the solution converges to give apparently erroneous stresses in some cases.

Faster convergence can usually be obtained if geometrical updating is done infrequently. This is because the curved, updated geometry of the beam causes the axial and bending displacements to be numerically "coupled" much more than they are in the initial flat, nonupdated geometry, and therefore slows the iterative convergence process. This suggests that solutions might be obtained at lower cost by applying the total load in the first load step. In this approach only one geometrical updating is required, corresponding to the final deflected state. Such a solution is shown on Table 3. The element #2 baseplane was updated in one step to the inclination of .193 radians, consistent with the zero$^{\underline{th}}$ iteration displacements, for which the displacement at the end of the beam is 2.75 in. (final convergence was to 2.69 in.). The computed displacements are almost identical to those of Table 2. The axial stresses are slightly different because the baseplane is updated to a slightly different angle than the .189 radians of Table 2. The results of Table 3 show two important facts: the converged result for large loads can be obtained in a single load step; certain aspects of the solution, such as the axial stress in the present problem, may be sensitive to the inclination of the updated baseplane, so that updating should not in general be neglected.

Geometrical updating is only required when the element rotation relative to the baseplane coordinate systems become large, e.g., greater than about 20°. Table 4 shows a case of delayed updating. Here the updating has been delayed until the load reaches 660 lbs. The deflection results are nearly the same as those of Table 2, but of course the axial stresses are not correctly computed until the baseplane is updated. Note that the final baseplane angles here have resulted from a less recent update, and hence differ slightly from those of Table 3.

The problem of Figure 1b has the global X displacement constrained at the right end. This problem is highly nonlinear, typical in character to many practically important cases involving end-or-edge-constrained beams and plates. The solution data are given in Table 5 and on Figure 3 for a load range of 0 to 240 lbs. The baseplane in this example is updated when the average slope of the beam (relative to the most recent baseplane update) exceeds .01 radians. Convergence is reasonably fast except at the 180 lb. load. The axial stresses are essentially constant over the lengths of the elements, being dominated by the effect of the end constraint. They are responsible for the nonlinear stiffening behavior illustrated by the force-deflection plot of Figure 3. For larger load levels, the degree of nonlinearity of this problem increases very rapidly.

The next example concerns a "conventional" beam element. This element is identical to the new nonlinear element except that the axial displacements at the interior nodes are constrained to take values defined by linear interpolation between the end node values. That is, these freedoms are in effect omitted from the problem solution by constraining the axial displacement shape to be the linear shape of the "simplex" type of element. In solving problems with the constrained element, fully nonlinear axial strains due to the bending displacements are retained, and the new nonlinear solution procedure is also used. Thus, the results provide a consistent comparison between the new type of element and one of conventional formulation, with all other aspects of the numerical processing kept the same. The results are tabulated in Table 6 for the load range of 0 to 240 lbs. and plotted on Figure 4. The solution at 720 lbs. was computed in a single load step. The final deflection at 720 lbs. is 1.57 in., which is considerably less than the 2.70 in. of the new nonlinear element. This error reflects the excessive and erroneous stiffness of the conventional element due to the axial stresses which are "locked" in this element by nonlinearity. The "simplex" axial displacements cannot remove these locked-in axial stresses because of deficiencies of their functional forms. The axial stresses are seen in the table to be very large. The error illustrated by this example is consistent with that discussed in Reference 1. The new nonlinear element has eliminated this type of error.

For displacements and rotations which are significantly larger than those of the previous examples, convergence difficulties were encountered with the static perturbation approach. As discussed in Section 2.2, the BFGS method was implemented to improve convergence. In addition, the transverse shear strain was modified to incorporate nonlinear terms. The primary effect of this modification is to rotate the resultant shear force on the element so that it is parallel to the deformed beam cross-section. This has a relatively small effect on problem solutions.

Figure 5 shows two cantilever beam problems which were solved with the BFGS approach. Figures 5a and 5b show a single element problem, with the support located both at the center of the element and at the left end. The problem of Figure 5a requires no geometrical updating because the baseplane does not rotate. It yields an exact solution for the rotations of the ends of the beam. In contrast, the problem of Figure 5b has significant element baseplane rotations requiring geometrical updating. The loading in both cases is a pure moment, and the purposes of the example are to investigate the rate and degree of convergence obtainable at large displacement and the influence of geometrical updating on convergence and accuracy. Figure 6 shows dimensioned sketches of the deflections for both cases. For each problem, the figure also shows the displaced condition referred to the coordinate system of the other problem. These data are shown in parentheses. The deflections are large, on the order of half the length of the beam. It is seen that the simple cantilever element converges to a solution having 4% to 5% more curvature than that for the doubly cantilevered (symmetrical) element. The cause of the difference is almost certainly the geometrical updating required for the simple cantilever case, which is not done in the symmetrical case. Convergence for the symmetrical case occurs on the first iteration; it is much slower for the case with geometrical updating. The values of the element shear and axial stresses for the simple cantilever differ from those of the symmetrical problem. This causes element (residual) loads which influence solution convergence and accuracy, particularly for multi-element problems.

The cause of the numerical differences between these two solutions has not been fully resolved. It appears doubtful that the differences arise due to the element formulation itself, because the symmetrical problem yields an exact solution. The simple cantilever case can clearly have exactly the same deformation state as the double cantilever, when referred to an updated baseplane. Thus, the determination of residual loads, which is based on only displacements referred to the updated baseplane, is potentially identical for the two problems. It is likely that the BFGS solution procedure (which is basically an optimum-seeking type) has become trapped along a solution path which has a false minimum error state. This view is strengthened by the fact that frequently a regeneration of the stiffness matrix in the deformed state, followed by subsequent BFGS calculations, yields a substantially improved solution accuracy. The extensive use of single precision arithmetic in the code may also be a factor in the minimum error problem. It is also noted that the geometrical updating of the axial displacement values at nodes 2 and 4 makes use of interpolated bending displacements at these nodes. The bending displacements themselves are not updated for these nodes. However, if they were updated, the resulting values would not adhere to the quadratic bending displacement form when referred to the updated baseplane. This is a source of inconsistency inherent in the updating process, due to the different function shapes used for the axial and bending displacements. This inconsistency should, however, be correctable by residual load iteration.

2.4 Suggested Further Work

The element development appears to have successfully controlled the problem of large nonlinear strains, as illustrated by the results shown on Figure 6 for the symmetrical cantilever. However, for more general cases, such as that of the unsymmetrical case on Figure 6 and a number of multi-element problems which have been solved, there remain unresolved problems in either the formulations or in the actual calculations. Displacements which are very large have been successfully computed despite the difficulties encountered, however, and the potential of the new type of element appears to have been adequately demonstrated. If the research is to be continued, the accuracy and convergence difficulties will have to be addressed as a first step. The four areas of study listed below are suggested candidates for this work.

- o eliminate the use of single precision arithmetic in all suspect calculations.

- investigate the possibility of the solution procedure being "trapped" by a false minimum of the error level.

- investigate whether the limitation of the element to constant curvature (w is quadratic) while simultaneously a quadratic rotation is allowed is a cause of inconsistency which could contribute to numerical problems.

It is expected that the numerical difficulties which still exist can be resolved. In this event it appears particularly important to develop and test the three-dimensional beam element. This work will evaluate the Euler angle deformation theory and the set of geometrical transformations inherent in this approach which is new in the field of nonlinear finite element stress analysis. The theory and procedural specifications have been completed and the computer coding partially completed for this task.

If the three-dimensional beam element work shows the Euler angle theory to be a valuable tool for large displacement finite element analysis, consideration should be given to a further task. This task would apply the Euler angle approach and the algorithms developed for the two-dimensional beam element to the development of plate and shell elements. It would result in a truly large deflection analysis method for plates and shells, leading eventually to a much needed large displacement shell buckling analysis capability.

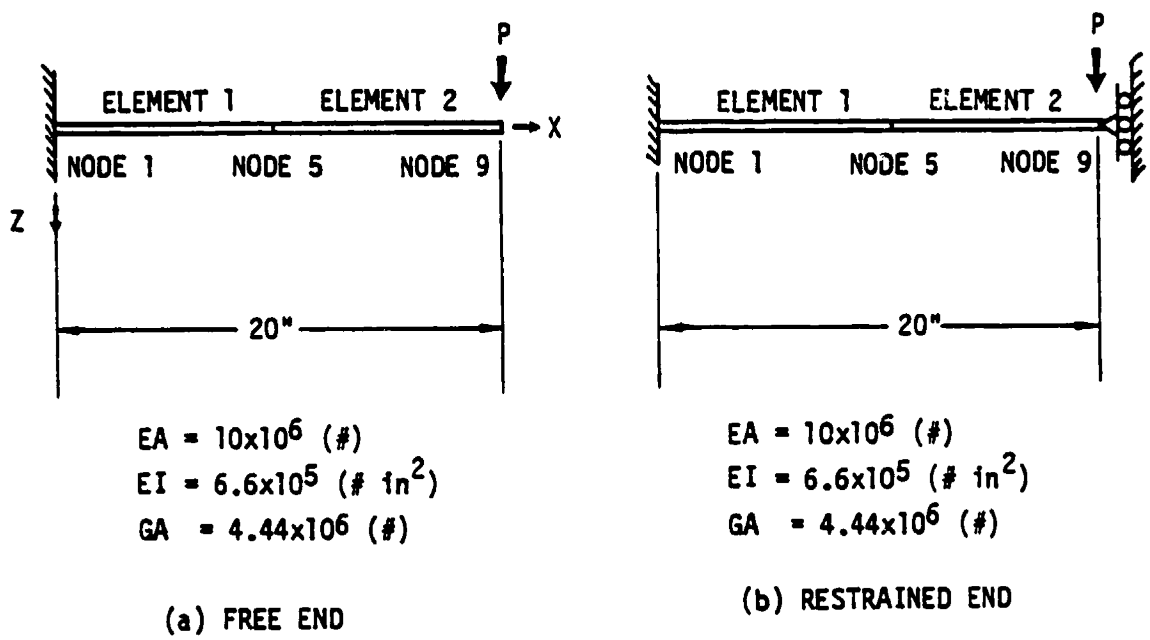
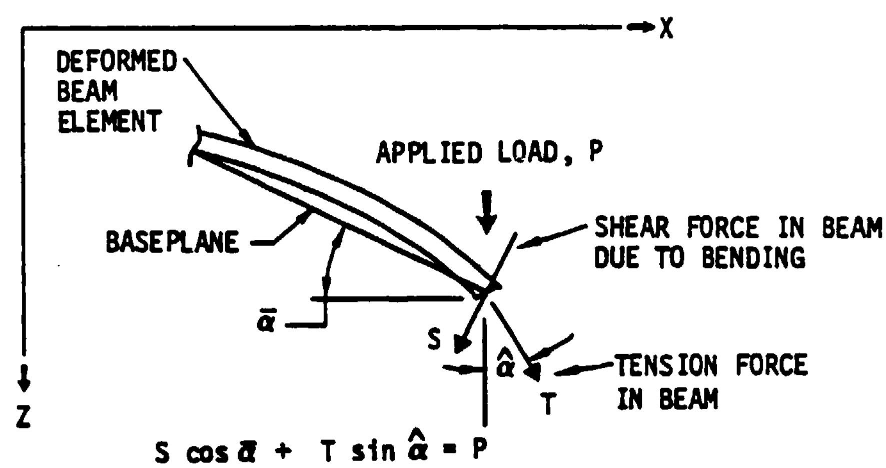


Figure 1: Illustrative Cantilever Beam Problems



$$S \cos \bar{\alpha} + T \sin \hat{\alpha} = P$$
$$-S \sin \bar{\alpha} + T \cos \hat{\alpha} = 0$$

Figure 2: Beam End Conditions for Large Rotations

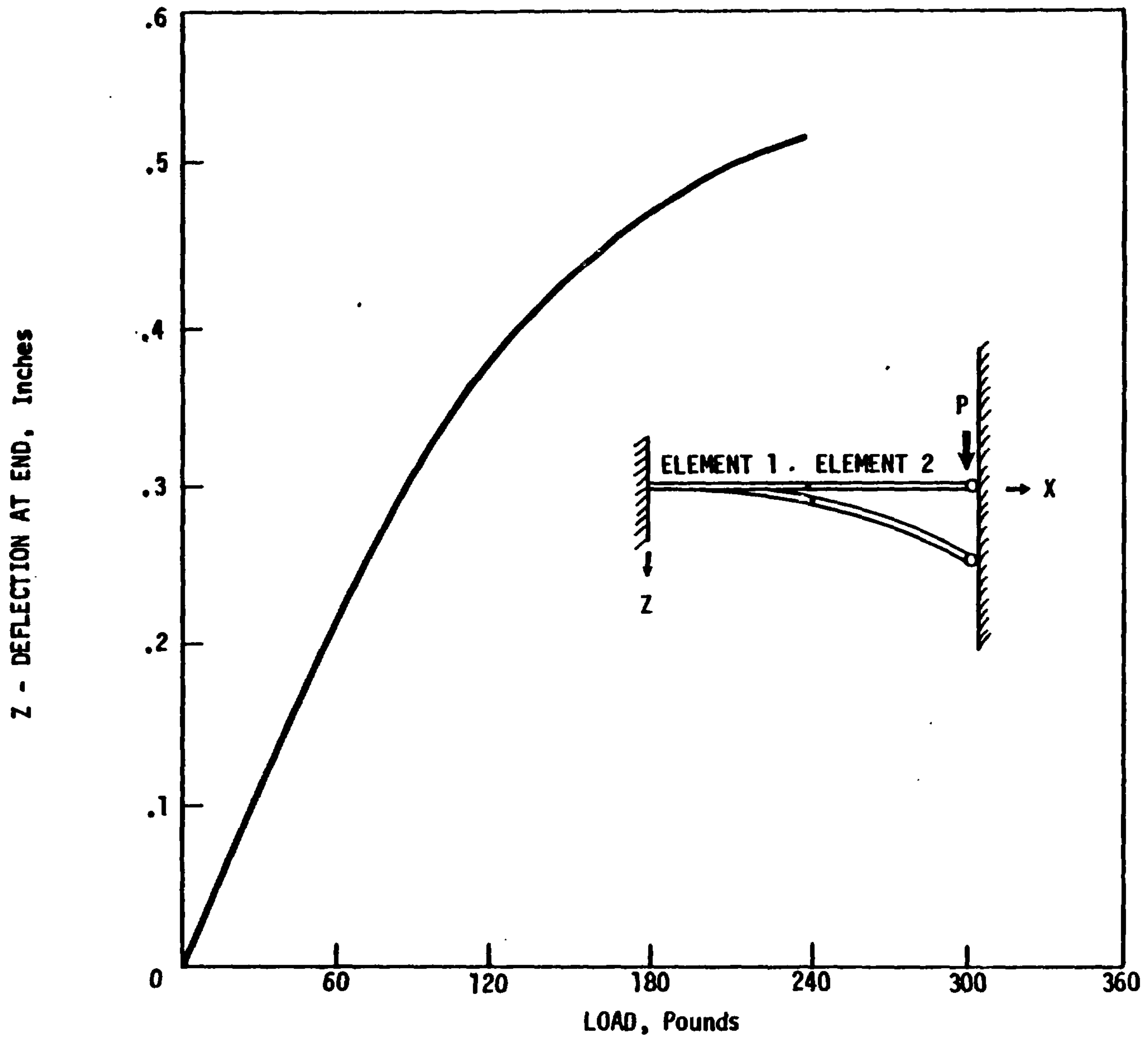


Figure 3: Load Deflection Behavior for Restrained - End Beam (Figure 1b)

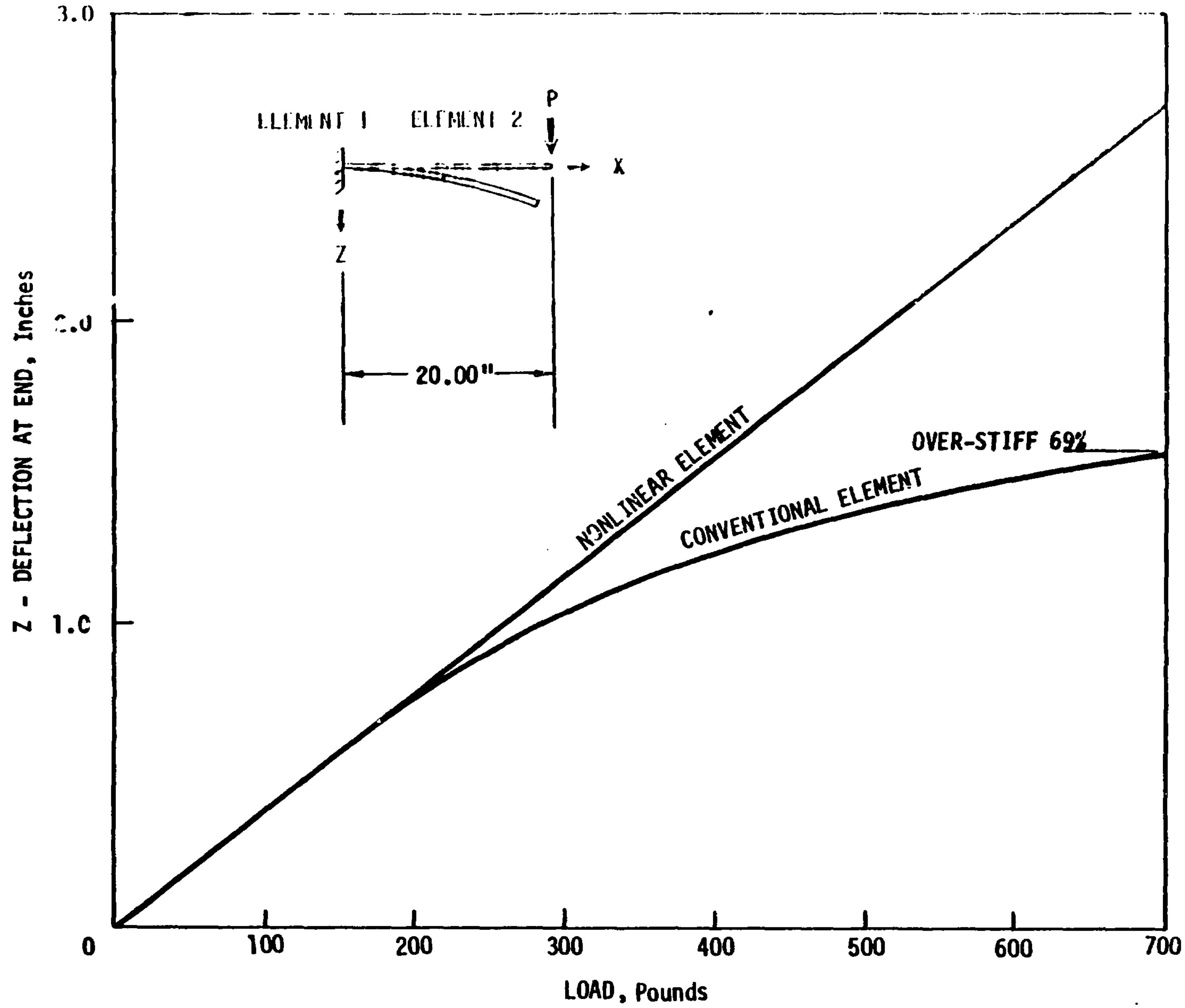
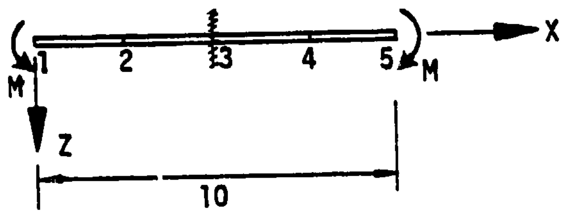
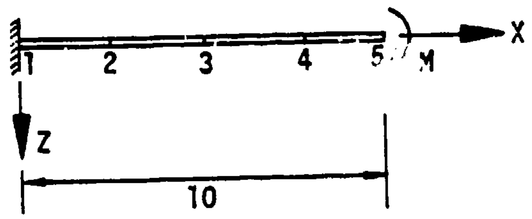


Figure 4: Comparison of Load-Deflection Behavior for Nonlinear and Conventional Elements



$EA = 1.0 \times 10^7$
$EI = 6.6 \times 10^5$
$GA = 1.11 \times 10^6$



(a) Double (Symmetrical) Cantilever

(b) Simple Cantilever

Figure 5: Moment Loaded Single Element Cantilever Beam Examples

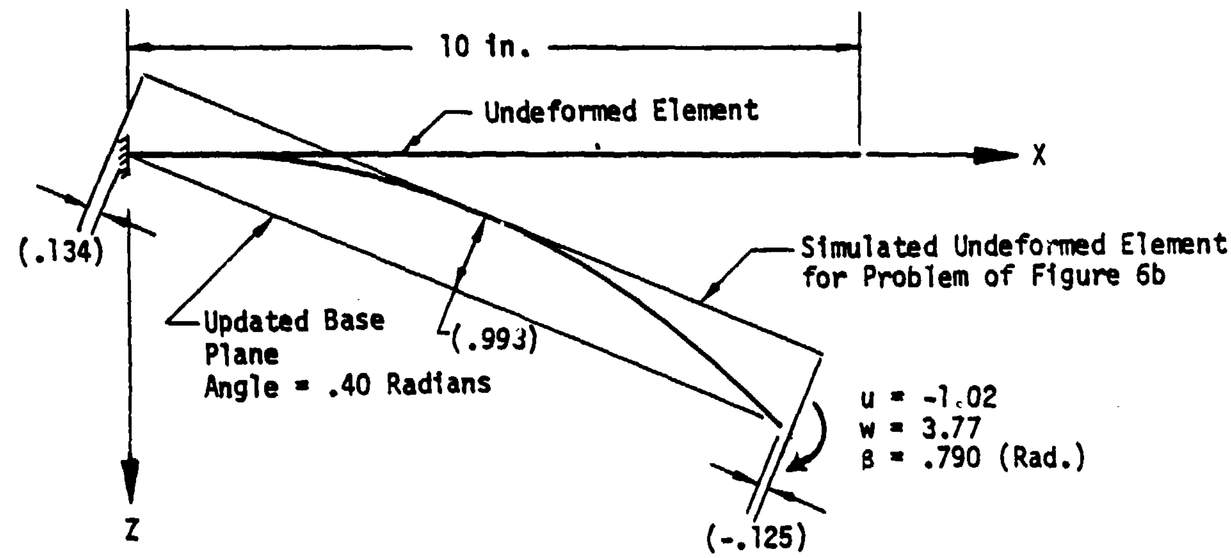


Figure 6a: Simple Cantilever

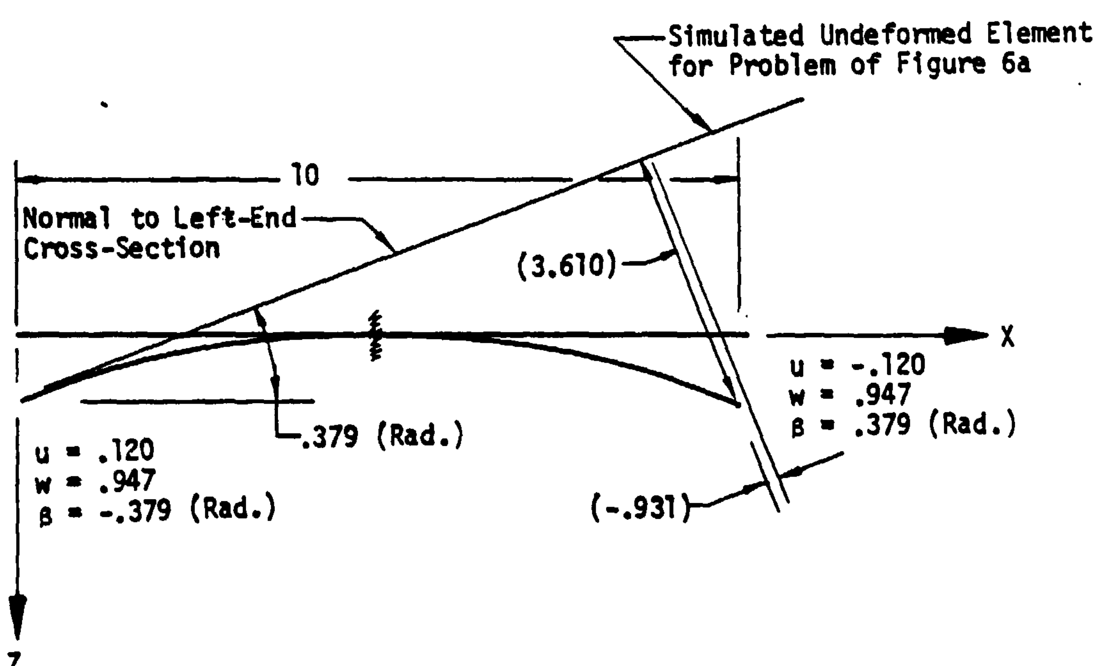


Figure 6b: Double (Symmetrical) Cantilever

Figure 6: Displacements of Single Element Cantilever Beam

Table 1: Summary of Nonlinear Solution Procedure Controls

(a) OPTION CONTROLS [1]

STATIC PERTURBATION ORDER	PATH PARAMETER		"ALTERNATE" ITERATION		LINEARIZED SOLUTION [3]	"CONVENTIONAL" ELEMENT
	LOAD TYPE	ENERGY TYPE	NO	YES		
1 [2]	[2] OPT2=1 OPT1=-2	No [2]	OPT1= -1	OPT1= 1	Yes OPT2=0	Yes OPT2=-1
2	Yes OPT2=20	Yes OPT2=2	OPT1= -1	OPT1= 1	No	No
3	Yes OPT2=30	Yes OPT2=3	OPT1= -1	OPT1= 1	No	No

[1] OPT1 and OPT2 are program input data.

[2] First order static perturbation is standard stepwise solution - Path parameter is always load-type in this case

[3] Linearized solution omits all types of nonlinearities.

[4] Partial list given; includes principal control inputs.

(b) NUMERICAL CONTROLS [4]

INPUT DATA NAME	PROCESS CONTROLLED BY INPUT DATA
EPSCON:	Allowable Error (Residual Loads) For Convergence
DALIM:	Rotation of Element From Base Plane at Which Geometrical Update is Performed.
EQCHK:	Rate of Divergence at Which Stiffness Matrix Update is Performed.
RACHK:	Error (Residual Loads) at Which Stiffness Matrix Update is Performed.
ITRLIM, UPLIM:	Iteration and Updating Counts at Which Solution is Aborted
DKCT:	Required Minimum Number of Iterations Between Stiffness Matrix Updates (Over-Rides EQCHK, RACHK)

Table 2: Solution Data for Stepwise Geometrical Updating - Cantilever Beam (Figure 1a)

LOAD (Pounds)	END DEFLECTION (Inches)	BASELINE INCLINATION (Radian)		AXIAL FORCE (Pounds)			NO. OF ITERATIONS	ERROR MEASURE (%)
		ELEMENT 1	ELEMENT 2	NODE 1	NODE 5 ▷	NODE 9		
0	0	0	0	0	0	0	—	—
120	.4577	0	0	0	0	0	0	.14
240	.913	.0276	.0639	6.59	6.57/15.34	15.33	3	.165
300	1.14	.0276	.0639	8.32	8.34/19.2	19.2	3	1.07
360	1.37	.0412	.0956	14.7	14.6/34.4	34.4	5	.56
420	1.59	.0412	.0956	16.9	16.7/40.1	40.1	5	1.29
480	1.81	.0548	.127	26.2	26.2/60.8	60.8	7	.209
540	2.04	.0548	.127	29.4	29.2/68.4	68.4	7	.623
600	2.26	.0683	.158	40.4	40.0/94.7	94.6	7	1.24
660	2.48	.0683	.158	45.6	46.0/103.9	103.9	9	.65
720	2.70	.0816	.189	60.0	61.0/134.8	135.1	9	1.01

Table 3: Solution Data for Single Large Load Step for Cantilever Beam (Figure 1a)

0	0	0	0	0	0	0	—	—
720	2.75	0	0	0	0/0	0	0	.14
720	2.69	.083	.193	58.9	58.3/138.5	138.2	5	1.4

▷ Node 5 Values Refer to Element 1 & 2, Figure 1.

Table 4: Solution Data for Delayed Geometrical Updating - Cantilever Beam (Figure 1a)

LOAD (Pounds)	END DEFLECTION (Inches)	BASELINE INCLINATION (Radian)		AXIAL FORCE (Pounds)			NO. OF ITERATIONS	ERROR MEASURE (%)
		ELEMENT 1	ELEMENT 2	NODE 1	NODE 5 ▷	NODE 9		
0	0	0	0	0	0	0		
120	.46	0	0	0	0	0	0	.14
240	.92	0	0	0	0	0	1	.08
300	1.14	0	0	0	0	0	1	.09
360	1.37	0	0	0	0	0	1	.06
420	1.60	0	0	0	0	0	1	.15
480	1.83	0	0	0	0	0	1	.07
540	2.06	0	0	0	0	0	1	.04
600	2.29	0	0	0	0	0	1	[illegible]
660	2.50	.0759	.176	51.3	51.3/116.0	116.0	7	.99
720	2.69	.0759	.176	55.2	55.2/126.2	126.2	11	.76

Table 5: Solution Data for Restrained - End Beam (Figure 1b)

0	0	0	0	0	0	0	—	—
60	.214	.0069	.0160	302	302 /302	302	7	.053
120	.382	.0114	.0262	844	844 /846	846	5	.318
180	.472	.0114	.0262	2000	2000 /2002	2002	13	1.38
240	.522	.0181	.0410	3239	3239 /3242	3242	5	.416

Table 6: Solution Data for "Conventional" Element (Figure 1a)

0	0	0	0	0	0	0	—	—
60	.22684	.00690	.0160	-315	550 /-300	336	3	.38
120	.447	.0130	.0304	-314	-54 /-2578	1445	5	.95
180	.6608	.0185	.0433	- 37	-1709/-6700	3255	9	.97
240	.868	.0239	.0557	440	-4214 /-12420	56541	13	.94

▷ Node 5 Values Refer to Element 1 & 2, Figure 1.

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APPENDIX A

Static Perturbation Path Parameter

This Appendix outlines briefly the use of the load-based and deformation-based path parameters in the static perturbation method. The equations given herein are programmed in the 2-D beam code (see Section 2.2).

The static perturbation method uses Taylor series to represent the incremental displacement vector ΔQ and the incremental load vector ΔP .

$$\Delta P = \dot{P} s + \frac{1}{2}\ddot{P} s^2 + \frac{1}{6}\dddot{P} s^3 + \cdots$$

$$\Delta Q = \dot{Q} s + \frac{1}{2}\ddot{Q} s^2 + \frac{1}{6}\dddot{Q} s^3 + \cdots$$

where $(\dot{\ })$ denotes differentiation with respect to the path parameter S. It is convenient to represent the P derivatives as follows:

$$\dot{P} = \dot{\lambda} P^{\circ}$$

$$\ddot{P} = \ddot{\lambda} P^{\circ}$$

$$\dddot{P} = \dddot{\lambda} P^{\circ}$$

—

—

so that

$$\Delta P = P^{\circ}\left(\dot{\lambda} s + \frac{1}{2}\ddot{\lambda} s^2 + \frac{1}{6}\dddot{\lambda} s^3 + \cdots\right)$$

and to set P^{0} equal to the load increment

$$P^{\circ} = \Delta P$$

such that

$$\dot{\lambda} S + \frac{1}{2}\ddot{\lambda} S^2 + \frac{1}{6}\dddot{\lambda} S^3 = 1$$

The values of the Q derivatives can be shown to be given by

$$\dot{Q} = K^{-1}\dot{\lambda} P^*$$
$$\ddot{Q} = K^{-1}(\ddot{\lambda} P^* - \dot{K}\dot{Q})$$
$$\dddot{Q} = K^{-1}(\dddot{\lambda} P^* - 2\dot{K}\ddot{Q} - \ddot{K}\dot{Q})$$
$$- \quad - \quad - \quad - \quad -$$

in which K is the tangent stiffness matrix and $\dot{K}$ and $\ddot{K}$ are, respectively, the first and second derivatives of the K matrix with respect to the path parameters (accomplished by chain-rule differentiation: $\partial K/\partial S = (\partial K/\partial Q)\cdot\dot{Q}$). The solution process requires solution first for $\dot{Q}$, followed by calculation of $\dot{K}$, followed by calculation of, in order, $\ddot{Q}$, $\ddot{K}$, and finally, $\dddot{Q}$. The Taylor series then gives the value of the vector ΔQ.

The load-based path parameter sets

$$\ddot{\lambda} = \dddot{\lambda} = 0 \quad , \quad \dot{\lambda} = 1$$

$$\Delta P = P^* \dot{\lambda} S = S P^*$$

$$S = 1$$

and allows calculations of ΔQ immediately that $\ddot{Q}$ and $\dddot{Q}$ are known. The first, second, and third order approaches retain, respectively, the terms S, S^2, and S^3. This is a relatively simple approach to implement.

The deformation - based path parameter is much more complex because the values of $\dot{\lambda}, \ddot{\lambda}, \dddot{\lambda}$, and S are unknown. This approach is based on the definition of S below, using the tangent stiffness matrix,

$$S^2 = \Delta Q^T K \Delta Q$$

where $(\ )^T$ denotes the row-vector (transpose). Thus, S is roughly proportional to the incremental displacement amplitude. To explain the solution process for this case it is necessary to make a number of definitions, as given below.

$$Q^\circ = K^{-1} P^\circ$$

$\dot{K}^\circ = \dot{K}(Q^\circ)$ (evaluate $\dot{K}$ with Q^0 in place of $\dot{Q}$)

$P1^\circ = \dot{K}^\circ Q^\circ \; ; \; Q1^\circ = K^{-1} P1^\circ$

$\ddot{K}1^\circ = \ddot{K}_1(Q^\circ)$ (evaluate $\ddot{K}$ with Q^0 in place of $\dot{Q}$ in the $\dot{Q}$ - dependent part of $\ddot{K}$, called $\ddot{K}_1$)

$\ddot{K}2^\circ = \ddot{K}_2(Q^\circ)$ (evaluate $\ddot{K}$ with Q^0 in place of $\ddot{Q}$ in the $\ddot{Q}$ - dependent part of $\ddot{K}$, called here $\ddot{K}_2$)

$\ddot{K}3^\circ = \ddot{K}_2(Q1^\circ)$ (evaluate $\ddot{K}$ with $Q1^0$ in place of $\ddot{Q}$ in $\ddot{K}_2$)

with these definitions,

$$\ddot{K} = \dot{\lambda}^2 \ddot{K}1^\circ + \ddot{\lambda} \ddot{K}2^\circ - \dot{\lambda}^2 \ddot{K}3^\circ$$

and (see equation for $\dddot{Q}$),

$$2\dot{K}\Delta\ddot{Q} + \ddot{K}\Delta\dot{Q} = 2\dot{\lambda}\dot{K}^\circ(\dot{\lambda}Q^\circ - \dot{\lambda}^2 Q1^\circ) + \dot{\lambda}Q^\circ(\dot{\lambda}^2 \ddot{K}1^\circ + \ddot{\lambda}\ddot{K}2^\circ - \dot{\lambda}^2 \ddot{K}3^\circ)$$

It is convenient to rewrite this as

$$2\dot{K}\Delta\ddot{Q} + \ddot{K}\Delta\dot{Q} = \dot{\lambda}\ddot{\lambda}(2P1^\circ + P4^\circ) + \dot{\lambda}^3(P3^\circ - 2P2^\circ - P5^\circ)$$

where the various "PN^O" are derived by substitutions for the various "K" type matrices above.

Finally, we define the vectors

$$QQ^\circ = K^{-1}(2P1^\circ + P4^\circ)$$

$$QQQ^\circ = K^{-1}(P3^\circ - 2P2^\circ - P5^\circ)$$

with the result

$$\Delta\dddot{Q} = \dddot{\Lambda}\, Q^\circ - \dot{\Lambda}^3\, QQQ^\circ - \dot{\Lambda}\ddot{\Lambda}\, QQ^\circ$$

It is seen that the entire evaluation depends on knowing $\dot{\Lambda}$, $\ddot{\Lambda}$, $\dddot{\Lambda}$ and S. These values can be computed (with much difficulty) from the basic definition of S^2 given above. The equations are complicated are are omitted here. The final result is

$$\Delta Q = Q^\circ - Q1^\circ\left(\tfrac{1}{2}\dot{\Lambda} S^2\right) - QQ^\circ\left(\tfrac{1}{6}\dot{\Lambda}\ddot{\Lambda} S^3\right) - QQQ^\circ\left(\tfrac{1}{6}\dot{\Lambda}^3 S^3\right)$$

The second order procedure keeps only S^2, and the third order procedure keeps both S^3 terms.

APPENDIX B

Euler Angle Theory for Beam Element

This Appendix describes the use of Euler angles in the determination of the deformations of beam elements, emphasizing the physical nature and basis for this approach. The derivation of the strain-displacement equations in terms of the Euler angles is also described briefly.

Figure B1 shows a beam cross-section in the initial undeformed state. The section is shown rectangular only to aid visual clarity. The xyz triad is oriented such that x is the beam centerline and the y and z axes are the axes of bending. The shear deformations associated with bending occur in the xy and xz planes.

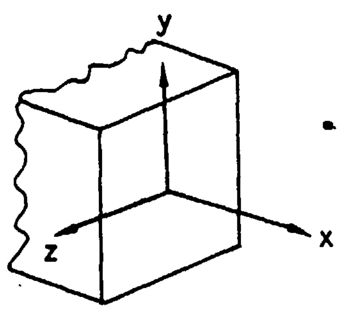


Figure B1 - Undeformed Section

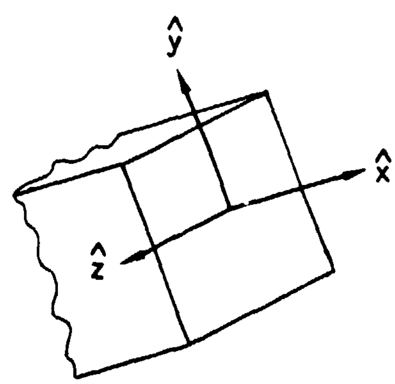
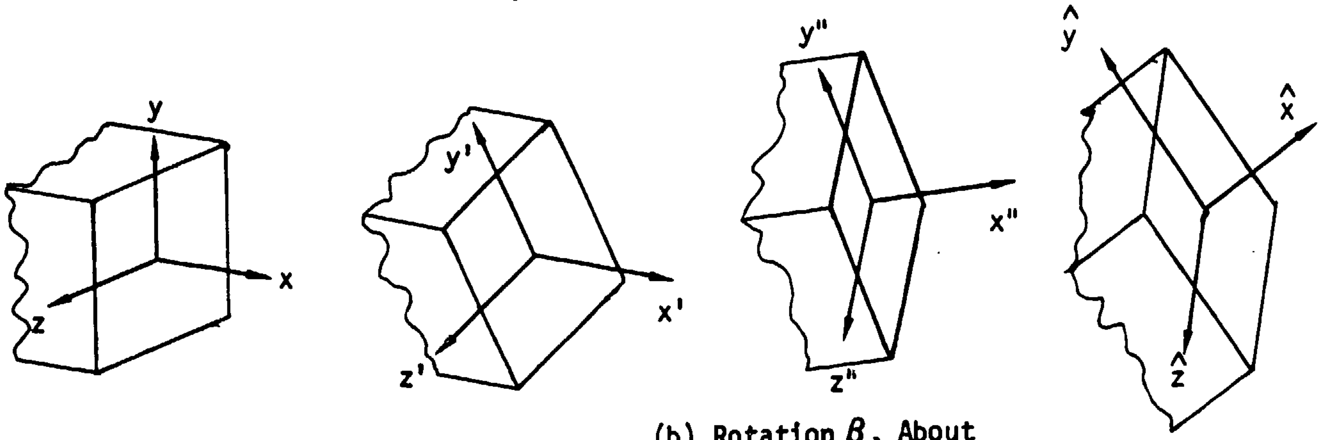


Figure B2 - Deformed Section

Figure B2 shows the cross-section after displacement. The triad $\hat{x}\hat{y}\hat{z}$ has followed the motion of the section, as described below. This "convected" $\hat{x}\hat{y}\hat{z}$ triad is orthogonal, but, as will be shown below, is not truly "imbedded" in the material. The xyz triad is carried into the $\hat{x}\hat{y}\hat{z}$ triad by means of the sequence of Euler angle rotations, in the following manner.

Allow first a rotation β_1 of the undeformed section about the x axis, as shown in Figure B3a. This results in a new triad, denoted in the figure by x'y'z'. Of course, x' and x are identical.



(a) Rotation β_1 About x Axis

(b) Rotation β_2 About y' Axis

(c) Rotation β_3 About z" Axis

Figure B3 - Euler Rotations

Next a rotation β_2 takes place about the y' axis, resulting in a new triad x"y"z". Finally, a rotation β_3 about z" results in the triad $\hat{x}\hat{y}\hat{z}$ which describes the deformed orientation of the cross-section. The angles β_1, β_2, and β_3 are Euler angles. They are restricted to the sequence of axes (1-2-3) in the special meaning and sequence illustrated. For the general case, there are 12 possible Euler angle sequences, but only the (1-2-3) sequence described above is needed for the present discussion. The importance of this rotation description in the present application is that it: (1) fully accounts for the effect of large rotations in reorienting the beam cross-sectional axes; (2) avoids any errors due to cartesian rotation incrementation.

The three Euler rotations are not in general those of conventional beam twist and bending, although for small total rotation magnitudes they are indistinguishable from these quantities. For large rotations they are not correctly viewed in this way, however, and for this reason it is incorrect to attribute the beam twisting and bending stiffness properties to the rotation values β_1, β_2, and β_3. Derivation of the beam twist and bending moments instead are derived by a rigorous process described later. While the Euler angle representation has the advantage of rigor, it does not provide a fully satisfactory deformation description from a physical viewpoint. To obtain the needed physical interpretation, it is necessary to compute small incremental rotations about the beam bending and twisting axes, superimposed on a previously accumulated large rotation state. Figure B4 attempts to illustrate this view of the deformation. The figure shows the deformed section with the associated triad $\hat{x}\hat{y}\hat{z}$. The triad is essentially identical with the beam section axes and centerline, deviating only slightly from these axes due to the shear strains (angles of the order of 0.3 degrees). The figure indicates that a small rotation superimposed on the section in its $\hat{x}\hat{y}\hat{z}$ orientation can be viewed in two ways: as a cartesian rotation taking place about the $\hat{x}$,$\hat{y}$, and $\hat{z}$ axes, or as an increment in the Euler angle values β_1, β_2, and β_3. The $\delta\beta_1$, $\delta\beta_2$, and $\delta\beta_3$ must be viewed as taking place about the axis systems which were defined <u>during</u> the entire deformation process—the xyz and x'y'z' and x"y"z" systems. This is indicated in the figure. These rotations are not physically meaningful, but can be used in a mathematically rigorous way to

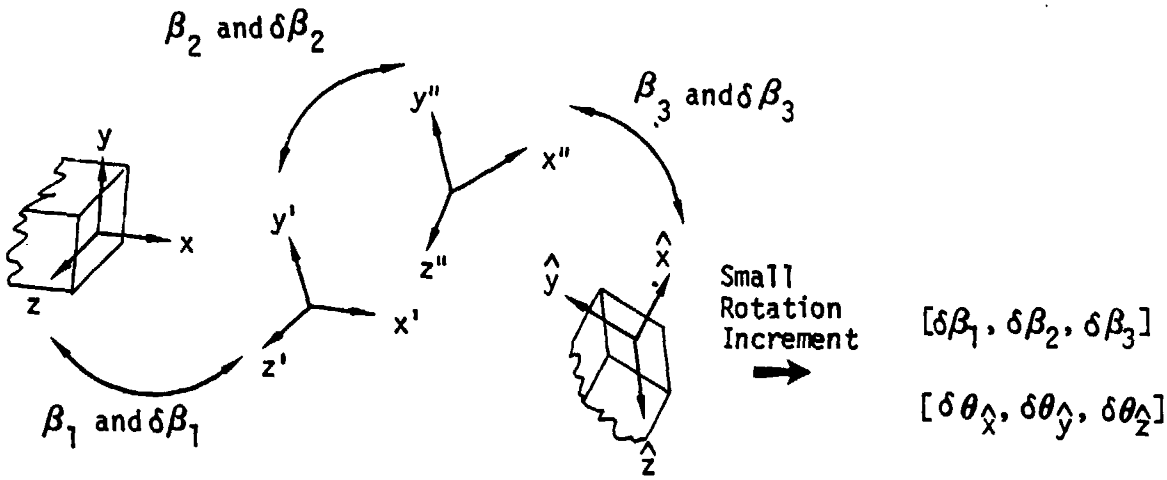


Figure B4 - Small Superimposed Rotations

describe the nonlinear deformation state. On the other hand, the cartesian small rotations $\delta\theta_{\hat{x}}$, $\delta\theta_{\hat{y}}$, and $\delta\theta_{\hat{z}}$ are physically meaningful because they indicate increments of twist and bending deformation; and in dynamic analysis they are correctly associated with the beam cross-section rotational inertia properties. To combine the rigorous nonlinear deformation description with the physically meaningful one a special type of geometrical transformation is available. This is expressed by

$$\begin{Bmatrix} \delta\theta_{\hat{x}} \\ \delta\theta_{\hat{y}} \\ \delta\theta_{\hat{z}} \end{Bmatrix} = [\pi] \begin{Bmatrix} \delta\beta_1 \\ \delta\beta_2 \\ \delta\beta_3 \end{Bmatrix}$$

The π matrix is a function of the total accumulated β_1, β_2, and β_3. A suitable nonlinear analysis approach must be based on deformations described rigorously by β_1, β_2, β_3 and $\delta\beta_1$, $\delta\beta_2$, $\delta\beta_3$, and in addition contain transformations to provide numerical results in the form $\delta\theta_{\hat{x}}$, $\delta\theta_{\hat{y}}$, $\delta\theta_{\hat{z}}$. The π matrix is the means for accomplishing this.

A refinement of the above approach can be made to eliminate the small approximation in the meaning of the $\delta\theta$ values which is due to the fact that the $\hat{x}\hat{y}\hat{z}$ triad is not truly identical to the conventional beam twist and bending axes.

Figure B5 shows a beam cross-section with a large shear deformation. The actual "material" cross-section is shown with the heavy

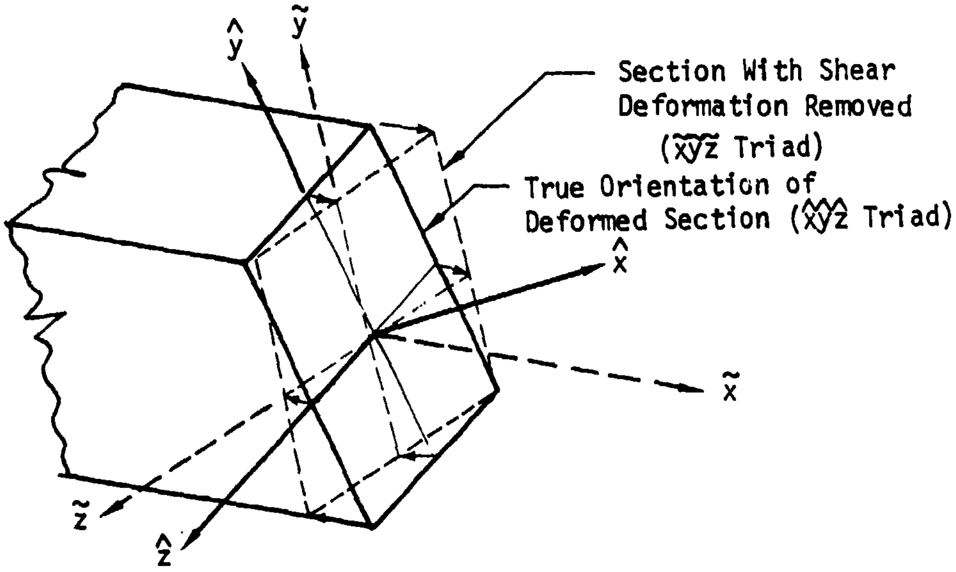


Figure B5 - Beam Cross-Section Local Coordinate Systems

lines. The light dashed lines show the section with the shear deformation removed; the displacement vectors show the small cross-section motions necessary to remove the shear angles. The section bisecting axes are shown for both cases as an aid to pictorial clarity. The triad $\hat{x}\hat{y}\hat{z}$ coincides with the "material-y and z" cross-section axes; it is shown by the very solid heavy coordinate axis lines. The $\hat{x}$ axis does not coincide with the beam material centerline. The triad $\tilde{x}\tilde{y}\tilde{z}$ coincides with the section material-y and z axes with the shear deformation removed. Since this section is normal to the beam centerline, the $\tilde{x}$ axis is colinear with the centerline. The $\tilde{x}\tilde{y}\tilde{z}$ triad is shown by the very heavy dashed lines. It is desirable to keep track of the $\tilde{x}\tilde{y}\tilde{z}$ triad in the solution process, because small cartesian rotation increments referred to this triad are precisely the "nominal" section twist and bending rotations. This is easily done as follows: first presume that Euler angles β_{M1}, β_{M2}, β_{M3} define the triad $\tilde{x}\tilde{y}\tilde{z}$; then (1) compute the (see fig. B4) rotation increment $\delta\theta_{\hat{x}}$, $\delta\theta_{\hat{y}}$, $\delta\theta_{\hat{z}}$; (2) compute the Euler increment $\delta\beta_1$, $\delta\beta_2$, $\delta\beta_3$ using the π matrix and sum to obtain the total β_1, β_2, β_3; (3) from the strain-displacement equations, compute the shear strain increment; (4) subtract this increment from the $\delta\theta_{\hat{y}}$ and $\delta\theta_{\hat{z}}$ values, calling the results $\delta\theta_{M\hat{y}}$ and $\delta\theta_{M\hat{z}}$ (these are the bending rotation increments which correspond to conventional beam theory); (5) using the π_M transformation (the π-type transformation which is computed using the Euler angles (β_{M1}, β_{M2}, β_{M3}), compute the incremental values $\delta\beta_{M1}$, $\delta\beta_{M2}$, $\delta\beta_{M3}$; (6) sum to compute the total values β_{M1}, β_{M2}, β_{M3}. This procedure maintains the totals of two sets of Euler angles: β_1, β_2, β_3 defining the xyz triad (needed to compute the deformations); and β_{M1}, β_{M2}, β_{M3} defining the xyz axes, which are the conventional (convected) axes of beam theory.

This Appendix closes with a brief outline of the procedure for the derivation of the nonlinear strain-displacement equations. These are derived in terms of the Euler angles β_1, β_2, β_3, because only these angles provide a rigorous representation of the total rotations of the beam material elements. The beam centerline has the displacement vector $\mathbf{u}$. Denoting by $\mathbf{a}_i$ the basis vectors of the undeformed beam element, and by x^i the initial undeformed coordinates of the element, the initial undeformed beam centerline has the position vector, respectively,

$$\mathbf{R}_{o_{\mathcal{L}}} = x^i \mathbf{a}_i$$

$$\mathbf{R}_{\mathcal{L}} = (x^i + u^i)\,\mathbf{a}_i$$

The vector $\mathbf{V}$ is defined to be the additional displacement of a material point which is off the centerline. Hence for an arbitrary point of material on a beam cross-section,

$$\mathbf{R} = (x^i + u^i)\,\mathbf{a}_i + \mathbf{V}$$

The Euler angles are used to define $\mathbf{V}$. Referring to Figure B3 and defining the basis vector sets $\mathbf{a}_i$, $\mathbf{a}'_i$, and $\mathbf{a}''_i$ as belonging, respectively, to triads xyz, x'y'z', and x"y"z", $\mathbf{V}$ can be written

$$\mathbf{V} = \beta_1(y\mathbf{a}_3 - z\mathbf{a}_2) + \beta_2(z'\mathbf{a}'_1) - \beta_3(y''\mathbf{a}''_1)$$

It is noted that actual values of y' and y", and z' and z", respectively, for a given point on the cross-section, are identical to the values of y and z. This is because all of these coordinates are defined by "following" the material. Hence, with y, and z interpreted as initial coordinates of a point on the beam cross-section, $\mathbb{V}$ is more simply written

$$\mathbb{V} = \beta_1 (y \mathbf{a}_3 - z \mathbf{a}_2) + \beta_2 z \mathbf{a}_1' - \beta_3 y \mathbf{a}_1'' .$$

The values of $\mathbf{a}_1'$ and $\mathbf{a}_1''$ can be represented in terms of the initial basis vectors $\mathbf{a}_i$, and the Euler rotations β_1, β_2, and β_3. The derivation is lengthy, and is not given here. The result is

$$\mathbb{V} = \beta_1 (y \mathbf{a}_3 - z \mathbf{a}_2) + \beta_2 z \mathbf{a}_1$$
$$- \beta_3 y \left(\mathbf{a}_1 \cos\beta_2 + \mathbf{a}_2 \sin\beta_1 \sin\beta_2 - \mathbf{a}_3 \sin\beta_2 \cos\beta_1 \right)$$

The value of the position vector $\mathbb{R}$ is now known for any point on the element, in terms of the centerline displacement u^i, the Euler rotations β_i, and the "material" coordinates x,y, and z. By differentiating $\mathbb{R}$ with respect to x, y, and z, there are obtained an important set of vectors, called the basis vectors of the deformed material coordinate system. These vectors contain a complete description of the deformation state. It is seen that these basis vectors contain derivatives of the β_i and the u^i with respect to the x, y, and z coordinates. The x-derivative, in particular, is important in defining the deformation of the beam element. The theory of the derivation process follows reference 13. Simplifying approximation can be made, and the details and results are too lengthy to include here. It is simply noted that this means of developing the strain equations is exact and includes all of the effects of nonlinearity.

The brief description of the nonlinear beam deformation given above has the purpose of illustrating that the deformation is determined in terms of the Euler angles rather than in terms of conventional cartesian rotations. This formulation is r   ous for large rotations and does not suffer any inaccuracies due to summing angular motions.

APPENDIX C

Summary of Proposal for Contract F49620-79-C-0057

This Appendix contains the Technical Approach (Section 2) and two appendices from "Technical Proposal - Program for Nonlinear Structural Analysis", submitted to AFOSR in August, 1978. This proposal is the basis of the current AFOSR contract on nonlinear finite element research. The discussions herein are intended to provide background and supplementary information supporting our present report.

2.0 TECHNICAL APPROACH

The technical approach builds on existing research results. The element technology to be used is basically that of the stability elements (References 1, 2, 3). The solution procedure technology will be based on the nonlinear-step static perturbation procedure (References 4, 5, 6). The static perturbation procedure has been demonstrated to be a superior solution method for strongly nonlinear static problems. For dynamic analysis, an extension of the procedure has been developed in the current AFOSR contract. Numerical data demonstrating the superiority of this approach are given in this proposal. The goal of this proposed research is to merge these two technologies into a working pilot computer program.

2.1 Technical Requirements

The principal features required of the overall approach, in regard to applicability to nonlinear problems, are listed below:

Element Technology (References 1, 2, 3):

1. Elements are required whose displacement function formulation prevents anomalous (overstiff) behavior due to nonlinear strains. These are called stability elements, and utilize extended forms of axial/membrane displacement functions, in conjunction with conventional bending deformation forms.

2. Element strain calculations must be made on a total strain basis, to avoid cumulative errors due to summing increments. This is required to allow the development of large rotations and nonlinearities.

3. Element displacement functions must be referred to convected coordinate systems. This avoids exchange of axial/membrane and bending displacement roles (e.g., u and w exchange meaning as the rotation becomes large) in the large rotation state, and permits the simplifying assumption of shallowness in forming the nonlinear strain equations for shells or highly deformed beams and plates.

4. Residual force evaluation and equilibrium corrections must include the effects of element strains and geometry changes.

Static Solution Procedure Technology (Referances 4, 5, 6)

1. The characteristic problem of excessive residual forces, with consequent slow convergence or divergence in problem solutions, must be avoided while retaining reasonably large step size. This requires the use of a nonlinear stepwise solution procedure.

2. The solution procedure must be compatible with the stability elements, in particular with the convected coordinate system approach.

3. The solution procedure should include a means of automatic, internal, computation of step size. Gains in solution economy from this feature can be very large.

Dynamic Solution Procedure Technology

1. As noted above, the method must incorporate a nonlinear step. Automatic, internal step size selection should be incorporated insofar as is possible.

2. The solution procedure must be compatible with the stability elements, particularly as regards the convected coordinate system and the residual load calculations.

3. The solution procedure must not require stepwise inversion of the structural stiffness matrix. Instead, inversion of the mass matrix must be used, for reasons of economy.

The proposed technical approach meets all of these goals. All of the technical developments required in the proposed research are reasonably well proven as regards accuracy and practicability. Hence, their merging into a single computer program appears to involve little risk. The major gains from the proposed work should be in the matter of evaluation of the overall technical approach on specialized problems. The particular types of problems for which this approach is required have the following characteristics:

- The equilibrium is governed primarily by nonlinear axial/membrane stresses induced by bending rotations.

- The axial/membrane stresses vary rapidly over the structure. An example is the type of buckle pattern which occurs typically in axially compressed cylinders, in higher vibration modes of beams and plates, and in short wave length vibration of shells. This includes also structures which undergo a near-uniform nonlinear axial/membrane stress, due to boundary constraint. However, this type of problem can often be solved adequately with conventional methods.

- Boundary constraints on stretched membranes, plates, and shells can cause rapidly varying local rotations and nonlinear strains at locations where the boundaries undergo sharp shape changes (corners, etc.). Hence, this type of problem requires the stability type of element in cases where accurate stress analysis within these zones is desired.

- Prediction of instability behavior in general requires a nonlinear-step type of solution procedure. The effects of nonlinear effects such as mode switching, limit points, snap-through, and buckling influenced by prior information, are not usually amenable to eigenvalue analysis. The alternative of asymptotic instability analysis involves very difficult calculations. For either case a competent nonlinear step procedure is required to obtain problem solutions. The case of follower loads also falls in this category.

- The important problems of nonlinear oscillations (e.g., limit cycle predictions) are not generally solvable analytically. The finite element approach with a competent nonlinear dynamic solution procedure probably offers the only practical approach to this problem. This approach can evaluate nonlinear responses for the "almost periodic" case as well as the true periodic case, and thus provide much information about dynamic behavior and potential large amplitude dynamic responses of nonlinear structures.

2.2 Technical Method Descriptions

This section outlines briefly some of the details, and proposed modifications, of the technical methodologies to be merged in this contract: the stability elements; and the static perturbation nonlinear stepwise method, as applied to static and dynamic problems.

Stability Elements: The present computer program (References 2, 3) for the stability elements (hereinafter called HMN elements, as in these references) has demonstrated superior accuracy, as compared to conventional elements, for the case of large bending deformations. In addition, the original work of Haftka, Mallett, and Nachbar (Reference 1) showed that a marked accuracy gain was obtained from the stability-type of beam element in application to buckling solutions for beam-columns. The basic cause of the accuracy improvement gained from the stability and HMN elements is that, due to the "strain-smoothing" enforced on the nonlinear strains by the high order membrane/axial displacement functions, the elements' strain energies are reduced to near minimum values, consistent with the magnitude of the overall deformation state. Since the improvement is effected through the membrane strains, to which correspond very large stiffness terms, the accuracy gain can be very large. In the case of stepwise linear, nonlinear problem solutions, the gain is effected through the residual load magnitudes. In the case of eigenvalue solutions, it occurs in the eigenvalue itself.

The extension from one-dimensional (Reference 1) to two-dimensional elements (References 2, 3) creates many difficulties in applying the original stability element concepts. This difficulty resides primarily in the fact that the added, higher order, membrane displacement functions (the basic approach of the stability elements) are nonzero over the entire two-dimensional element, including its boundaries. If one attempts to minimize the strain energy on the elemental level, which would be a relatively simple task, in general inter-element displacement incompatibilities will be created. The alternative is to derive specific, explicit constraints on the added functions, such that specific higher order terms in the strains (ε_x, ε_y, γ_{xy}) are set to zero, without violating inter-element compatibility. This alternative becomes very complicated, but nevertheless was the one adopted in the HMN element work of references 2 and 3. The work was very successful for large bending deformations, and less so for large torsional deformations. The reason for this is that the specific higher order membrane functions which compensate for large torsion (nonlinear γ_{xy}) may in some cases cause undesirable higher order direct strains ε_x and ε_y. The requirement to allow arbitrary element orientations relative to any structural deformation pattern causes this difficulty to go both ways: HMN compensations for large torsion may create undesirable direct strains; HMN compensation for large bending may cause undesirable shear strains. The physical meaning of this situation is that either large bending or large torsion will in actual practice cause a trade to occur between higher order shear and direct membrane strains, such that the structural potential energy is minimized. The failing of the HMN elements of references

2 and 3 is that they deal with the strains separately, rather than with the total deformation state.

There are several alternatives for continuing work on the present elements. First, it is recognized that they perform well as they are currently formulated. They might perform better with the torsion-membrane shear interactions removed, which would be very simple to accomplish. Finally, a method for obtaining the shear-direct strain "trade" could be devised and implemented. It appears that before any of these alternatives are pursued, another option should be investigated. Figure 1 shows an isoparametric quadrilateral (of the general type of Reference 9) which has a special relationship between nodes and displacement freedoms. The element has 17 nodes, of which only 8 nodes are used to define the bending freedoms, and all 17 are used to define the membrane freedoms. This element will have higher order membrane strains, to compensate the nonlinear strains due to bending and torsion, by virtue of its extra 9 membrane only nodes. Thus it is basically a stability element in the sense defined by reference 1. The element has an advantage over the HMN elements of references 2 and 3 because its higher order freedoms are nodally defined, and thus can be committed to the global solution process without creating inter-element incompatibilities. The displacement functions for the 8 node bending behavior will be those of references 2 and 9. Those for the 17 node membrane functions will follow the conventional forms for isoparametric-elements. It is proposed to use this element in the research described herein.

Figure 1 also shows a beam element which will be developed. This element differs from conventional beam (cubic displacement) elements. It has identical displacements to those of one side of the quadrilateral. This will make the two elements nodally compatible in problem solutions.

The work of references 2 and 3 includes many features which are not dependent on the explicit strain constraints of the HMN elements. These include the developed solution procedure details, geometrical transformations, and nonlinear shell equations. All of these are applicable to the element of Figure 1, and will be retained in the proposed work.

The updating (or convection) of the element coordinate system is presently done for every iteration of every solution step. This is costly, and is not always necessary, as this set of transformations is only important when significant rotations have occurred during the step. It is proposed to make this updating conditional on the rotation magnitudes. The residuals will be referred to the start-of-step coordinate system unless the updating is found to be required. In addition, it appears that when it is necessary (rotations are large) to update the element coordinate system, also the solution coordinate systems and the stiffness matrix should be updated. The programs have this feature already and it is simply necessary to make the implementation conditional on the coordinate system updating. The changes to be made will cause the updating to be done infrequently, conditional on the rotation magnitudes being of the order of 15°. This will reduce costs considerably without degradation of accuracy.

Several features of the present nonlinear element formulations which have proved particularly effective and will be retained are listed below:

- The iteration procedure which alternates axial/membrane and all-freedom iterative corrections will be retained (unless it is shown to be unnecessary due to the use of the nonlinear step solution procedure).

- The conditional updating of the geometric stiffness matrix, based on the magnitudes of the residual stresses, will be retained.

- The convected coordinate system approach will be retained.

- The user-option of over-riding the internally computed solution coordinate systems is needed for generality of boundary condition specification in the nonlinear case, and will be retained.

o The shallow-shell formulation will be retained.

Further development of the triangular HMN element is not proposed herein. This element has the recognized difficulty of inter-element bending slope discontinuities. While this effect is not always necessarily a bad one, it has complicated the handling and interpretation of residual loads in the stepwise solution of nonlinear problems. It is noted however, that the triangular element appears to be nearly free of the difficulty regarding bending/torsion and shear/direct strain interactions which are described above. Thus, it may ultimately turn out that the triangular HMN (BCIZ-Reference 8) element merits further work.

Nonlinear Step Static Solution Procedure: The nonlinear step capability will be developed based on the "Static Perturbation" method. This method was described by Sewell (Reference 4,) and extended in a cost effective manner to finite element applications by Vos (References 5, 6). In this procedure the nodal displacement vector is expressed in Taylor series form in terms of a path parameter. Displacement derivative vectors for use in the Taylor series are determined from solutions of successive differentiations of the equilibrium equations, using the system tangent stiffness matrix. Problem solutions are determined from the Taylor series expansion. The residual load method is still used to assure close conformance to the equilibrium path.

This nonlinear step approach allows solutions to be continued through limit point instabilities. The method can incorporate both material and geometric nonlinearities, as well as the effects of nonconservative follower-type forces. The only matrix decomposition required is that of the system stiffness matrix, and this is only required once per step. Techniques will be developed for selecting appropriate step sizes. It is proposed that both second order (quadratic step) and third order (cubic step) approaches be incorporated and compared for relative efficiency. Appendix A gives the basic equations of the static perturbation method for the case in which quadratically varying in step solution variables are retained. Appendix B gives formulas for the nonlinear stiffness matrices, in terms of element displacement forms and material property matrices.

Figure 2 shows results computed for a simple nonlinear problem, comparing quadratic static perturbation solutions and Newton-Raphson (piecewise linear) solutions for two step sizes. The static perturbation procedure is seen to converge, with decreasing step size, much faster than the conventional piecewise linear method. Also shown is a result computed with the static perturbation method using automatically varied step sizes, computed during the solution by the formula ΔS = constant x $(\dot{Q}/\ddot{Q})$. The results are excellent. The figure notes the numbers of steps computed for each plotted curve.

For use with the convected coordinate system procedure (updated total-Langrangian formulation), the static perturbation method must accomplish coordinate transformations on the in-step nonlinearity matrix ($\dot{K}_T$ - see Appendix A). The proposed method for accomplishing this is as follows:

Solution variable rates are computed in solution or global system:

$$\dot{Q}$$

Transform to element baseplane system

$$\dot{Q} \xrightarrow{T} \dot{q}$$

Evaluate elemental matrix $P1k \cdot \dot{q}$

(see Appendix A, Equations A4, A5)

Transform to solution or global coordinates

$$P1k \cdot \dot{q} \xrightarrow{T} P1K \cdot \dot{Q}$$

Form $P1 = (P1K \cdot \dot{Q})\ \dot{Q}$

This procedure avoids the requirement to transform the third order tensor quantity P1k. The transformation of $P1k \cdot \dot{q}$ is a simple stiffness matrix transformation, using a conventional coordinate transformation matrix, T.

Nonlinear Step Dynamic Solution Procedure: There are many different forms of discrete step solution procedures in use for solving transient dynamic analysis problems. For the most part these methods are based on using the set of previously computed solution steps, together with the differential equations of motion, to predict the solution values at the end of the current computation step. The stepwise equations used are based on either difference formula representation of timewise derivatives of the unknown variables (using past and future solution sets), or on interpolation formula representation of these variables (again using past and future solution sets) with corresponding analytical representations of the time derivatives. In all cases the equilibrium equations are forced to be satisfied, in terms of solution variables at discrete time points, at a particular point in time. The choice of this time point is such that the unknowns to be determined, i.e., displacements at the $(n+1)^{\underline{st}}$ time point, appear in the discretized equations. The difference formula and interpolation formula approaches are closely related, but in general lead to different equations, and hence to somewhat different numerical results in applications. Other distinctions between these methods include whether the equations are implicit (solution requires iterations at each time point, because equation coefficients are dependent on future points), or explicit (solution steps do not require iteration because equation coefficients are only dependent on past points); and also what order of derivatives are employed in the equations of motion. Regarding the latter options, one can, for example, simply use the second order equation of motion,

$$M\ddot{Q} = P - K_S\, Q \qquad 1$$

or employ further differentiations to obtain, in addition,

$$M\dddot{Q} = \dot{P} - K_T\, \dot{Q} \qquad 2$$

$$M\ddddot{Q} = \ddot{P} - K_T\, \ddot{Q} - \dot{K}_T\, \dot{Q} \qquad 3$$

In these equations, K_S and K_T are, respectively, the structural secant and tangent stiffness matrices. A further important consideration relative to these solution procedures is whether, at each solution step (and perhaps at each iteration of implicit methods), the computations require only the decomposition of the structural mass matrix, or alternatively, a decomposition involving the mass and stiffness matrices. The latter is generally the case for implicit methods, and is very costly in practical numerical work.

The implicit methods in some instances have the advantage of unconditional stability as the time steps are increased in size, while the explicit methods become unstable for particular step sizes (of the order of the half-period of the highest frequency components of the structural system). The advantage of the unconditional stability is that the highest frequency structural actions of a finite element model (which can be of very high frequency for fine discretizations) will be "damped" to a near zero amplitude in problem solutions. However, particularly for nonlinear problems, obtaining good solution accuracy may require smaller time steps for properly representing rapidly varying structural behavior than would be required to satisfy stability criteria for the integration procedure. Thus it appears that the implicit methods, requiring costly stiffness matrix decomposition, may not be optimum for nonlinear dynamic analysis. In addition, the implicit methods impart a numerically-induced artificial damping to problem solutions, which in itself requires the use of small time steps to avoid excessive energy loss due to the artificial damping effects.

References 7, 12-16 discuss various solution procedures of the general types described above. The discussions in these references are for the most part mathematical in approach. In order to put such methods in perspective, a particular procedure, called the Houbolt method (Reference 16) and generally considered to be a superior method, has been used to solve a simple nonlinear problem. Figure 3 shows the numerical results for several time step sizes. The solution involves iterations at each time point, and the data shown are iterated to obtain fully converged results. It will be seen in later discussions that the accuracy of the Houbolt method, at least in this particular nonlinear problem, is not particularly good, and can be improved on by simpler methods. The Houbolt method is described in Appendix A.

A basically different type of formulation starts from the representation of the solution as a Taylor series. In this case the solution at the $(n+1)^{\underline{st}}$ time point is based on its derivatives at the nth time point. This approach offers a number of advantages: complete freedom to vary time step size during the solution; solution behavior governed by the most recent structural behavior, rather than by extrapolation from past behavior; simple extension to higher orders of approximation, even during a problem solution, without changing the basic solution equations; ability to handle in-step nonlinearity without the use of an implicit/iterative solution method (only the mass matrix needs to be decomposed). This approach is analogous to the static perturbation procedure, and the relevant equations are given in Appendix A. This approach is proposed for the subject research and computer program development.

The Taylor series representation approach, called herein the "dynamic perturbation method", can be formulated to make use of the second order equilibrium equation, plus an arbitrary succession of higher order equations obtained by differentiating the basic equation. Through the higher order derivatives, more complete information describing the variation of the forces acting during the computation time step is incorporated into the solution. This is clearly seen in Equations 1-3, in which $K_T\dot{Q}$ represents the effect of variable force at constant stiffness, and $\dot{K}_T\,\dot{Q}$, is the first term which represents the effect of in-step structural nonlinearity. Equations 1-3 can be solved for $\ddot{Q}$, $\dddot{Q}$, Q^{iv}, ... etc., requiring only decomposition of the mass matrix, M. These derivatives, evaluated at time t_n are used in the Taylor series (about t_n) through which the solution at time t_{n+1} is computed. The simplest option, using only $\ddot{Q}$, does not generally provide accurate problem solutions. Including $\dddot{Q}$, or $\dddot{Q}$ and Q^{iv}, causes the results to be very accurate, even for time steps approaching the stability limit ($\Delta t \sim 1/2$ period) of the formulation.

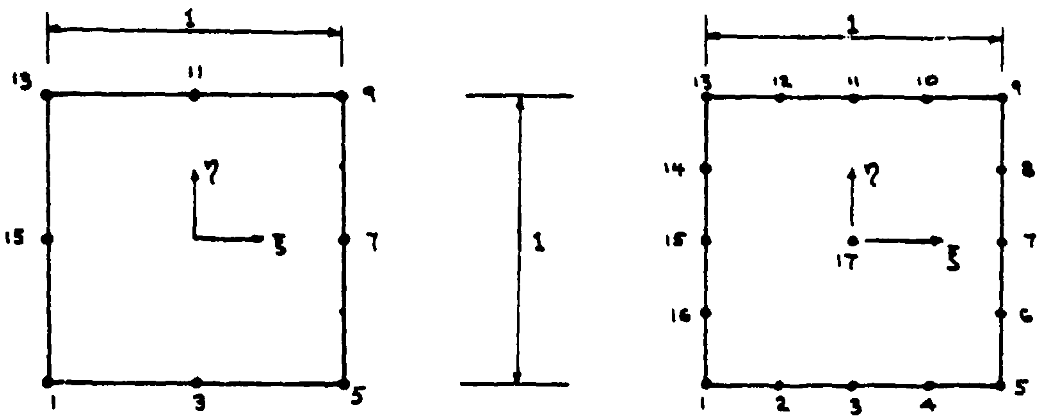
The inadequacy of the lowest order "dynamic perturbation" method is easily remedied by a slight change in formulation, described in Reference 15. The resulting particularly simple method, called the "acceleration-pulse" method (Reference 15), offers probably the most cost effective of the available solution procedures. This method achieves its excellent accuracy by compensating errors, as described in Appendix A. Since it is only a zero'th order method (based on $\ddot{Q}$ only), only simple calculations are involved, and the method does not include effects of in-step nonlinearities. Nevertheless, because of its ease of use, economy and good accuracy, it is felt that this method should be included in the subject program development, and it is proposed to include it as a user-optional choice, along with the Taylor series, or "dynamic perturbation", method.

Figure 4 illustrates the acceleration pulse method and the "dynamic perturbation" method through the 4th derivative for a simple problem. In this problem, the approach keeping w^{iv} is essentially exact, as proved by solutions obtained with a set of smaller time steps. The data illustrate elastic, plastic, material failure, and load discontinuity induced behavior. The superiority of the higher order method, which includes both linear and nonlinear in-step force variation, is seen to be greatest when some degree of discontinuity of load or stiffness behavior is present, particularly when the discontinuity is an added, positive load. Even in this case, however, the excellent accuracy of the acceleration pulse method, in relation to its simplicity, is clearly seen. The Houbolt method (Figure 3) was seen to provide mediocre results in comparison with the "dynamic perturbation" method, even for the simple elastic case.

It should be noted that the simple, one-degree-of-freedom example may be somewhat misleading. Judgement suggests that more severe calculation difficulty, with attendant greater accuracy requirements, would be present in multi-degree-of-freedom problems, particularly when material yield or failure occurs, resulting in growth and contraction (unloading) of failure/yield zones. Difficulties related to this type of behavior have previously been encountered with the "acceleration-pulse" method (Reference 17).

Reference 7 uses an implicit, interpolation-type, solution procedure retaining $\dddot{u}$ to solve inelastic problems of beams and shells. The method is related to the Houbolt method. Despite the complexity and inherent cost of the method, small time steps were apparently required to obtain accurate solutions. This may suggest that some sort of "dynamic residual load" concept would be a valuable asset with this, and probably other, solution procedures. Such a residual load method will be investigated as an option in the proposed computer program.

The goal of the proposed research is to handle nonlinear dynamic problems with relatively large time steps (of the order permissible for linear problems, governed by solution stability criteria), while using a solution procedure which only requires decomposing the mass matrix. The latter assures a method which is both fast and simple. The approaches proposed (dynamic application of static perturbation procedure, and the acceleration pulse method, Appendix A) provide these desirable features. In addition, the first method lends itself to the automatic computation of time step size, based on specified accuracy criteria (using ratios of time derivatives of solution quantities). In most problems, this can yield considerable savings in computing costs.



PARENT ISOPARAMETRIC ELEMENT FOR BENDING DEFORMATIONS (NODES NEED NOT BE EQUALLY SPACED)

PARENT ISOPARAMETRIC ELEMENT FOR MEMBRANE DEFORMATIONS (NODES NEED NOT BE EQUALLY SPACED)

DISPLACEMENT FUNCTIONS:

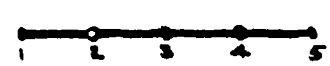
BENDING (w, θ_ξ, θ_η) — LAGRANGE INTERPOLATION POLYNOMIAL OF THIRD DEGREE (THRU $\xi^2\eta$, $\eta^2\xi$), NO. DEGREES OF FREEDOM = 24 (TERMS ξ^3, η^3 OMITTED)

MEMBRANE (u, v) — LAGRANGE INTERPOLATION POLYNOMIAL OF FOURTH DEGREE (THRU ξ^4, η^4, $\xi^3\eta$, $\eta^3\xi$, $\xi^2\eta^2$), PLUS FIFTH DEGREE TERMS $\xi^4\eta$, $\eta^4\xi$: 34 FREEDOMS

NOTES: NODE 17 MAY NOT BE NEEDED; IT MAY BE POSSIBLE TO OMIT θ_η AT NODES 3,11; AND θ_ξ AT NODES 7,15.

FIGURE 1a: QUADRILATERAL SHELL ELEMENT

BEAM ELEMENT FOR BENDING (QUADRATIC, FIVE DEGREES OF FREEDOM PER NODE)



BEAM ELEMENT FOR AXIAL DEFORMATION (QUARTIC, ONE FREEDOM PER NODE)

FIGURE 1b: BEAM ELEMENT

FIGURE 1: STABILITY ELEMENT DESCRIPTIONS

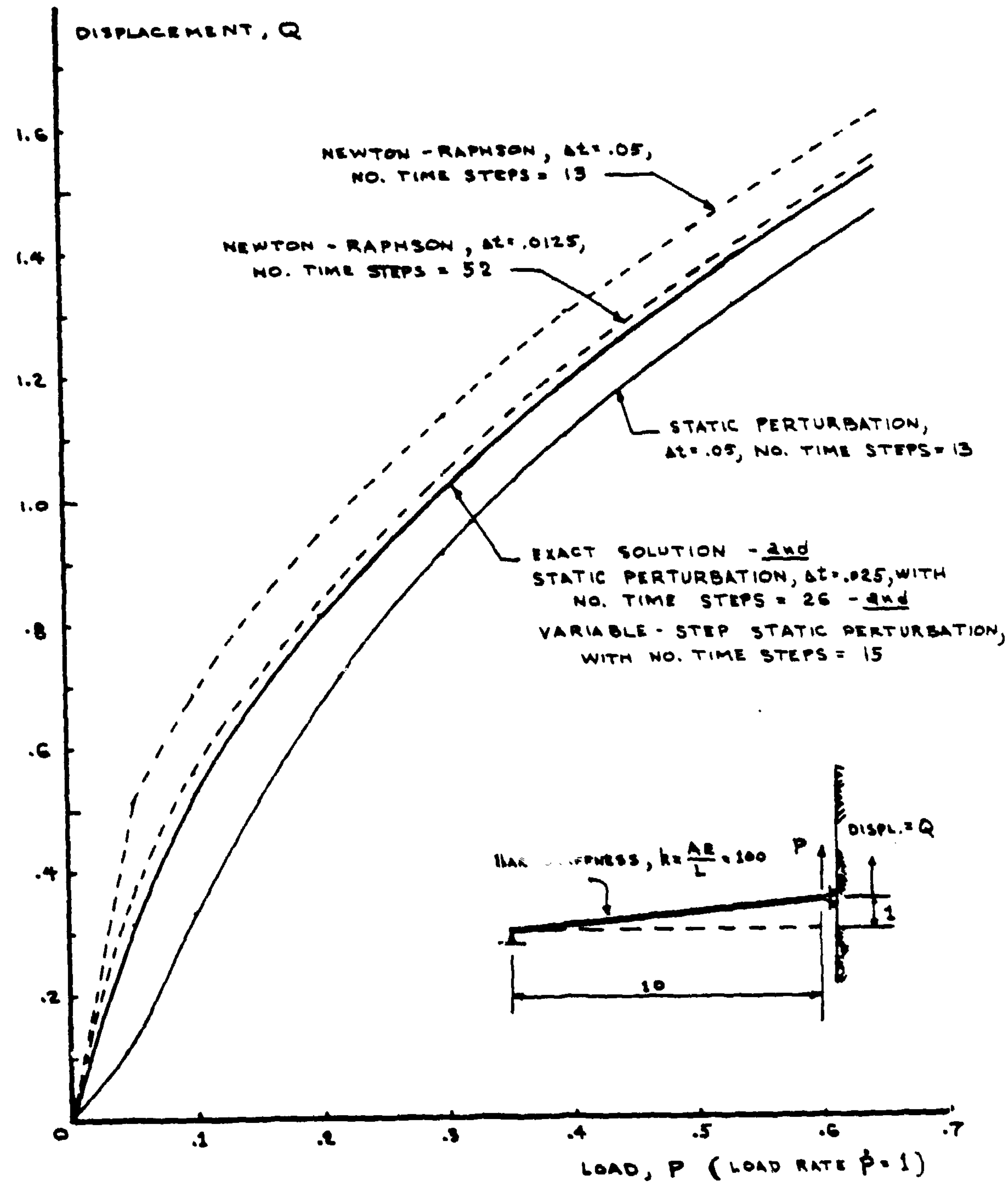


FIGURE 2 : COMPARISONS OF STATIC NONLINEAR ANALYSES

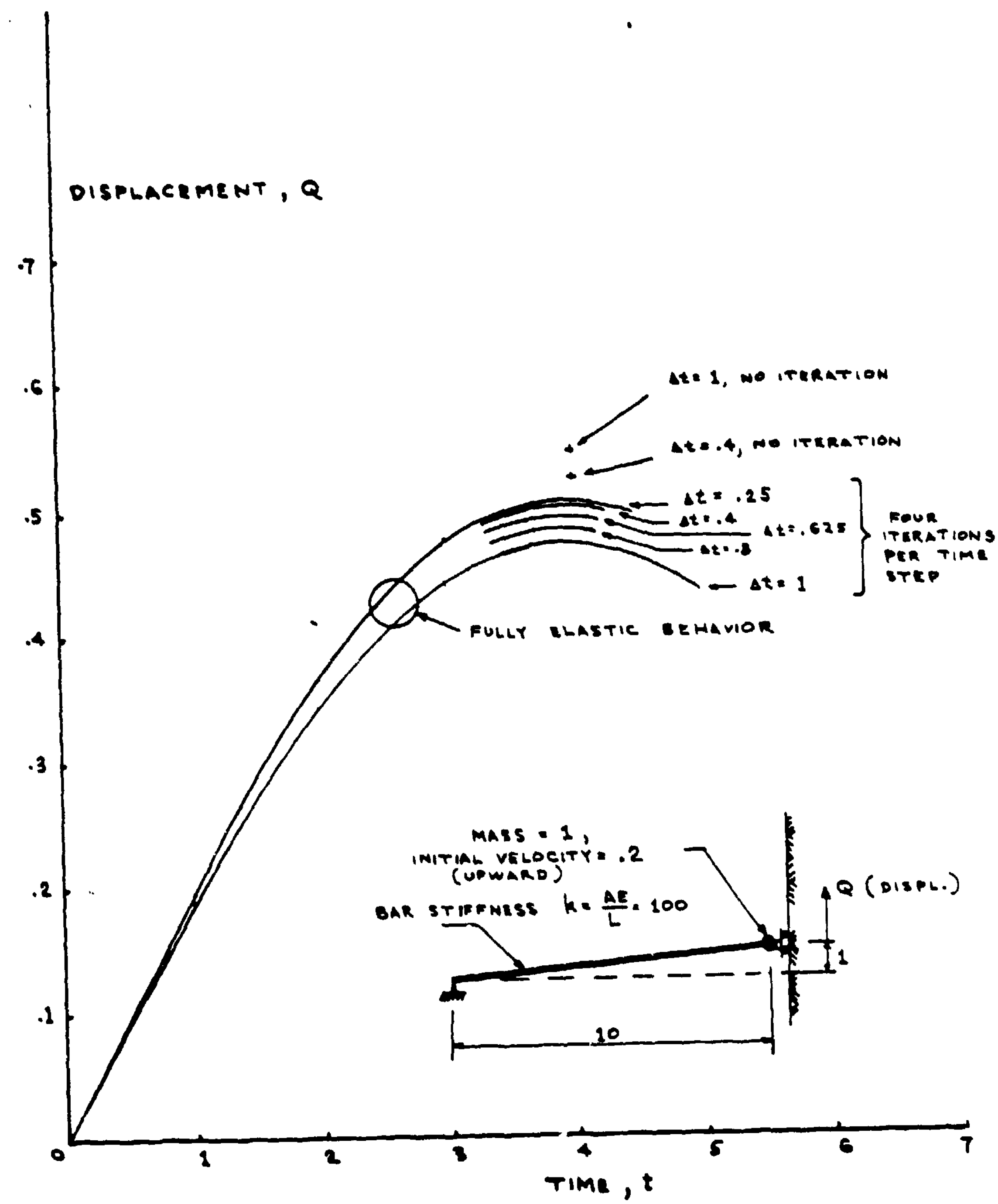


FIGURE 3 : NONLINEAR DYNAMIC RESPONSE - HOUBOLT METHOD

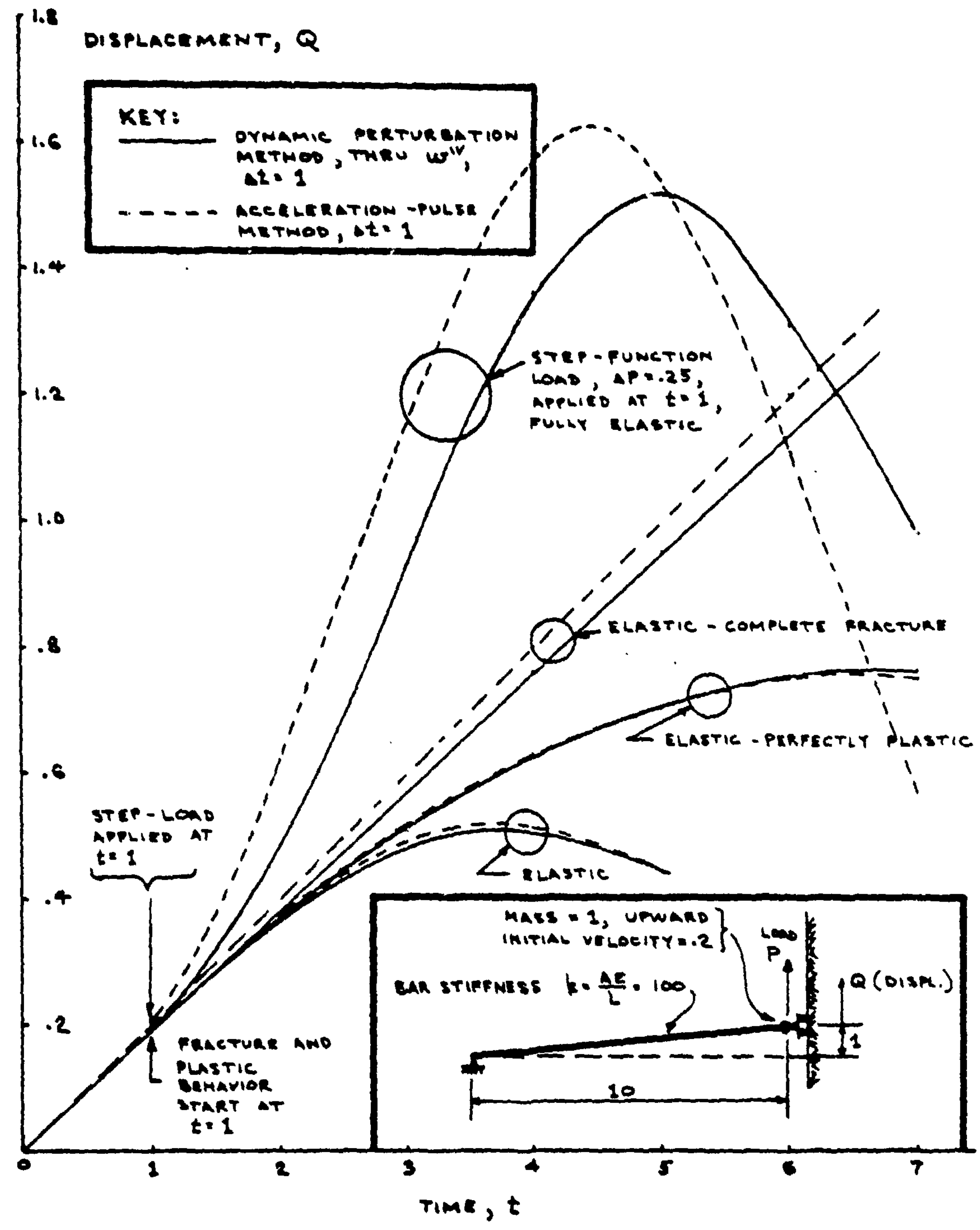


FIGURE 4: COMPARISONS OF DYNAMIC NONLINEAR ANALYSES

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APPENDIX A

This appendix briefly describes the equations of the numerical stepping procedures (static and dynamic) considered in this proposal. In the equations given, the following definitions hold (matrix notation omitted).

Q = solution vector (column matrix)

P = load vector (column matrix)

M = mass (square matrix)

K_S, K_T = secant and tangent stiffness (square matrix)

C = damping coefficient (square matrix)

Δt = incremental time or incremental path parameter

P1 = load vector which accounts for in-step internal structural loads due to nonlinearity (column matrix)

P1K = the rate of change of K_T, due to nonlinearity (third order tensor, or "cubic matrix array")

$(\dot{\ }), (\ddot{\ })$, etc. = denotes time or "path parameter" derivatives

$(\ )_u$ = denotes the n^{th} time point or path parameter point

<u>Static Perturbation Method (through $\ddot{Q}$)</u>

The starting equation is the equilibrium equation

$$P = K_S Q \qquad \text{A1}$$

Successive differentiations yield

$$\dot{P} = \dot{K}_S Q + K_S \dot{Q} \equiv K_T \dot{Q} \qquad \text{A2}$$

$$\ddot{P} = K_T \ddot{Q} + \dot{K}_T \dot{Q} \equiv K_T \ddot{Q} + P1 \qquad \text{A3}$$

where P1 can be written as

$$P1 = (P1K \cdot \dot{Q})\ \dot{Q} \qquad \text{A4}$$

and

$$\dot{K}_T = P1K \cdot \dot{Q} \qquad \text{A5}$$

Solving for the derivatives of Q,

$$\dot{Q} = K_T^{-1}\ \dot{P} \qquad \text{A6}$$

$$\ddot{Q} = K_T^{-1}\ (\ddot{P} - P1) \qquad \text{A7}$$

It is noted that only K_T needs to be inverted (decomposed), even though the equations contain the effects (through P1) of structural nonlinearity.

The final solution is obtained by a Taylor series stepping process in which Q_{n+1} is computed from the previous step solution Q_n and the start-of-step derivatives (Equations A6, A7) $\dot{Q}_n$, $\ddot{Q}_n$

$$Q_{n+1} = Q_n + \dot{Q}_n\ \Delta t + 1/2\ \ddot{Q}_n\ (\Delta t)^2 \qquad \text{A8}$$

The procedure can be used retaining only $\dot{Q}_n$, in which case it is equivalent to the simple, and usually inadequate, piecewise linear Newton-Raphson procedure. The real accomplishment of the static perturbation methods lies in including the higher derivatives. It is noted that, by retaining $\dot{P}1$, $\dddot{Q}$ can be included, and similarly even higher derivatives can easily be included. See Appendix B for closed form equations for nonlinear stiffness matrices.

Dynamic Perturbation Method (through Q^{iv})

The starting equation is the second order equation of motion, with time the path parameter,

$$M\ddot{Q} = P - K_S Q - C\dot{Q} \qquad \text{A9}$$

Differentiating, and solving for successive derivatives,

$$\ddot{Q} = M^{-1} [P - K_S Q - C\dot{Q}] \qquad \text{A10}$$

$$\dddot{Q} = M^{-1} [\dot{P} - K_T\dot{Q} - C\ddot{Q} - \dot{C}\dot{Q}] \qquad \text{A11}$$

$$Q^{iv} = M^{-1} [\ddot{P} - K_T\ddot{Q} - P1 - C\dddot{Q} - 2\dot{C}\ddot{Q} - \ddot{C}\dot{Q}] \qquad \text{A12}$$

The Taylor series about time t_n gives the solution at t_{n+1}

$$Q_{n+1} = Q_n + \dot{Q}_n \Delta t + \ddot{Q}_n \frac{(\Delta t)^2}{2} + \dddot{Q}_n \frac{(\Delta t)^3}{6} + Q_n^{iv} \frac{(\Delta t)^4}{24} \qquad \text{A13}$$

$$\dot{Q}_{n+1} = \dot{Q}_n + \ddot{Q}_n \Delta t + \dddot{Q}_n \frac{(\Delta t)^2}{2} + Q_n^{iv} \frac{(\Delta t)^3}{6} \qquad \text{A14}$$

It is noted that in the dynamic case, both Q_{n+1} and $\dot{Q}_{n+1}$ are solved for, in order that the succeeding steps can be handled as an initial value problem.

If terms are only retained through $\ddot{Q}$, the method is not accurate. The physical reason for this is that the computed value of $\dot{Q}_{n+1}$ consider:

only a constant acceleration through the step - i.e., a constant-force step. This is not adequate for even linear analysis. If terms are retained through $\dddot{Q}$, the method has effectively retained in-step linear force variation, through the term $K_T\dot{Q}$. This level of approximation has been found to be very accurate for moderately nonlinear behavior. Retention of Q^{iv} includes in-step nonlinearity and further improves accuracy for strongly nonlinear behavior.

"Acceleration Pulse" Method

This method can be derived from Equation A9 (without the damping term) by using a central difference formula for $\ddot{Q}$, and representing the start-of-step velocity $\dot{Q}_n$ by a backward difference formula. The result is equivalent for a rather surprising modification of Equations A13 and A14, as follows:

$$Q_{n+1} = Q_n + \dot{Q}_n^* \Delta t + \ddot{Q}_n \frac{(\Delta t)^2}{2} + \ddot{Q}_n \frac{(\Delta t)^2}{2} \qquad \text{A15}$$

$$\dot{Q}_{n+1}^* = (Q_{n+1} - Q_n)/\Delta t \qquad \text{A16}$$

The starred quantities indicate approximate velocities. The appearance of the extra acceleration term in equation A15 compensates for the error incurred by the backward difference representation of the velocity in Equation A16. It can be rather easily seen that

$$\dot{Q}_n \approx \dot{Q}_n^* + \frac{1}{2} \ddot{Q}_n (\Delta t)^2 \qquad \text{A17}$$

with the result

$$Q_{n+1} = Q_n + \dot{Q}_n \Delta t + \frac{1}{2} (\Delta t)^2 \ddot{Q}_n \qquad \text{A18}$$

$$\dot{Q}_{n+1}^* = (Q_{n+1} - Q_n)/\Delta t \qquad \text{A19}$$

Equation A18 differs from Equation A13 retaining only through $\ddot{Q}$ in that $\dot{Q}_n$ in A18, obtained from Equation A17, is much more accurate than the corresponding term in Equation A13.

The acceleration pulse method achieves truly outstanding accuracy, in consideration of its simplicity, even for quite large step sizes. The means of including damping while maintaining the internal error compensation feature of the method is not developed as yet.

Houbolt Method (four point backward difference)

The Houbolt method uses a four point interpolation formula for Q, based on the unique cubic polynomial passed through four equally spaced points

$$Q = Q_{n-3} + \frac{t}{\Delta t}\left[-\frac{11}{6}Q_{n-3} + 3Q_{n-2} - \frac{3}{2}Q_{n-1} + \frac{1}{3}Q_n\right]$$
$$+ \left(\frac{t}{\Delta t}\right)^2\left[Q_{n-3} - \frac{5}{2}Q_{n-2} + 2Q_{n-1} + \frac{1}{2}Q_n\right]$$
$$+ \left(\frac{t}{\Delta t}\right)^3\left[-\frac{1}{6}Q_{n-3} + \frac{1}{2}Q_{n-2} - \frac{1}{2}Q_{n-1} + \frac{1}{2}Q_n\right] \qquad \text{A20}$$

Differentiating this formula to obtain $\dot{Q}_{n+1}$ and $\ddot{Q}_{n+1}$, and substituting in Equation A9 yields

$$\left[M + \frac{11}{12}\Delta t \cdot C_{n+1} + \frac{1}{2}(\Delta t)^2 \cdot K_{S_{n+1}}\right] Q_{n+1} = \frac{1}{2}(\Delta t)^2 P_{n+1}$$
$$+ Q_n\left(\frac{5}{2}M + \frac{3}{2}\Delta C \cdot C\right) - Q_{n-1}\left(2M + \frac{3}{4}\Delta t \cdot C\right)$$
$$+ Q_{n-2}\left(\frac{1}{2}M + \frac{1}{6}\Delta t \cdot C\right) \qquad \text{A21}$$

Use of this equation requires inversion (decomposition) of the quantity

$$\left[M + \frac{11}{12}\Delta t\, C_{n+1} + \frac{1}{2}(\Delta t)^2 K_{S_{n+1}}\right],$$

which makes the method implicit, as $K_{S_{n+1}}$ is not generally known until Q_{n+1} is known. Solutions are obtained by iteration.

APPENDIX B

This appendix briefly describes the equations required for the elemental level forces and stiffness matrices, and their derivatives, for nonlinear problems. In the equations given, the following definitions hold (matrix notation omitted):

q = element freedoms (column matrix)

σ = stresses or stress resultants (column matrix)

ε = Lagrangian strains or curvatures (column matrix)

θ = spatial derivatives of displacements (column matrix)

D = material stress/strain relation (square matrix)

$A0$ = linear terms for Lagrangian strain definitions (rectangular matrix)

$A1$ = nonlinear terms for Lagrangian strain definition (3rd order tensor, or "cubic matrix array")

A = terms for nonlinear Lagrangian strain rate definition (rectangular matrix)

B = strain/displacement rate relation (rectangular matrix)

G = element shape function derivatives (rectangular matrix)

k_T = element tangent stiffness (square matrix)

$p1$ = element nonlinear structural force terms (column matrix)

$(\dot{\ })$, $(\ddot{\ })$, etc., denote time or "path parameter" derivatives

The equations for elemental level force and stiffness quantities can be derived in a straightforward manner, based on virtual work. The basic relations are provided here. The stress/strain relation is given by

$$\sigma = D \varepsilon \qquad \text{B1}$$

This equation is formulated so as to include treatment of sandwich and nonisotropic materials. The quantities in B1, as well as the element displacement (shape) functions and displacement derivatives, are evaluated at a series of numerical integration points within the element. The displacement derivatives θ are given by

$$\theta = G q \qquad \text{B2}$$

Strains are defined by

$$\varepsilon = (A0 + 1/2\ A1\ \theta)\ \theta \qquad \text{B3}$$

and strain rates by

$$\dot{\varepsilon} = (A0 + A1\ \theta)\ \dot{\theta} \equiv A\ \dot{\theta} \qquad \text{B4}$$

Combining B2 and B4 provides

$$\dot{\varepsilon} = A\ G\ \dot{q} \equiv B\ \dot{q} \qquad \text{B5}$$

The virtual work formulation leads to the expression for element nodal forces, in terms of a numerically integrated volume integral

$$p = \int_V G\ A\ \sigma\ dV = \int_V B\ \sigma\ dV \qquad \text{B6}$$

Substituting B1 and B3 into B6, and differentiating, provides the first order (tangent stiffness) relation

$$\dot{p} = k_T \dot{q} \qquad B7$$

where

$$k_T = \int_V G(\sigma A1 + ADA)\, G\, dV \qquad B8$$

Here k_T is a symmetric matrix due to symmetry properties of A1. Differentiating B7 provides the second order relation for the element

$$\ddot{p} = k_T \ddot{q} + \dot{k}_T \dot{q} \equiv k_T \ddot{q} + p1 \qquad B9$$

where

$$p1 = \dot{k}_T \dot{q} \qquad B10a$$

In an expanded but perhaps more computationally advantageous form

$$p1 = \int_V GD\,(2A1\varepsilon\theta + AA1\theta\theta)dV \qquad B10B$$