LEVEL II

THE DEVELOPMENT OF A DYNAMIC ELASTOPLASTIC FINITE ELEMENT ANALYSIS FOR FAST FRACTURE UNDER IMPACT LOADING

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Recent evidence has pointed to the possible inadequacy of elastodynamic treatments of rapid crack propagation and crack arrest. This paper describes the development of a dynamic elastic-plastic finite element capability designed to address this concern by taking direct account of crack tip plasticity. Comparisons with known dynamic fracture mechanics solutions and with experimental data are made to demonstrate the fidelity of the approach. A comparison with an elastodynamic solution in an impact loaded 4340 steel bend specimen is also made. This result reveals that a significant effect of crack tip plasticity may exist even for high strength materials.

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INTRODUCTION

Fracture mechanics researchers are becoming aware that the applicability of even rigorous dynamic analyses of unstable crack propagation and crack arrest may be more limited than was previously realized [1]. An important contributor to this dilemma is the still unexplained difference in the crack propagation behavior when crack growth is initiated under impact loading rather than under conventional quasi-static conditions. Specifically, as reported by Kanninen et al,[2] the use of the K_{ID} values obtained for 4340 steel in quasi-static initiation gave decidedly poor agreement when used to predict the crack length-time data obtained in an impact test. In fact, the K_{ID} value inferred from the latter test was roughly a factor of two greater! Added to the geometry-dependence that cast doubt on the validity of $K_{ID} = K_{ID}$ (V) as a unique material property, there is some concern about the presently accepted elastodynamic treatments of fast fracture. This paper describes a first step towards a possible resolution of these difficulties via the development of an elastoplastic dynamic finite element model for the future treatment of fast fracture problems.

Besides providing a more realistic model of the specimen geometry and the boundary conditions, a finite element method is particularly suitable for modeling nonlinear material behavior. To avoid the use of an extremely small mesh size in the evaluation of the dynamic stress intensity factor, a conservation integral, $\hat{J}$ [3] can be utilized. The $\hat{J}$-integral is essentially a consequence of several attempts[4-6] aimed at extending the regime of applicability of the path-independent contour integral J[7] to include dynamic, elastic-plastic, body force, and thermal contributions to the energy release rate under mixed-mode conditions. Then, crack propagation can be simulated via a gradual crack tip nodal-force release technique using either crack-length vs time data (generation-phase analysis) or a given fracture criterion (application-phase analysis). Finally, to obtain more realistic material modelling, a strain-rate independent constitutive relation based on a Von Mises plasticity potential with an isotropic hardening rule has been included.

In this paper, a general background discussion of current problems in dynamic fracture is followed by a description of the salient features of the finite element based computational procedure. The validity of the computational procedure is ascertained by solving problems for which reliable analytical or experimental results are available. Results for both stationary and propagating cracks are presented. Also presented is a comparison of the results of linear elastodynamic vs elastoplastic dynamic analyses performed on a three-point-bend specimen of AISI 4340 steel under impact loading conditions.

BASIS OF THE COMPUTATIONAL PROCEDURE

Background

Until recently there has been a controversy between the use of static or dynamic treatments for the arrest of a rapidly propagating crack. On the basis of results obtained by Hahn et al[8] and Kalthoff[9], it is now widely believed that a dynamic based analysis is the more correct. Nevertheless, workers in dynamic fracture mechanics are now faced with other problems. Analytical studies, which have been until recently based predominantly on linear elastodynamic analyses, have brought increased understanding of the problem. But, they have also brought to light some new problems.

The problem identified by Kanninen et al [2] concerns the marked differences in the initiation and growth of cracks initiated under different loading conditions. Their key result is shown in Figure 1. This experiment was performed on an impact loaded three-point-bend specimen of AISI 4340 steel (yield strength ≈ 200 ksi). It can be seen that, by using K_{ID} values obtained from quasi-static initiation tests (i.e., $K_{ID} = 65 + .044\ V$), a very poor prediction is obtained. Instead, the value $K_{ID} = 170$, which has no apparent connection with the "established" value, is needed for good agreement.

Because the analytical results in this study were obtained by a relatively simple elastodynamic finite difference scheme, one possible reason for the discrepancy would be the analysis procedure itself. Solving the same problem with an entirely different - and preferably improved - analytical procedure should remove any such doubt. For this purpose a finite element computer code with isoparametric element formulation was developed. To further depart from the previously used global energy balance approach for the calculation of stress intensity factor, the $\hat{J}$-integral approach was implemented.

A second possible reason for the discrepancy could be the assumption of linear elastic material behavior. The finite element code was, therefore, enriched to model the material behavior in accordance with a user supplied uniaxial stress-strain curve. This, and other reasons for the discrepancy, will be dealt with in a later section.

Outline of the Solution Procedure

The approach used for the solution of the equations of motion in the analytical procedure presented here employs a displacement-based finite element method. An isoparametric finite element formulation with linear and quadratic shape functions in a two-dimensional space is used.

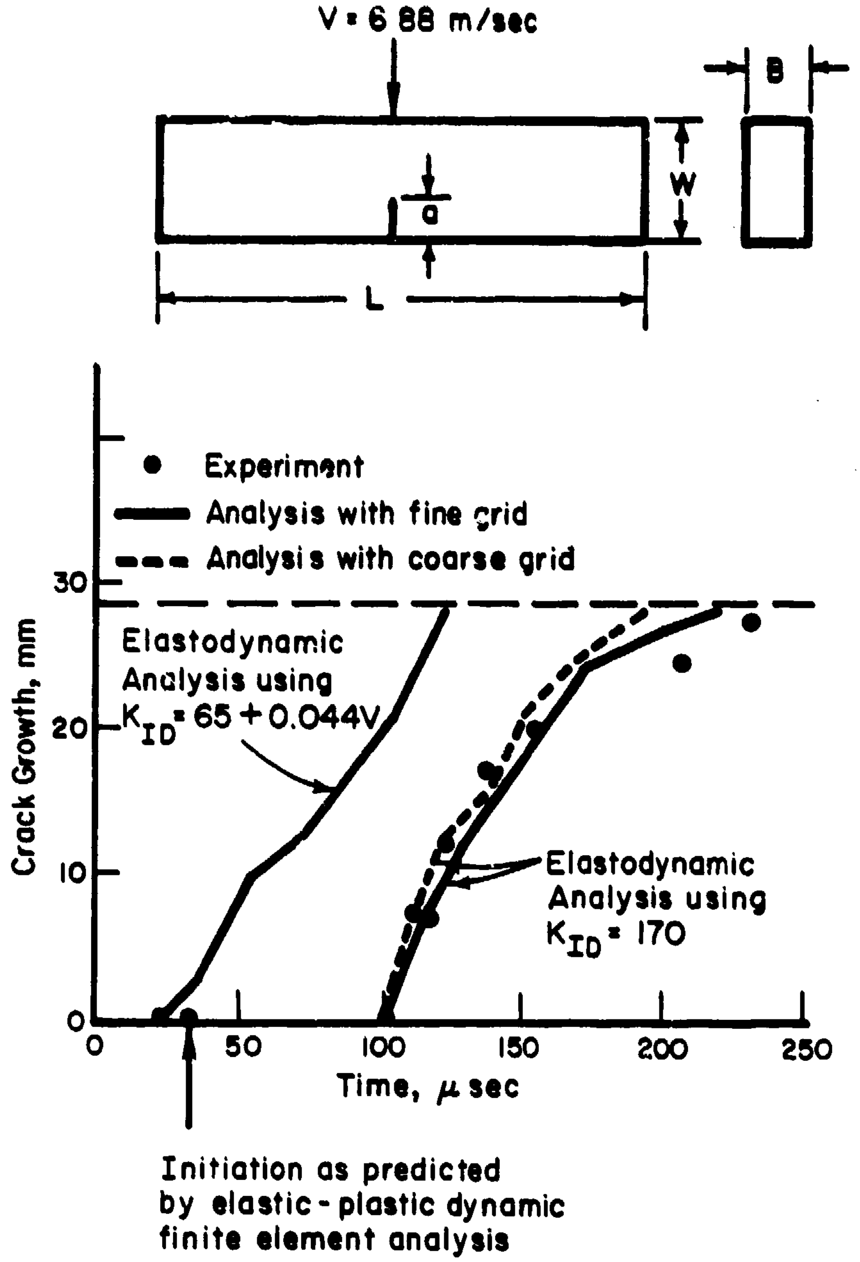


FIGURE 1. CRACK GROWTH IN 4340 STEEL UNDER IMPACT LOADING
L = 0.181 m, a =.0095 m, B = .0158 m, W = .038 m

General quadrilateral elements with a variable number of nodes in both plane-stress and plane-strain conditions may be used. If so desired the $1/\sqrt{r}$ or $1/r$ stress singularity at the crack tip may be imposed by using the quarter point approach[10]. However, this feature was not utilized in any of the analyses presented in this paper.

The modified Newton-Raphson approach[11] is used for elastic-plastic analysis. Von Mises yield condition with isotropic strain hardening is assumed. For time integration, either an explicit (central difference) or an implicit (Newmark-Beta) scheme may be used. Due to it being inherently more stable, the implicit scheme offers computational advantages in the solution of dynamic fracture problems. All the results included in the present paper were obtained by using the Newmark-Beta time integration scheme[12].

Crack growth is modelled by gradually releasing the force experienced by the crack tip at a given instant of time in several steps. Details of the crack growth modelling scheme are described in a paper by Jung, Ahmad, Kanninen, and Popelar[13]. The scheme allows for modelling crack growth in both generation and application phases of analysis. For application-phase analysis where the crack tip is advanced according to a selected fracture parameter, a choice of fracture parameters is necessary. Currently, crack-tip-opening displacement (CTOD), crack-tip-opening angle (CTOA), Mode I and Mode II dynamic stress intensity factors (K_I and K_{II}), and the $\hat{J}$ family of conservation integrals[3] are available. Since CTOD and CTOA are obtained directly from the finite element displacement solution, and K_I and K_{II} are obtained by first calculating $\hat{J}$ for linear elastic material, only a description of the $\hat{J}$-integrals is included in the following.

The $\hat{J}$-Integral

The mathematical details involved in the derivation of $\hat{J}$ are available in a paper by Kishimoto et al[3]. Here, a general expression for $\hat{J}$ is taken in a form which makes the parameter physically more tractable and convenient to implement in a computational scheme. This is done by defining

$$\hat{J}_k = J_{k_e} + J_{k_d} + J_{k_t} + J_{k_p} + J_{k_b} \quad (1)$$

Where the lower case letter subscripts stand for the elastic (e), dynamic (d), thermal (t), plastic (p), and body force (b), contributions to the $\hat{J}_k$-integrals and K (=1,2) indicates that each term in Equation (1) is a vector.

Kishimoto et al [3] define $\hat{J}$ as follows:

$$\hat{J} = \hat{J}_1 \cos \Theta + \hat{J}_2 \sin \Theta , \quad (2)$$

where Θ is the angle of crack extension measured anticlockwise from the crack line (Figure 2). The integrals J_{k_e} thru K_{k_b} in Equation (2) may be expressed as follows:

$$J_{ke} = \int_{\Gamma + \Gamma s} \left(W_e n_k - T_i \frac{\partial u_i}{\partial x_k} \right) d\Gamma$$

$$J_{kd} = \iint_A \rho \, \ddot{u}_i \frac{\partial u_i}{\partial x_k} \, dA$$

$$J_{kp} = \iint_A \sigma_{ij} \frac{\partial \varepsilon^*_{ij}}{\partial x_k} \, dA$$

$$J_{k_t} = \iint_A \alpha \, \varepsilon_{ii} \frac{\partial T}{\partial X_k} \, dA - \int_{\Gamma + \Gamma_s} 1/2 \, \alpha \, T \, \varepsilon_{ii} \, n_k d\Gamma$$

$$J_{k_b} = - \iint_A F_i \frac{\partial ui}{\partial X_k} \, dA$$

$$\alpha = \frac{E}{1-2\nu} \text{ --- Plane Strain } = \frac{E}{1-\nu} \text{ ---Plane Stress}$$

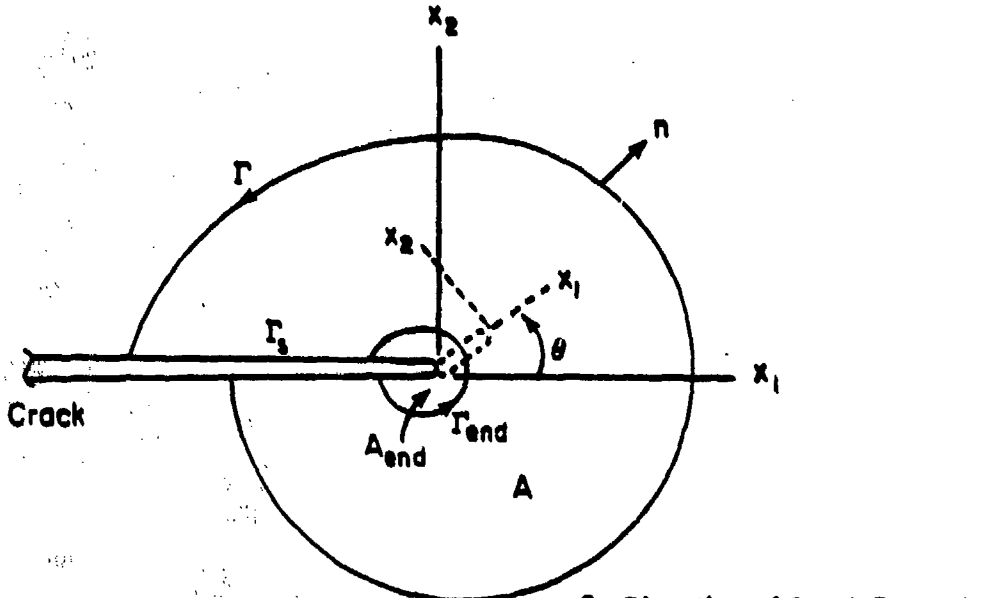


FIGURE 2. CRACK TIP COORDINATES USED IN THE DEFINITION OF $\hat{J}_k$

where

W_e = elastic strain energy density
T_i = traction vector
σ_{ij} = stress tensor
ϵ^*_{ij} = plastic strain tensor
u_i = displacement vector
$\ddot{u}_i$ = acceleration vector
T = temperature increment
F_i = body force

In Figure 2, A is the area enclosed between contours Γ and Γ_{end} and $\hat{J}$ is defined as the area Aend approaches zero. For running cracks, it is assumed that the so-called "process-region"[3] shown in Aend in Figure 2 remains constant in dimensions and moves with the crack tip.

While there may be some uncertainty regarding the use of $\hat{J}$ as a fracture criterion, it is highly appealing from a computational viewpoint. The strength of the idea is in the fact that other proposed forms of energy release rate expressions such as J of Rice [7], J^* of Blackburn [6], $\tilde{G}$ of Eftis and Liebowitz[14], and expressions proposed by Wilson and Yu[15], Freund [5], and Bui [4], all can be shown to be specialized versions of the $\hat{J}$-integral[3]. Therefore, $\hat{J}$ is at least equally valid as a fracture criterion as any of the above-mentioned parameters.

For a linear elastic material (J_{k_p} = 0) and in the absence of body forces (J_{k_b} = 0), it can be shown[16] that:

$$\hat{J}_1 = \frac{\kappa+1}{8\mu}\left[K_I^2(t) + K_{II}^2(t)\right] + \frac{1}{2\mu} K_{III}^2(t) \tag{3}$$

$$\hat{J}_2 = \frac{\kappa+1}{4\mu} K_I(t)\, K_{II}(t) \tag{4}$$

where μ is the shear modulus, and $\kappa = 3-4\nu$ for plane strain and $(3-\nu)/(1+\nu)$ for plane stress.

In a single mode situation the appropriate stress intensity factor can be readily obtained from Equation (3).

NUMERICAL RESULTS

Solutions to some representative problems solved by using the analytical procedure described above are now presented. The first four problems were chosen primarily for ensuring the validity of the solutions procedure by comparing the results with available analytical solutions and with experimental results. In the second and the fourth problems, comparisons with the previously used finite difference scheme[2] are also made.

The fifth and sixth problems were selected to demonstrate the differences in K_I obtained by using quasi-static analysis vs the dynamic analysis, and to illustrate the effect of plasticity even in a high strength material, AISI 4340 steel. Note that, in all cases involving linear-elastic-material assumption, the stress intensity factors were calculated by the $\hat{J}$ approach.

Problem 1: Stationary Crack in an Impulsively-Loaded Center-Cracked Panel

The problem considered here is of a center-cracked plate (Figure 3) loaded dynamically in a suddenly-applied uniform tension σ. The material was taken to be linear elastic ($E = 200$ GPa, $\nu = 0.3$) having a density of 5000 Kg/m^3. This problem has been solved by a number of other authors. Some of these results are shown in Figure 3 along with the results of the present analysis. The good correlation that is evident indicates that the present finite element procedure with the $\hat{J}$-integral provides sufficiently accurate dynamic stress intensity factors for stationary cracks.

Problem 2: Unrestrained Impact-Loaded Bend Specimen

A bend specimen totally unrestrained (no supports) is considered (Figure 4). In an actual experiment the specimen was struck by a hammer at an average velocity of 6.88 m/sec and was allowed to fly freely. In Figure 4, the variation of the dynamic stress intensity factor with time,

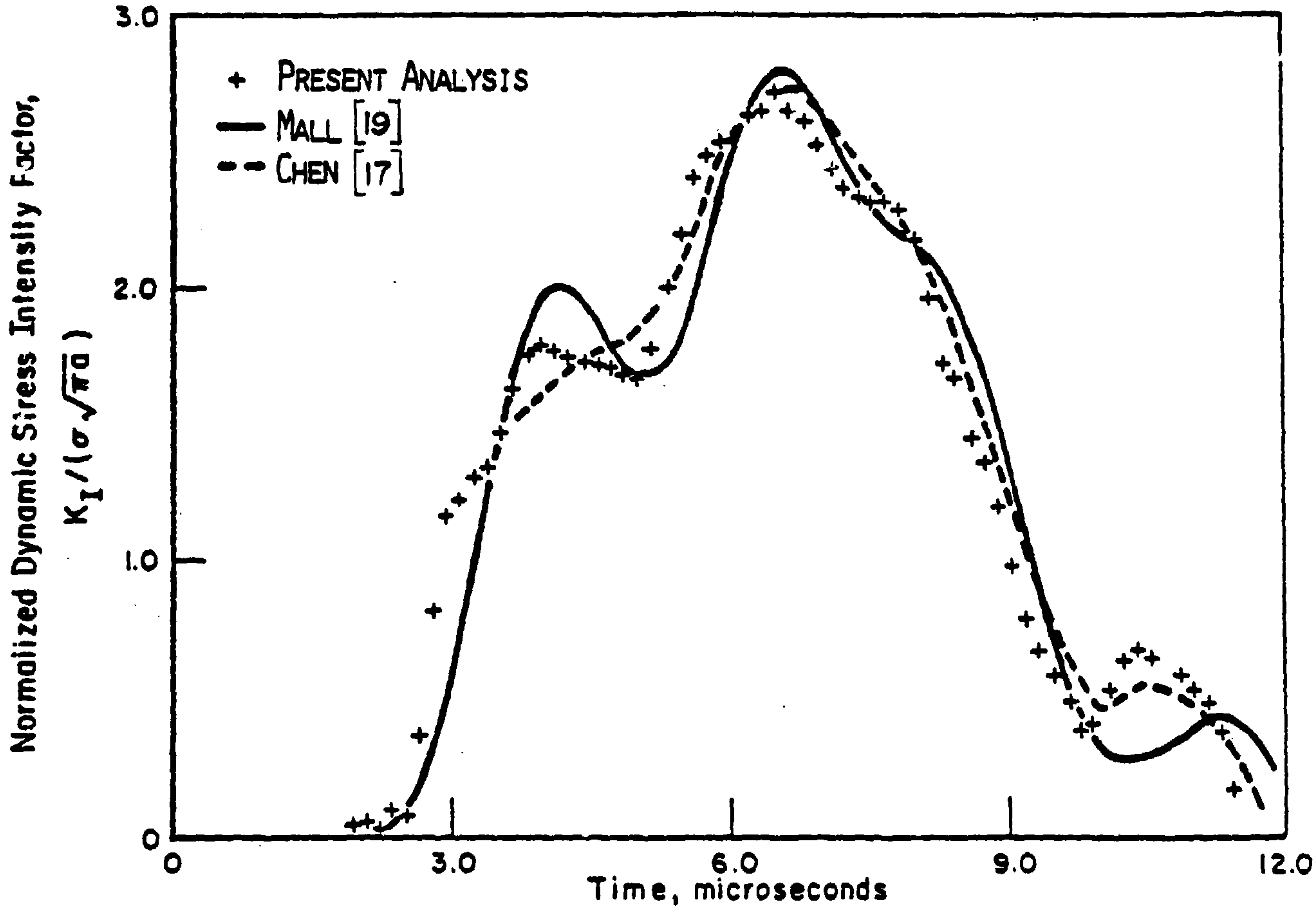
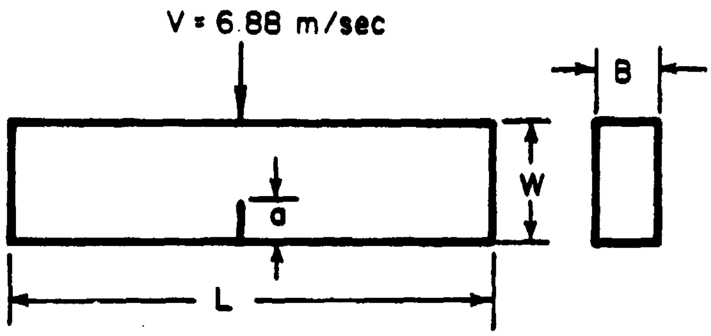


FIGURE 3. DYNAMIC STRESS INTENSITY FACTOR VERSUS TIME FOR AN IMPULSIVELY LOADED CENTER-CRACKED PANEL WITH A STATIONARY CRACK.



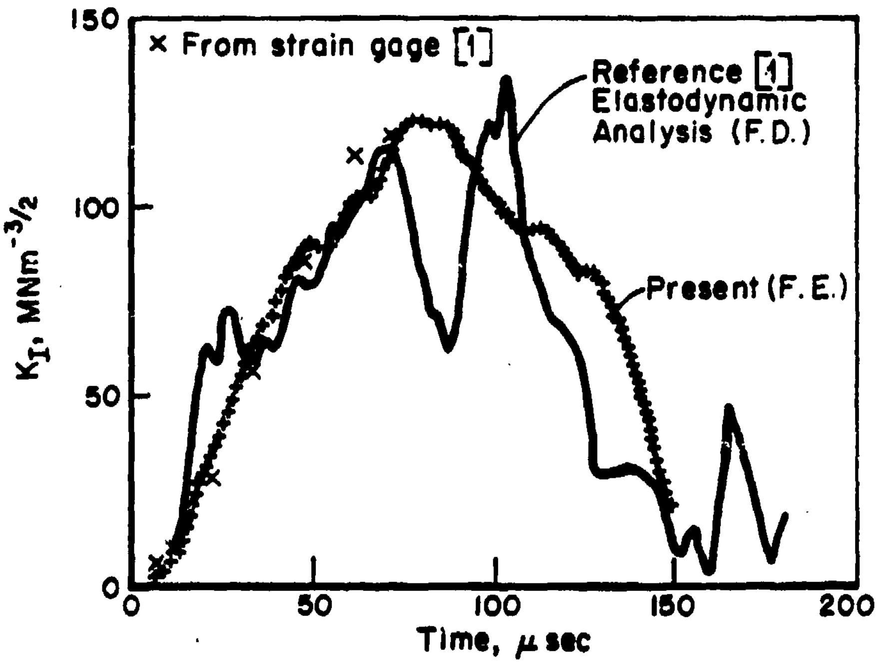


FIGURE 4. RESULTS FOR UNRESTRAINED 1 SPECIMEN

L = 0.181 m, W = 0.038 m,
a = 0.0095 m, B = 0.0158 m

as obtained by the present analysis, is compared with K_I obtained by both the finite difference calculation (2) and with experimental strain measurements. The material properties used in this analysis were those of AISI 4340 steel. Good agreement is obtained which demonstrates that the $\hat{J}$-integral approach provides essentially the same values for dynamic stress intensity factors as the global energy balance approach used in the finite difference scheme and, in view of the agreement with the strain gage data, that both are correct.

Problem 3: Propagating Crack in an Infinite Medium

The problem of a crack propagating at a constant velocity of 0.2 times the shear wave speed (C_s) in a center-cracked panel was analyzed for an initial crack length to specimen width ratio (a/w) of 0.2. The results are compared with analytical solution by Broberg [18] in Figure 5.

The results indicate that the $\hat{J}$-integral provides an effective means of calculating dynamic stress intensity factors for propagating cracks. However, it should be noted that the crack velocity in this comparison was held constant; NB, analytical solutions for a crack propagating at changing velocity do not exist. In the next problem, such a comparison is made with experimental, as well as numerical results, obtained by finite difference method.

Problem 4: Application-Phase Analysis of a Propagating Crack in a Bend Specimen

An elastodynamic crack growth analysis in a three-point-bend specimen of AISI 4340 steel was performed. Experimental as well as finite difference analysis results for this problem were first obtained by Kanninen et al [2] using $K_{ID} = 65 + 0.044\ V$ as a fracture criterion. In the present finite element computations the same criterion was used. Figure 6 shows the specimen geometry used in the analysis and a comparison of the new results with those of Reference (2).

This application-phase analysis does indicate an equivalence between the $\hat{J}$-integral approach of calculating K_I and the approach used in the finite

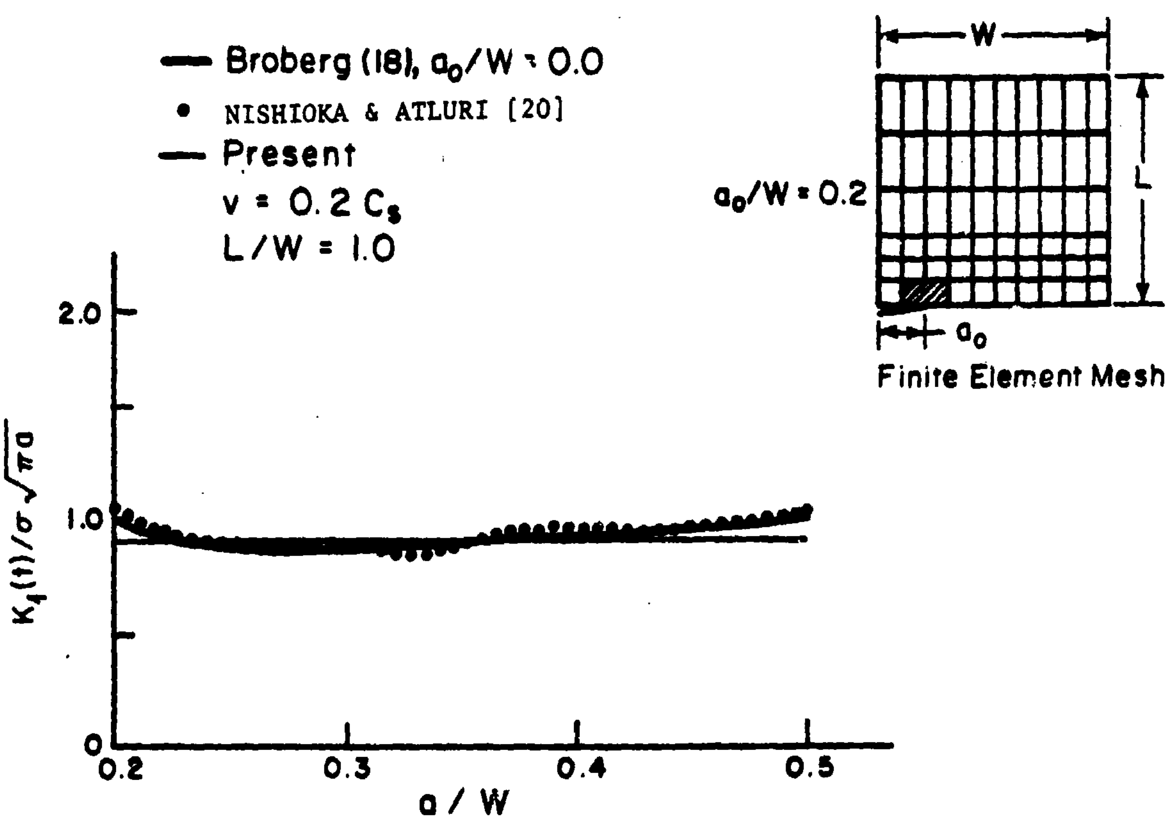
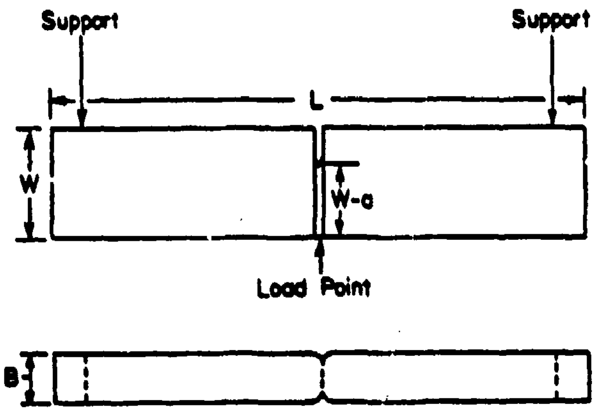


FIGURE 5. ELASTODYNAMIC ANALYSIS OF CONSTANT VELOCITY CRACK PROPAGATION



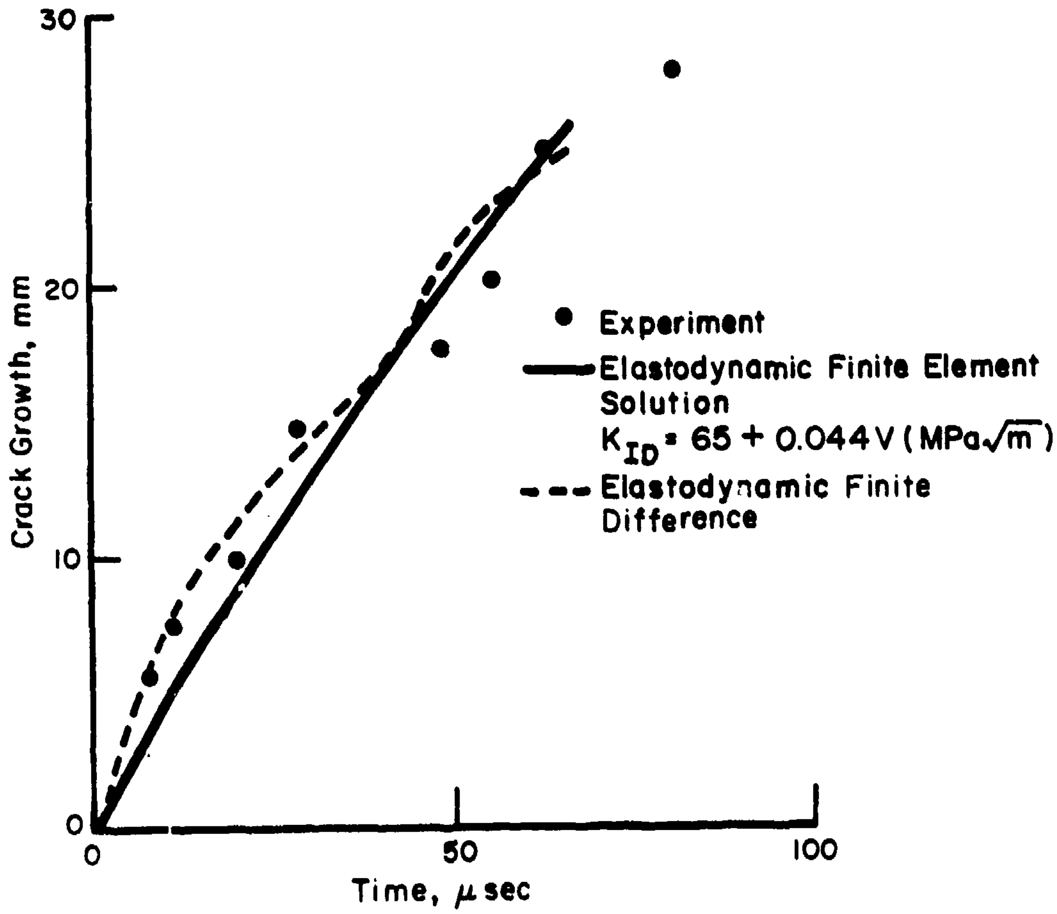


FIGURE 6. A COMPARISON OF FINITE ELEMENT ANALYSIS, FINITE DIFFERENCE ANALYSIS AND EXPERIMENT FOR A QUASI-STATICALLY INITIATED CRACK IN A 4340 DYNAMIC TEAR SPECIMEN.
L = 181 mm, W = 38 mm, B = 15.8 mm, (W-a) = 28.5 mm.

difference scheme even for a crack propagating at a nonconstant speed. The results also provide an increased level of confidence in the accuracy of numerical results.

Problem 5: Generation-Phase Analysis of a Propagating Crack in a Bend Specimen

A generation phase analysis, in which an experimentally obtained crack length vs time record (Figure 7) from a dynamic tear test experiment on HY130 steel was provided as an input to the computer code, was performed. The elastodynamic finite element analysis then gave values of K_D as a function of time. These are plotted in Figure 8. Also shown are K_0 values inferred by using a handbook formula for three-point-bend specimen in which inertia forces are ignored. As might be expected, the dynamic values oscillate around a mean value given by the quasi-static solution.

Problem 6: Elastic-Plastic Dynamic Analysis of a Stationary Crack in a Bend Specimen

The three-point-bend specimen of AISI 4340 steel was analyzed under an impact load with and without the elastic material assumption. For the elastic-plastic dynamic analysis, the material behavior was described by an experimentally obtained static stress-strain curve from a uniaxial tension test. Strain rate effects on material property are not included in the analysis.

Figure 9 shows the variation with time of the crack-opening displacement (COD), measured at 0.68 mm behind the crack-tip for both elastic and elastic-plastic analyses. In Figure 10, values obtained are plotted against time. It can be seen that, even in a high-strength material, the effect of plasticity appears to be significant. The effect of plasticity seems to damp-out the oscillations in COD (which is directly proportional to K_I and $\sqrt{J}$ in the linear elastic case); cf, Figure 8.

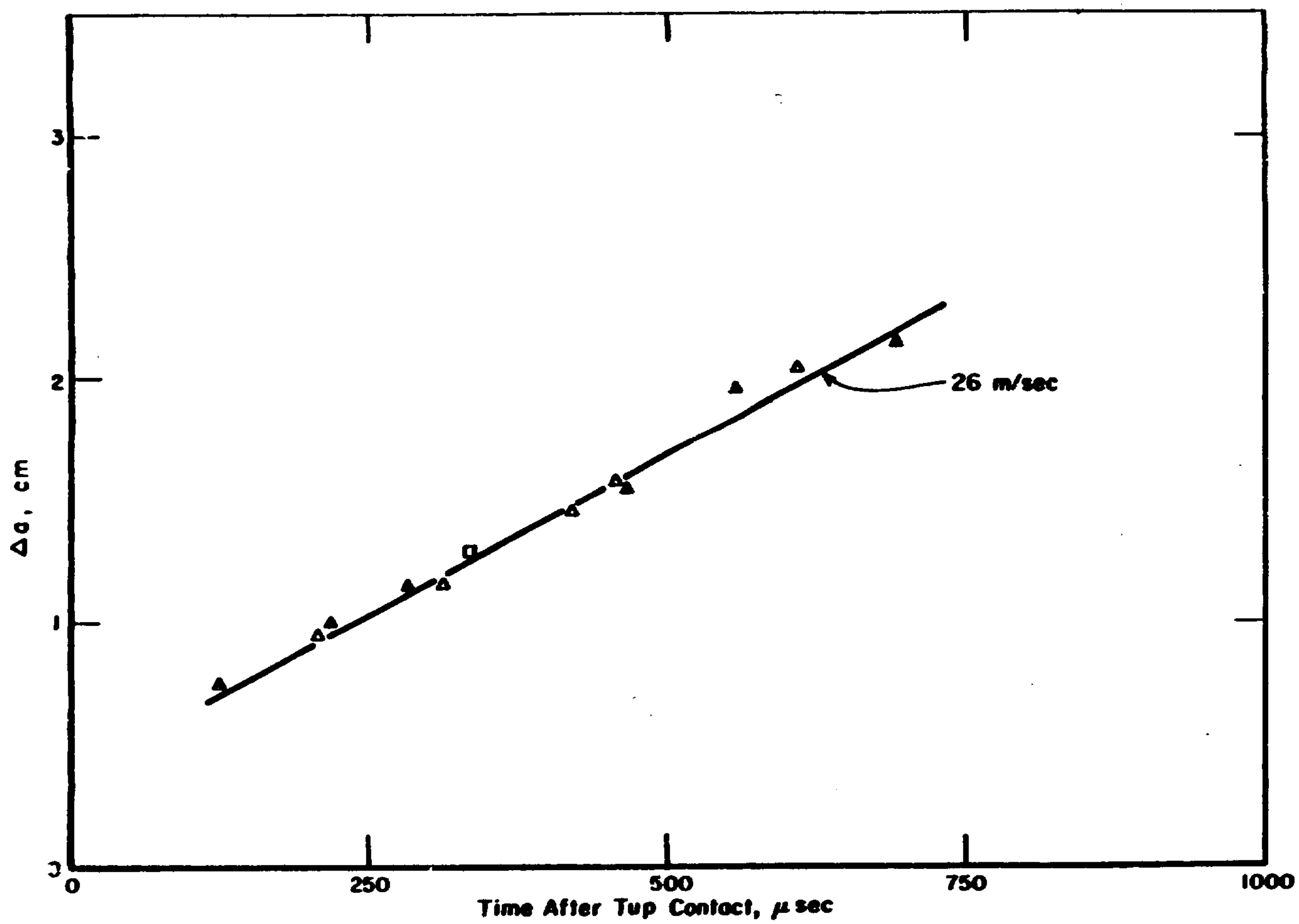


FIGURE 7. MEASURE CRACK EXTENSION-TIME RECORD FOR HY-130 STEEL FROM DYNAMIC TEAR TEST EXPERIMENT.

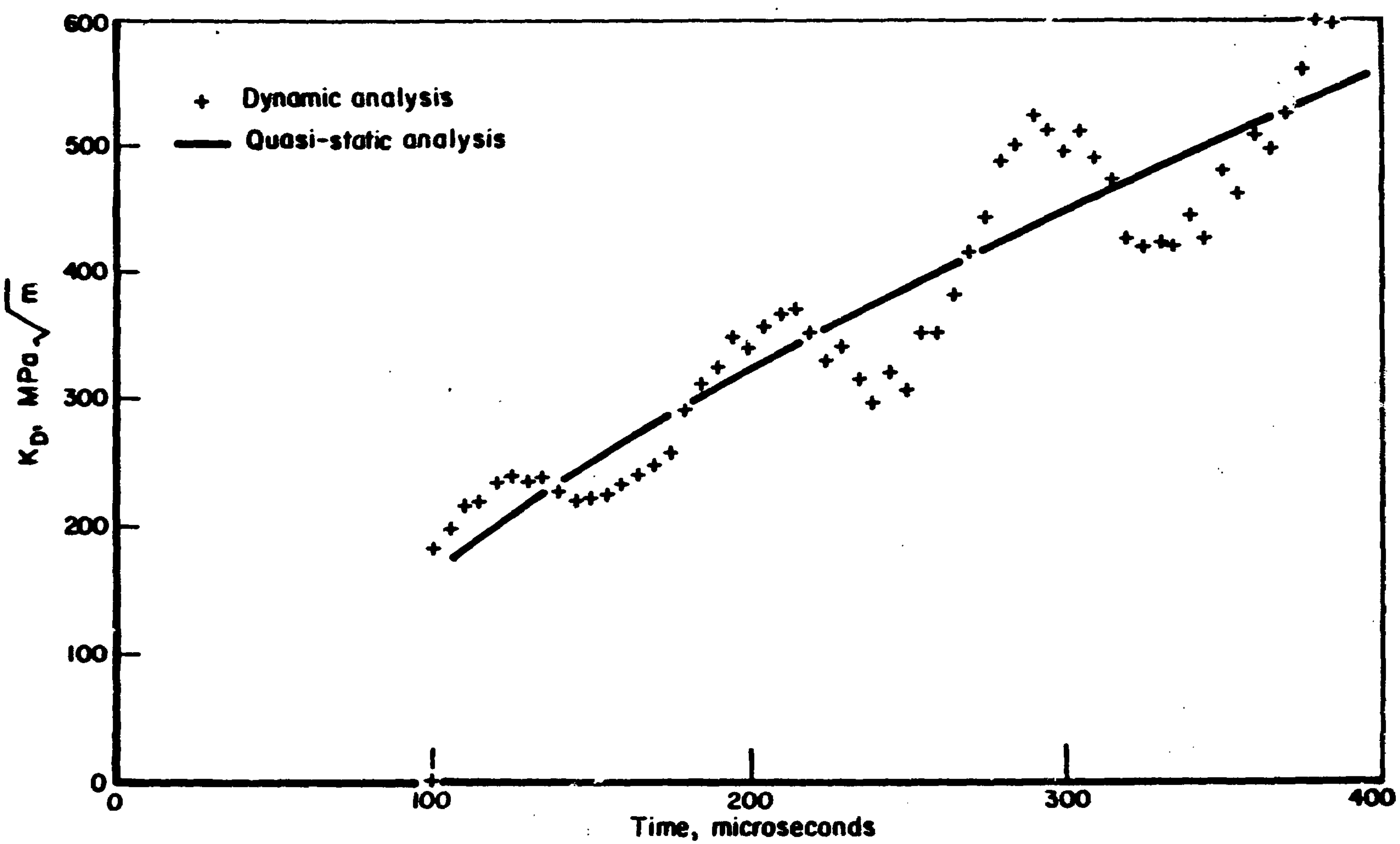


FIGURE 8. COMPARISON OF ELASTIC FRACTURE TOUGHNESS VALUES DEDUCED BY QUASI-STATIC AND DYNAMIC CALCULATIONS FOR A HY-130 DYNAMIC TEAR TEST EXPERIMENT

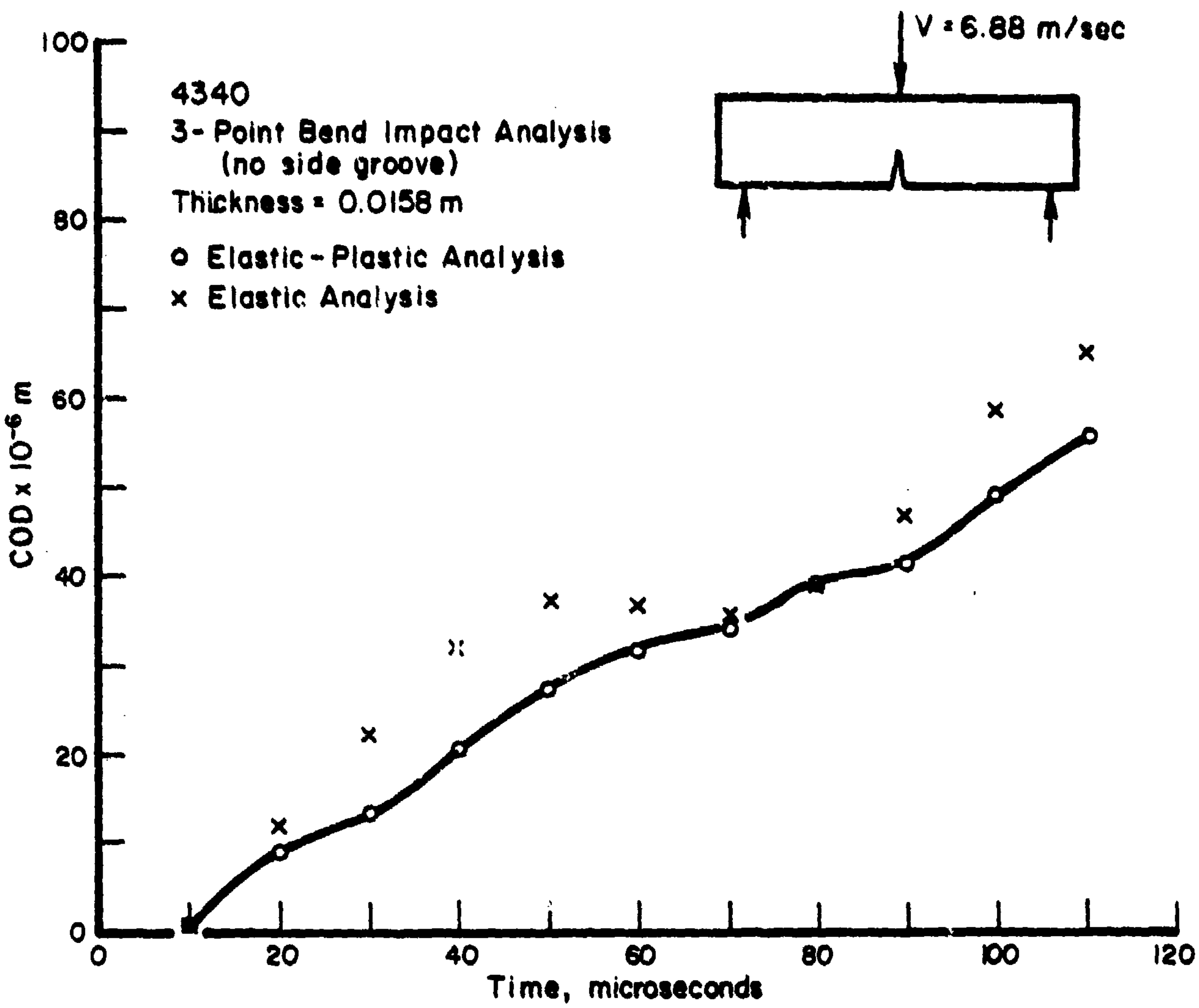


FIGURE 9. CRACK-OPENING DISPLACEMENT (0.00068 m BEHIND THE CRACK TIP) VS TIME.

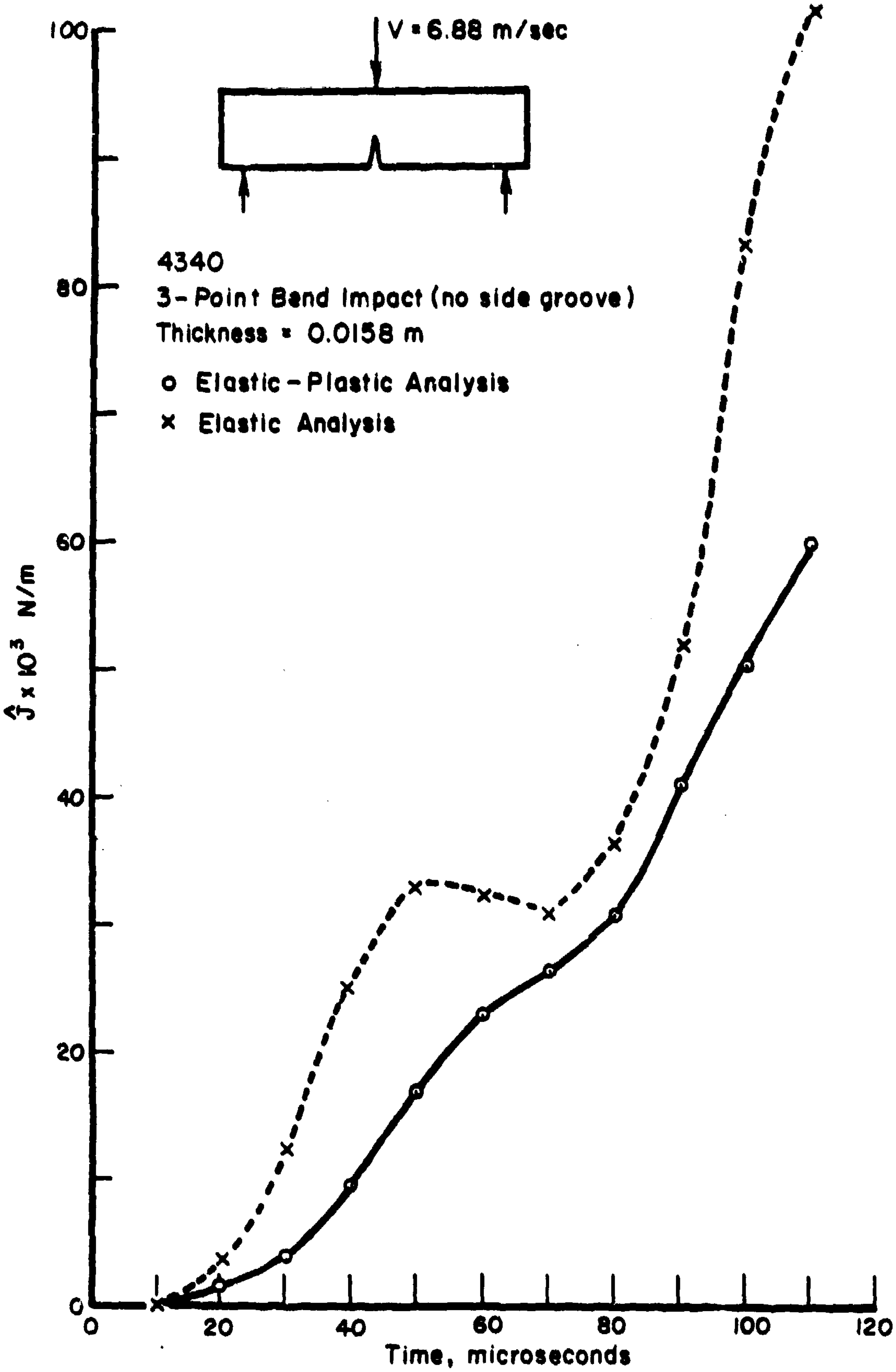


FIGURE 10. $\hat{J}$-INTEGRAL VS. TIME

DISCUSSION AND CONCLUSIONS

Uncertainties now exist concerning the adequacy of elastodynamic solution procedures for unstable crack propagation and crack arrest analyses. The most obvious reason for the existence of these uncertainties is the neglect of crack tip plasticity in such formulations. To address this possibility, an elastic-plastic dynamic finite element capability has been developed. It has been shown in this paper, by comparisons with existing dynamic fracture mechanics solutions and with experimental data, that this capability is more than adequate for its intended purposes.

Some preliminary calculations comparing the elastic and the elastic-plastic predictions for a stationary crack in an impact loaded bend specimen were also obtained. These indicate that the effect of crack tip plasticity is significant even for a high-strength material. The present capability will next be extended to treat a propagating crack whereupon an even more prominent effect is expected to be revealed. But, whether or not taking direct account of crack tip plasticity during rapid crack propagation will resolve the questions that now exist on the geometry and initiation mode dependence of K_{ID} remains to be seen.

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