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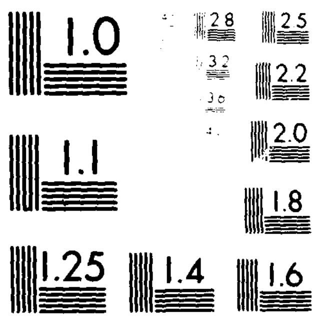
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GROWTH OF THE MAXIMUM IN A CRITICAL

AGE-DEPENDENT BRANCHING PROCESS

By

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Growth of the maximum

in a critical age-dependent branching process

by Howard Weiner

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I. Introduction.

Let $Z(t)$ denote the number of cells alive at time $t \geq 0$ in a critical age-dependent branching process with offspring generating function $h(s)$ and absolutely continuous cell lifetime distribution $G(t)$. It is assumed that $h(s)$ and $G(t)$ have finite second moments. The process evolves by starting with one newborn cell at time $t = 0$. At the end of its life, the distribution of offspring cells follows $h(s)$. Each new cell proceeds independently and identically as every other cell and independent and identically of the parent cell.

Let

$$M(t) \equiv \max_{0 \leq s \leq t} Z(t). \tag{1.1}$$

It is shown by a comparison method that $EM(t) \sim c(t) \log t$, and $\operatorname{Var} M(t) \sim b(t)t$, $0 < d < c(t)$, $b(t) < c < \infty$ for t sufficiently large.

II. Galton-Watson process.

Let $\{Z_n\}$, $n \geq 0$, with $Z_0 \equiv 1$ denote the number of cells in a critical Galton-Watson branching process in discrete time with (Athreya, Ney (1970), pp. 6-7)

$$k_n(s) \equiv Es^{Z_n} = \frac{np - (np-q)s}{np + q - nps} \tag{2.1}$$

where $p > 0$, $q > 0$, $p + q = 1$ and are to be chosen in Section III.

Let

$$M_n = \max_{1 \leq \ell \leq n} Z_\ell. \tag{2.2}$$

Lemma 1. For $1 \ll r \leq n$

$$\frac{b}{r} \leq P[M_n \geq r] \leq \frac{1}{r} \tag{2.3}$$

for some $0 < b \leq 1$.

For $r \geq n$,

$$P[M_n > r] \leq \left(1 + \frac{q}{np}\right)^{-r} \tag{2.4}$$

Proof. Since $\{Z_n\}$ is a non-negative martingale with $EZ_n = 1$, the right side of (2.3) follows by Kolmogorov's maximal inequality.

The left side of (2.3) follows from

$$P[M_n \geq r] \geq P[Z_r \geq r] = \frac{q}{rp+q}\left(1 + \frac{q}{rp}\right)^{r-1} \tag{2.5}$$

so that

$$P[M_n \geq r] \geq \frac{q}{rp+q} e^{-q/p} \simeq \frac{b}{r},$$

where

$$b = \frac{q}{p} e^{-q/p}. \tag{2.6}$$

To prove (2.4), let

$$T = \begin{cases} \min\{k, 1 \leq k \leq n \text{ such that } Z_k > r\} \\ n \text{ if } Z_k \leq r, \text{ all } 1 \leq k \leq n \end{cases}$$

Then

$$E[Z_T;Z_T > r] \geq rP[Z_T > r] = rP[M_n > r], \tag{2.7}$$

and repeated use of the martingale property,

$$E[Z_T;Z_T > r] = E[Z_n;Z_T > r] \leq E[Z_n; \bigcup_{\ell=1}^{n} \{Z_\ell > r\}]$$

so that

$$E[Z_T;Z_T > r] \leq \sum_{\ell=1}^{n} E[Z_\ell;Z_\ell > r]. \tag{2.8}$$

Hence (2.7), (2.8) yield

$$P[M_n > r] \leq \frac{1}{r} \sum_{\ell=1}^{n} E[Z_\ell;Z_\ell > r]. \tag{2.9}$$

By a computation using (2.1),

$$E[Z_\ell; Z_\ell > r] = \left(\frac{\ell p}{\ell p+q}\right)^{r-1}\left(\frac{rq+\ell p}{\ell p+q}\right) \tag{2.10}$$

Then (2.9), (2.10) suffice for (2.4).

<u>Lemma 2</u>. Under the hypotheses of Lemma 1,

$$EM_n \sim a_n \log n, \; 0 < d < a_n < c \tag{2.11}$$

$$EM_n^2 \sim b_n n, \qquad 0 < d < b_n < c \tag{2.12}$$

for n sufficiently large, and c, d denote positive finite constants.

<u>Proof</u>.

$$\sum_{r=1}^{n} P[M_n \geq r] \leq EM_n = \sum_{r=1}^{n} P[M_n \geq r] + \sum_{r=n+1}^{\infty} P[M_n \geq r]. \tag{2.13}$$

By (2.3), for $1 \leq r \leq n$, there exist positive constants a, d such that

$$a \leq rP[M_n \geq r] \leq d. \tag{2.14}$$

By (2.4), the second sum on the right of (2.13) is a convergent series. Then (2.14) applied to the other sum on both sides of (2.13) suffices for (2.11).

The expression (2.12) follows by the same argument using the expression

$$EM_n^2 \simeq 2 \sum_{r=1}^{\infty} rP[M_n \geq r]. \tag{2.15}$$

III. Age-Dependent case.

Theorem. Let a critical age-dependent branching process with absolutely continuous lifetime distribution function G(t) and offspring generating function h(s), both have finite second moments, then

$$EM(t) = a(t)\log t \tag{3.1}$$

$$\mathrm{Var}\, M(t) = b(t)t \tag{3.2}$$

where $0 < d < a(t) < c < \infty$, $0 < d < b(t) < c < \infty$, for c, d constants.

Proof. Since G(t) is absolutely continuous, the split times of the Z(t) process are distinct a.s.

Let $\{W_n\}$, $n \geq 1$ denote a critical Galton-Watson process with given offspring generating function h. By Spitzer's comparison lemma (Athreya and Ney, 1970, p. 22) and its extension to joint distributions (see, e.g. Weiner (1978), pp. 216-217), there exist critical Galton-Watson processes $\{Z_{0n}\}$, $\{Z_n\}$ each with fractional-linear offspring generating function with positive variances, an $0 < s_0 < 1$, and an $M > 0$ so that for $s_0 < s_\ell < 1$, $1 \leq \ell \leq m$, and $M < n_1 < n_2 < \cdots < n_m$,

$$E\left[\prod_{\ell=1}^{m}\left(s_\ell^{Z_{0n_\ell}}\right)\right] \leq E\left[\prod_{\ell=1}^{m}\left(s_\ell^{W_{n_\ell}}\right)\right] \leq E\left[\prod_{\ell=1}^{m}\left(s_\ell^{Z_{n_\ell}}\right)\right]. \tag{3.3}$$

From the form of the terms $P[Z_n = j]$ where $\{Z_n\}$ is a critical Galton-Watson process with fractional linear offspring generating function, one may conclude that

$$P[\bigcup_{\ell=1}^{m} Z_\ell > j] \text{ is increasing in } 0 < \sigma^2 \equiv E(Z_1 - 1)^2 < \infty. \tag{3.4}$$

From (3.3), (3.4) it follows that there are critical Galton-Watson processes with fractional linear offspring generating functions with positive variances $\{Z_{in}\}$, $1 \leq i \leq 4$, such that for all n sufficiently large, with

$$
\begin{aligned}
M_n &= \max_{1 \leq \ell \leq n} W_\ell \\
M_{in} &= \max_{1 \leq \ell \leq n} Z_{i\ell}, \qquad 1 \leq i \leq 4
\end{aligned}
\tag{3.5}
$$

such that

$$
\begin{aligned}
EM_{1n} &\leq EM_n \leq EM_{2n} \\
\mathrm{Var}\, M_{3n} &\leq \mathrm{Var}\, M_n \leq \mathrm{Var}\, M_{4n}.
\end{aligned}
\tag{3.6}
$$

From (1.3), (1.4), (2.4) of (Esty (1975) pp. 49-50), an induction yields that for $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m$, $\int_0^\infty t dG(t) = \mu > 0$, $0 < s_\ell < 1$, $1 \leq \ell \leq m$, and $n_\ell = [\alpha_\ell t/\mu]$, $t_\ell = \alpha_\ell t$, $1 \leq \ell \leq m$, that

$$\lim_{t \to \infty} t \left| E\left[\prod_{\ell=1}^{m} \left(s_\ell^{Z(t_\ell)}\right)\right] - E \prod_{\ell=1}^{m} \left(s_\ell^{W_{n_\ell}}\right) \right| = 0. \tag{3.7}$$

It follows that for t sufficiently large, and $n = [t/\mu]$, that

$$
\begin{aligned}
&|EM_n - EM(t)| \to 0 \\
&|\mathrm{Var}\, M_n - \mathrm{Var}\, M(t)| \to 0
\end{aligned}
\tag{3.8}
$$

so that (3.6), (3.8) imply that for $n = [t/\mu]$, and $t \to \infty$, that

$$\begin{aligned} EM_{1n} &\le EM(t) \le EM_{2n} \\ \mathrm{Var}\, M_{3n} &\le \mathrm{Var}\, M(t) \le \mathrm{Var}\, M_{4n}. \end{aligned} \tag{3.9}$$

Then (2.11), (2.12) and (3.9) yield (3.1), (3.2), upon replacing n by $[t/\mu]$. This completes the theorem.

IV. Remarks.

The distribution of the absolute maximum of a critical Galton-Watson process over all time until extinction and the application of this result to critical age-dependent processes has been obtained in (Lindvall (1976)) by different methods.

This approach, combining easily estimable quantities from the critical Galton-Watson process with fractional-linear offspring generating function with the asymptotic approximations in (Esty (1975), pp. 49-50) may be used to obtain asymptotic results for critical age-dependent branching processes. For example, if T = time to extinction, then $tP[T > t] \to \alpha > 0$.

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GROWTH OF THE MAXIMUM IN A CRITICAL
AGE-DEPENDENT BRANCHING PROCESS

Let $Z(t)$ denote the number of cells alive at time t in a critical age-dependent branching process starting with one new cell at $t = 0$ and let $M(t) = \max_{0 \le s \le t} Z(s)$. Under suitable moment assumptions and an absolutely continuous lifetime distribution function it is shown that $EM(t) \sim c(t) \log t$, $\mathrm{Var}\, M(t) \sim b(t)t$, $0 < d < b(t), c(t) < c < \infty$ for t sufficiently large. The method is by comparison with critical fractional-linear Galton-Watson processes.

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