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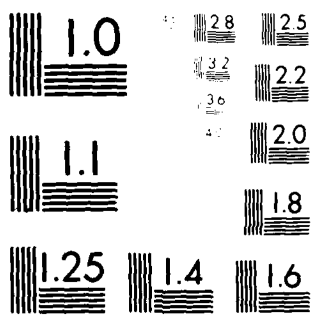

MICROCOPY RESOLUTION TEST CHART

CHOOSING THE BEST OF THE CURRENT CROP

By

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Choosing the Best of the Current Crop

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1. INTRODUCTION AND SUMMARY

This paper presents a new version of the so-called "best choice" problem - that is, the problem of sequential selection of the best one of a sequence of n rankable items which appear in random order.

Two standard versions of the problem are the "no-information" problem, in which only the successive relative ranks of the items can be observed, and the "full information" problem in which observations are successive values in a random sample from a known continuous distribution. Optimal stopping rules and their probabilities of success (i.e. choosing the best) are contained in Gilbert and Mosteller (1966). As $n \to \infty$ the optimal probabilities are asymptotically $e^{-1} \approx .37$ and $v^* \approx .58$ in the two problems.

These two versions are not only inherently oversimplified models for selection, but may be criticized as being in one case too constraining (based only on relative ranks) and in the other too demanding (the distribution must be known precisely).

A number of intermediate versions have been proposed, which feature some form of partial prior knowledge of the distribution. Petruccelli (1978) showed that if the distribution is known to be normal, then the maximin success probability is asymptotically v^*; while if it is uniform on an interval of known length the limiting probability is strictly between e^{-1} and v^*. However for the class of all uniform distributions, Samuels (1979) showed that the maximin stopping rule is based only on relative ranks; hence the limiting success probability is e^{-1}. The same limit applies to a closely related Bayesian problem of Stewart (1978). Giving the end-points (a,b) of the uniform distribution a conjugate prior density of the form

$$f(a,b) = m(m-1)(\beta-\alpha)^{m-1}/(b-a)^{m+1} \qquad -\infty<a<\alpha<\beta<b<\infty$$

he found that when the risk is 1 - . P(item selected is largest in the sample and > β) the Bayes stopping rule is based only on relative ranks with respect to β as well as to preceding observations.

Campbell (1977) placed a Dirichlet process prior on the space of all distributions. In the situation in which the values themselves are observed, this leads to strategies and success probabilities that depend only on the nonatomic measure of the process. In the event that only relative ranks are available, Campbell (1978) showed the optimal strategy depends only on the mass of the measure. In that the process selects only discrete distributions, ties are inevitable and the mass of the measure regulates these. The improvement over the "no-information" probability vanishes as the mass tends to infinity.

It should be noted that to achieve the "full information" success probability it suffices to observe only $\{F(X_i)\}$ where F is the underlying continuous distribution. For large i, $F(X_i)$ is well approximated by $i^{-1}Y_i$ where Y_i is the rank of X_i among $X_1, \ldots, X_i$. This suggests representing prior information in the form of a preliminary or training sample of, say, size m, and observations as relative ranks with respect to the entire training sample as well as to predecessors in the current sample. This is the version presented here.

As an application of this version one may consider this year's Ph.D.'s in Statistics who seek academic employment as the current crop, and those in the interviewer's previous years of experience as the training sample. That there is some oversimplification here is readily conceded.

The form of optimal stopping rules for this problem turns out to be particularly simple; their parameters depend on m (the training sample size) and n (the size of the current crop) only through $m + n$. Specifically, there are integers $\{S_k^*(m+n)\}$ non-decreasing in k such that the optimal (m,n)-policy is to stop at $i > m$ if the i-th item is the best so far of the current crop and k-th best of all i, provided that $i \geq S_k^*(m+n)$.

Not only are those parameters readily computable from (2.6) and (2.8) and closely approximated from (2.12), but also the limits s_k^* of $S_k^*(m+n)/(m+n)$ exist as $m+n \to \infty$, are themselves computable from (3.14) and can serve as asymptotically optimal approximations to the optimal (m,n) parameters. (See Table 2).

Although the optimal success probabilities $P^*(m,n)$ can be obtained from the algorithm given by (2.5) and (2.6) their true form is best revealed by the following asymptotic results: For each fixed n, as m increases from 0 to ∞, $P^*(m,n)$ increases from the "no-information" optimal success probability to the "full-information" optimal value for sample size n, which is denoted by v_n^* (Theorem 4.1). And for $m/(m+n) \to t$, as m and n become infinite, $P^*(m,n) \to p^*(t)$, the function $p^*(\cdot)$ being given not only implicitly as the limit of solutions to a system of piece-wise differential equations (3.7) but explicitly in Theorem 3.1 from which not only is numerical evaulation a simple matter (see Table 4) but also $\lim_{t\to 1} p^*(t) = v^*$ can be proved (Theorem 3.3).

The s_k^*'s and $p^*(\cdot)$ are introduced in section 3 as, respectively, the parameters of the optimal policies and the optimal success probabilities for an infinite version of the problem in which the arrival times of the best item, second best, etc. are independent random variables, each uniform on $(0,1)$. This model provides, among other things, a convenient vehicle for showing in Theorem 4.2 that the difference between $P^*(m,n)$ and $p^*(m/(m+n))$ is of an order no bigger than $(m+n)^{-\frac{1}{2}}\log(m+n)$. (Numerical evidence - see Table 4 - suggests that the actual order is $(m+n)^{-1}$.)

Since we know of no way to deduce the explicit form of $p^*(\cdot)$ directly from the differential equations (3.7), it is in fact obtained as a special case of the explicit formula for the success probability of any "plausible" stopping rule. The formula is given in Theorem 3.2 which is proved in Section 5. This formula also facilitates evaluation of suboptimal rules of simplified form analagous to ones considered in Gilbert and Mosteller (1966), though we have not done so.

2. THE FINITE PROBLEM

The problem of "choosing the best of the current crop with maximal probability" may be formulated as follows:

Let $X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+n}$ be a random permutation of $\{1, 2, \ldots, m+n\}$; i.e. all $(m+n)!$ permutations are equally likely; and let $Y_1, Y_2, \ldots, Y_{m+n}$ denote the corresponding sequence of relative ranks; i.e. $Y_i = j$ if X_i is the j-th smallest of $\{X_1, \ldots, X_i\}$. Let $\mathcal{J} = \mathcal{J}(m,n)$ be the class of stopping rules τ based on the Y_i's and $P^*(m,n)$ be the maximal attainable probability of selecting the smallest of $X_{m+1}, \ldots, X_{m+n}$; i.e.,

$$(2.1)\qquad P^*(m,n) = \max_{\tau \in \mathcal{J}} P(X_\tau = \min\{X_i : m < i \le m+n\}).$$

(We write "max" rather than "sup" because standard results in optimal stopping, as presented in Chow, Robbins, and Siegmund (1971), insure that the supremum is attained.) The object is to evaluate $P^*(m,n)$ and find the optimal stopping rule which attains this value.

As in other problems of optimal stopping based on relative ranks, the distributional properties of the X_i's are exploited, notably the fact that the Y_i's are independent with each Y_i uniformly distributed on $\{1, 2, \ldots, i\}$. This leads to two expressions which greatly simplify the form of the standard backward induction algorithm for the optimal rule and its expected payoff, as described in Chow, Robbins, and Siegmond (1971). First the conditional probability of success for selecting the i-th arrival satisfies

$$(2.2)\quad P(X_i = \min\{X_j : m<j\leq m+n\} \mid Y_1, \ldots, Y_i)$$
$$= \sum_{k=1}^{i} \Big[\prod_{j=i+1}^{m+n} \Big(1 - \frac{k}{j}\Big)\Big] I_{\{Y_i=\min\{Y_r: m<r\leq i\}=k\}}$$
$$= \sum_{k=1}^{i} [(i)_k/(m+n)_k]\, I_{\{Y_i=\min\{Y_r: m<r\leq i\}=k\}}; \qquad m<i\leq m+n$$

where $(x)_k \equiv x(x-1)\ldots(x-k+1)$.

Second, defining

$$(2.3)\quad P_k^*(i;m+n) \equiv \max_{\tau>i} P(X_\tau\leq k \;\&\; X_\tau = \min\{X_j : i<j\leq m+n\}) \qquad 1\leq k\leq i\leq m+n,$$

the optimal success probability with stopping ranks which don't select any of the first i arrivals satisfies

$$(2.4)\quad \max_{\tau>i} P(X = \min\{X_j : m<j\leq m+n\} \mid Y_1, \ldots, Y_i)$$
$$= \sum_{k=1}^{m+1} P_k^*(i;\, m+n)\, I_{\{\min\{Y_j:\, m<j\leq i\} = k\}} \qquad m<i<m+n.$$

It can then be shown that

$$(2.5)\qquad P^*(m,n) = (m+1)^{-1}\sum_{k=1}^{m+1} \max\{P_k^*(m+1;m+n),\ (m+1)_k/(m+n)_k\}$$

$$m=0,\ 1,\ 2,\ \ldots\ ;\ n=1,\ 2,\ \ldots$$

where the $P_k^*(i,\ m+n)$'s are computed iteratively from

$$(2.6a)\qquad P_k^*(i-1;\ m+n) = (1-k/i)\ P_k^*(i;\ m+n) + i^{-1}\sum_{j=1}^{k} \max\{P_j^*(i;\ m+n),\ (i)_j/(m+n)_j\} \qquad k<i<m+n$$

together with the boundary conditions

$$(2.6b)\qquad P_k^*(m+n-1;\ m+n) = k/(m+n) \qquad k=1,\ 2,\ \ldots,\ m+n-1.$$

Furthermore the optimal policy is to stop at the first $i > m$ (if any) for which

$$Y_i = \min\ \{Y_j:\ m<j\leq i\}$$

and

$$(2.7)\qquad P_{Y_i}^*(i;\ m+n) \leq (i)_{Y_i}/(m+n)_{Y_i}.$$

It is noteworthy that (2.7) depends on m and n only through m+n. This means that all of the optimal policies for a given value of m+n combine the obvious <u>distinct</u> prescriptions, "don't stop if the present arrival is not best so far of the current crop ...", with the <u>common</u> criterion, "... and, if it is best so far of the current crop, and k-th best among all i seen so far, stop if and only if

$$i \in I_k(m+n) \subset \{1,\ 2,\ \ldots,\ m+n\}."$$

What is the structure of the sets $I_k(m+n)$? Since the right side of (2.7) is decreasing in Y_i while, by its very definition, the left side is increasing in Y_i it follows that

$$I_k(m+n) \supset I_{k+1}(m+n).$$

To show that each $I_k(m+n)$ is actually an interval of the form

$$I_k(m+n) = \{i: S_k^*(m+n) \le i \le m+n\}$$

-- and, necessarily,

$$(2.8) \quad S_k^*(m+n) = \min \{i: P_k^*(i; m+n) \le (i)_k/(m+n)_k\}$$

-- it would suffice to establish that each $P_k^*(i; m+n)$ is decreasing in i. While this is not true, the change of variables

$$(2.9) \quad G_k(i; m+n) = P_k^*(i; m+n)/[(i)_k/(m+n)_k]$$

accomplishes the same end because then (2.6) becomes

$$(2.10a) \quad G_k(i-1; m+n) = G_k(i; m+n)$$

$$+ (i-k)^{-1} \sum_{j=1}^{k} [\frac{(m+n-j)_{k-j}}{(i-j)_{k-j}}] \max\{G_j(i; m+n), 1\}$$

$$k<i<m+n,$$

$$(2.10b) \quad G_k(m+n-1; m+n) = k/(m+n-k),$$

which shows that $G_k(k; m+n)$ is decreasing in i. Since (2.7) is equivalent to

$$G_{Y_i}(i; m+n) \le 1$$

this proves

<u>Proposition 2.1</u>. <u>The optimal stopping rule in the finite problem is</u>

$$(2.11) \quad \tau^*(m,n) = \min\{i>m: Y_i = \min\{Y_j: m<j\leq i\} \ \& \ i \geq S_{Y_i}{}^*(m+n)\}$$

$$= m+n \text{ if no such } i<m+n$$

<u>where</u> $S_k{}^*(m+n)$ <u>is given by</u> (2.8) <u>and is non-decreasing in</u> k.

The parameters $\{S_k{}^*(m+n)\}$ can easily be computed from (2.6) and (2.8). Some values are given in Table 2. In addition they may be estimated from the inequalities

$$(2.12) \qquad (1-k^{-1})(m+n) \leq S_k{}^*(m+n) \leq (.5)^{1/k}(m+n) + k$$

the derivation of which is similar to (but somewhat less tidy than) that for (3.15) in the next section.

3. THE INFINITE PROBLEM

As in Gianini and Samuels (1976), let the arrival times $U_1, U_2, \ldots$, of the best, second best, etc., be IID, each uniformly distributed on (0,1). This is motivated by the fact that in the finite problem not only the ranks of successive arrivals, but also the arrival positions of the best, second best, etc., form a random permutation of $\{1, 2, \ldots, m+n\}$.

For each $t \in (0,1)$, let $Z_1(t), Z_2(t), \ldots$, be the arrival times of the best, second best, etc. among those which have arrived by time t; $\mathfrak{F}(t)$ be the σ-field generated by $\{Z_i(t): i=1, 2, \ldots\}$; and, noting that $s < t$ implies $\mathfrak{F}(s) \subset \mathfrak{F}(t)$, $\mathcal{J}$ be the class of all (possibly defective) stopping rules τ adapted to $\{\mathfrak{F}(t): 0<t\leq 1\}$ with the property that $\tau \in$ [illegible]$_i$: i=1, 2, ...} on the set where τ is defined.

In this framework the problem of interest can be described as follows. For any $t \in [0,1)$ define

(3.1a) $K_1(t) \equiv \min\{j: U_j > t\}$

(3.1b) $K_{i+1}(t) \equiv \min\{j > K_i: U_j > t\}$ $\qquad$ $i=1, 2, \ldots$

(so $K_1(t)$, $K_2(t)$, ... are the (unobservable) absolute ranks of the best, second best, etc. post-t arrivals), and let

(3.2) $T_i(t) \equiv U_{K_i(t)}$ $\qquad$ $i=1, 2, \ldots$

(so $T_i(t)$ is the arrival time of the i-th best post-t arrival). The object is to evaluate

(3.3) $p^*(t) \equiv \sup_{\tau \in \mathcal{T}} P(\tau = T_1(t))$

and to find for each t a stopping rule $\tau^*(t)$ which attains this maximal probability of selecting the best post-t arrival.

To do so it is helpful to consider quantities analagous to the $P_k^*(i; m+n)$'s in the finite problem, namely

(3.4) $p_k^*(t) \equiv \sup_{\tau \in \mathcal{T}} P(\tau = T_1(t) \;\&\; K_1(t) \leq k)$ $\qquad$ $k = 1, 2, \ldots$,

and the associated optimal stopping rules $\tau_k^*(t)$. Just as in the finite problem, $p^*(t)$ and $\tau^*(t)$ are obtained from the $p_k^*(t)$'s and $\tau_k^*(t)$'s. It is evident from their definition that $p_k^*(t)$ is increasing in k; also, for any $\tau \in \mathcal{T}$,

(3.5) $P(\tau = T_1(t)) - P(\tau = T_1(t) \;\&\; K_1(t) \leq k) \leq P(K_1(t) > k) = t^k$.

Hence

$$(3.6)\quad p^*(t) - t^k \le p_k^*(t) \le p^*(t) \qquad k=1, 2, \ldots, 0\le t<1.$$

For the problem of finding (3.4), the conditional success probability for stopping at a particular arrival time, say u, given $\mathcal{F}(u)$, is

$$\sum_{j=1}^{k} u^j I_{\{Z_i(u)\le t,\ i<j;\ Z_j(u)=u\}}$$

if $u > t$;, zero otherwise. Since this is not a function only of u and the relative rank at u of the current arrival, the standard results of Mucci (1973a and b), Gianini and Samuels (1976) and Lorenzen (1977) do not apply. Nevertheless, identical arguments lead to the following system of differential equations:

$$(3.7a)\quad \frac{d}{dt} p_k^*(t) = t^{-1}[kp_k^*(t) - \sum_{j=1}^{k} \max\{p_j^*(t), t^j\}] \qquad k=1, 2, \ldots;\quad 0<t<1$$

with the boundary conditions

$$(3.7b)\quad p_k^*(1-) \equiv 0.$$

This formula is strongly suggested by letting the arrival times in the finite problem be $\{k/(m+n): k=1, 2, \ldots, m+n\}$ and rewriting (2.6a) as

$$\frac{P_k^*(i;m+n) - P_k^*(i-1;m+n)}{1/(m+n)} = [i/(m+n)]^{-1}[kP_k^*(i;\ m+n) - \sum_{j=1}^{k} \max\{P_j^*(i;\ m+n),\ (i)_j/(m+n)_j\}].$$

From the form of (3.7) it follows that $p_k^*(t)$ is attained by the stopping rule

(3.8) $\tau_k^*(t) = \min\{U_i > t$: for some $r \leq k, Z_j(U_i) < t$ for $j < r$,

$Z_r(U_i) = U_i$, and $U_i^{\,r} \geq p_r^*(U_i)\}$

= (undefined) if no such U_i;

also

(3.9) $\tau^*(t) = \lim_{k\to\infty} \tau_k^*(t)$.

Just as in the finite problem, a close examination of (3.7) leads to explicit expressions for these optimal rules. The bonus here is that the solution to (3.7) itself is also obtained. Making the change of variables

$$g_k(t) = t^{-k} p_k^*(t)$$

transforms (3.7) to

(3.10a) $g_k'(t) = -\sum_{j=1}^{k} t^{-(k+1-j)} \max\{g_j(t), 1\}$ $\qquad k=1, 2, \ldots; 0<t<1$

(3.10b) $g_k(1^-) \equiv 0$.

Thus the $g_k(\cdot)$'s are strictly decreasing and unbounded in (0, 1) so one can define

(3.11) s_k^*: $g_k(s_k^*) = 1$ $\qquad k=1, 2, \ldots$.

From (3.10) it is clear that for each t in (0, 1), $g_k(t)$ is increasing in k; hence so are the s_k^*'s. Now, on $[s_k^*, 1]$, (3.10) is simply

$$g_k'(t) = -\sum_{j=1}^{k} t^{-j}$$

$$g_k(1^-) = 0.$$

Hence

$$(3.12) \qquad g_k(t) = |\ln t| + \sum_{j=1}^{k-1} j^{-1}(t^{-j}-1)I_{\{k>1\}} \qquad s_k^* \le t < 1.$$

Thus a complete description of the optimal stopping rules $\tau^*(t)$ can be summarized by

<u>Proposition 3.1</u>. <u>In the infinite problem, there is a single increasing sequence</u> $\{s_k^*: k=1, 2, \ldots\}$ <u>such that, for any</u> t <u>in</u> $[0,1)$,

$$(3.13) \qquad \tau^*(t) = \min \{U_i > t: \text{ for some } r, Z_j(U_i) < t \text{ for } j<r,\ Z_r(U_i) = U_i \ge s_r^*\}$$
$$= \text{(undefined) if no such } U_i.$$

<u>The sequence</u> $\{s_k^*\}$ <u>is given by</u>

$$(3.14a) \qquad s_1^* = e^{-1}$$

$$(3.14b) \qquad |\ln s_k^*| + \sum_{j=1}^{k-1} j^{-1}[(s_k^*)^{-j}-1] = 1 \qquad k=2, 3, \ldots .$$

<u>Corollary</u>: <u>For any</u> $t \in [0,1)$ <u>and</u> $k = 1, 2, \ldots$ <u>the value</u> $p_k^*(t)$ <u>is attained by</u> (3.13) <u>with</u> s_r^* <u>replaced by one for each</u> $r > k$.

Some values of s_k^* are given in Table 1 together with the upper and lower bounds provided by the following

<u>Proposition 3.2</u>. <u>For each</u> $k=1, 2, \ldots$

$$(3.15) \qquad 1- k^{-1} < s_k^* < (1/2)^{1/k}.$$

<u>Proof</u>: On the one hand, $p_k^*(s_k^*) = (s_k^*)^k$. On the other hand, taking $t = s_k^*$ in (3.4),

$$(3.16) \qquad P(K_1(s_k^*) < k+1 < K_2(s_k^*)) \le p_k^*(s_k^*) \le P^*(K_1(s_k^*) \le k).$$

The second inequality is immediate; the first holds because the first probability is smaller than the success probability for the rule "select the first post-s_k^* arrival with relative rank $\leq k$". The left and right sides of (3.16) are just $k(1-s_k^*)(s_k^*)^k$ and $1-(s_k^*)^k$ respectively. The result then follows by substitution. □

Remark: The bounds in (3.15) are the limits obtained by dividing the left and right sides of (2.12) by m+n and letting $m+n \to \infty$.

As for the optimal probabilities, $p_k^*(t)$, these can be obtained iteratively from (3.10) and (3.11) since (3.10) is given by (3.12) on $[s_k^*,1)$ and can be transformed to

$$(3.17) \qquad \frac{d}{dt}(t\, g_k(t)) = -\sum_{j=1}^{k-1} t^{-(k-j)} \max\{g_j(t),1\}\, I_{\{k>1\}}$$

on $(0, s_k^*)$. While one could certainly contemplate numerical evaluation of the $p_k^*(t)$'s via (3.17), this is unnecessary, for we have -- by indirect means -- arrived at an explicit expression for the solution to (3.7), namely the following:

Theorem 3.1. Let

$$(3.18a) \qquad r^*(t) = \max\{j: s_j^*<t\}$$
$$= 0 \text{ if } t\leq s_1^* \qquad\qquad 0<t<1$$

$$(3.18b) \qquad r_k^*(t) = \min\,(r^*(t),k) \qquad\qquad k=1,\, 2,\, \ldots$$

Then $p^*(t) = \lim_{k\to\infty} p_k^*(t)$ where (with $r\equiv r_k^*(t)$)

$$(3.19)\quad p_k^*(t) = t^r[|\ln t|\, I_{\{r\geq 1\}} + \sum_{j=1}^{r-1} j^{-1}(t^{-j}-1)\, I_{\{r\geq 2\}}$$

$$+ \sum_{j=r}^{k} t^j h_j^*(s_{j+1}^*) \qquad 0<t<1,$$

and

$$(3.20a)\quad h_0^*(s_1^*) = e^{-1}$$

$$(3.20b)\quad h_k^*(s_{k+1}^*) = k^{-1}[(s_{k+1}^*)^{-k}-1] - (1-s_{k+1}^*) \qquad k=1, 2, \ldots\ .$$

Remark: From the first inequality of (3.15) we conclude that

$$h_k^*(s_{k+1}^*) < (e-1)/k < 2k^{-1}$$

and hence

$$p^*(t) = \sum_{j=K+1}^{\infty} t^j h_j^*(s_{j+1}^*) < 2t^{K+1}/K(1-t) \qquad K=1, 2, \ldots, \quad 0<t<1.$$

This inequality facilitates computation of $p^*(t)$ to any desired degree of accuracy and was used in preparing Table 3.

It is straightforward to verify that (3.19) is a solution to (3.7) which, strictly speaking, proves the theorem. But, without some indication of how (3.19) might be deduced directly from (3.7), this is hardly satisfactory. And we know of no such deductive method.

Our method of obtaining (3.19) begins by considering the class of all families of stopping rules of the form

$$(3.21)\quad \tau(t;\, \underline{s}) = \min\{U_i>t\colon Z_j(U_i)<t \text{ for } j<r,\ Z_r(U_i) = U_i \geq s_r\}$$
$$= (\text{undefined}) \text{ if no such } U_i$$

where $\underline{s} = (s_1, s_2, \ldots)$ is any sequence with

(3.22) $\quad 0 < s_1 \leq s_2 \leq \ldots \leq 1.$

The optimal stopping rules are of this form; namely

$$\tau^*(t) = \tau(t; \underline{s}^*)$$
$$\tau_k^*(t) = \tau(t; \underline{s}_k^*)$$

where

$$s^* = (s_1^*\ s_2^*, \ldots)$$
$$s_k^* = (s_1^*, \ldots, s_k^*, 1, \ldots).$$

For any $\underline{s}$ satisfying (3.22), denote

$$\underline{s}_k = (s_1, \ldots, s_k, 1, 1, \ldots)$$

and also

(3.23) $$r(t; \underline{s}) = \sup \{j: s_j < t\}\ I_{\{t > s_1\}}$$
$$= \infty \text{ if } \sup s_k \leq t$$

(3.24a) $\quad p(t; \underline{s}) = P(\tau(t; \underline{s}) = T_1(t)),$

(3.24b) $\quad p_k(t; s) \equiv p(t; s_k).$

Then it can be shown that on $(0, \sup s_k) - \{s_j\}$, with $r \equiv r(t; \underline{s}_k)$,

(3.25) $$\frac{d}{dt} p_k(t; \underline{s}) = t^{-1}[k\ p_k(t; \underline{s}) - \sum_{j=1}^{r} t^j I_{\{r \geq 1\}}$$
$$- \sum_{j=r+1}^{k} p_j(t; \underline{s})\ I_{\{r<k\}}]$$

$$k=1, 2, \ldots$$

which has a unique system of continuous solutions satisfying the boundary conditions $p_k(\sup s_k; \underline{s}) = 0$. That solution is given by the following:

<u>Theorem 3.2</u>. <u>For any s satisfying</u> (3.22) <u>and for</u> $t < \sup s_k$,

$$(3.26) \qquad p(t;\, \underline{s}) = t^r[|\ln t|\, I_{\{r \geq 1\}} + \sum_{j=1}^{r-1} j^{-1}(t^{-j}-1) I_{\{r \geq 2\}}] + \sum_{k=r}^{\infty} t^k\, h_k(s_{k+1})$$

<u>where</u> $r \equiv r(t;\, \underline{s}) < \infty$ <u>and</u>

$$(3.27a) \qquad h_0(s_1) = s_1\, |\ln s_1|$$

$$(3.27b) \qquad h_k(s_{k+1}) = k^{-1}[(s_{k+1})^{-k}-1] - (1-s_{k+1})\{|\ln s_{k+1}| + \sum_{j=1}^{k} j^{-1}[(s_{k+1})^{-j}-1]\} \qquad k=1,\, 2,\, \ldots\, .$$

<u>In particular</u> $p_k(t;\, \underline{s})$ <u>is given by the right side of</u> (3.28) <u>with</u> $r = r(t;\, s_k)$.

As with (3.7) and (3.19), (3.25) can be verified in a straightforward manner for $p_k(t;\, \underline{s})$ as given above, but the latter cannot be directly deduced from the former. A detailed proof of Theorem 3.2 without using (3.25) is deferred to section 5, while the relationship of Theorem 3.1 to Theorem 3.2 will be presented here.

First, the two theorems are consistent because, for each k, $h_k(s_{k+1}^*) = h_k^*(s_{k+1}^*)$ and $h_k(1) = 0$; hence $p_k(t;\, \underline{s}_k^*) = p_k^*(t)$. Second, Theorem 3.2 has the

<u>Corollary</u>; <u>For each</u> t in (0,1), $p(t;\, \underline{s})$ <u>and</u> $\{p_k(t;\, \underline{s}),\, k=1,\, 2,\, \ldots,\}$ <u>are all maximized by</u> $\underline{s} = \underline{s}^*$.

<u>Remark</u>: This corollary states only that $\tau(t;\, \underline{s}^*)$ is optimal in the subclass of all rules of the form (3.21). Its optimality in the class of <u>all</u> stopping rules is asserted by Proposition 3.1. Thus it is the combination of two methods which yields the solution to the

infinite problem. One is the so-called "method of backward induction" which leads to (3.7); the other is the probabilistic "conditioning" argument used in section 5.

<u>Proof of Corollary</u>: Strictly speaking, no proof is needed since the corollary follows immediately from Proposition 3.1 and its corollary. So we shall merely indicate briefly how it follows directly from (3.26) and (3.27). There are two key facts to be checked. First, each $h_k(x)$ is concave in (0,1) and maximized at s_{k+1}^*, so

$$r(t;\ \underline{s}') = r(t;\ \underline{s})\ \&\ 0 \le (s'_k - s_k^*)/(s_k - s_k^*) \le 1 \qquad k=1,\ 2,\ \ldots$$

$$\Rightarrow \quad p(t;\ \underline{s}') \ge p(t;\ \underline{s})$$

Second, if $\underline{s}$ and $\underline{s}'$ are such that $s_k = s_k'$ whenever $s_k > t$ or $s_k' > t$, then

$$p(t;\ \underline{s}') = p(t;\ \underline{s})\ .$$

From just these two facts it follows that any $\underline{s}$ can be replaced by $\underline{s}^*$ without decreasing $p(t;\ \underline{s})$. □

We return now to Theorem 3.1 to investigate the optimal best choice probability $p^*(t)$ for $t = 0$ and as $t \to 1$. The former case is immediate:

$$p^*(0) = h_0^*(s_1^*) = e^{-1};$$

this is the familiar limiting no-information value. As $t \to 1$ one would expect $p^*(t)$ to converge to the limiting full-information value v^*. The following expression for v^* was obtained by Samuels (1980):

$$v^* \equiv e^{-c} + (e^c - c - 1) \int_1^\infty x^{-1} e^{-cx} dx \tag{3.28}$$

where $c \approx .804352$ satisfies

$$(3.29)\qquad \sum_{j=1}^{\infty} c^j/j!j = 1.$$

From the formula

$$\int_1^{\infty} x^{-1}e^{-cx}dx = |\log c| - \gamma - \sum_{j=1}^{\infty} (-c)^j/j!j$$

where γ is Euler's constant ($\gamma \approx .577216$), v^* can easily be evaluated. The value is $v^* \approx .580164$, in agreement with that in Gilbert and Mosteller (1966).

Theorem 3.3. The following limits hold:

$$(3.30)\qquad \lim_{t\to 1} p^*(t) = v^*$$

$$(3.31)\qquad \lim_{k\to\infty} k(1-s_k^*) = c$$

$$(3.32)\qquad \lim_{t\to 1} (1-t)r^*(t) = c$$

where v^* and c are given by (3.28) and (3.29) respectively.

Proof of Theorem: To establish (3.31), write

$$(3.33)\qquad s_k^* = k/(k+c_k)$$

and use the identity

$$\sum_{j=1}^{k} j^{-1}(x^{-j}-1) = \sum_{j=1}^{k} j^{-1}\binom{k}{j}(1-x)^j x^{-j}$$

with $x = s_k^*$. It follows from (3.14b) that

$$\sum_{j=1}^{k} j^{-1}\binom{k}{j}(c_k)^j k^{-j} - k^{-1}[(1+k^{-1}c_k)^k-1] + \ln(1+k^{-1}c_k) = 1.$$

Now, by the left side of (3.15), $c_k < k/(k-1)$, so the second and third terms on the left side go to zero with k. Thus,

$$\lim_{k\to\infty} \sum_{j=1}^{k} j^{-1}\binom{k}{j}(c_k)^j k^{-j} = 1.$$

Writing the sum as

$$\sum_{j=1}^{k} [(c_k)^j \prod_{i=1}^{j-1} (1-k^{-1}i)]/j!j$$

the dominated convergence theorem can be employed to show that for any convergent subsequence $c_{k_j} \to c$, c must satisfy (3.29). Thus $c_k \to c$ and, since $s_k^* \to 1$ by (3.15), (3.31) holds.

Formula (3.32) is obtained from (3.31) by noting that $r^*(t) = k$ if $s_k^* \leq t < s^*_{k+1}$, and by using (3.33), which gives

$$s_k^* \leq t < s^*_{k+1} \Rightarrow t\, c_{k+1} - (1-t) < (1-t)k \leq t\, c_k.$$

Note that (3.32) implies

$$\text{(3.34)} \qquad \lim_{t\uparrow 1} t^{r^*(t)} = e^{-c} .$$

Also, substituting (3.33) into (3.20b) gives

$$\text{(3.35)} \qquad h^*_{k-1}(s_k^*) = \frac{1}{k-1}\{(1+\frac{c_k}{k})^{k-1}-1\} - (\frac{c_k}{k-1})(\frac{k-1}{k+c_k}) \approx \frac{1}{k}(e^c-c-1).$$

Thus,

$$\text{(3.36)} \qquad \sum_{k=r^*(t)}^{\infty} t^j\, h_k^*(s^*_{k+1}) \approx (e^c-c-1) \sum_{k=r^*(t)}^{\infty} k^{-1}t^k.$$

The remainder of the proof is based on the following two calculus lemmas which are straightforward to prove.

Lemma 1: *If* $n(t) \to \infty$ *as* t *increases to* 1, *then*

$$\sum_{k \geq n(t)} k^{-1} t^k \approx \int_{n(t)}^{\infty} x^{-1} t^x dx = \int_1^{\infty} y^{-1} t^{n(t)y} dy;$$

hence if $n(t) \ln t \to -b$ *as* $t \to 1$, *then*

$$\sum_{k \geq n(t)} k^{-1} t^k \to \int_1^{\infty} x^{-1} e^{-bx} dx.$$

Lemma 2: *If* $n(t) \to \infty$ *as* t *increases to* 1 *but* $n(t) \ln t = O(1)$, *then*

$$\sum_{1 \leq k \leq n(t)} k^{-1} (t^{-k} - 1) \approx \int_1^{n(t)} x^{-1} (t^{-x} - 1) dx.$$

Hence, if $n(t) \ln t \to -b$ *as* $t \to 1$, *then*

$$\sum_{1 \leq k \leq n(t)} k^{-1} (t^{-k} - 1) \approx \int_{b/n(t)}^{b} y^{-1} (t^{-n(t)y/b} - 1) dy$$

$$\to \int_0^b y^{-1} (e^y - 1) dy = \sum_{j=1}^{\infty} \frac{b^j}{j!j} .$$

From (3.34) observe that $r^*(t)$ and $r^*(t) - 1$ satisfy all of the conditions on $n(t)$ in both Lemmas, with $b = c$. Thus from Lemma 2, with $n(t) = r^*(t) - 1$, and from (3.34)

$$t^{r^*(t)} \sum_{j=1}^{r^*(t)-1} j^{-1} (t^{-j} - 1) \to e^{-c} \sum_{j=1}^{\infty} c^j / j!j = e^{-c}$$

and from Lemma 1 with $n(t) = r^*(t)$, together with (3.34),

$$\sum_{j=r^*(t)}^{\infty} t^j h_j^*(s_{j+1}^*) \to (e^c - c - 1) \int_1^{\infty} x^{-1} e^{-cx} dx .$$

The only remaining term in (3.19) is $t^{r^*(t)}\,|\ln t|$ which goes to zero. This completes the proof of Theorem 3.3. $\square$

4. ASYMPTOTIC RESULTS

The following results will be proved in this section:

Theorem 4.1. For each fixed n, $P^*(m,n)$ *increases to* $v^*(n)$ *as* $m \to \infty$.

Theorem 4.2. As $m + n \to \infty$,

$$|P^*(m,n) - p^*(m/(m+n))| = 0\,((m+n)^{-\frac{1}{2}} \log(m+n))$$

uniformly in $\varepsilon < m/(m+n) < 1 - \varepsilon$ *for any* $\varepsilon > 0$.

Corollary. As $m + n \to \infty$,

$$(m+n)^{-1} S_k^*(m+n) \to s_k^* \qquad k=1,\ 2,\ \ldots\ .$$

Since $v^*(n)$ is the full-information success probability, Theorem 4.1 confirms the fact that a large prior sample size provides enough information to enable one to choose the best of a current crop of size n with probability nearly as high as if sampling from a known continuous distribution.

Theorem 4.2 and its corollary show that the infinite problem success probabilities and optimal stopping rules can serve as approximations to the finite problem ones. Also, it can easily be checked that the finite-problem policies $S_k(m+n) = [(m+n)s_k^*]$ are asymptotically optimal.

Proof of Theorem 4.1. Let $\{Z_i\colon 1 \le i \le m+n\}$ be independent random variables each uniform on $(0,1)$; $\{X_i\colon 1 \le i \le m+n\}$ be the ranks of the

Z_i's, $\{Y_i: 1 \le i \le m+n\}$ be the relative ranks, and, if $m \ge 1$, $\{Y_i': 2 \le i \le m+n\}$ be the relative ranks of $\{Z_i: 2 \le i \le m+n\}$. Let $\mathcal{T}(m+n)$ be the class of all stopping rules adapted to $\{Y_i\}$.

Each Y_i' is a function of $\{Y_j: j \le i\}$, so $\mathcal{T}(m+n)$ contains all stopping rules adapted to $\{Y_i'\}$. But the distribution of $\{Y_i': 2 \le i \le m+n\}$ is the same as that of $\{Y_i: 1 \le i \le (m-1)+n\}$. Thus the optimal success probability in this subclass of $\mathcal{T}(m+n)$ -- when $m \ge 1$ -- is precisely $P^*(m-1,n)$, which is necessarily no greater than $P^*(m,n)$. This proves monotonicity of $P^*(m,n)$ in m.

Since each Y_i is a function of $\{Z_j: j \le i\}$, it follows that $v_n^* \ge P^*(m,n)$ for each m and n. Hence, to complete the proof, it suffices to show that $\liminf_m P^*(m,n) \ge v_n^*$. This is accomplished by exhibiting a sequence of stopping rules, τ_m, for which $P(\tau_m = \tau_m^*) \to 1$ where τ_m^* is the optimal rule based on $\{Z_i: 1 \le i \le m+n\}$ which, as Gilbert and Mosteller (1966) showed, is of the form

$$\tau_m^* = \min \{i: Z_{m+i} \le b_{n-i}\}.$$

We can then let

$$\tau_m = \min \{i: m^{-1} Y_{m+i} \le b_{n-i}\},$$

which is satisfactory because, as $m \to \infty$, $\{m^{-1} Y_{m+i}: 1 \le i \le n\}$ converges in distribution to $\{Z_{m+i}: 1 \le i \le n\}$. □

The proof of Theorem 4.2 is based on the following propositions:

<u>Proposition 4.1</u>: <u>For any</u> $t \ge m/(m+n)$,

$$P^*(m,n) \ge p^*(m/(m+n)) - [1 - E(X/n)I_{\{X \le n\}}]$$

<u>where</u> X <u>is</u> <u>a</u> <u>binomial</u> <u>random</u> <u>variable</u> <u>with</u> <u>parameters</u> $m+n$ <u>and</u> $1-t$.

<u>Proposition 4.2</u>: For any $t \leq m/(m+n)$,

$$p^*(m/(m+n)) \geq E(n/Y)I_{\{Y \geq n\}}P^*(m,n)$$

<u>where</u> Y <u>is</u> <u>a</u> <u>negative</u> <u>binomial</u> <u>random</u> <u>variable</u> <u>with</u> <u>parameters</u> $m+1$ <u>and</u> t (i.e., Y has the distribution of the number of failures before the $(m+1)$-th success).

Before proving these two propositions let us show how the theorem follows from them by a slight adaptation of a familiar large deviation result for binomial distributions.

The method in Feller (1968; p. 193) yields this

<u>Lemma</u>: <u>If</u> Z <u>is</u> <u>binomial</u> (N,p), <u>then</u>, <u>for</u> <u>any</u> $\varepsilon > 0$,

$$\sup_{\varepsilon<p<1-\varepsilon} P(|Z-EZ| > N^{\frac{1}{2}} \log N) = O(N^{-\frac{1}{2}}/\log N).$$

To use this lemma with Proposition 4.1, choose $t = m/(m+n) + (m+n)^{-\frac{1}{2}}\log(m+n)$ so $EX = n - (m+n)^{\frac{1}{2}} \log(m+n)$. Then

$$E(X/n)I_{\{X\leq n\}} \geq [1-2n^{-1}(m+n)^{\frac{1}{2}}\log(m+n)] \cdot P(n-2(m+n)^{\frac{1}{2}}\log(m+n) \leq X \leq n)$$
$$= [1-2(1+mn^{-1})(m+n)^{-\frac{1}{2}}\log(m+n)] \cdot P(|X-EX| \leq \sqrt{m+n}\log(m+n)).$$

Now $(1+mn^{-1})$ is uniformly bounded if, for some $\varepsilon > 0$, $\varepsilon < m/(m+n) < 1-\varepsilon$; so, for t as given above,

$$\sup_{\varepsilon<m/(m+n)<1-\varepsilon} [1-E(X/n)I_{\{X\leq n\}}] = O((m+n)^{-\frac{1}{2}}\log(m+n)). \tag{4.1}$$

The application of the lemma to Proposition 4.2 is similar. Choose $t = m/(m+n) - (m+n)^{-\frac{1}{2}} \log(m+n)$. Then for any $c > 0$

$$E(n/Y)I_{\{Y\geq n\}} \geq [1-2\,cn^{-1}(m+n)^{\frac{1}{2}}\log(m+n)] \cdot P(n\leq Y\leq n+2c(m+n)^{\frac{1}{2}}\log(m+n))$$

$$= [1-2c(1+mn^{-1})(m+n)^{-\frac{1}{2}}\log(m+n)] \cdot [P(V>m)-P(U>m)]$$

where U and V are binomial random variables with parameter m+n and t, and the integer part of $[m+n+2c(m+n)^{\frac{1}{2}}\log(m+n)]$ and t, respectively. Then $EU = m - (m+n)^{\frac{1}{2}}\log(m+n)$, and, taking $c = t^{-1}$, $EV = m+(m+n)^{\frac{1}{2}}\log(m+n)-\delta$ where $\delta < 1$ and can be neglected. Hence

$$P(V>m) - P(U>m) = P(V-EV\geq -(m+n)^{\frac{1}{2}}\log(m+n)) - P(U-EU\geq (m+n)^{\frac{1}{2}}\log(m+n)).$$

Since $c(1+mn^{-1})$ is uniformly bounded if, for some $\varepsilon > 0, \varepsilon < m/(m+n)<1-\varepsilon$, conclude that

$$(4.2) \qquad \sup_{\varepsilon<m/(m+n)<1-\varepsilon} [1-E(n/Y)I_{\{Y\geq n\}}] = O((m+n)^{-\frac{1}{2}}\log(m+n)) \cdot$$

The theorem now follows immediately from (4.1), (4.2), and the two propositions.

Proof of Proposition 4.1: Modify the infinite model of section 3 by augmenting the σ-fields $\mathfrak{F}_t$ to $\mathfrak{F}_t \vee \mathfrak{F}_0$ where $\mathfrak{F}_0$ is generated by the order statistics, $U_{(1)}, U_{(2)}, \ldots, U_{(m+n)}$ of $U_1, U_2, \ldots, U_{m+n}$, the arrival times of the m+n best. Denote by $\bar{\mathcal{J}}$ the augmented class of stopping rules which includes all those adopted to $\{\mathfrak{F}_t \vee \mathfrak{F}_0\}$. Letting

$$I = \min\{i\colon U_i = U_{(j)} \text{ for some } j > m\},$$

U_I is the arrival time of the best of the last n to arrive among the m+n best.

Clearly

$$\sup_{\tau\in\bar{\mathcal{J}}} P(\tau=U_I) = P^*(m,n)$$

which is attained by the rule, say $\bar{\tau}(m,n)$, which corresponds to $\tau^*(m,n)$. Hence, in particular, for any $t \geq m/(m+n)$

$$
\begin{aligned}
(4.3) \qquad P^*(m,n) &= P(\bar{\tau}(m,n)=U_I) \\
&\geq P(\tau^*(t)=U_I) \\
&\geq P(\tau^*(t)=Z_1(t)) - P(Z(t)\neq U_I) \\
&\equiv p^*(t) - P(Z_1(t)\neq U_I) \\
&\geq p^*(m/(m+n)) - P(Z_1(t)\neq U_I).
\end{aligned}
$$

Let $X = \max\{j\colon U_{(m+n-j)} > t\}$. Then X has a binomial distribution with parameters $m+n$ and $1-t$,

$$(4.4) \qquad \{X \leq n \;\&\; U_I > t\} \subset \{Z_1(t) = U_I\},$$

and

$$(4.5) \qquad P(X \leq n \;\&\; U_I > t) = E(X/n)I_{\{X\leq n\}}.$$

The proposition follows immediately from (4.3), (4.4), and (4.5). □

Proof of Proposition 4.2: Returning to the unaugmented infinite model of section 3, for fixed m, n, and t let Y be the number of post-t arrivals which are better than the $(m+1)$-th best pre-t arrival. Then Y has the negative binomial distribution with parameters $m+1$ and t.

Now let $R_1, R_2, \ldots, R_Y$ denote the successive relative ranks of these Y arrivals. On the event $\{Y\geq n\}$, $R_1, \ldots, R_n$ have the same joint distribution as the last n relative ranks in the finite (m,n) problem; namely, they are independent with R_i uniformly distributed on $\{1, 2, \ldots, m+i\}$. Moreover the R_i's are "observable", so there is a stopping rule τ in $\mathcal{T}$ which on $Y\geq n$ is based on $R_1, \ldots, R_n$ in exactly the same way as the optimal (m,n)-policy is based on the last n relative ranks in the (m,n)-problem.

Note that on the event $\{Y \geq n\}$ the best of the first n of the Y arrivals is the best post-t arrival if and only if the best of all Y is among the first n of them. Hence, if $t \leq m/(m+n)$,

$$(4.6) \quad p^*(m/(m+n)) \geq p^*(t) \geq P(\tau \text{ selects best post-t arrival})$$

$$\geq P(Y \geq n \text{ \& best of all } Y \text{ is among first } n)\, P(\tau \text{ selects best of first } n \mid Y \geq n \text{ \& best of } Y \text{ is among first } n).$$

The first factor on the right side of (4.6) is $E(n/Y)I_{\{Y \geq n\}}$ because whatever the value of Y the ranks of the Y arrivals relative to each other, are just a random permutation of $\{1, \ldots, Y\}$. The second factor is $P^*(m,n)$ because, given $Y \geq n$, the distribution of $R_1, \ldots, R_n$ does not depend on whether or not the best of all Y arrivals is among the first n. Substituting these values onto (4.6) completes the proof. □

5. PROOF OF THEOREM 3.2

Let $t \in (0,1)$ and $\underline{s} = (s_1, s_2, \ldots)$, with $0 < s_1 \leq s_2 \leq \ldots$, be fixed. The abbreviations

$$\tau \equiv \tau(t; \underline{s})$$
$$K_i \equiv K_i(t)$$
$$T_i \equiv T_i(t)$$
$$r \equiv r(t; \underline{s})$$

will be used throughout the proof.

Begin by partitioning the event $\{\tau = T_1\}$ according to the value of the smallest i for which $t < T_i < T_1$; so

$$(5.1) \qquad P(\tau = T_1) = \sum_{i=2}^{\infty} P(\tau = T_1 \text{ \& } T_i < T_1 < \min_{1<j<i} T_j).$$

Note that on the event $\{T_i = \min_{j \le i} T_j\}$ the relative rank of the arrival at T_i is $K_i-(i-1)$. It can then be shown that on $\{T_i < T_1 < \min_{1<j<i} T_j\}$

$$(5.2)\qquad T_i > s_{K_i-(i-1)} \Rightarrow \tau \le T_i \text{ so } \tau \ne T_1$$

$$T_i < s_{K_i-(i-1)} \Rightarrow \tau \ge T_1 \text{ or } \tau \text{ undefined.}$$

The first implication is immediate; the second is true because (a) $\tau \neq T_j$ by definition of τ; (b) $\tau \nless T_i$ since

$$T_k < T_i \;\&\; T_k = \min_{\ell \le k} T_\ell \Rightarrow k > i$$

$$\Rightarrow K_k-(k-1) \ge K_i-(i-1)$$

so

$$s_{K_k-(k-1)} \ge s_{K_i-(i-1)} > T_i > T_k;$$

and (c) $\tau \notin (T_i,T_1)$ since on $\{T_i < T_1 < \min_{1<j<i} T_j\}$ there is no k for which both $T_k = \min_{\ell \le k} T_\ell$ and $T_k \in (T_i,T_1)$.

From (5.2) it follows immediately that

$$(5.3)\qquad \{\tau = T_1 \;\&\; T_i<T_1<\min_{1<j<i}T_j\} =$$

$$\{T_i<T_1<\min_{1<j<i}T_j \;\&\; T_1>s_{K_1} \;\&\; T_i<s_{K_i-(i-1)}\},$$

and, from the independence of $\{T_i\}$ and $\{K_i\}$,

$$(5.4)\qquad P(T_i<T_1<\min_{1<j<i}T_j \;\&\; T_1>s_{K_1} \;\&\; T_i<s_{K_i-(i-1)})$$

$$= \sum_{i=2}^{\infty} P(T_i<T_1<\min_{1<j<i}T_j) \sum_{k=1}^{\infty} \sum_{\ell=k}^{\infty} P(K_1=k,K_i=i-1+\ell)P(Z_{(1)}<s_\ell \;\&\; Z_{(2)}>s_k)$$

where $Z_{(1)}$ and $Z_{(2)}$ denote the smallest and second smallest order statistics from a random sample $Z_1, Z_2, \ldots, Z_i$, IID, each uniform on $(t,1)$. Now it is straightforward to evaluate the terms on the right side of (5.4). They are

(5.5) $$P(T_i<T_1<\min_{1<j<i}T_j) = 1/i(i-1);$$

(5.6) $$P(K_1=k, K_i=i-1+\ell) = P(K_1=k, K_i-K_1=i-1+\ell-k) = \binom{i-2+\ell-k}{i-2}t^{\ell-1}(1-t)^i$$

and

(5.7) $$P(Z_{(1)}<s_\ell \;\&\; Z_{(2)}>s_k) =$$

$$\begin{cases} 0 \text{ if } t > s_\ell \qquad \text{since } Z_{(1)} > t \\ 1-(1-s_\ell)^i/(1-t)^i = 1-P(Z_j>s_\ell, j=1, 2, \ldots, i) \text{ if } s_k < t < s_\ell \\ (1-t)^{-i}\{i(s_k-t)(1-s_k)^{i-1}+(1-s_k)^i-(1-s_\ell)^i\} \\ \quad = P(\text{exactly one of } Z_1, \ldots, Z_n \in (t,s_k) \;\&\; \text{all others} > s_k) \\ \quad + P(s_k < \min_{1\le j\le i}Z_j<s_\ell) \quad \text{if } t < s_k. \end{cases}$$

Substituting (5.5), (5.6), and (5.7) into (5.4), then (5.4) into (5.1) using the identity (5.3), gives

$$P(\tau=T_1) = A(r) + B(r) + C(r) + D(r)$$

where

$$A(r) = \sum_{i=2}^{\infty} i^{-1}(i-1)^{-1} \sum_{k=1}^{r} \sum_{\ell=r+1}^{\infty} \binom{i-2+\ell-k}{i-2} t^{\ell-1}(1-t)^{i} I_{\{r\geq 1\}}$$

$$B(r) = -\sum_{k=r+1}^{\infty} t^{k-1} \sum_{i=2}^{\infty} i^{-1}(i-1)^{-1} \sum_{\ell=1}^{r} \binom{i-2+k-\ell}{i-2} (1-s_k)^{i} I_{\{r\geq 1\}}$$

$$C(r) = \sum_{k=r+1}^{\infty} t^{k-1} \sum_{i=2}^{\infty} i^{-1}(i-1)^{-1} \sum_{\ell=0}^{\infty} \binom{i-2+\ell}{i-2} t^{\ell}(1-t)^{i-1} \left[\frac{i(s_k-t)(1-s_k)^{i-1}+(1-s_k)^{i}}{(1-t)^{i-1}}\right]$$

$$D(r) = -\sum_{k=r+1}^{\infty} t^{k-1} \sum_{i=2}^{\infty} i^{-1}(i-1)^{-1} \sum_{\ell=r+1}^{k} \binom{i-2+k-\ell}{i-2} (1-s_k)^{i}.$$

Now from the identity

$$\sum_{j=0}^{\infty} \binom{i-2+j}{i-2} u^{j}(1-u)^{i-1} = 1 \qquad i = 2, 3, \ldots; \quad 0<u<1$$

B, C, and D are easily simplified to

$$C(r) = \sum_{k=r+1}^{\infty} t^{k-1}(1-s_k)$$

$$B(r) + D(r) = \sum_{k=r+1}^{\infty} t^{k-1}[h_{k-1}(s_k)-(1-s_k)]$$

with $h_k(s_{k+1})$ as defined in (3.27). Then

$$B(r) + C(r) + D(r) = \sum_{j=r}^{\infty} t^{j} h_j(s_{j+1}) \quad ,$$

so, from (3.26), to complete the proof it must be shown that $A(r) \equiv E(r)$ where

$$E(r) \equiv t^{r}[|\ln t| I_{\{r\geq 1\}} + \sum_{j=1}^{r-1} j^{-1}(t^{-j}-1) I_{\{r\geq 2\}}].$$

This is trivally true for $r = 0$ and can be proved by induction on r for $r \geq 1$ by verifying the following equalities:

$$A(1) = E(1)$$

$$E(r+1) = t[E(r) + r^{-1} (1-t)^r] \qquad r=1, 2, \ldots$$

(5.8) $$A(r+1) = t[A(r) + u(r)] \qquad r=1, 2, \ldots$$

-- where

$$u(r) = \sum_{i=2}^{\infty} i^{-1}(i-1)^{-1} \sum_{j=r+1}^{\infty} \binom{i-2+j}{i-2} t^{j-1}(1-t)^i$$

-- and

$$u(1) = 1-t$$

$$u(r-1) - u(r) = (r-1)^{-1}(1-t) - r^{-1}(1-t)^r \qquad r=1, 2, \ldots .$$

Hence

$$u(r) = r^{-1}(1-t)^r \qquad r=1, 2, \ldots .$$

Substitute this into (5.8) to conclude that $A(r) \equiv E(r)$ and complete the proof. □

It should be noted that the common value $A(r) = E(r)$ is in fact just $E\, X\, I_{\{X \geq 1\}}$ where X is the number of failures before the r-th success in independent Bernoulli trials with success probability t. This can easily be seen by verifying that

$$w(r) \equiv \sum_{j=1}^{\infty} j^{-1}\binom{j+r-1}{j} t^r(1-t)^j$$

satisfies

$$w(1) = A(1) = E(1)$$

and

$$w(r+1) = t[w(r) + r^{-1}(1-t)^r]. \qquad r=1, 2, \ldots$$

(See Feller [1968; p. 241 problem 33 and solution on p. 493] for a related result.)

TABLE 1

Parameters of Optimal Policies for the Infinite Problem

k	Lower Bound $(1-k^{-1})$	Exact Value s_k^*	Upper Bound $(.5)^{1/k}$
1	0	.36788	.500
2	.500	.64200	.707
3	.667	.75181	.794
4	.750	.81018	.841
5	.800	.84636	.871
6	.833	.87098	.891
7	.857	.88880	.906
8	.875	.90230	.917
9	.889	.91287	.926
10	.900	.92139	.933
11	.909	.92838	.939
12	.917	.93424	.944
13	.923	.93921	.948
14	.929	.94348	.952
15	.933	.94719	.955
16	.938	.95044	.958
17	.941	.95332	.960
18	.944	.95588	.962
19	.947	.95817	.964
20	.950	.96024	.966
21	.952	.96211	.968
22	.955	.96382	.969
23	.957	.96537	.970
24	.958	.96680	.972
25	.960	.96812	.973

TABLE 2

Parameters $S_k^*(m+n)$ and Estimates $(m+n)s_k^*$ (in parenthesis) of Optimal Rules

k \ m+n	4	5	6	10	15	20	25	30
1	2(1.5)	3(1.8)	3(2.6)	4(3.7)	6(5.5)	8(7.4)	10(9.2)	12(11.0)
2			5(4.5)	7(6.4)	11(9.6)	14(12.8)	17(16.1)	20(19.3)
3					12(11.3)	16(15.0)	20(18.8)	24(22.6)
4					13(12.2)	17(16.2)	21(20.3)	25(24.3)
5						18(16.9)	22(21.2)	26(25.4)
6						18(17.4)	23(21.8)	27(26.1)
7							23(22.2)	28(26.7)
8								28(27.1)
9								28(27.4)

TABLE 3

Optimal Success Probabilities, P*(m,n), and Limits, p*(t), for Fixed t=m/(m+n)

$t = \frac{m}{m+n}$ \ min(m,n)	2	4	6	8	10	14	20	24	30	p*(t)
$\frac{1}{6}$	.439	.421	.416	.413	.412	.410	.409			.405
$\frac{1}{5}$	.453	.435	.428	.424	.422	.420	.418	.418		.414
$\frac{1}{4}$	.486	.456	.446	.442	.439	.436	.434	.433	.432	.428
$\frac{1}{3}$	.544	.497	.481	.475	.471	.466	.463	.461	.460	.455
$\frac{2}{5}$	.600	.532	.514	.505	.499	.493	.488	.487	.485	.478
$\frac{1}{2}$	.667	.583	.554	.541	.533	.524	.517	.515	.512	.502
$\frac{3}{5}$	.700	.593	.569	.557	.549	.540	.534	.532	.529	.520
$\frac{2}{3}$	.700	.610	.581	.569	.561	.553	.547	.554	.542	.533
$\frac{3}{4}$	.714	.621	.594	.582	.574	.566	.560	.557	.555	.546
$\frac{4}{5}$	.722	.628	.602	.589	.581	.573	.567	.565		.553
$\frac{5}{6}$	.727	.633	.606	.594	.587	.578	.572			.558

TABLE 4

Optimal Success Probabilities, $P^*(m,n)$, and Limits, v_n^* for Fixed n

n \ m	2	3	5	10	20	50	v_n^*
2	.667	.700	.714	.727	.738	.745	.750
3	.600	.608	.631	.652	.667	.677	.684
5	.505	.540	.562	.593	.613	.627	.639
10	.439	.455	.487	.533	.561	.587	.609
20	.404	.413	.431	.471	.517	.554	.594
50	.382	.386	.394	.412	.446	.508	.586

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CHOOSING THE BEST OF THE CURRENT CROP

A best choice problem is presented which is intermediate between the constraints of the "no-information" problem (observe only the sequence of relative ranks) and the demands of the "full information" problem (observations from a known continuous distribution). In the intermediate problem prior information is available in the form of a "training sample" of size m and observations are the successive ranks of the n current items relative to their predecessors in both the current and training samples.

Optimal stopping rules for this problem depend on m and n essentially only through $m + n$; and, as $m/(m+n) \to t$, their success probabilities, $P^*(m,n)$, converge rapidly to explicitly derived limits $p^*(t)$ which are the optimal success probabilities in an infinite version of the problem. For fixed n, $P^*(m,n)$ increases with m from the "no information" optimal success probability to the "full information" value for sample size n. And as t increases from 0 to 1, $p^*(t)$ increases from the "no information" limit $e^{-1} \approx .37$ to the "full information" limit $\approx .58$. In particular $p^*(.5) \approx .50$.

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