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APPLICATION OF THE FINITE ELEMENT METHOD TO VERTICALLY STRATIFI--ETC(U)  
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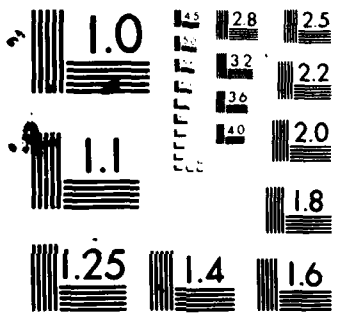
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**APPLICATION OF THE FINITE ELEMENT METHOD  
TO VERTICALLY STRATIFIED HYDRODYNAMIC  
FLOW AND WATER QUALITY**

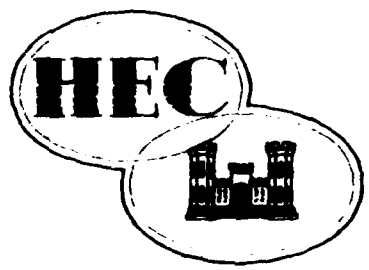
by

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and dissolved oxygen concentrations for Lake Taneycomo, Missouri: Comparison of measured and computed results from both applications shows that the RMA-7 model provides reasonably accurate results for both flow circulating and water quality.

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APPLICATION OF THE FINITE ELEMENT METHOD TO VERTICALLY  
STRATIFIED HYDRODYNAMIC FLOW AND WATER QUALITY<sup>1</sup>

R. C. MacArthur<sup>2</sup> and W. R. Norton<sup>3</sup>

INTRODUCTION

Continuing interest in the internal processes of reservoirs, lakes and estuaries has intensified the development of mathematical models for simulation of vertically stratified flow. Motivated primarily by the long term desire to predict the water quality response to system modifications, current modeling efforts have focused on the need to describe flow coupled with temperature and/or salinity fields in order to forecast the influence of density induced flows. Computational algorithms have shown sufficient promise that efforts are under way to collect prototype data which can be used for calibration and verification of both flow and water quality models.

Among the several models which have been formulated to simulate density induced flows and water quality is one called RMA-7. This model, which was originally developed for the Office of Water Resources Research, (King, 1973) has existed for several years but has received relatively little use in prototype applications. It is the intent of the work reported herein to demonstrate the operation of the current version of RMA-7 and to offer appropriate information and comments on the use and implementation of the model.

Specifically, the paragraphs which follow contain a brief description of the mathematical basis of RMA-7 as well as an example of its implementation on a prototype system which approximates the physical dimensions of the so-called GRH flume at the Corps' Waterways Experiment Station in Vicksburg. Also included are some results obtained by the model from the simulation of temperature and dissolved oxygen for Lake Taneycomo in Missouri. Several statistical comparisons are included between simulated and measured values for Lake Taneycomo which are designed to quantify, to some extent, the accuracy of the model.

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<sup>1</sup>Presented at the 3rd International Conference on Finite Elements in Water Resources, May 19-23, 1980, the University of Mississippi, Oxford, MS.

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## GOVERNING EQUATIONS - DIFFERENTIAL FORMS

RMA-7 is a two dimensional mathematical model which describes the behavior of velocity, pressure, temperature and dissolved oxygen in the vertical plane with homogeneity assumed in the third (lateral) direction; the model will accommodate width gradients in both the X and Y directions. This model is formulated on the classical concepts of conservation of mass, momentum, and energy although it is somewhat unusual as it retains the complete vertical momentum equation and does not make the assumption that pressure must be hydrostatic.

### Hydrodynamic Model

The equations used in RMA-7 have the following differential forms:

#### Velocity equations.

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} - \epsilon_{xx} \frac{\partial^2 u}{\partial x^2} - \epsilon_{xy} \frac{\partial^2 u}{\partial y^2} = 0 \quad (1)$$

$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} + \rho g - \epsilon_{yx} \frac{\partial^2 v}{\partial x^2} - \epsilon_{yy} \frac{\partial^2 v}{\partial y^2} = 0 \quad (2)$$

#### Continuity equation.

$$\frac{\partial}{\partial x} (wu) + \frac{\partial}{\partial y} (wv) = 0 \quad (3)$$

#### Temperature equation.

$$C_p \frac{\partial T}{\partial t} + C_{pu} \frac{\partial T}{\partial x} + C_{pv} \frac{\partial T}{\partial y} - D_x \frac{\partial^2 T}{\partial x^2} - D_y \frac{\partial^2 T}{\partial y^2} - \phi_2 = 0 \quad (4)$$

where

- u = X direction velocity
- v = Y direction velocity
- p = pressure
- T = temperature, degrees C
- t = time
- $\rho$  = fluid density
- = f(T)
- g = gravitational acceleration
- C = specific heat
- $\phi_2$  = thermal heat source or sink
- w = width
- $\epsilon_{xx}, \epsilon_{xy}$  = eddy viscosity coefficients
- $\epsilon_{yx}, \epsilon_{yy}$
- $D_x, D_y$  = eddy dispersion coefficients

### Water Quality Model

The equations used in RMA-7 to describe the behavior of dissolved oxygen have the differential forms:

#### Dissolved oxygen equation.

$$\frac{\partial c_1}{\partial t} + u \frac{\partial c_1}{\partial x} + v \frac{\partial c_1}{\partial y} - D_x \frac{\partial^2 c_1}{\partial x^2} - D_y \frac{\partial^2 c_1}{\partial y^2} + K_2 c_2 - \alpha \mu c_3 = 0 \quad (5)$$

#### Carbonaceous biochemical oxygen demand (BOD) equation.

$$\frac{\partial c_2}{\partial t} + u \frac{\partial c_2}{\partial x} + v \frac{\partial c_2}{\partial y} - D_x \frac{\partial^2 c_2}{\partial x^2} - D_y \frac{\partial^2 c_2}{\partial y^2} + K_2 c_2 = 0 \quad (6)$$

#### Phytoplankton equation.

$$\frac{\partial c_3}{\partial t} + u \frac{\partial c_3}{\partial x} + v \frac{\partial c_3}{\partial y} - D_x \frac{\partial^2 c_3}{\partial x^2} - D_y \frac{\partial^2 c_3}{\partial y^2} - \mu c_3 = 0 \quad (7)$$

- where  $c_1$  = dissolved oxygen concentration  
 $c_2$  = carbonaceous BOD concentration  
 $c_3$  = phytoplankton concentration (dry weight)  
 $K_2$  = BOD decay rate  
 $\alpha$  = the mass of oxygen produced per unit mass of phytoplankton growth  
 $\mu$  = the local rate of phytoplankton growth  
=  $\mu_{\max} \left( \frac{I}{I + K_I} \right) - r$

where  $\mu_{\max}$  = the maximum specific phytoplankton growth rate at the local temperature  
 $I$  = the intensity of light at the local depth  
 $K_I$  = the light half saturation constant  
 $r$  = the phytoplankton respiration rate at the local temperature  
and all other terms as previously defined.

Equations 1 through 7 represent the mathematical basis for the model RMA-7. It is recognized that these equations provide an approximate representation of the governing processes in that certain second order terms with respect to width have been dropped and that the viscosity/dispersion terms are largely empirical. Notwithstanding these shortcomings, however, the above relationships have shown promise in simulation of observed phenomena, and provide a general framework for testing and improving the procedures necessary for simulation of vertically stratified flow.

As can be seen, the first four equations and the second three equations each form a closed set which must be solved simultaneously. The coupling between the two sets is manifest in the convective velocities  $u$  and  $v$ , and in the water temperature,  $T$ . Fortunately, the coupling between the

two sets is of the "feed forward" type in that equations 1 through 4 may be solved independently of equations 5, 6 and 7, with the results of the hydrodynamic solution acting like coefficients in the solution of the water quality model.

In addition to the volumetric terms shown above, the model contains a number of additional features to account for surface effects and source/sink terms. Most important of these are: 1) bottom friction at the soil-water interface calculated as a function of bottom velocity and a Chezy coefficient; 2) surface heat exchange as a function of the water temperature, the equilibrium temperature and an exchange coefficient; 3) internal absorption of short wave solar radiation as a function of depth; and 4) surface oxygen exchange as a function of the local oxygen deficit and an exchange coefficient which is a function of wind speed.

#### SOLUTION TECHNIQUE

The governing equations are solved by the finite element method using Galerkin's criteria for the method of weighted residuals. The formulation employs a mixed set of basis functions, with quadratic functions used for all state variables except pressure where a linear function is used. The linear pressure function implies a constant element density, which is calculated as a function of average nodal temperatures. Green's Theorem is used to lower the order of all second derivatives in the viscosity/dispersion terms, resulting in surface integrals which must be evaluated (either implicitly or explicitly) along system extremities. Green's Theorem is also used on the pressure terms of equations 1 and 2 permitting a surface integral to be used as a discharge boundary condition rather than a specified pressure value; this procedure permits retention of all nodal continuity equations and substantially improves the model's performance.

RMA-7 uses an implicit, Newton-Raphson computation scheme to achieve a solution to the set of nonlinear equations which define the model. The computer program accommodates either triangular or quadrilateral isoparametric elements with numerical integration used to evaluate all surface and area integrals. The use of the isoparametric formulation with interelement geometric slope continuity allows flow to move parallel to boundaries at all points while at the same time providing a means for reasonable representation of real physical systems.

#### EXAMPLE PROBLEMS

The model described above has been applied to several physical situations, two of which are summarized below. The first example shows results calculated from tests conducted with the geometry of the GRH flume at WES. The second shows results calculated from conditions found in Lake Taneycomo in Missouri. The first example was run using only the flow and temperature portions of the model, while the second includes flow, temperature and dissolved oxygen. The networks used for each of these example problems are reproduced in figure 1.

### Example Problem 1 - GRH Flume

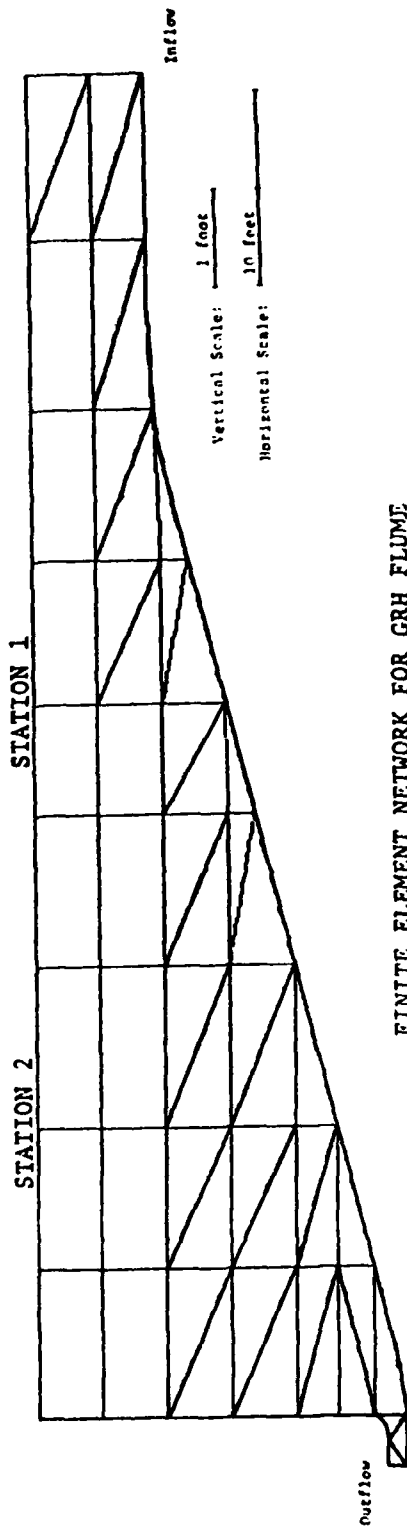
The GRH flume is a hydrodynamic test facility which is 80 feet long and varies in depth from approximately 1 foot at its upper (inflow) end to 3 feet at its lower (outflow) end, with two different cross sections along its length. The inflow section, which is 20 feet long, has a horizontal bottom and varies linearly in width from 1.0 foot to 2.85 feet. The lower section has a constant width of 2.85 feet, but a bottom which drops 2 feet over its 60 foot length.

RMA-7 was applied to the GRH flume geometry by construction of a finite element network containing 57 elements and 158 node points as shown in figure 1. In constructing this network it was felt desirable to allow flow to move parallel to the flume bottom at all locations. For this reason continuous curves were passed through the breakpoint on the flume's bottom 20 feet from the upstream end and at the transition to the outlet. This type of construction permits continuous velocities to exist along the flume bottom and completely eliminates artificial stagnation points. The small discharge nozzle at the outlet has been included for each boundary condition specification and does not exist on the physical flume.

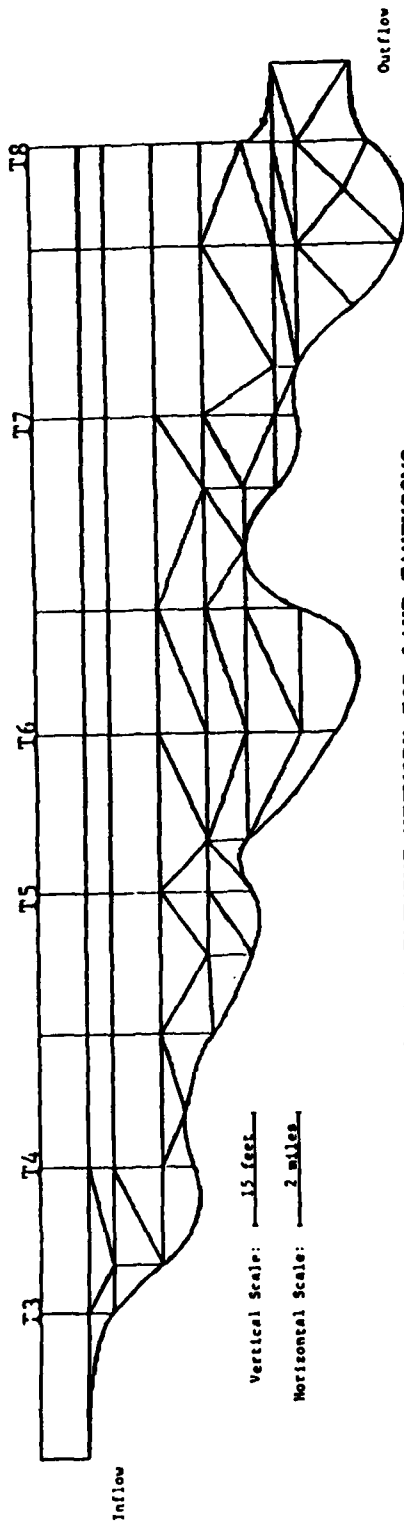
To run RMA-7 it is necessary to specify values for eddy viscosity and eddy diffusion coefficients. At present, this is done by experience and numerical testing of problems which have known or assumed velocity and temperature distributions. In the GRH flume case, a series of numerical tests were conducted on a steady state problem to determine a set of satisfactory coefficients, and the network's sensitivity to the various coefficients. The values determined for the GRH flume had the relative values of  $\epsilon_{xx} = 0.05$ ,  $\epsilon_{xy} = 0.0005$ ,  $\epsilon_{yx} = 0.01$ ,  $\epsilon_{yy} = 0.1$ ,  $D_x = 0.1$ , and  $D_y = 0.0005$ , after accounting for element distortion and size.

Results from two examples are shown, one for a homogeneous flow and one for a nonhomogeneous flow. In each case a flow of 10 gpm was introduced into a still flume with a linear velocity distribution in the lower element at the inflow end. The homogeneous case was run isothermally at a temperature of 10.3°C, while the nonhomogeneous case was started with an initial temperature of 10.3°C and an inflow of 5°C.

Velocity distributions produced by each condition are graphically compared at three times in figure 2. The effects of the density stratification are quite evident between the two cases with the colder, more dense water underflowing the lighter and warmer water near the surface. The results shown are representative of ongoing work with the GRH flume, although no measured data is currently available for model/prototype comparisons under the conditions simulated. The general shape of velocity distributions, and the arrival time of the temperature front (15-18 min) are in general agreement with measured data, however, and suggest the model will perform well when suitable data become available.



FINITE ELEMENT NETWORK FOR GRH FLUME  
EXAMPLE PROBLEM NUMBER 1



FINITE ELEMENT NETWORK FOR LAKE TANEYCONONG  
EXAMPLE PROBLEM NUMBER 2

Figure 1.--Example Problem Networks

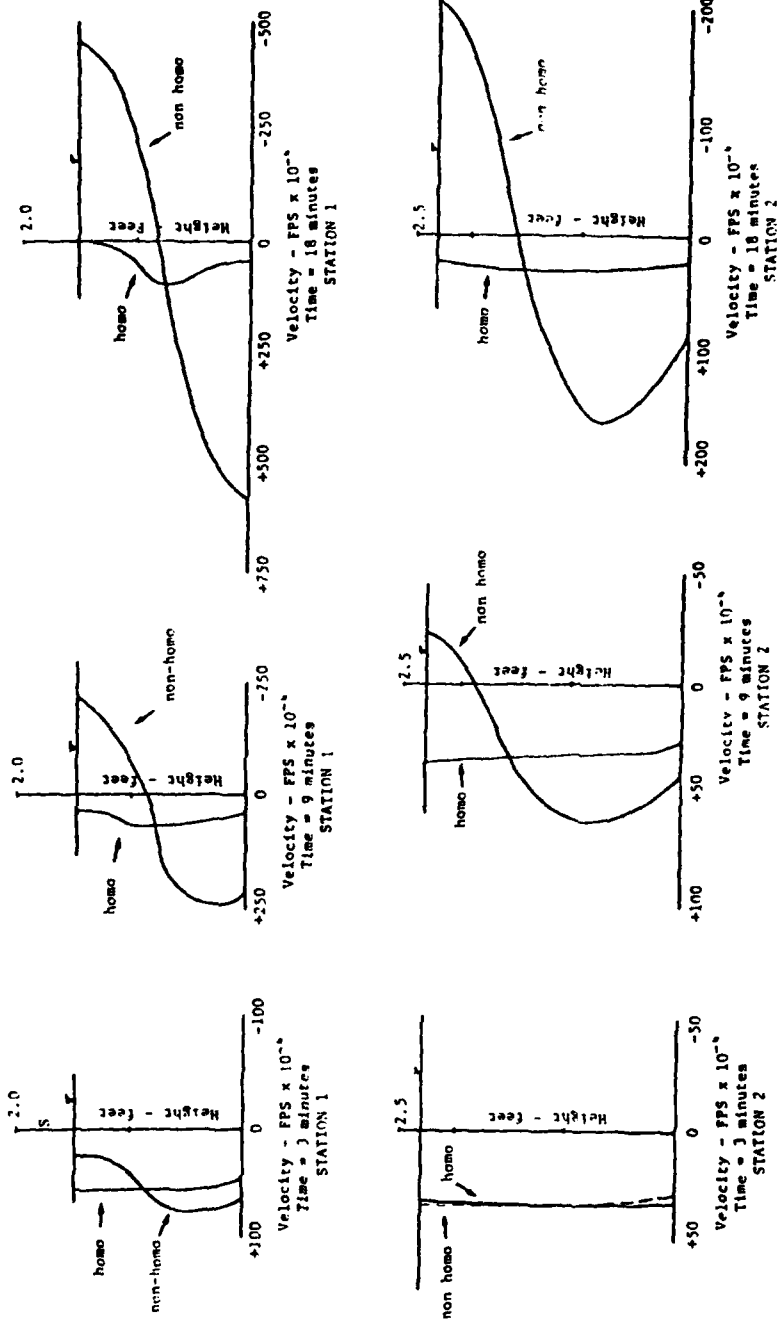


Figure 2.--Comparison of Homogeneous and Non Homogeneous Simulated Velocity Distributions for the GRH Flume Example Problem

### Example Problem 2 - Lake Taneycomo

The second example problem presents results obtained from the simulation of Lake Taneycomo in southern Missouri. Lake Taneycomo is a 26-mile-long reservoir bounded by Table Rock Dam upstream and Bull Shoals Reservoir downstream, and is used for power production and recreation among other things. As low dissolved oxygen has been observed in the lake, RMA-7 was applied with the goal of evaluating the impact of changes in reservoir operation on the ambient levels of dissolved oxygen.

To conduct the required simulations a network of 92 elements and 246 node points was constructed as shown in figure 1; lateral width varied from about 200 to 1000 feet with depth at a typical cross section. Diffusion coefficients and eddy viscosity coefficients were chosen to be consistent with those used in the GRH flume when scaled for element distortion.

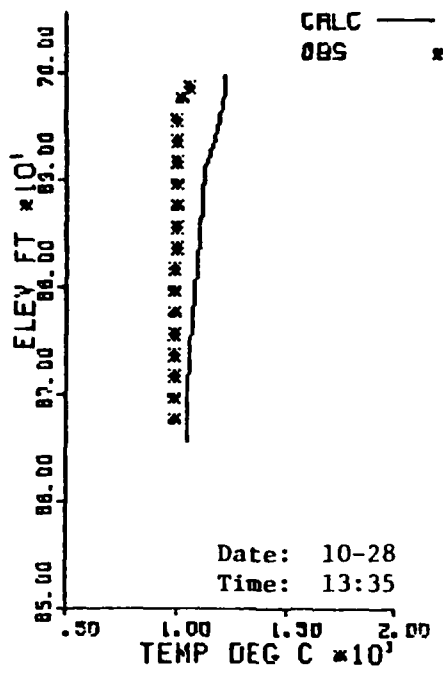
Detailed water temperature and dissolved oxygen measurements were available for three separate week long periods in the fall of 1977. RMA-7 was calibrated against two of these periods and verified against the third. Typical measured and simulated vertical profiles for temperature and dissolved oxygen are presented for stations T6 and T8 in figure 3 for the earliest calibration period. It should be noted that the model had simulated over five days of operation by the time shown in these figures, and that the inflow hydrograph varied from 0 to 7000 cfs in a four to six hour period on a regular basis.

In order to bring a degree of quantification to the accuracy of the model, a linear least squares regression of simulated to observed temperature and dissolved oxygen has been made as shown in figure 4, with the statistics for each fit given in table 1.

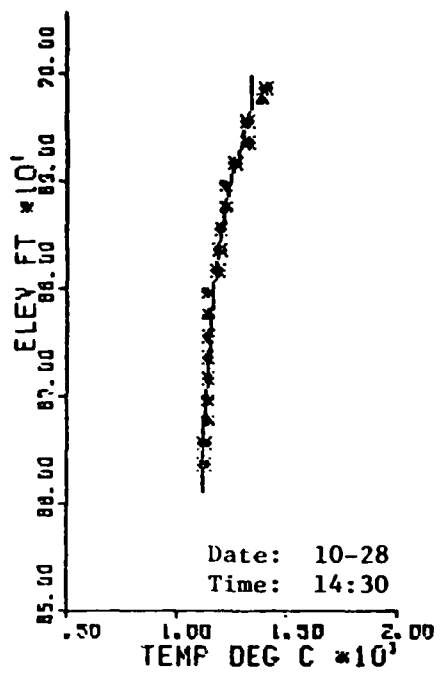
As can be seen, these statistics indicate a fairly good fit of both temperature and dissolved oxygen, and seem reasonable in light of the uncertainty in both the model's upstream inputs (BOD, dissolved oxygen, meteorological data, etc.) and the usual measurement errors.

Table 1.--Regression Statistics

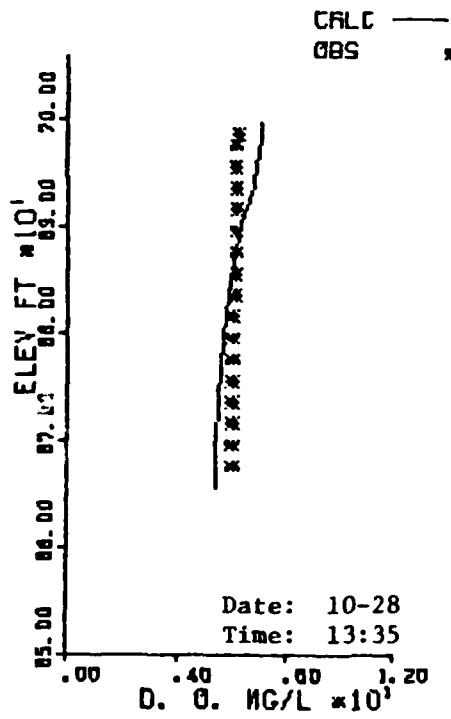
Regression Statistic	STATION T6		STATION T8	
	Temperature	Oxygen	Temperature	Oxygen
intercept	4.77	3.27	0.29	3.30
slope	0.58	0.45	0.96	0.39
s.d. error	0.65	0.49	0.43	0.30
correlation	0.44	0.30	0.89	0.81



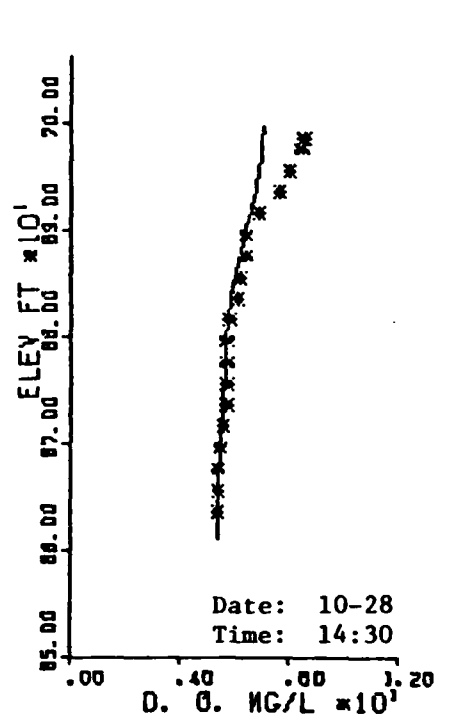
Station T6



Station T8

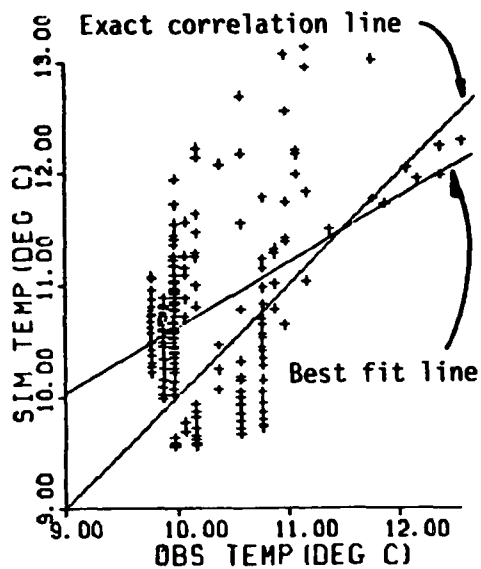


Station T6

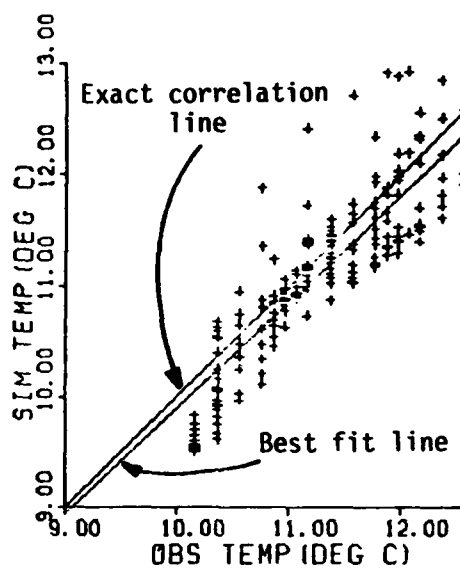


Station T8

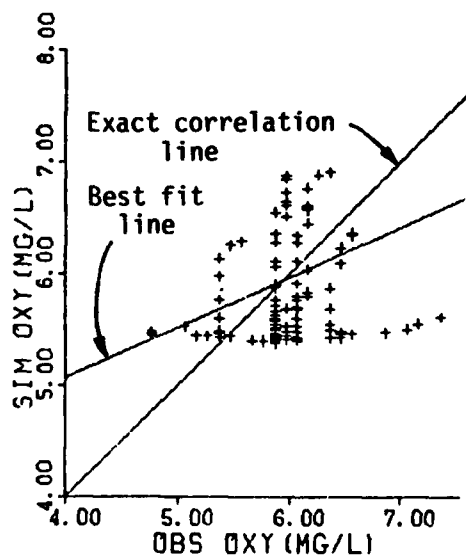
Figure 3.--Observed and Simulated Water Quality for Lake Taneycomo



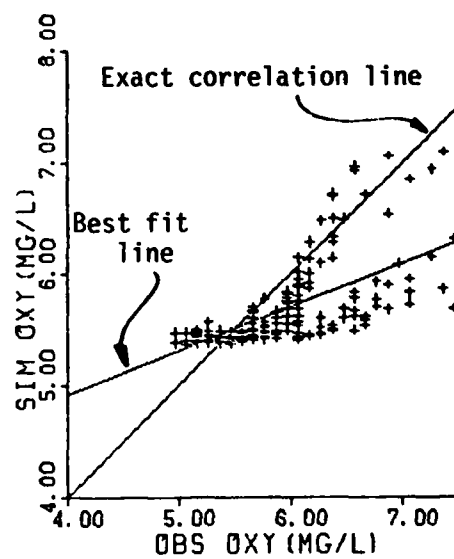
Station T6



Station T8



Station T6



Station T8

Figure 4.--Statistical Regressions for Lake Taneycomo Temperature and Dissolved Oxygen

## SUMMARY AND CONCLUSIONS

The above information summarizes the application of a two dimensional finite element model (RMA-7) to two prototype stratified flow situations. In each application a stratified flow was simulated as a result of density differences arising from temperature gradients. In the second example dissolved oxygen, BOD and phytoplankton were also routed in accordance with the overall flow fields.

Based on the data contained herein it seems reasonable to conclude:

- . the finite element method in general and RMA-7 in particular, is capable of simulating vertically stratified two dimensional flow.
- . the proper definition of eddy viscosity and dispersion coefficients is essential to proper model operation, and that there may be transferability of values from problem to problem if element size and distortion is accounted for.
- . the RMA-7 model appears to give reasonable answers, but it is not currently possible to evaluate its accuracy in relation to velocity due to a lack of prototype data; initial comparisons of temperature and dissolved oxygen data are promising and will improve as calibrations become more rigorous and quantified.

## REFERENCE

- King, I. P.; W. R. Norton; G. T. Orlob  
1973. A finite element solution for two-dimensional density stratified flow. Prepared for the U. S. Dept. of the Interior, Office of Water Resources Research. Water Resources Engineers.

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