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DAA629-80-C-0041

MRC-TSR-2279

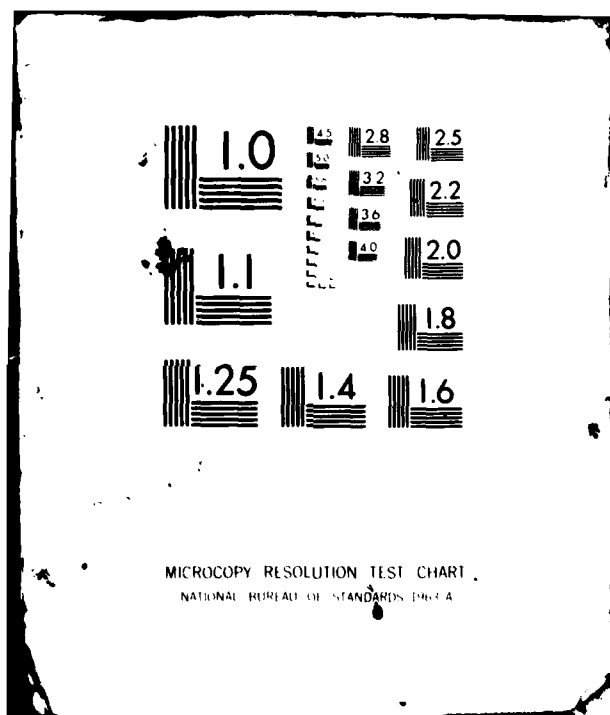
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MRC Technical Summary Report #2279

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R. Creighton Buck

DTIC
FEB 4 1982

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September 1981

(Received August 6, 1981)

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ABSTRACT

A nomographic function of k variables is one that can be represented by the format

$$f(x_1, x_2, \dots, x_k) = h(\phi_1(x_1) + \phi_2(x_2) + \dots + \phi_k(x_k))$$

where the ϕ_i and h are continuous. Any individual nomographic function is very special in nature, since it is constructed from functions of one variable and addition alone. However, Kolmogorov showed in 1957 that every continuous function of k variables has a representation as a sum of not more than $2k+1$ nomographic functions. The present paper throws additional light on this, and settles a conjecture, by giving a constructive proof that the nomographic functions form a nowhere dense subset of the space $C[I^k]$ of continuous real valued functions on the k -cell.

AMS (MOS) Subject Classifications: 41A30, 26A72.

Key Words: approximations of functions of several variables; superpositions;
nowhere dense sets

Work Unit Number 3 - Numerical Analysis and Computer Science

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

An important topic in approximation theory is the study of ways to approximate complicated functions of many variables by combinations of simpler functions. One important type of the latter are the nomographic functions, which can be written entirely in terms of addition and functions of one variable. The present paper shows that these are inherently a very sparse subset of the class of all continuous functions; this places a severe limitation upon their use as single functions, but not if they are added together.

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NOMOGRAPHIC FUNCTIONS ARE NOWHERE DENSE

R. Creighton Buck

Let $I = [-1, 1]$. The class $\mathcal{N}^k$ of nomographic functions on the k -cube I^k are those that can be represented there in the special format

$$(1) \quad \begin{aligned} f(x) &= f(x_1, x_2, \dots, x_k) \\ &= h(\phi_1(x_1) + \phi(x_2) + \dots + \phi_k(x_k)) \end{aligned}$$

where the ϕ_i are real valued continuous functions on I and h is continuous on $-\infty < t < \infty$. Interest in $\mathcal{N}^k$ revived when Kolmogorov used them in 1957 to settle Hilbert's 13th problem by showing that every continuous real function on I^k could be written as a sum of $2k+1$ nomographic functions. [5], [6]

An individual nomographic function is quite special. For example, if $f \in \mathcal{N}^2$ and the component functions h , ϕ_1 and ϕ_2 are at least of class C^3 , then f must be a solution of a specific third order nonlinear PDE, characteristic for the class $\mathcal{N}^2$. (See [3].) The conjecture is that in general $\mathcal{N}^k$ is a rather sparse subset of the banach space $C[I^k]$ of all real valued continuous functions on I^k ; in the present paper, we present a constructive proof that $\mathcal{N}^k$ is nowhere dense.

This fact does not conflict with the Kolmogorov property of the set $\mathcal{N}^k$, for example, the interval $[0,1]$ can be written easily as the algebraic sum of two (or m for any $m \geq 2$) copies of a nowhere dense subset E .

Our proof follows a familiar pattern. To make this explicit we give a general approach to such proofs, and then verify later that the special property used in the proof holds for $\mathcal{N}^k$.

Let D be a compact set in $\mathbb{R}^n$ with non void interior and $C[D]$ the space of real valued continuous functions F on D , with the usual uniform norm

$$\|F\|_D = \max_{p \in D} |F(p)|.$$

Let $\mathcal{F}$ be a subset of $C[D]$ which we wish to prove nowhere dense. The key property used is the existence of special functions in $C[D]$ that fail to belong, locally, to the closure of $\mathcal{F}$.

Theorem 1. Suppose that it is true that there is a point p_0 interior to D such that for any real c there exists $g \in C[D]$ such that for every compact neighborhood V of p_0 ,

$$(2) \quad \inf_{f \in \mathcal{F}} \|f - g\|_V > 0$$

and such that $g(p_0) = c$. Then, $\mathcal{F}$ is nowhere dense in $C[D]$.

Proof: Let U be any non void open set in $C[D]$. We will produce a non void open subset of U that is disjoint from $\mathcal{F}$. Choose $G_0 \in U$ and $r > 0$ such that $\|G - G_0\| < r$ implies $G \in U$. Let $c = G_0(p_0)$, and let g be the special function in $C[D]$ satisfying (2) whose existence is predicated.

Let B be a closed ball in D , centered at p_0 , such that

$$(3) \quad |G_0(p) - c| < r/3$$

for all $p \in B$. Then, choose a ball B_0 of smaller radius, also centered at p_0 , such that

$$(4) \quad |g(p) - c| < r/3$$

for all $p \in B_0$. Using Tietze, construct a continuous function G_1 on B such that

$$G_1(p) = \begin{cases} G_0(p) & \text{for } p \in \partial B \\ g(p) & \text{for } p \in B_0 \end{cases}$$

and obeying

$$(5) \quad |G_1(p) - c| < r/3$$

for all $p \in B$. Then, define $G_2(p)$ to be $G_0(p)$ off B and $G_1(p)$ on B . Observe that G_2 is continuous on D and agrees with G_0 except on a small neighborhood of p_0 where it has been modified to agree with the special function g locally. If $p \in B$, then

$$\begin{aligned} |G_2(p) - G_0(p)| &= |G_1(p) - G_0(p)| \\ &\leq |G_1(p) - c| + |c - G_0(p)| \\ &\leq r/3 + r/3 < r. \end{aligned}$$

Thus, $\|G_2 - G_0\|_D < r$ so that $G_2 \in U$.

Choose $\delta > 0$ so that $\|f - g\|_{B_0} > \delta$ for all $f \in \mathcal{F}$ and let

$$U_1 = \{\text{all } F \in U \text{ with } \|F - G_2\| < \delta/2\}.$$

Suppose that $F \in U_1$ and $f \in \mathcal{F}$. Then,

$$\begin{aligned} \|f - F\|_D &\geq \|f - F\|_{B_0} \\ &\geq \|f - G_2\|_{B_0} - \|F - G_2\|_{B_0} \\ &\geq \|f - g\|_{B_0} - \|F - G_2\|_{B_0} \\ &\geq \delta - \delta/2 = \delta/2. \end{aligned}$$

Thus, U_1 is an open subset of U that contains G_2 and is disjoint from $\mathcal{F}$. Hence, $\mathcal{F}$ is nowhere dense in $C[D]$.

We now return to the nomographic functions and show that $\mathcal{N}^2$ is nowhere dense in $C[I^2]$, where $I = [-1, 1]$. We must exhibit functions g not in $\mathcal{N}^2$ that have the special property (2). We choose p_0 to be $(0,0)$, and do not need to require the extra condition that $g(p_0) = c$ since $\mathcal{N}^2$ has

the property that if $f \in \eta^2$, so does $f + c$. For any $r > 0$, let V_r be the compact neighborhood of p_0 consisting of those (x,y) with $|x| \leq r$, $|y| \leq r$.

Theorem 2. The function $g(x,y) = x^2 + xy + y^2 + 2x + y$ has the property that

$$(6) \quad r^2 > \inf_{f \in \eta^2} \|f - g\|_{V_r} > \frac{3}{r} / 10$$

for all $r < .01$.

We begin by quoting one of the characterization theorems for nomographic functions, modified to match the notation and needs of the present proof.

(See p. 293 of [2])

[Theorem 12] Let g be of class C' on the set $V_r = J^2$ where $J = [-r, r]$, and suppose that g_x and g_y are bounded below by $\sigma > 0$. Let $(u,v) = p_0 = (0,0)$ and suppose that the distance from g to η^2 in the space $C[V_r]$ is less than e , where $e < r\sigma/12$. Then, one of the following systems of inequalities must have a solution as indicated:

(i) For some choice of x_1 and x_2 in $[-r, r]$

$$(7) \quad \begin{aligned} |g(x_1, -r) - g(-r, 0)| &< 2e \\ |g(x_2, -r) - g(-r, r)| &< 2e \\ |g(x_1, r) - g(x_2, 0)| &< 2e \end{aligned}$$

(ii) For some choice of y_1 and y_2 in $[-r, r]$

$$(8) \quad \begin{aligned} |g(-r, y_1) - g(0, -r)| &< 2e \\ |g(-r, y_2) - g(r, -r)| &< 2e \\ |g(r, y_1) - g(0, y_2)| &< 2e \end{aligned}$$

If we apply this result to the function $g(x,y)$ given above, and set $x_i = rs_i$, $y_i = rt_i$, then if the distance from g to η^2 is less than e , either there exist s_1 and s_2 with $|s_i| \leq 1$ such that

$$\begin{aligned} & |2s_1 + 1 + (s_1^2 - s_1)r| < 2e/r \\ (9) \quad & |2s_2 + (s_2^2 - s_2)r| < 2e/r \\ & |2s_1 - 2s_2 + 1 + (s_1^2 - s_2^2 + s_1 + 1)r| < 2e/r \end{aligned}$$

or there exist t_1 and t_2 with $|t_i| \leq 1$ such that

$$\begin{aligned} & |t_1 - 1 + (t_1^2 - t_1)r| < 2e/r \\ (10) \quad & |t_2 - 3 + (t_2^2 - t_2)r| < 2e/r \\ & |t_1 - t_2 + 2 + (t_1^2 - t_2^2 + t_1 + 1)r| < 2e/r. \end{aligned}$$

To prove theorem 2, we now show that if $r < .01$ and $e = r^3/10$ then neither (9) nor (10) have a solution as specified. For (10) this is immediate. From the second inequality in (10) and the fact that $|t_i| \leq 1$ we see that

$$|t_2 - 3| \leq 2r + 2e/r = 2r + r^2/5$$

which would contradict $|t_2| \leq 1$. To show that (9) also fails to have a solution, we proceed as follows. Set

$$A = 2s_1 + 1 + (s_1^2 - s_1)r$$

$$B = 2s_2 + (s_2^2 - s_2)r$$

$$C = 2s_1 - 2s_2 + 1 + (s_1^2 - s_2^2 + s_1 + 1)r$$

so that the desired inequalities are $|A| < r^2/5$, $|B| < r^2/5$ and $|C| < r^2/5$.

From the first, we obtain

$$|2s_1 + 1| \leq 2r + r^2/5 < 3r$$

which shows that $s_1 = -1/2 + ar$, where $|a| \leq 3/2$. Substituting this into A, we find that

$$|2a + 3/4| \leq 3r + r/5 + \frac{9}{4}r^2 < 4r$$

so that $s_1 = -1/2 - (3/8)r + br^2$ where $|b| \leq 2$. In the same way, starting from $|B| < r^2/5$, we find that $s_2 = cr^2$ where $|c| \leq 1$.

Finally, observing that $C = A - B + (2s_1 + 1 - s_2)r$, we arrive at $|2s_1 + 1 - s_2| \leq (3/5)r$ and substituting the values for s_1 and s_2 , obtain

$$|-3/4 + (2b - c)r| \leq 3/5$$

which clearly cannot hold if $r < .01$.

To obtain the left side of formula (6), we observe that the special function f_0 defined by

$$f_0(x, y) = (x + y/2 + 1)^2 - 1$$

belongs to $\mathcal{N}^2$ and obeys $\|f_0 - g\|_{V_r} < r^2$.

Corollary. $\mathcal{N}^2$ is a nowhere dense subset of $C[I^2]$.

To show now that this is also true for the class $\mathcal{N}^k$ of nomographic functions of k variables, for any $k > 2$, we will prove that one obtains no better nomographic approximation to the function $g(x, y) = x^2 + xy + y^2 + 2x + y$ by using functions $f \in \mathcal{N}^k$.

Theorem 3 For any continuous function $g(x_1, x_2)$ of two variables,

$$\inf_{f \in \mathcal{N}^k} \|f - g\|_{I^k} = \inf_{f \in \mathcal{N}^2} \|f^* - g\|_{I^2}.$$

Proof: Suppose that d is the distance in $C[I^k]$ from g to $\mathcal{N}^k$. Given $\delta > 0$, choose $f_0 \in \mathcal{N}^k$ so that

$$d_0 = \|f_0 - g\|_{I^k} < d + \delta$$

and choose $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k) \in I^k$ so that

$$\begin{aligned} d_0 &= |f_0(\bar{x}) - g(\bar{x})| \\ &= |h(\phi_1(\bar{x}_1) + \phi_2(\bar{x}_2) + \dots + \phi_k(\bar{x}_k)) - g(\bar{x}_1, \bar{x}_2)|. \end{aligned}$$

Define a function ψ on I by

$$\psi(x_2) = \phi_2(x_2) + \phi_3(\bar{x}_3) + \dots + \phi_k(\bar{x}_k)$$

and set

$$f^*(x_1, x_2) = h(\phi_1(x_1) + \psi(x_2)).$$

Then, $|f^*(\bar{x}_1, \bar{x}_2) - g(\bar{x}_1, \bar{x}_2)| = d_0 = \|f_0 - g\|_{I^k}$ while for every (x_1, x_2) in I^2

$$\begin{aligned} |f^*(x_1, x_2) - g(x_1, x_2)| &= |f_0(x_1, x_2, \bar{x}_3, \dots, \bar{x}_k) - g(x_1, x_2)| \\ &< \|f_0 - g\|_{I^k}. \end{aligned}$$

Accordingly, $\|f^* - g\|_{I^2} = \|f_0 - g\|_{I^k} < d + \delta$ and since this holds for any $\delta > 0$,

$$\inf_{f \in \mathcal{N}^2} \|f - g\|_{I^2} \leq d = \inf_{f \in \mathcal{N}^k} \|f - g\|_{I^k}.$$

Since $\mathcal{N}^2 \subset \mathcal{N}^k$, the reverse inequality also holds, proving the theorem.

It is known that four copies of $\mathcal{N}^2$ are not enough to give $C[I^2]$ as their algebraic sum. [4]. It would be of interest to know if the sum of four copies of $\mathcal{N}^2$ is dense in $C[I^2]$.

The argument used above will not suffice here since the special function $g(x, y)$ in fact already belongs to $\mathcal{N}^2 + \mathcal{N}^2$, for

$$g(x, y) = \{(x^2 - x - 6) + (y^2 - y)\} + \exp\{\log(x+2) + \log(y+3)\}.$$

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1. REPORT NUMBER # 2279	2. GOVT ACCESSION NO. AD-A110 466	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Nomographic Functions are Nowhere Dense		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) R. Creighton Buck		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Computer Science
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE September 1981
		13. NUMBER OF PAGES 8
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) approximations of functions of several variables; superpositions; nowhere dense sets		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A nomographic function of k variables is one that can be represented by the format $f(x_1, x_2, \dots, x_k) = h(\phi_1(x_1) + \phi_2(x_2) + \dots + \phi_k(x_k))$ where the ϕ_i and h are continuous. Any individual nomographic function is very special in nature, since it is constructed from functions of one variable and addition alone. However, Kolmogorov showed in 1957 that every continuous function of k variables has a representation as a sum of not more than $2k+1$		

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