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MRC Technical Summary Report #2270

ON AN AXISYMMETRIC  
FREE BOUNDARY PROBLEM

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August 1981

(Received April 8, 1981)

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Shu-Zi Zhou\*

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ABSTRACT

The axisymmetric elastic-plastic torsion of a shaft of general shape subject to the Hencky consistency condition with the von Mises yield function is considered. It is proved that the Haar-Kármán principle is valid in this case, and that the problem is essentially two-dimensional. The problem is reformulated as a variational inequality, and the existence and uniqueness of the solution is studied.

AMS (MOS) Subject Classifications: 35J20; 35J65; 35R35; 73C05; 73E99.

Key Words: Torsion; elastic-plastic; axisymmetric; free boundary problem; variational inequalities; Haar-Kármán principle.

Work Unit Number 1 - Applied Analysis

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and the National Science Foundation under Grant No. MCS77-26732.

SIGNIFICANCE AND EXPLANATION

When a shaft of circular cross section is subjected to a torque, a plastic enclave may appear. The boundary between the elastic region and the plastic region is unknown. It is a so-called free boundary problem.

Assuming that Hencky's consistency condition with the von Mises yield function is satisfied, we can prove that the Haar-Kármán principle is valid, which means that the strain energy must be minimized subject to the constraint that the stress should not exceed its permissible limit. We show that the problem is essentially two-dimensional, and give two kinds of variational formulations of the problem: one for the stress field, the other for the stress function. The existence and uniqueness of the solution of the variational inequality for the stress function is proved.

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ON AN AXISYMMETRIC FREE BOUNDARY PROBLEM

Shu-Zi Zhou\*

1. Introduction

The elastic-plastic torsion of shafts is one of the classic free boundary problems. For the case of that the cross-section is constant, it has been studied deeply by using variational inequalities during the last decade (see, for instance, Ting [1971, 1976], Brezis and Sibony [1971], Lanchon [1974], Friedman [1980], Pozzi [1980]). Recently, Cryer [1980] has considered the case in which the shaft has variable cross-section and rotational symmetry. He has proved the existence, uniqueness and regularity of the solution of the variational inequality problem for the stress function under some assumptions. He has assumed that the function it describes the generator of the rotational shaft is monotone. In this paper we consider the case in which the generator may have more general shape; give two kinds of variational formulation of the problem: one for the stress field, the other for the stress function; prove that Haar-Kármán principle is valid and the problem is essentially two-dimensional under the so-called Hencky's conditions; study the existence and uniqueness of the solution.

I am grateful to Professor C. Cryer for many valuable discussions.

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and  
the National Science Foundation under Grant No. MCS77-26732.

## 2. Classical problem

Let Figure 1 represent a shaft of circular cross section with equal and opposite pure torques  $T$  applied at the ends. Assume that the material is homogeneous, isotropic, and elastic-perfectly plastic, that there are no body forces, and that there are no external tractions on the lateral surface. Our aim is to find the resulting stress distribution.

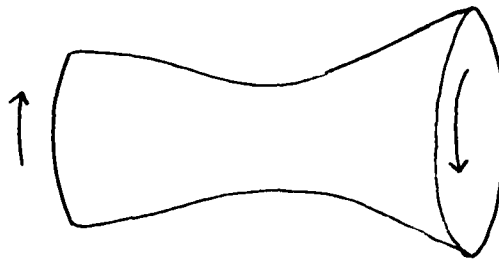


Figure 1

Setting up a cylindrical coordinate system in which the  $z$ -axis coincides with the center line of the shaft and the origin lies in a end of the shaft, we assume that the equation for the generator is  $r = R(z)$ . Let (see Figure

2)

$$\Omega = \{(z,r): 0 < z < L, 0 < r < R(z)\}$$

$$\Gamma_0 = \{(z,r): 0 \leq z \leq L, r = 0\}$$

$$\Gamma_1 = \{(z,r): 0 \leq z \leq L, r = R(z)\}$$

$$\Gamma_{21} = \{(z,r): 0 < r < R(0), z = 0\}$$

$$\Gamma_{22} = \{(z,r): 0 < r < R(L), z = L\}$$

$$\Gamma_2 = \Gamma_{21} \cup \Gamma_{22} .$$

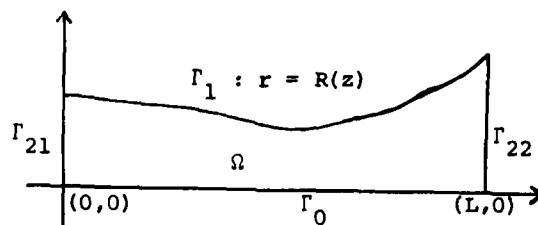


Figure 2

Then we have (Eddy and Shaw [1949]).

Classical Problem: Find stress function  $v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$  such that

$$|\nabla v| < kr^2 \quad \text{in } \Omega \quad (k \text{ is a constant given})$$

$$\Delta v \equiv -\frac{\partial}{\partial z} \left( r^{-3} \frac{\partial v}{\partial z} \right) - \frac{\partial}{\partial r} \left( r^{-3} \frac{\partial v}{\partial r} \right) = 0$$

$$\text{in } \Omega \cap \{(r,z) : |\nabla v| < kr^2\}$$

$$v = 0 \quad \text{on } \Gamma_0, \quad v = T/2\pi \quad \text{on } \Gamma_1$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma_2.$$

This formulation of the problem is based on the von Mises yield criteria, and on the so-called semi-inverse method given by Saint-Venant (see, for instance, Timoshenko et al. [1951, p. 259, p. 306]) in which one assumes a priori that there are no radial and axial displacement. This fact will be proved in section 4 of this paper.

### 3. Haar-Kármán principle

The argument we use in these two sections is similar to that in Lanchon [1974] for the case of constant cross section.

From now on we assume that

$R(z)$  is piecewisely, continuously differentiable

$$R'(0) \neq \infty, \quad R'(L) \neq \infty \quad (3.1)$$

which implies that  $\Omega$  is strongly Lipschitz domain. Then the three dimensional region  $\Omega^*$  occupied by the shaft is also strongly Lipschitz domain. Its boundary is

$$\partial\Omega^* = \Gamma_1^* \cup \Gamma_{21}^* \cup \Gamma_{22}^*$$

where  $\Gamma_1^*$  is the lateral surface while  $\Gamma_{21}^*$  and  $\Gamma_{22}^*$  are the end surfaces.

Denote by  $u_r$ ,  $u_\theta$  and  $u_z$  the components of the displacements in the radial, tangential and axial directions respectively. Let

$$u = [u_r, u_\theta, u_z]^T, \quad \sigma = [\sigma_r, \sigma_\theta, \sigma_z, \sigma_{\theta z}, \sigma_{rz}, \sigma_{r\theta}]^T, \\ \epsilon = [\epsilon_r, \epsilon_\theta, \epsilon_z, \epsilon_{\theta z}, \epsilon_{rz}, \epsilon_{r\theta}]^T$$

where  $\sigma$ -stress field,  $\epsilon$ -strain field. Then we have (see, for instance, Timoshenko et al. [1951, pp. 305-308])

$$\epsilon_r = \frac{\partial u_r}{\partial r}, \quad \epsilon_\theta = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad \epsilon_z = \frac{\partial u_z}{\partial z} \\ \epsilon_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta}, \quad \epsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \quad \epsilon_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \\ \text{in } \Omega^* \quad (3.2)$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \quad \text{in } \Omega^* \quad (3.3)$$

We have the boundary condition as follows:

$$\sigma_r n_r + \sigma_{rz} n_z = 0 \\ \sigma_{r\theta} n_r + \sigma_{\theta z} n_z = 0 \\ \sigma_{rz} n_r + \sigma_z n_z = 0 \quad \text{on } \Gamma_1^* \quad (\text{no external tractions}) \quad (3.4)$$

$$\sigma_{rz} = \sigma_{\theta z} = \sigma_z = 0 \quad \text{on } \Gamma_{21}^* \cup \Gamma_{22}^* \quad (\text{pure torque}) \quad (3.5)$$

$$\int_{\Gamma_{21}^*} r \sigma_{\theta z} ds = \int_{\Gamma_{22}^*} r \sigma_{\theta z} ds = T \quad (3.6)$$

where  $(n_r, n_\theta, n_z) = n$  is the outer normal of  $\partial\Omega^*$ . (Obviously,  $n$  is well-defined a.e. on  $\partial\Omega^*$  because of (3.1), and  $n_\theta = 0$  on  $\partial\Omega^*$ ,  $n_r = 0$  on  $\Gamma_{21}^* \cup \Gamma_{22}^*$ ).

Assume that the resulting stress field  $\sigma^0$  satisfies the Hencky's consistency conditions (Lanchon [1974])

$$\begin{aligned} F(\sigma^0) &< 0 \\ \epsilon^0 &= A\sigma^0 + \lambda \\ \lambda^T(\sigma - \sigma^0) &< 0, \quad \forall \sigma \in M_1 \end{aligned} \quad (3.7)$$

where  $F(\sigma)$  is the von Mises yield function

$$F(\sigma) = \frac{1}{2} (\sigma_r^2 + \sigma_\theta^2 + \sigma_z^2 + \sigma_{rz}^2 + \sigma_{r\theta}^2) - \frac{1}{6} (\sigma_r + \sigma_\theta + \sigma_z)^2 - k^2, \quad (3.8)$$

$\lambda = [\lambda_r, \dots, \lambda_{rz}]^T$ ,  $M_1 = \{\sigma : F(\sigma) < 0\}$ , and  $A$  is the matrix in the Hooke's law (Timoshenko et al. [1951, p. 7, p. 66])

$$A = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & 1 & & & \\ & & & 2(1+\nu) & & \\ & & & & 2(1+\nu) & \\ & & & & & 2(1+\nu) \end{bmatrix} \quad (3.9)$$

where  $E$  is the modulus of elasticity and  $\nu$  is the Poisson's ratio with  $0 < \nu < 1/2$ .

Let

$$M_2 = \{\sigma \in M_1 : \sigma \in [H^1(\Omega^*)]^6, \sigma \text{ satisfies (3.3) - (3.6)}\}$$

where the  $\sigma$  in (3.4) - (3.6) is the trace of  $\sigma$  on  $\partial\Omega^*$  (see, for instance, Nečas [1967, p. 15]).

**Proposition 3.1.** If (3.2) and (3.7) are valid for  $\sigma^0 \in M_2$ , then the Haar-Kármán principle is valid in this case, that is

$$J_1(\sigma^0) = \min_{\sigma \in M_2} J_1(\sigma)$$

where

$$J_1(\sigma) = \frac{1}{2} \int_{\Omega^*} \sigma^T A \sigma dV \quad (3.10)$$

Proof:  $\forall \sigma \in M_2$ , we have (noting that  $A$  is symmetric)

$$\begin{aligned} J_1(\sigma) - J_1(\sigma^0) &= \frac{1}{2} \int_{\Omega^*} (\sigma^T A \sigma - (\sigma^0)^T A \sigma^0) dV \\ &= \frac{1}{2} \int_{\Omega^*} (\sigma - \sigma^0)^T A (\sigma - \sigma^0) dV + \int_{\Omega^*} (\sigma^0)^T A (\sigma - \sigma^0) dV \end{aligned}$$

It follows from Gerschgorin's theorem (Varga [1962, p. 16]) that all of the eigenvalues of  $A$  are not less than  $(1-2\nu)/E > 0$ . Hence

$$x^T A x > (1-2\nu)x^2/E, \quad \forall x \in R^6 \quad (3.11)$$

and we have

$$\begin{aligned} J_1(\sigma) - J_1(\sigma^0) &> \int_{\Omega^*} (\sigma^0)^T A (\sigma - \sigma^0) dV \\ &> \int_{\Omega^*} (\varepsilon^0)^T (\sigma - \sigma^0) dV \quad (\text{since (3.7)}) \\ &= \int_{\Omega^*} \left[ \frac{\partial u_r^0}{\partial r} (\sigma_r - \sigma_r^0) + \left( \frac{u_r^0}{r} + \frac{1}{r} \frac{\partial u_\theta^0}{\partial \theta} \right) (\sigma_\theta - \sigma_\theta^0) + \frac{\partial u_z^0}{\partial z} (\sigma_z - \sigma_z^0) \right. \\ &\quad + \left. \left( \frac{\partial u_\theta^0}{\partial z} + \frac{1}{r} \frac{\partial u_z^0}{\partial \theta} \right) (\sigma_{\theta z} - \sigma_{\theta z}^0) + \left( \frac{\partial u_r^0}{\partial z} + \frac{\partial u_z^0}{\partial r} \right) (\sigma_{rz} - \sigma_{rz}^0) + \right. \\ &\quad \left. + \left( \frac{1}{r} \frac{\partial u_r^0}{\partial \theta} + \frac{\partial u_\theta^0}{\partial r} - \frac{u_\theta^0}{r} \right) (\sigma_{r\theta} - \sigma_{r\theta}^0) \right] r dr d\theta dz \quad (\text{since (3.2)}) \\ &= \int_{\partial \Omega^*} (u_r^0 ((\sigma_r - \sigma_r^0) n_r + (\sigma_{rz} - \sigma_{rz}^0) n_z) + u_\theta^0 ((\sigma_{r\theta} - \sigma_{r\theta}^0) n_r + (\sigma_{\theta z} - \sigma_{\theta z}^0) n_z) \\ &\quad + u_z^0 ((\sigma_{rz} - \sigma_{rz}^0) n_r + (\sigma_z - \sigma_z^0) n_z)) dS - \int_{\Omega^*} (u_r^0 \frac{\partial}{\partial r} (\sigma_r - \sigma_r^0) + \\ &\quad \frac{1}{r} \frac{\partial}{\partial \theta} (\sigma_{r\theta} - \sigma_{r\theta}^0) + \frac{\partial}{\partial z} (\sigma_{rz} - \sigma_{rz}^0) + \frac{(\sigma_r - \sigma_r^0) - (\sigma_\theta - \sigma_\theta^0)}{r}) + u_\theta^0 \frac{\partial}{\partial r} (\sigma_{r\theta} - \sigma_{r\theta}^0) + \end{aligned}$$

$$\frac{1}{r} \frac{\partial}{\partial \theta} (\sigma_{\theta} - \sigma_{\theta}^0) + \frac{\partial}{\partial z} (\sigma_{\theta z} - \sigma_{\theta z}^0) + \frac{2(\sigma_{r\theta} - \sigma_{r\theta}^0)}{r} + u_z^0 \left( \frac{\partial}{\partial r} (\sigma_{rz} - \sigma_{rz}^0) + \frac{1}{r} \frac{\partial}{\partial \theta} (\sigma_{\theta z} - \sigma_{\theta z}^0) + \frac{\partial}{\partial z} (\sigma_z - \sigma_z^0) + \frac{1}{r} (\sigma_{rz} - \sigma_{rz}^0) \right) dV \quad (\text{Green's formula})$$

$$= 0 \quad . \quad (\text{since (3.3) - (3.5)})$$

Q.E.D.

About more general discussion on the Haar-Kármán principle, see Martin [1975, pp. 733-736].

#### 4. Variational formulation of the problem

Proposition 3.1 suggests the following variational formulation of our problem.

Problem (A) Find  $\sigma^0 \in M_2$  such that

$$J_1(\sigma^0) = \min_{\sigma \in M_2} J_1(\sigma)$$

where  $J_1$  is defined by (3.10).

It is easy to show the following assertion.

Proposition 4.1. Problem (A) is equivalent to the variational inequality

$$\sigma^0 \in M_2 \quad (4.1)$$

$$a_1(\sigma^0, \sigma - \sigma^0) > 0 \quad \forall \sigma \in M_2$$

where

$$a_1(\sigma^1, \sigma^2) = \int_{\Omega^*} (\sigma^1)^T A \sigma^2 dV \quad . \quad (4.2)$$

Proposition 4.2. Problem (A) has at most one solution.

Proof: If  $\sigma^0$  and  $\sigma^1$  are solutions, then we have by (4.1)

$$a_1(\sigma^0, \sigma^1 - \sigma^0) > 0$$

$$a_1(\sigma^1, \sigma^0 - \sigma^1) > 0 \quad .$$

By virtue of (3.11) we obtain

$$\begin{aligned} (1-2\nu) \int_{\Omega^*} (\sigma^1 - \sigma^0)^2 dv/E &< a_1(\sigma^1 - \sigma^0, \sigma^1 - \sigma^0) \\ &= -a_1(\sigma^1, \sigma^0 - \sigma^1) - a_1(\sigma^0, \sigma^1 - \sigma^0) < 0 . \end{aligned}$$

Hence  $\sigma^1 = \sigma^0$  a.e. in  $\Omega^*$ .

Q.E.D.

**Theorem 4.3.** Assume problem (A) has solution  $\sigma^0$ . Then

$$(a) \quad \sigma_r^0 = \sigma_\theta^0 = \sigma_z^0 = \sigma_{rz}^0 = 0 \quad \text{a.e. in } \Omega^*$$

$$(b) \quad \frac{\partial \sigma^0}{\partial \theta} = 0 \quad \text{a.e. in } \Omega^*$$

**Proof:** Let  $\sigma^* = [0, 0, 0, \sigma_{\theta z}^*, 0, \sigma_{r\theta}^*]^T$  in  $\overline{\Omega^*}$  and

$$\begin{aligned} \sigma_{\theta z}^* &= \frac{1}{2\pi} \int_0^{2\pi} \sigma_{\theta z}^0 d\theta && \text{in } \overline{\Omega^*} \\ \sigma_{r\theta}^* &= \frac{1}{2\pi} \int_0^{2\pi} \sigma_{r\theta}^0 d\theta && \text{in } \overline{\Omega^*} . \end{aligned}$$

If we can prove  $\sigma^* \in M_2$  and

$$J_1(\sigma^*) < J_1(\sigma^0) , \quad (4.3)$$

then the conclusion of the theorem is clear by Proposition 4.2.

At first we prove that  $\sigma^* \in [H^1(\Omega^*)]^6$ . Recall that  $\Omega$  is defined by (2.1) and it is the cross-section of  $\Omega^*$  by plane  $\theta = \text{constant}$ . Define space

$$L_r^2(\Omega) = \{v : v \text{ measurable in } \Omega, \|v\|_{L_r^2(\Omega)} < \infty\}$$

with norm

$$\|v\|_{L_r^2(\Omega)}^2 = \int_{\Omega} v^2 r dr dz ,$$

and define a distribution

$$\langle \sigma_{\theta z}^*, v \rangle_{\Omega, r} = \int_{\Omega} \sigma_{\theta z}^* v r dr dz, \quad \forall v \in C_0^\infty(\Omega) .$$

$\forall \varphi \in C_0^\infty(\Omega)$ , define

$$\begin{aligned} \Phi(r, \theta, z) &= \varphi(r, z) && \text{in } \Omega^* \{r = 0\} \\ \Phi(0, \theta, z) &= 0 . \end{aligned}$$

Then we have

$$|\langle \sigma_{\theta z}^*, \varphi \rangle_{\Omega, r}| = \left| \frac{1}{2\pi} \int_{\Omega^*} \sigma_{\theta z}^0 \varphi \, dV \right|$$

$$\leq \frac{1}{2\pi} \|\sigma_{\theta z}^0\|_{L^2(\Omega^*)} \cdot \|\varphi\|_{L^2(\Omega^*)} = \frac{1}{\sqrt{2\pi}} \|\sigma_{\theta z}^0\|_{L^2(\Omega^*)} \cdot \|\varphi\|_{L^2(\Omega)}$$

Hence  $\sigma_{\theta z}^*$  (more precisely, its restriction in  $\Omega$ ) belongs to the dual space of  $L^2_r(\Omega)$  and then

$$\sigma_{\theta z}^* \in L^2_r(\Omega) \quad .$$

So

$$\int_{\Omega^*} (\sigma_{\theta z}^*)^2 \, dV = \int_0^{2\pi} d\theta \int_{\Omega} (\sigma_{\theta z}^*)^2 r \, dr \, dz = 2\pi \|\sigma_{\theta z}^*\|_{L^2_r(\Omega)}^2 < \infty$$

i.e.  $\sigma_{\theta z}^* \in L^2(\Omega^*)$ . Similarly, we have

$$v \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \sigma_{\theta z}^0}{\partial r} \, d\theta \in L^2(\Omega^*) \quad .$$

Given  $\varphi \in C_0^\infty(\Omega)$ , we have

$$\begin{aligned} \langle v, \varphi \rangle_{\Omega, r} &= \int_{\Omega} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \sigma_{\theta z}^0}{\partial r} \, d\theta \right) \varphi r \, dr \, dz \\ &= \frac{1}{2\pi} \int_{\Omega^*} \frac{\partial \sigma_{\theta z}^0}{\partial r} \varphi \, dV = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{\Omega} \frac{\partial \sigma_{\theta z}^0}{\partial r} \varphi r \, dr \, dz \\ &= -\frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{\Omega} \sigma_{\theta z}^0 \frac{\partial(\varphi r)}{\partial r} \, dr \, dz \\ &= -\frac{1}{2\pi} \int_{\Omega} \left( \int_0^{2\pi} \sigma_{\theta z}^0 \, d\theta \right) \frac{\partial(\varphi r)}{\partial r} \, dr \, dz \\ &= - \int_{\Omega} \sigma_{\theta z}^* \frac{\partial(\varphi r)}{\partial r} \, dr \, dz = \int_{\Omega} \frac{\partial \sigma_{\theta z}^*}{\partial r} \varphi r \, dr \, dz \\ &= \langle \frac{\partial \sigma_{\theta z}^*}{\partial r}, \varphi \rangle_{\Omega, r} \quad . \end{aligned}$$

It means that

$$\frac{\partial \sigma_{\theta z}^*}{\partial r} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \sigma_{\theta z}^0}{\partial r} \, d\theta \in L^2(\Omega^*) \quad .$$

Similarly, we obtain

$$\frac{\partial \sigma_{\theta z}^*}{\partial z} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \sigma_{\theta z}^0}{\partial z} d\theta \in L^2(\Omega^*) .$$

But

$$\frac{\partial \sigma_{\theta z}^*}{\partial \theta} = 0 .$$

Hence  $\sigma_{\theta z}^* \in H^1(\Omega^*)$ . The same argument indicates that  $\sigma_{r\theta}^* \in H^1(\Omega^*)$ . and we obtain

$$\sigma^* \in [H^1(\Omega^*)]^6 .$$

Turn to (3.3) - (3.6). Check the second equations of (3.3) and (3.6) for  $\sigma^*$ :

$$\begin{aligned} \frac{\partial \sigma_{r\theta}^*}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta}^*}{\partial \theta} + \frac{\partial \sigma_{\theta z}^*}{\partial z} + \frac{2\sigma_{r\theta}^*}{r} &= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\partial \sigma_{r\theta}^0}{\partial r} + \frac{\partial \sigma_{\theta z}^0}{\partial z} + \frac{2\sigma_{r\theta}^0}{r} \right) d\theta \\ &= -\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r} \frac{\partial \sigma_{\theta}^0}{\partial \theta} d\theta \quad (\text{since (3.3)}) \\ &= 0 \quad (\text{periodicity of } \sigma \text{ in } \theta) \end{aligned}$$

$$\begin{aligned} \int_{\Gamma_{22}^*} r \sigma_{\theta z}^* dS &= \int_{\Gamma_{22}^*} r \left( \frac{1}{2\pi} \int_0^{2\pi} \sigma_{\theta z}^0 d\theta \right) r d\theta dr \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{R(L)} dr \int_0^{2\pi} r^2 \sigma_{\theta z}^0 d\theta \\ &= \int_0^{R(L)} dr \int_0^{2\pi} r^2 \sigma_{\theta z}^0 d\theta = \int_{\Gamma_{22}^*} r \sigma_{\theta z}^0 dS = T . \end{aligned}$$

The rest of (3.3) - (3.6) is clear.

Now we prove

$$F(\sigma^*) = (\sigma_{\theta z}^*)^2 + (\sigma_{r\theta}^*)^2 - k^2 < 0 . \quad (4.4)$$

We have

$$(\sigma_{\theta z}^*)^2 + (\sigma_{r\theta}^*)^2 = \sigma_{\theta z}^* \sigma_{\theta z}^* + \sigma_{r\theta}^* \sigma_{r\theta}^*$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_0^{2\pi} (\sigma_{\theta z}^0 \sigma_{\theta z}^* + \sigma_{r\theta}^0 \sigma_{r\theta}^*) d\theta \\
&< \frac{1}{2\pi} \left( \int_0^{2\pi} ((\sigma_{\theta z}^*)^2 + (\sigma_{r\theta}^*)^2) d\theta \right)^{1/2} \cdot \left( \int_0^{2\pi} ((\sigma_{\theta z}^0)^2 + (\sigma_{r\theta}^0)^2) d\theta \right)^{1/2} \\
&= \frac{1}{\sqrt{2\pi}} [(\sigma_{\theta z}^*)^2 + (\sigma_{r\theta}^*)^2]^{1/2} \cdot \left( \int_0^{2\pi} [(\sigma_{\theta z}^0)^2 + (\sigma_{r\theta}^0)^2] d\theta \right)^{1/2}
\end{aligned}$$

i.e.

$$(\sigma_{\theta z}^*)^2 + (\sigma_{r\theta}^*)^2 < \frac{1}{2\pi} \int_0^{2\pi} [(\sigma_{\theta z}^0)^2 + (\sigma_{r\theta}^0)^2] d\theta \quad (4.5)$$

On the other hand, we have

$$(\sigma_{\theta z}^0)^2 + (\sigma_{r\theta}^0)^2 < F(\sigma^0) + k^2 < k^2$$

Therefore, we obtain

$$(\sigma_{\theta z}^*)^2 + (\sigma_{r\theta}^*)^2 < \frac{1}{2\pi} \int_0^{2\pi} k^2 d\theta = k^2$$

i.e., (4.4) is valid, and  $\sigma^* \in M_2$ .

Finally, we compute  $J_1(\sigma^0) - J_1(\sigma^*)$ . Clearly,

$$(\sigma^*)^T A \sigma^* = \frac{2(1+\nu)}{E} [(\sigma_{\theta z}^*)^2 + (\sigma_{r\theta}^*)^2]$$

$$(\sigma^0)^T A \sigma^0 = \frac{1}{E} [(\sigma_r^0)^2 + (\sigma_\theta^0)^2 + (\sigma_z^0)^2 - 2\nu(\sigma_r^0 \sigma_\theta^0 + \sigma_r^0 \sigma_z^0 + \sigma_\theta^0 \sigma_z^0)]$$

$$+ \frac{2(1+\nu)}{E} [(\sigma_{r\theta}^0)^2 + (\sigma_{\theta z}^0)^2 + (\sigma_{rz}^0)^2] > \frac{2(1+\nu)}{E} [(\sigma_{\theta z}^0)^2 + (\sigma_{r\theta}^0)^2]$$

(since  $0 < \nu < \frac{1}{2}$ ).

It follows from (4.5) that

$$\int_{\Omega^*} [(\sigma_{\theta z}^*)^2 + (\sigma_{r\theta}^*)^2] dV < \int_{\Omega^*} [(\sigma_{\theta z}^0)^2 + (\sigma_{r\theta}^0)^2] dV$$

Hence

$$2(J_1(\sigma^0) - J_1(\sigma^*)) = \int_{\Omega^*} [(\sigma^0)^T A \sigma^0 - (\sigma^*)^T A \sigma^*] dV$$

$$> \frac{2(1+\nu)}{E} \left( \int_{\Omega^*} [(\sigma_{\theta z}^0)^2 + (\sigma_{r\theta}^0)^2] dV - \int_{\Omega^*} [(\sigma_{\theta z}^*)^2 + (\sigma_{r\theta}^*)^2] dV \right) > 0$$

(4.3) has been proved, and the proof of the theorem is completed.

Q.E.D.

The basic idea of the above proof is the same as that of the so-called semi-inverse method.

This theorem enables us to take the set

$$N = \{\sigma \in M_2 : \sigma_r = \sigma_\theta = \sigma_z = \sigma_{rz} = 0 \text{ in } \Omega^*\}$$

as the set of the admissible stress vectors instead of  $M_2$  in problem (A).

Then we have the following problem.

Problem (B). Find  $\sigma^0 \in N$  such that

$$J_1(\sigma^0) = \min_{\sigma \in N} J_1(\sigma) .$$

Obviously, problem (B) has at most one solution; if  $\sigma^0$  is the solution of problem (A), then it is also the solution of problem (B).

Remark 4.1. If  $\sigma \in N$ , then it follows from (3.3) that

$$\frac{\partial \sigma_{\theta z}}{\partial \theta} = \frac{\partial \sigma_{r\theta}}{\partial r} = 0 .$$

Therefore,

$$J_1(\sigma) = \frac{(1+\nu)2\pi}{E} J_0(\sigma)$$

where

$$J_0(\sigma) = \int_{\Omega} (\sigma_{r\theta}^2 + \sigma_{\theta z}^2) r \, dr dz .$$

##### 5. The variational problem for stress function

If problem (B) has solution  $\sigma^0 \in [C^0(\overline{\Omega^*})]^6$ , then we have by remark 4.1:

$$\sigma^0 \in N_1, \quad J_0(\sigma^0) = \min_{\sigma \in N_1} J_0(\sigma) \quad (C)$$

where  $N_1$  is a subset of  $N$ :

$$N_1 = \{\sigma \in N : \sigma \in [C^0(\overline{\Omega^*})]^6\}$$

$\forall \sigma \in N_1$ , by virtue of (3.3) we have

$$\frac{\partial}{\partial r} (r^2 \sigma_{r\theta}) + \frac{\partial}{\partial z} (r^2 \sigma_{\theta z}) = 0 .$$

Then it is easy to show that there exists  $v^* \in H^2(\Omega) \cap C^1(\overline{\Omega})$  such that

$$\frac{\partial v^*}{\partial r} = r^2 \sigma_{\theta z}, \quad \frac{\partial v^*}{\partial z} = -r^2 \sigma_{r\theta} \quad \text{in } \bar{\Omega} .$$

Hence  $\frac{\partial v^*}{\partial z} = 0$  on  $\Gamma_0$ , and  $v^* = C_1$  on  $\Gamma_0$ . Let  $v = v^* - C_1$ . Then

$$\frac{\partial v}{\partial r} = r^2 \sigma_{\theta z}, \quad \frac{\partial v}{\partial z} = -r^2 \sigma_{r\theta} \quad \text{in } \bar{\Omega} \quad (5.1)$$

$$v = 0 \quad \text{on } \Gamma_0 \quad (5.2)$$

$$J_0(\sigma) = \int_{\Omega} r^{-3} \left[ \left( \frac{\partial v}{\partial r} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 \right] dr dz \equiv J(v) . \quad (5.3)$$

It follows from (3.4) and (5.1) that

$$\frac{dv}{ds} = -\frac{\partial v}{\partial z} \cos(n, r) + \frac{\partial v}{\partial r} \cos(n, z) = 0 \quad \text{on } \Gamma_1$$

where  $s$  is the tangential direction of  $\Gamma_1$ . Therefore,

$$v = C_2 \quad \text{on } \Gamma_1 . \quad (5.4)$$

Since (3.6), (5.1) and (5.2), we have

$$T = \int_0^{2\pi} d\theta \int_0^{R(0)} r^2 \sigma_{\theta z} dr d\theta = 2\pi \int_0^{R(0)} \frac{\partial v}{\partial r} dr = 2\pi v(0, R(0)) , \quad (5.5)$$

Combine (5.4) and (5.5) we obtain

$$v = T/2\pi \quad \text{on } \Gamma_1 . \quad (5.6)$$

By (3.5) and (5.1) we have

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma_2 . \quad (5.7)$$

It follows from  $F(\sigma) < 0$  that

$$\begin{aligned} |\nabla v|^2 &= r^4 (\sigma_{r\theta}^2 + \sigma_{\theta z}^2) < k^2 r^4 \\ |\nabla v| &< kr^2 \quad \text{in } \Omega . \end{aligned} \quad (5.8)$$

Since  $\sigma \in [H^1(\Omega^*)]^6$ , we have

$$\sigma_{r\theta}, \sigma_{\theta z} \in H_r^1(\Omega)$$

where

$$H_r^1(\Omega) = \{v \in L_r^2(\Omega) : \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L_r^2(\Omega)\}$$

$$L_r^2(\Omega) \text{ is defined by (4.3).}$$

Hence

$$r^{-2} \frac{\partial v}{\partial r}, r^{-2} \frac{\partial v}{\partial z} \in H_r^1(\Omega) , \quad (5.9)$$

particularly,

$$\int_{\Omega} r^{-3} \left(\frac{\partial v}{\partial r}\right)^2 dr dz, \int_{\Omega} r^{-3} \left(\frac{\partial v}{\partial z}\right)^2 dr dz < \infty. \quad (5.10)$$

Finally by (5.1) and (5.2) we have

$$v^2 = \left(\int_0^r r^2 \sigma_{\theta z} dr\right)^2 < \int_0^r r^4 dr \cdot \int_0^r \sigma_{\theta z}^2 dr < Cr^5$$

so

$$\int_{\Omega} r^{-5} v^2 dr dz < \infty. \quad (5.11)$$

Now let

$$N_2 = \{v : v \in H^2(\Omega) \cap C^1(\bar{\Omega}), \text{ (5.2) and (5.6) - (5.9) are valid} \}.$$

Then it is easy to see that (5.1) - (5.2) defines a biunivocal map from  $N_1$  onto  $N_2$ , and problem (C) is equivalent to the following problem:

$$v_0 \in N_2, \quad J(v_0) = \min_{v \in N_2} J(v) \quad (D)$$

where (noting (5.3))

$$J(v) = \int_{\Omega} \rho \left[ \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 \right] dr dz \quad (5.12)$$

$$\rho = r^{-3}. \quad (5.13)$$

Now we enlarge the set of admissible functions of the variational problem (D) for solving the problem on the existence of the solution. Noting (5.11) and that only the derivatives of first order appear in the functional  $J(v)$ , we introduce a set as follows

$$K = \{v : v \in H_{\rho}^1(\Omega), v = T/2\pi \text{ on } \Gamma_1, |\nabla v| < kr^2 \text{ in } \Omega\} \quad (5.14)$$

where

$$H_{\rho}^1(\Omega) = \{v \in L_{\rho}^2(\Omega) : \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L_{\rho}^2(\Omega)\} \quad (5.15)$$

$$L_{\rho}^2(\Omega) = \{v : v \text{ measurable, } |v| \in L_{\rho}^2(\Omega) < +\infty\} \quad (5.16)$$

with norm, respectively,

$$\|v\|_{L_{\rho}^2(\Omega)} = \left[ \int_{\Omega} \rho v^2 dr dz \right]^{1/2}$$

$$\|v\|_{H_{\rho}^1(\Omega)} = \left( \|v\|_{L_{\rho}^2(\Omega)}^2 + \left\| \frac{\partial v}{\partial r} \right\|_{L_{\rho}^2(\Omega)}^2 + \left\| \frac{\partial v}{\partial z} \right\|_{L_{\rho}^2(\Omega)}^2 \right)^{1/2},$$

and  $v = T/2\pi$  is in the sense of trace (see, for instance, Nečas [1967, p. 15]).  $K$  is a natural extension of  $N_2$  because of that according to theorem 2.2 in Cryer [1980], if  $u \in H_0^1(\Omega)$ , then (5.2) (in the sense of trace) and (5.11) are valid. Therefore, we have the variational problem for stress function as follows.

Problem (E). Find  $v_0 \in K$  such that

$$J(v_0) = \min_{v \in K} J(v)$$

and, equivalently,

Problem (F). Find  $v_0$  such that

$$\begin{cases} v_0 \in K \\ a(v_0, v - v_0) > 0, \quad \forall v \in K \end{cases}$$

where

$$a(v', v'') = \int_{\Omega} \rho \left( \frac{\partial v'}{\partial r} \frac{\partial v''}{\partial r} + \frac{\partial v'}{\partial z} \frac{\partial v''}{\partial z} \right) dr dz .$$

Similarly to the case of constant cross-section, we introduce the obstacle problems relevant to problem (F). There are two obstacle problems to be considered.

Problem (F1). Find  $v_1$  such that

$$\begin{cases} v_1 \in K_1 \\ a(v_1, v - v_1) > 0 \quad \forall v \in K_1 \end{cases}$$

where

$$K_1 = \{v \in H_0^1(\Omega) : v = T/2\pi \text{ on } \Gamma_1, v > \psi_1 \text{ in } \Omega\} ,$$

and  $\psi_1$  is the solution of the Cauchy problem

$$\psi_1 \in C^2(\bar{\Omega})$$

$$|\nabla\psi_1|^2 = k^2 r^4, \psi_1 < T/2\pi \text{ in } \Omega \quad (5.17)$$

$$\psi_1 = T/2\pi \quad \text{on } \Gamma_1 .$$

Problem (F2). Find  $v_2$  such that

$$\begin{cases} v_2 \in K_2 \\ a(v_2, v-v_2) > 0 \quad \forall v \in K_2 \end{cases}$$

where

$$K_2 = \{v \in H_p^1(\Omega) : v = T/2\pi \text{ on } \Gamma_1, v < \psi_2 \text{ in } \Omega\} ,$$

and  $\psi_2$  is the solution of the Cauchy problem

$$\psi_2 \in C^2(\bar{\Omega})$$

$$|\nabla\psi_2|^2 = k^2 r^4, \psi_2 > 0 \text{ in } \Omega \quad (5.18)$$

$$\psi_2 = 0 \quad \text{on } \Gamma_0 .$$

The problem (F1) is just the problem (4.7) in Cryer [1980], there the solution for (5.17) is also discussed.

6.  $\psi_2$  - solution of the Cauchy problem (5.18)

Assume  $\psi_2$  is the solution of the Cauchy problem (5.18). Let  $p = \frac{\partial\psi_2}{\partial z}$ ,  $q = \frac{\partial\psi_2}{\partial r}$ . Then the equation is

$$F \equiv p^2 + q^2 - k^2 r^4 = 0 .$$

We have along the characteristics parametrized by  $s$  in  $\bar{\Omega} \setminus \Gamma_0$  (Courant and Hilbert [1962, p. 78])

$$\frac{dz}{ds} = \frac{\partial F}{\partial p} = 2p \quad (6.1)$$

$$\frac{dr}{ds} = \frac{\partial F}{\partial q} = 2q \quad (6.2)$$

$$\frac{d\psi_2}{ds} = p \frac{\partial F}{\partial p} + q \frac{\partial F}{\partial q} = 2(p^2 + q^2) = 2k^2 r^4 \quad (6.3)$$

$$\frac{dp}{ds} = -\left(p \frac{\partial F}{\partial \psi_2} + \frac{\partial F}{\partial z}\right) = 0 \quad (6.4)$$

$$\frac{dq}{ds} = -\left(q \frac{\partial F}{\partial \psi_2} + \frac{\partial F}{\partial r}\right) = 4k^2 r^3 \quad (6.5)$$

Since  $\psi_2 \in C^2(\bar{\Omega})$ , (6.1) - (6.5) are also valid on  $\Gamma_0$ . Given  $(z_0, 0) \in \Gamma_0$ , consider the characteristic passing this point. Let the parameter value of this point be  $s = 0$ . We have initial conditions (since  $\psi_2 = 0$  on  $\Gamma_0$  and  $F = 0$ )

$$z(0) = z_0, \quad r(0) = p(0) = q(0) = \psi_2(0) = 0.$$

It follows from (6.4) that  $p(s) \equiv 0$ . Hence  $z \equiv z_0$  (since (6.1)), and  $q = kr^2$  (since  $F = 0$  and  $\psi_2 > 0$ ). Therefore, we obtain by integrating along the characteristic

$$\begin{aligned} \psi_2(z_0, r) - \psi_2(s) &= \int_0^s \frac{d\psi_2}{ds} ds = \int_0^s \left[ \frac{\partial \psi_2}{\partial z} \frac{dz}{ds} + \frac{\partial \psi_2}{\partial r} \frac{dr}{ds} \right] ds \\ &= \int_0^s q \frac{dr}{ds} ds = \int_0^r q dr = \int_0^r kr^2 dr = kr^3/3. \end{aligned}$$

Since  $(z_0, 0) \in \Gamma_0$  is arbitrary, we obtain the solution

$$\psi_2(z, r) = kr^3/3 \quad \text{in } \bar{\Omega}. \quad (6.6)$$

## 7. Properties of the set K

Denote by  $C^{0,1}(\bar{\Omega})$  the set of functions which satisfy Lipschitz conditions, that is, if  $v \in C^{0,1}(\bar{\Omega})$ , then

$$h_1(v, \bar{\Omega}) \equiv \sup_{\substack{p_1, p_2 \in \bar{\Omega} \\ p_1 \neq p_2}} \frac{|v(p_1) - v(p_2)|}{|p_1 - p_2|} < +\infty \quad (7.1)$$

where  $p_1 = (z_1, r_1)$ ,  $p_2 = (z_2, r_2)$ , and  $|p_1 - p_2| = [(z_1 - z_2)^2 + (r_1 - r_2)^2]^{1/2}$ .

Lemma 7.1. The following statements are equivalent:

(a)  $v \in C^{0,1}(\bar{\Omega})$

(b)  $v \in H^1(\Omega)$ , and  $\frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L^\infty(\Omega)$ .

Proof: It has been shown (cf. Adams [1975, pp. 133-134]) that if  $v, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L^\infty(\Omega)$  then

$$h_1(v, \Omega) < C_1 \left( \|v\|_{L^\infty(\Omega)} + \left\| \frac{\partial v}{\partial r} \right\|_{L^\infty(\Omega)} + \left\| \frac{\partial v}{\partial z} \right\|_{L^\infty(\Omega)} \right). \quad (7.2)$$

By similar argument we can prove that if  $\frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L^\infty(\Omega)$ , and  $v \in L^1(\Omega)$  then

$$h_1(v, \Omega) < C_2 \left( \left\| \frac{\partial v}{\partial r} \right\|_{L^\infty(\Omega)} + \left\| \frac{\partial v}{\partial z} \right\|_{L^\infty(\Omega)} \right). \quad (7.3)$$

Therefore, it is clear that (b) implies (a).

Now assume  $v \in C^{0,1}(\bar{\Omega})$ . Then  $v$  possesses a total differential a.e. in  $\Omega$  (Morrey [1966, p. 65]), i.e.

$$v(p_1) - v(p_2) = \left[ \frac{\partial v}{\partial z} \right] (z_1 - z_2) + \left[ \frac{\partial v}{\partial r} \right] (r_1 - r_2) + o(|p_1 - p_2|) \quad (7.4)$$

a.e. in  $\Omega$

where  $\left[ \frac{\partial v}{\partial z} \right], \left[ \frac{\partial v}{\partial r} \right]$  are the partial derivatives in the usual sense. Clearly, they are measurable in  $\Omega$ . It follows from (7.1) and (7.4) that

$$\left| \left[ \frac{\partial v}{\partial z} \right] \right|, \left| \left[ \frac{\partial v}{\partial r} \right] \right| < h_1(v, \bar{\Omega}) \quad \text{a.e. in } \Omega. \quad (7.5)$$

Hence  $v \in H^1(\Omega)$  and (Morrey [1966, p. 63])

$$\left( \frac{\partial v}{\partial z} \right) = \frac{\partial v}{\partial z}, \quad \left( \frac{\partial v}{\partial r} \right) = \frac{\partial v}{\partial r}. \quad (7.6)$$

Then by (7.5) we obtain  $\frac{\partial v}{\partial z}, \frac{\partial v}{\partial r} \in L^\infty(\Omega)$ , i.e. (b) is valid.

Q.E.D.

Proposition 7.2.  $K \subset K_1 \cap K_2$ .

Proof: Given  $v \in K$ , it is enough to prove that

$$\psi_1 < v < \psi_2 . \quad (7.7)$$

By Lemma 7.1 we have  $v \in C^{0,1}(\bar{\Omega})$ . Then (7.6) is valid. Since  $v \in H^1(\Omega)$ ,  $v$  is absolutely continuous in  $r$  on  $0 < r < R(z)$  for almost all value  $z$  (Morrey [1966, p. 66]). Therefore, noting that  $v = 0$  on  $\Gamma_0$ , we obtain

$$\begin{aligned} v(r,z) &< |v(r,z)| = \left| \int_0^r \left[ \frac{\partial v}{\partial r} \right] dr \right| = \left| \int_0^r \frac{\partial v}{\partial r} dr \right| \\ &< \int_0^r |\nabla v| dr < \int_0^r kr^2 dr = \psi_2 \text{ a.e. in } \Omega . \end{aligned}$$

The second part of (7.7) has been proved. Now prove the first part. The system of the characteristic equations for the Cauchy problem (5.17) has the same form as (6.1) - (6.5). Let the parameter value  $s = 0$  correspond to the point on  $\Gamma_1$ . Then it follows from (6.2) that the point in  $\Omega$  corresponds to the negative value of  $s$ . Noting that  $v$  has a total differential a.e. in  $\Omega$  and that (7.6) is valid, we have along the characteristic

$$\begin{aligned} v(z(0),r(0)) - v(z(s),r(s)) &= \int_s^0 \frac{dv}{ds} ds = \int_s^0 \left( \frac{\partial v}{\partial z} \frac{dz}{ds} + \frac{\partial v}{\partial r} \frac{dr}{ds} \right) ds \\ &= \int_s^0 \left( \frac{\partial v}{\partial z} 2p + \frac{\partial v}{\partial r} 2q \right) ds < 2 \int_s^0 |\nabla v| \cdot (p^2 + q^2)^{1/2} ds \\ &< 2 \int_s^0 k^2 r^4 ds = \int_s^0 \frac{d\psi_1}{ds} ds = \psi_1(z(0),r(0)) - \psi_1(z(s),r(s)) , \end{aligned}$$

But  $v(z(0),r(0)) = \psi_1(z(0),r(0)) = T/2\pi$ . Hence  $v > \psi_1$ .

Q.E.D.

Proposition 7.3. The sets  $K$ ,  $K_1$  and  $K_2$  are closed, convex subsets in  $H_p^1(\Omega)$ .

Proof: Given  $v, w \in K$  and  $\lambda$  with  $0 < \lambda < 1$ , we have

$$|\nabla[\lambda v + (1-\lambda)w]| = \left[ \left( \lambda \frac{\partial v}{\partial z} + (1-\lambda) \frac{\partial w}{\partial z} \right)^2 + \left( \lambda \frac{\partial v}{\partial r} + (1-\lambda) \frac{\partial w}{\partial r} \right)^2 \right]^{1/2}$$

$$\begin{aligned}
&< [(\lambda \frac{\partial v}{\partial z})^2 + (\lambda \frac{\partial v}{\partial r})^2]^{1/2} + [(1-\lambda)^2 (\frac{\partial w}{\partial z})^2 + (1-\lambda)^2 (\frac{\partial w}{\partial r})^2]^{1/2} \\
&= \lambda |\nabla v| + (1-\lambda) |\nabla w| < kr^2 .
\end{aligned}$$

By the linearity of the trace operator (Nečas [1967, p. 15]) we have

$$\lambda v + (1-\lambda)w = \lambda T/2\pi + (1-\lambda)T/2\pi = T/2\pi \text{ on } \Gamma_1 .$$

Hence  $\lambda v + (1-\lambda)w \in K$  and  $K$  is convex.

Let  $\{v_n\}$  be a Cauchy sequence in  $K$ . Since  $H^1_\rho(\Omega)$  is a Banach space (Cryer [1980]), there exists  $v \in H^1_\rho(\Omega)$  such that  $v_n \rightarrow v$  in  $H^1_\rho(\Omega)$ . It follows from the continuity of trace operator (Nečas [1967, p. 15]) that  $v = T/2\pi$  on  $\Gamma_1$ . By well-known subsequence argument we obtain that  $|\nabla v| < kr^2$  in  $\Omega$ . Then  $K$  is closed.

The conclusion about  $K_1$  and  $K_2$  can be proved by similar argument.

Q.E.D.

### 8. Solution of the problem (F)

At first we solve the problem  $(F_2)$  it suggests the solution of the problem (F). Let (cf. Cryer [1980, Remark 5.3])

$$\bar{R} = \min_{0 \leq z \leq L} R(z) \tag{8.1}$$

$$k_0 = (3T/2\pi)\bar{R}^{-3} . \tag{8.2}$$

We need a lemma it may easily be shown by a well-know theorem (Adams [1975, p. 54]).

Lemma 8.1. If  $u \in C^0(\bar{\Omega}) \cap H^1(\Omega)$ , then

$$\text{tr } u = u \text{ on } \partial\Omega$$

where  $\text{tr } u$  is the trace of  $u$  on  $\partial\Omega$ .

Proposition 8.2. If  $k < k_0$  then the problem  $(F_2)$  has no solution. If

$k > k_0$  then it has a unique solution.

Proof: There exists a  $\bar{z}$  such that

$$0 < \bar{z} < L, \quad R(\bar{z}) = \bar{R} .$$

If  $k < k_0$  then by (6.6) and (8.2) we have

$$\psi_2(\bar{z}, \bar{R}) = k\bar{R}^3/3 < k_0\bar{R}^3/3 = T/2\pi . \quad (8.3)$$

Thus, there exists a real number  $c$  and an open neighborhood  $\Sigma$  of the point  $(\bar{z}, \bar{R})$  such that

$$\psi_2 < c < T/2\pi \text{ in } \Sigma_1 \equiv \Sigma \cap \Omega .$$

Assume that  $K_2$  is nonempty. Take  $v \in K_2$ . Then

$$v < c < T/2\pi \text{ in } \Sigma_1 ,$$

and there exists a sequence  $\{v_n\} \subset C^\infty(\bar{\Omega})$  such that

$$\|v_n - v\|_{H^1(\Omega)} \rightarrow 0 \quad (n \rightarrow \infty) . \quad (8.4)$$

From the construction of  $v_n$  (Adams [1975, pp. 54-56]) we see that there exists  $\Sigma_2 \subset \Sigma_1$  and a real number  $c^* > c$  such that  $\text{meas } \Gamma^* > 0$ , where  $\Gamma^* = \Gamma_1 \cap \bar{\Sigma}_2$ , and that

$$v_n < c^* < T/2\pi \text{ in } \Sigma_2 . \quad (8.5)$$

Then by the continuity of trace operator and (8.4) we have

$$\|v_n - \text{tr } v\|_{L_2(\Gamma^*)} < \|v_n - \text{tr } v\|_{L_2(\Gamma_1)} < C_1 \|v_n - v\|_{H^1(\Omega)} \rightarrow 0 .$$

Hence there exists a subsequence  $\{v'_n\}$  which converge to  $\text{tr } v$  a.e. on  $\Gamma^*$ ,

and by (8.5) we obtain

$$\text{tr } v < c^* < T/2\pi \text{ on } \Gamma^* .$$

This contradicts that  $\text{tr } v = T/2\pi$  on  $\Gamma_1$ . Therefore,  $K_2$  is empty, and the problem  $(P_2)$  has no solution.

If  $k > k_0$ , then let

$$v = \min(\psi_2, T/2\pi) \text{ in } \bar{\Omega} . \quad (8.6)$$

Show that  $v \in K_2$ . Clearly,  $v < \psi_2$ , and

$$\begin{aligned} v &= kr^3/3 && \text{in } \Omega \cap \{r < d\} \\ v &= T/2\pi && \text{in } \Omega \cap \{r > d\} \end{aligned}$$

where  $d = (3T/2\pi k)^{1/3}$ . So we have (Gilbarg and Trudinger [1977, p. 145])

$$\begin{aligned} v \in H^1(\Omega), \quad \frac{\partial v}{\partial z} &= 0 \quad \text{in } \Omega \\ \frac{\partial v}{\partial r} &= kr^2 \quad \text{in } \Omega \cap \{r < d\} \\ \frac{\partial v}{\partial r} &= 0 \quad \text{in } \Omega \cap \{r > d\} . \end{aligned} \quad (8.7)$$

It is easy to see that

$$\|v\|_{H^1(\Omega)} < \infty .$$

Hence  $v \in H^1(\Omega)$ . Clearly,  $v \in C^0(\bar{\Omega}) \cap H^1(\Omega)$ , and  $v = T/2\pi$  on  $\Gamma_1$  in the usual sense. So  $v = T/2\pi$  on  $\Gamma_1$  in the sense of trace of lemma 8.1, and  $v \in K_2$ .

Thus,  $K_2$  is a closed, convex, nonempty set of  $H^1(\Omega)$ ; and  $a(v', v'')$  is a continuous, coercive, real bilinear form on  $H^1(\Omega) \times H^1(\Omega)$  (Cryer [1980, p. 549]); hence the problem  $(F_2)$  has unique solution (Stampacchia [1964]).

Q.E.D.

**Theorem 8.3.** If  $k < k_0$  then the problem  $(F)$  has no solution. If  $k > k_0$  then it has a unique solution.

**Proof:** If  $k < k_0$  then  $K_2$  is empty. By proposition 7.2  $K$  is also empty, and the problem  $(F)$  has no solution.

If  $k > k_0$ , then take  $v$  as in (8.6). We have known that  $v \in H^1(\Omega)$  and  $v = T/2\pi$  on  $\Gamma_1$ . But from (8.7) we see that  $|\nabla v| < kr^2$  in  $\Omega$ . Hence  $v \in K$ , and  $K$  is nonempty. By the similar argument to that in the proof of Proposition 8.2 we obtain that the problem  $(F)$  has a unique solution.

Q.E.D.

**Remark 8.1.** By virtue of theorem 8.3 and Proposition 7.2 we obtain that the theorem 6.2 in Cryer [1980] means that the problem  $(F)$  and  $(F_1)$  are equivalent under the conditions described there.

Remark 8.2. The conjecture that the problems (F) and (F<sub>2</sub>) are equivalent is not right. The numerical experiment we have made for the case  $R(z) \equiv 1$  indicates that  $v_0 \neq v_1$ . This fact may be shown by analytical method in this case.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2270	2. GOVT ACCESSION NO. AD-A110479	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On An Axisymmetric Free Boundary Problem		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Chu-Zi Zhou		8. CONTRACT OR GRANT NUMBER(s) MCS77-26732 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS  (see Item 18 below)		12. REPORT DATE August 1981
		13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)  UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office and National Science Foundation P. O. Box 12211 Washington, DC 20550 Research Triangle Park North Carolina 27709		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) torsion; elastic-plastic; axisymmetric; free boundary problem; variational inequalities; Haar-Kármán principle.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The axisymmetric elastic-plastic torsion of a shaft of general shape subject to the Hencky consistency condition with the von Mises yield function is considered. It is proved that the Haar-Kármán principle is valid in this case, and that the problem is essentially two-dimensional. The problem is reformulated as a variational inequality, and the existence and uniqueness of the solution is studied.		

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