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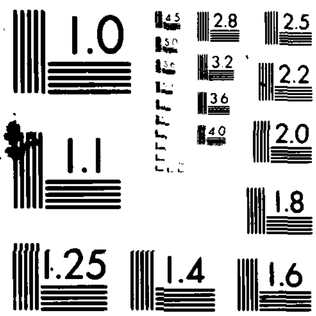

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SINGULAR PERTURBATION PROBLEMS WITH A SINGULARITY OF THE SECOND KIND

Peter A. Markowich† and Ch. A. Ringhofer*

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ABSTRACT

This paper deals with systems of singularly perturbed ordinary differential equations posed as boundary value problems on an infinite interval. The system is assumed to consist of singularly perturbed (fast) components and unperturbed (slow) components and to have a singularity of the second kind at ∞. Under the assumption that there is no turning point we derive uniform asymptotic expansions (as the perturbation parameter tends to zero) for the fast and slow components uniformly on the whole infinite line. The second goal of the paper is to derive convergence estimates for the solutions of 'finite' singular perturbation problems obtained by cutting the infinite interval at a finite (far out) point and by substituting appropriate additional boundary conditions at the far end. Using a suitable choice for these boundary conditions the order of convergence is shown to depend only on the decay property of the infinite solution.

AMS(MOS) Subject Classification 34B15, 34C05, 34C11, 34D15, 34E05, 34A45

Key words Nonlinear boundary value problems, singular points, boundedness of solutions, singular perturbations, asymptotic expansion, theoretical approximation of solutions.

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SIGNIFICANCE AND EXPLANATION

This paper deals with singularly perturbed differential equations posed as boundary value problems on infinite intervals of the general form

$$\left.\begin{array}{l} \epsilon y' = t^{\omega} f(y,z,t,\epsilon) \\ z' = t^{\alpha} g(y,z,t,\epsilon) \end{array}\right\} \quad \begin{array}{l} 1 \le t < \infty \\ \alpha > -1 \end{array}$$

$$R(\epsilon)\begin{pmatrix} y(1,\epsilon) \\ z(1,\epsilon) \end{pmatrix} = b(\epsilon)$$

$$y,z \in C([1,\infty]).$$

The last condition means that we only admit solutions which tend to a finite limit as $t \to \infty$. It is assumed that $0 < \epsilon \ll 1$ holds; y is called the fast component, z is called the slow component. Problems of this kind frequently occur in fluid and thermodynamics; for example, the problem

$$\left.\begin{array}{l} \epsilon y' = -zy + \epsilon \frac{n-1}{t} z \\ z' = y \end{array}\right\} \quad 1 \le t < \infty$$

$$z(1,\epsilon) = \gamma > 0, \qquad z(\infty,\epsilon) = 1$$

is a mathematical model for flow past an obstacle; ϵ represents the viscosity of the fluid and z is the velocity of the fluid. ϵ small corresponds to a high Reynolds number flow. We are interested in the form of solution $y(t,\epsilon)$, $z(t,\epsilon)$ as $\epsilon \to 0$ for $t \in [1,\infty]$.

The second problem addressed in this paper is the numerical solution of such problems. We cut the infinite interval $[1,\infty]$ at $t = T \gg 1$ obtaining a finite singular perturbation problem approximating the infinite one. For the above example this is

$$\left.\begin{array}{l} \epsilon \tilde{y}' = -\tilde{z}\tilde{y} + \epsilon \frac{n-1}{t} \tilde{z} \\ \tilde{z}' = \tilde{y} \end{array}\right\} \quad 1 \le t \le T$$

$$\tilde{z}(1,\epsilon) = \gamma, \qquad \tilde{z}(T,\epsilon) = 1.$$

We are interested in the question how well do $\tilde{y},\tilde{z}$ approximate y,z as $T \to \infty$. In order to derive convergence estimates we use the derived 'asymptotic' properties of $y(t,\epsilon)$, $z(t,\epsilon)$ as $t \to \infty$, $\epsilon \to 0$.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

SINGULAR PERTURBATION PROBLEMS
WITH A SINGULARITY OF THE SECOND KIND

PETER A. MARKOWICH[†] AND CH. A. RINGHOFER[*]

1. INTRODUCTION

In this paper we deal with the singular perturbation problem

$$\left.\begin{aligned} &(1.1) \qquad \varepsilon y' = t^{\alpha} h(y,z,t,\varepsilon) \\ &(1.2) \qquad z' = t^{\alpha} g(y,z,t,\varepsilon) \end{aligned}\right\} \quad \alpha > -1, \quad t \in [1,\infty)$$

$$(1.3) \qquad \begin{aligned} &\binom{y}{z} \in C([1,\infty]) :\iff \binom{y}{z} \in C([1,\infty)) \quad \text{and} \\ &\lim_{t\to\infty} \binom{y(t,\varepsilon)}{z(t,\varepsilon)} = \binom{y(\infty,\varepsilon)}{z(\infty,\varepsilon)} \quad \text{is finite} \end{aligned}$$

$$(1.4) \qquad F(\varepsilon)\binom{y(1,\varepsilon)}{z(1,\varepsilon)} = \beta(\varepsilon)$$

where $0 < \varepsilon \ll 1$; y,h are n-vectors z,g are m-vectors. y is called fast component and z is called slow component. $F(\varepsilon)$ is a $k \times (n+m)$ matrix, $\beta(\varepsilon) \in \mathbb{R}^k$ where $k < n+m$ holds if the matrix

$$(1.5) \qquad \frac{\partial \begin{pmatrix} \frac{1}{\varepsilon} h \\ g \end{pmatrix}}{\partial \binom{y}{z}} (y(\infty,\varepsilon), z(\infty,\varepsilon), \infty, \varepsilon)$$

has at least one eigenvalue with positive real part. For $\alpha > 1$ the system (1.1), (1.2) has a singularity of the second kind of $t = \infty$.

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Problems of this kind frequently occur in fluid mechanics, especially in boundary layer theory (see for example Schlichting (1959) for the Orr-Sommerfeld problem and Lagerstrom (1961) for a model of flow past an obstacle) whenever flows of high Reynolds number ($R = \frac{1}{\varepsilon}$) over infinite media are investigated. Other applications occur in thermodynamics (see Lagerstrom and Casten (1972)).

Our assumptions on the problems (1.1), (1.2) are the following. We assume that $h, g \in C^2(\mathbb{R}^{n+m} \times [1,\infty] \times [0,\varepsilon_0])$ where ε_0 is sufficiently small and that the system is quasilinear in the sense of [illegible] ([illegible]), which means that

$$h(y,z,t,\varepsilon) = A(z,t)y + f(z,y,t,\varepsilon) \tag{1.6}$$

and

$$\frac{\partial f}{\partial y} = O(\varepsilon) \tag{1.7}$$

for $t \in [1,\infty]$, $\varepsilon \in [0,\varepsilon_0]$ and y,z in compact subsets of $\mathbb{R}^{n+m}$. Here $A(z,t)$ is an $n\times n$-matrix which is in block diagonal form

$$A(z,t) = \begin{array}{cc} \begin{bmatrix} A_+(z,t) & 0 \\ 0 & A_-(z,t) \end{bmatrix} & \begin{array}{l} \}n_+ \\ \}n_- \end{array} \\ \begin{array}{cc} \quad n_+ \qquad & \qquad n_- \end{array} & \end{array} \tag{1.8}$$

such that the real parts of the eigenvalues of $A_+(z,t)$ are strictly positive and the real parts of the eigenvalues of $A_-(z,t)$ are strictly negative for $t \in [1,\infty]$ and z in a neighborhood of the solution of (1.1), (1.2), (1.3), (1.4). So we exclude turning-point problems (eigenvalues change sign).

The first goal of our analysis is to study the asymptotic behavior of the solutions of (1.1), (1.2), (1.3) as $\varepsilon \to 0$ globally on $[1,\infty)$ and to find conditions on $F(\varepsilon)$ which guarantee the locally unique solvability of the singular boundary value problems (1.1), (1.2), (1.3), (1.4).

For this we use techniques already developed for 'finite' singular perturbation problems as for example matched asymptotic expansions (see O'Malley (1978), (1979), Ringhofer (1980), (1981)) and the theory of singular boundary value problems (see de Hoog and Weiss (1980a,b), Markowich (1980a,b,c) and Lentini and Keller (1980)).

We show that the solutions y,z of (1.1), (1.2), (1.3), (1.4) fulfill

(1.9) $$z(t,\varepsilon) = \bar{z}(t) + O(\varepsilon), \qquad t \in [1,\infty]$$

(1.10) $$y(t,\varepsilon) = \sigma\left(\frac{t-1}{\varepsilon}\right) + \bar{y}(t) + O(\varepsilon), \qquad t \in [1,\infty]$$

where $\bar{y},\bar{z}$ are solutions of (1.1), (1.2), (1.3) with $\varepsilon = 0$ (reduced problem) fulfilling appropriate boundary conditions. Here $\sigma(\tau)$ decays exponentially to zero as $\tau \to \infty$ (boundary layer term) and $\bar{z},\bar{y}$ decay to a finite limit $\bar{z}_\infty,\bar{y}_\infty$ as $t \to \infty$ fulfilling

(1.11) $$0 = h(\bar{y}_\infty,\bar{z}_\infty,\infty,0)$$

(1.12) $$0 = g(\bar{y}_\infty,\bar{z}_\infty,\infty,0).$$

This result generalizes the results by O'Malley (1979) and Ringhofer (1980), (1981) obtained for finite interval singular perturbation problems.

Singularly perturbed initial value problems on the infinite line have been investigated by Hoppensteadt (1966) by imposing severe stability assumption on the reduced problem.

The second goal is to study approximating 'finite' singular perturbation problems, which are set up by cutting the infinite interval $[1,\infty]$ at a finite point $T \gg 1$ and by substituting (for the continuity condition (1.3) at $t = \infty$) additional, so called asymptotic boundary conditions obtaining a 'finite' singular perturbation problem

$$\left.\begin{aligned} &(1.13) \qquad \varepsilon y_T' = t^\alpha h(y_T,z_T,t,\varepsilon) \\ &(1.14) \qquad z_T' = t^\alpha g(y_T,z_Tt,\varepsilon) \end{aligned}\right\} \quad 1 \le t \le T$$

$$(1.15) \qquad F(\varepsilon)\binom{y_T(1,\varepsilon)}{z_T(1,\varepsilon)} = \beta(\varepsilon)$$

$$(1.16) \qquad S(T,\varepsilon)\binom{y_T(T,\varepsilon)}{z_T(T,\varepsilon)} = \gamma(T,\varepsilon)$$

where $S(T,\varepsilon)$ is an $(n+m-k) \times (n+m)$ matrix, $\gamma(T,\varepsilon) \in \mathbb{R}^{n+m-k}$. The condition (1.16) shall reflect the asymptotic behaviour of the 'infinite' solution (y,z) as $t \to \infty$.

'Finite' approximating two point boundary value problems (for unperturbed infinite problems) have been studied extensively by de Hoog and Weiss (1980a), Markowich (1980b) and Lentini and Keller (1980).

We show that under rather mild assumptions on the 'infinite' problem (a certain 'wellposedness' is required) there is a choice of $S(T,\varepsilon) \equiv S$ and $\gamma(T,\varepsilon) \equiv \gamma$ only depending on the reduced $(\varepsilon = 0)$ infinite problem such that the 'finite' (perturbed) problem has a unique solution y_T, z_T for T sufficiently large and ε sufficiently small (but T and ε independent) which fulfills the convergence estimate

$$(1.17) \qquad \left\|\binom{y-y_T}{z-z_T}\right\|_{[1,T]} \le \mathrm{const}\left(\exp\left(-\frac{c}{\alpha+1}T^{\alpha+1}\right) + \varepsilon\right), \quad c > 0$$

($\|\cdot\|_{[a,b]}$ denotes the sup-norm on $[a,b]$) where the constants are independent of T and ε.

The 'finite' singular perturbation problem (1.13), (1.14), (1.15), (1.16) can then be solved by polynomial collocation methods (see Kreiss and Nichols (1975), Ringhofer (1981), Ascher and Weiss (1981)). An exponential mesh size strategy for 'long interval' problems has been developed for the Box-scheme by Markowich and Ringhofer (1981). This can be used on $[\omega,T]$, $\omega > 1$ while within the boundary layer (on $[1,1+O(\varepsilon|\ell n\varepsilon|)]$) a very fine grid (see Ascher and

Weiss (1981)) has to be used. Since the solution of (1.13), (1.14), (1.15), (1.16) is smooth (has uniformly (in ε) bounded derivatives) on $[1+O(\varepsilon|\ell n\varepsilon|),\omega]$ standard techniques can be used there.

The paper is organized as follows. In chapter two linear constant coefficient problems are treated, in chapter three variable coefficients are admitted and chapter four is concerned with nonlinear problems.

2. Constant-Coefficient Problems.

At first we study the problem

$$\left.\begin{array}{ll}(2.1) & \varepsilon y' = t^{\alpha}A(\varepsilon)y + t^{\alpha}B(\varepsilon)z + t^{\alpha}f(t,\varepsilon) \\ (2.2) & z' = t^{\alpha}C(\varepsilon)y + t^{\alpha}D(\varepsilon)z + t^{\alpha}g(t,\varepsilon)\end{array}\right\} \quad \begin{array}{l} 1 \le t < \infty \\ \alpha > -1 \end{array}$$

$$(2.3) \qquad F(\varepsilon)\binom{y(1,\varepsilon)}{z(1,\varepsilon)} = \beta(\varepsilon)$$

$$(2.4) \qquad \binom{y}{z} \in C([1,\infty]).$$

Here y,f are n-vectors, z and g are m-vectors, $A(\varepsilon)$ is an $n\times n$-matrix, $B(\varepsilon)$ an $n\times m$-matrix, $C(\varepsilon)$ an $m\times n$-matrix and $D(\varepsilon)$ is an $m\times m$-matrix. The dimension of the matrix $F(\varepsilon)$ and the vector $\beta(\varepsilon)$ will be discussed in the sequel (obviously $F(\varepsilon)$ has $n+m$ columns). We assume that

$$(2..5) \qquad f,g \in C([1,\infty] \times [0,\varepsilon_0]), \qquad \varepsilon_0 > 0$$

$$(2.6) \qquad A,B,C,D,F,\beta \in C([0,\varepsilon_0]).$$

and that A,B,C,D,f,g,F,β are (uniformly) Lipschitz continuous at $\varepsilon = 0$. It is convenient to decouple the system (2.1), (2.2) by a linear transformation such that the unperturbed equation of the transformed system does not contain the fast component. We use the transformation given by Ascher and Weiss (1981):

$$(2.7) \qquad \binom{y}{z} = \begin{bmatrix} I & 0 \\ \varepsilon L(\varepsilon) & I \end{bmatrix}\binom{u}{v}.$$

Assuming that $A^{-1}(\varepsilon)$ exists for $\varepsilon \in [0,\varepsilon_0]$ and that $\|A^{-1}(\varepsilon)\| \le$ const. for $\varepsilon \in [0,\varepsilon_0]$ we determine $L(\varepsilon)$ from the equation

$$(2.8) \qquad L(\varepsilon) = C(\varepsilon)A^{-1}(\varepsilon) - \varepsilon(D(\varepsilon)L(\varepsilon)A^{-1}(\varepsilon) + L(\varepsilon)B(\varepsilon)L(\varepsilon)A^{-1}(\varepsilon)).$$

A simple contraction argument assures the unique solvability of (2.8) such that

$$(2.9) \qquad L(\varepsilon) = C(\varepsilon)A^{-1}(\varepsilon) + O(\varepsilon)$$

holds. The new system (with u,v as dependent variables) has the form

$$(2.10) \qquad \varepsilon u' = t^{\alpha}\bar{A}(\varepsilon)u + t^{\alpha}B(\varepsilon)v + t^{\alpha}f(t,\varepsilon)$$

$$(2.11) \qquad v' = t^{\alpha}\bar{D}(\varepsilon)v + t^{\alpha}\bar{g}(t,\varepsilon)$$

$$(2.12) \qquad \binom{u}{v} \in C([1,\infty])$$

where $\bar{A} = A + \varepsilon BL$, $\bar{D} = D - LB$, $\bar{g} = g - Lf$ holds.

Now we make assumptions on the eigenvalues of $A(0)$, $\bar{D}(0) = D(0) - C(0)A^{-1}(0)B(0)$:

(2.13) $A(0)$ has r_+ eigenvalues with positive real part and r_- eigenvalues with negative real part (counting algebraic multiplicities) and $r_+ + r_- = n$.

(2.14) $\bar{D}(0)$ has $\tilde{r}_+$ eigenvalues with positive real part and $\tilde{r}_-$ eigenvalues with negative real part and $\tilde{r}_+ + \tilde{r}_- = m$.

We investigate the perturbed problem

$$(2.15) \qquad \varepsilon u_1' = t^{\alpha}A(0)u_1 + t^{\alpha}B(\varepsilon)v_1 + t^{\alpha}f_1(t,\varepsilon)$$

$$(2.16) \qquad v_1' = t^{\alpha}\bar{D}(0)v_1 + t^{\alpha}g_1(t,\varepsilon)$$

$$(2.17) \qquad \binom{u_1}{v_1} \in C([1,\infty]).$$

The assumptions (2.13), (2.14) guarantee that there are transformation matrices E_1, E_2 such that the matrices J_1, J_2 defined by

$$(2.18) \qquad \text{(a)}\ A(0) = E_1J_1E_1^{-1} \qquad \text{(b)}\ \bar{D}(0) = E_2J_2E_2^{-1}$$

are in block diagonal form:

$$(2.19)\qquad (a)\quad J_1 = \begin{bmatrix} J_1^+ & 0 \\ 0 & J_1^- \end{bmatrix}, \quad\text{with } \underbrace{}_{r_+}\ \underbrace{}_{r_-}, \qquad (b)\quad J_2 = \begin{bmatrix} J_2^+ & 0 \\ 0 & J_2^- \end{bmatrix}, \quad \underbrace{}_{\tilde r_+}\ \underbrace{}_{\tilde r_-}$$

where J_1^+, J_2^+ (J_1^-, J_2^-) only have eigenvalues with positive (negative) real parts. We substitute

$$(2.20)\qquad \binom{u_1}{v_1} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \binom{w}{x}$$

and get

$$(2.21)\qquad \varepsilon w' = t^\alpha J_1 w + t^\alpha \tilde B(\varepsilon)x + t^\alpha \tilde f_1(t,\varepsilon)$$

$$(2.22)\qquad x' = t^\alpha J_2 x + t^\alpha \tilde g_1(t,\varepsilon)$$

$$(2.23)\qquad \binom{w}{x} \in C([1,\infty])$$

where $\tilde B(\varepsilon) = E_1^{-1}B(\varepsilon)E_2$, $\tilde f_1 = E_1^{-1}f_1$, $\tilde g_1 = E_2^{-1}g_1$ holds. We solve (2.22), (2.23) defining

$$(2.24)\qquad \phi(t,\delta) = \exp\left(\frac{t^{\alpha+1}-\delta^{\alpha+1}}{\alpha+1} J_2\right)$$

such that for the slow component

$$(2.25)\qquad x(t,\varepsilon) = \phi(t,1)\begin{bmatrix} 0 \\ I_{\tilde r_-} \end{bmatrix}\xi + (H_1\tilde g_1(\cdot,\varepsilon))(t)$$

holds where $\xi \in C^{\tilde r_-}$ and the solution operator H_δ for $\delta \geq 1$ is defined by

$$(2.26)\qquad (H_\delta g(\cdot,\varepsilon))(t) = \int_\infty^t \phi(t,\delta)\tilde D_+\phi^{-1}(s,\delta)s^\alpha g(s,\varepsilon)ds + \int_\delta^t \phi(t,\delta)\tilde D_-\phi^{-1}(s,\delta)s^\alpha g(s,\varepsilon)ds.$$

Here $\tilde{D}_+, \tilde{D}_-$ are diagonal projections

$$(2.27) \qquad (a)\ \tilde{D}_+ = \begin{bmatrix} I_{\tilde{r}_+} & \\ & 0 \end{bmatrix}, \qquad (b)\ \tilde{D}_- = \begin{bmatrix} 0 & \\ & I_{\tilde{r}_-} \end{bmatrix}.$$

H_δ was used by de Hoog and Weiss (1980a,b) and they showed

$$(2.28)(a) \qquad H_\delta : C([\delta,\infty]) \to C([\delta,\infty]), \qquad \delta \geq 1$$

$$(2.28)(b) \qquad \|H_\delta\|_{[\delta,\infty]} \leq \text{const. independently of } \delta$$

$$(2.28)(c) \qquad (H_\delta f(\cdot,\varepsilon))(\infty) = -J_2^{-1} f(\infty,\varepsilon).$$

Estimates of the asymptotic behaviour of $(H_\delta f)(t)$ as $t \to \infty$ are given in Markowich (1980a).

Inserting (2.25) into (2.21) we can regard $h(t,\varepsilon) = \tilde{B}(\varepsilon)x(t,\varepsilon) + \tilde{f}_1(t,\varepsilon)$ as inhomogenity and are left with solving for the fast component:

$$(2.29) \qquad \varepsilon w' = t^\alpha J_1 w + t^\alpha h(t,\varepsilon), \qquad w \in C([1,\infty]).$$

We define

$$(2.30) \qquad \psi(t,\delta,\varepsilon) = \exp\left(\frac{t^{\alpha+1} - \delta^{\alpha+1}}{\varepsilon(\alpha+1)} J_1\right)$$

$$(2.31) \qquad \begin{aligned} (G_{\varepsilon,\delta} h(\cdot,\varepsilon))(t) = &\int_\infty^t \psi(t,\delta,\varepsilon) D_+ \psi^{-1}(s,\delta,\varepsilon) s^\alpha h(s,\varepsilon)\,ds + \\ &+ \int_\delta^t \psi(t,\delta,\varepsilon) D_- \psi^{-1}(s,\delta,\varepsilon) s^\alpha h(s,\varepsilon)\,ds \end{aligned}$$

where

$$(2.32) \qquad (a)\ D_+ = \begin{bmatrix} I_{r_+} & \\ & 0 \end{bmatrix}, \qquad (b)\ D_- = \begin{bmatrix} 0 & \\ & I_{r_-} \end{bmatrix}$$

holds. Then the solution of (2.29) is

$$(2.33) \qquad w(t,\varepsilon) = \psi(t,1,\varepsilon) \begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix} \zeta + (G_{\varepsilon,1} h(\cdot,\varepsilon))(t)$$

for $\zeta \in C^{r_-}$.

It is easy to show that $G_{\varepsilon,\delta}$ has the following properties:

(2.34)(a) $\quad G_{\varepsilon,\delta} : C([\delta,\infty]) \to C([\delta,\infty]), \quad \delta \geq 1$

(2.34)(b) $\quad \|G_{\varepsilon,\delta}\|_{[\delta,\infty]} \leq$ const. independently of δ,ε

(2.34)(c) $\quad (G_{\varepsilon,\delta}h(\cdot,\varepsilon))(\infty) = -J_1^{-1}h(\infty,\varepsilon).$

If $h(\cdot,\varepsilon), \dfrac{h'(t,\varepsilon)}{t^{\alpha}} \in C([1,\infty])$ for all ε sufficiently small then

$$\text{(2.34)(d)} \qquad (G_{\varepsilon,\delta}h(\cdot,\varepsilon))(t) = -J_1^{-1}h(t,\varepsilon) + \psi(t,\delta,\varepsilon)D_-J_1^{-1}h(\delta,\varepsilon) + \varepsilon(\Delta_{\varepsilon,\delta}h(\cdot,\varepsilon))(t)$$

where

$$\text{(2.35)} \qquad \|\Delta_{\varepsilon,\delta}h(\cdot,\varepsilon)\|_{[\delta,\infty]} \leq \text{const}\left(\|h(\cdot,\varepsilon)\|_{[\delta,\infty]} + \max_{s\in[\delta,\infty]} \left\|\frac{h'(s,\varepsilon)}{s^{\alpha}}\right\|\right)$$

holds. In the sequel we use the space

$$\text{(2.36)} \qquad C_{\alpha}^{1}([\delta,\infty]) = C([\delta,\infty]) \cap C^{1}([\delta,\infty)) \cap \{f \mid \max_{s\in[\delta,\infty]} \left\|\frac{f'(s)}{s^{\alpha}}\right\| < \infty\}.$$

and as norm we take $\|f\|^{\alpha}_{[\delta,\infty]} := \|f\|_{[\delta,\infty]} + \max\limits_{s\in[\delta,\infty]} \|f'(s)/s^{\alpha}\|.$

We rewrite (2.11) obtaining

$$\text{(2.37)} \qquad v' = t^{\alpha}\bar{D}(0)v + t^{\alpha}(\bar{D}(\varepsilon) - \bar{D}(0))v + t^{\alpha}\bar{g}(t,\varepsilon), \quad t \geq 1$$

and get by integration (using (2.20), (2.25))

$$\text{(2.38)} \qquad v(t,\varepsilon) = E_2\phi(t,1)\begin{bmatrix} 0 \\ I_{\tilde{r}_1} \end{bmatrix}\xi + E_2(H_1E_2^{-1}(\bar{D}(\varepsilon) - \bar{D}(0))v(\cdot,\varepsilon))(t) + E_2(H_1E_2^{-1}\bar{g}(\cdot,\varepsilon))(t).$$

where $H_1 : C([1,\infty]) \to C_{\alpha}^{1}([1,\infty])$ is bounded.

We conclude from (2.28)(b) that $(I - E_2H_1E_2^{-1}(\bar{D}(\varepsilon) - \bar{D}(0))^{-1} = I + O(\varepsilon)$ as an operator on $C([1,\infty])$ such that

$$v(t,\varepsilon) = E_2\phi(t,1)\begin{bmatrix} 0 \\ I_{\tilde{r}_-} \end{bmatrix}\xi + O(\varepsilon)\xi + E_2(H_1E_2^{-1}\bar{q}(\cdot,\varepsilon))(t) + \tag{2.39}$$

$$+ O(\varepsilon\|\bar{g}(\cdot,\varepsilon)\|_{[1,\infty]})$$

holds uniformly on $[1,\infty]$. (The asymptotics for $v'(t,\varepsilon)/t^\alpha$ follow immediately). Rewriting (2.10) gives

$$\varepsilon u' = t^\alpha A(0)u + t^\alpha(\bar{A}(\varepsilon) - A(0))u + t^\alpha B(\varepsilon)v + t^\alpha f(t,\varepsilon), \qquad t > 1 \tag{2.40}$$

and we obtain (using (2.20), (2.33))

$$u(t,\varepsilon) = E_1\psi(t,1,\varepsilon)\begin{bmatrix} 0 \\ I_r \end{bmatrix}\zeta + E_1(G_{\varepsilon,1}E_1^{-1}(\bar{A}(\varepsilon) - A(0))u(\cdot,\varepsilon)(t) + \tag{2.41}$$

$$+ E_1(G_{\varepsilon,1}E_1^{-1}(B(\varepsilon)v(\cdot,\varepsilon) + f(\cdot,\varepsilon)))(t).$$

From (2.34)(b) we conclude that $(I - E_1G_{\varepsilon,1}E_1^{-1}(\bar{A}(\varepsilon) - A(0)))^{-1} = I + O(\varepsilon)$ as operator on $C([1,\infty])$ such that

$$u(t,\varepsilon) = E_1\psi(t,1,\varepsilon)\begin{bmatrix} 0 \\ I_r \end{bmatrix}\zeta + O(\varepsilon)\zeta + \tag{2.42}$$

$$+ E_1(G_{\varepsilon,1}E_1^{-1}(B(\varepsilon)v(\cdot,\varepsilon) + f(\cdot,\varepsilon)))(t) +$$

$$+ O(\varepsilon(\|v(\cdot,\varepsilon)\|_{[1,\infty]} + \|f(\cdot,\varepsilon)\|_{[1,\infty]}).$$

Resubstitution in (2.7) gives y,z. Since y,z depend on $r_- + \tilde{r}_-$ parameters (ζ and ξ) we assume that the matrix $F(\varepsilon)$ as in (2.3) has $r_- + \tilde{r}_-$ rows. By collecting the terms of y,z which depend on ξ,ζ we get

Theorem 2.1. Let $f,g \in C_\alpha^1([1,\infty])$ uniformly for small $\varepsilon > 0$ and assume that the $(r_- + \tilde{r}_-) \times (r_- + \tilde{r}_-)$-matrix

$$(2.43)\qquad F(0)\begin{bmatrix} E_1\begin{bmatrix}0\\ I_{r_-}\end{bmatrix} & E_1D_+E_1^{-1}A^{-1}(0)E_2\begin{bmatrix}0\\ I_{\tilde{r}_-}\end{bmatrix} \\ 0 & E_2\begin{bmatrix}0\\ I_{\tilde{r}_-}\end{bmatrix}\end{bmatrix}$$

is nonsingular. Then, under the given assumption on A, B, C, D, F, β the boundary value problem (2.1), (2.2), (2.3), (2.4) has for all ε sufficiently small and for all $\beta(\varepsilon) \in R^{r_-+\tilde{r}_-}$ a unique solution $\binom{y}{z}$ which depends uniformly continuous (in ε) on $\beta(\varepsilon)$ and on $f,g \in C([1,\infty])$ when regarded as dwelling in $C([1,\infty])$. Moreover $\binom{y}{z} \in C^1_\alpha([\delta,\infty])$ for $\delta > 1$ depends uniformly continuous on $f,g \in C^1([1,\infty])$ and

$$(2.44)\qquad \begin{aligned} z(t,\varepsilon) = E_2\phi(t,1)\begin{bmatrix}0\\ I_{\tilde{r}}\end{bmatrix}\xi + E_2(H_1E_2^{-1}(g(\cdot,0) \\ - C(0)A^{-1}(0)f(\cdot,0)))(t) + O(\varepsilon)\end{aligned}$$

$$(2.45)\qquad \begin{aligned} y(t,\varepsilon) = E_1\psi(t,1,\varepsilon)\left(\begin{bmatrix}0\\ I_{r_-}\end{bmatrix}\zeta + D_-J_1^{-1}(E_1^{-1}B(\varepsilon)z(1,\varepsilon) + \right. \\ \left. + E_1^{-1}f(1,\varepsilon))\right) - A^{-1}(0)(B(0)z(t,0) + f(t,0)) + O(\varepsilon)\end{aligned}$$

holds uniformly on $[1,\infty]$.

From (2.28)(c), (2.34)(c) we derive

$$(2.46)\quad z(\infty,\varepsilon) = -(D(0) - C(0)A^{-1}(0)B(0))^{-1}(g(\infty,0) - C(0)A^{-1}(0)f(\infty,0)) + O(\varepsilon)$$

$$(2.47)\qquad y(\infty,\varepsilon) = -A^{-1}(0)(B(0)z(\infty,0) + f(\infty,0)) + O(\varepsilon).$$

The first term in (2.45) is the exponentially decreasing boundary layer contribution (the thickness of the boundary layer is $O(\varepsilon|\ln\varepsilon|)$ and the second term is the solution of the reduced problem (2.1) (with $\varepsilon = 0$).

If we drop the restriction that $\bar{D}(0) = D(0) - C(0)A^{-1}(0)B(0)$ has no eigenvalue on the imaginary axis (see (2.14)) we have to assume that $f(t,\varepsilon), g(t,\varepsilon)$ converge to zero algebraically as $t \to \infty$. A sufficient order of decay is $t^{-(\alpha+1)r-\gamma}$, where r is the dimension of the largest Jordan block of $\bar{D}(0)$ which has an eigenvalue on the imaginary axis and $\gamma > 0$ (see Markowich (1980a)). For the contraction arguments algebraically weighted $C([1,\infty])$ resp. $C^1_\alpha([1,\infty])$ spaces have to be used.

For the numerical solution of (2.1), (2.2), (2.3), (2.4) we cut the infinite interval at a finite point $T >> 1$ and replace the continuity requirement (2.4) by $r_+ + \tilde{r}_+$ boundary condition at $t = T$. These boundary conditions shall reflect the asymptotic behaviour of y,z as $t \to \infty$. So we get the 'finite' singular perturbation problem

$$\left.\begin{aligned} &(2.48) \qquad \varepsilon y_T' = t^\alpha A(\varepsilon)y_T + t^\alpha B(\varepsilon)z_T + t^\alpha f(t,\varepsilon) \\ &(2.49) \qquad z_T' = t^\alpha C(\varepsilon)y_T + t^\alpha D(\varepsilon)z_T + t^\alpha g(t,\varepsilon) \end{aligned}\right\} \quad 1 < t < T$$

$$(2.50) \qquad F(\varepsilon)\binom{y_T(1,\varepsilon)}{z_T(1,\varepsilon)} = \beta(\varepsilon)$$

$$(2.51) \qquad S(T,\varepsilon)\binom{y_T(T,\varepsilon)}{z_T(T,\varepsilon)} = \gamma(T,\varepsilon).$$

Here $S(T,\varepsilon)$ is an $(r_+ + \tilde{r}_+) \times (n+m)$-matrix and $\gamma(T,\varepsilon) \in \mathbb{R}^{r_+ + \tilde{r}_+}$.

A possible choice is

$$(2.52) \qquad S \equiv S(T,\varepsilon) = \begin{bmatrix} [I_{r_+},0]E_1^{-1} & [I_{r_+},0]E_1^{-1}A^{-1}(0)B(0) \\ 0 & \left[I_{\tilde{r}_+},0\right]E_2^{-1} \end{bmatrix}$$

$$(2.53) \qquad \gamma(T) \equiv \gamma(T,\varepsilon) = \begin{bmatrix} [I_{r_+},0]E_1^{-1}A^{-1}(0)f(T,0) \\ -\left[I_{\tilde{r}_+},0\right]E_2^{-1}z(\infty,0) \end{bmatrix}.$$

This 'asymptotic' boundary condition has been used by de Hoog and Weiss (1980a), Markowich (1980b) and Lentini and Keller (1980) for unperturbed problems on infinite intervals. Let $\mathcal{A}(t,\varepsilon)$ denote the fundamental matrix of (2.1), (2.2) ($\mathcal{A}(1,\varepsilon)= I$). Then by proceeding as de Hoog and Weiss(1980a) did we can easily show that $S\mathcal{A}(T,\varepsilon)$ does not contain exponentially decreasing terms. Therefore the countary condition $S\binom{y(T,\varepsilon)}{z(T,\varepsilon)} = 0$ sets the exponentially increasing solution components of the homogenous problem (2.1), (2.2) to zero. $\gamma(T)$ as in (2.53) is the necessary (boundary) correction term for the inhomogenous problem.

By proceeding as in de Hoog and Weiss (1980a) we find the stability estimate for the solution of (2.48), (2.49), (2.50), (2.51) when using S as in (2.52) and assuming that the matrix (2.43) is nonsingular

$$(2.54) \qquad \left\|\binom{y_T}{z_T}\right\|_{[1,T]} \le \text{const}(\|\beta(\varepsilon)\| + \|\gamma(T,\varepsilon)\| + \|f(\cdot,\varepsilon)\|_{[1,T]} + \|g(\cdot,\varepsilon)\|_{[1,T]})$$

for all $\beta \in R^{r_- + \tilde{r}_-}$, $\gamma \in R^{r_+ + \tilde{r}_+}$, $f,g \in C([1,T])$ where the constant is independent of T and ε. By subtracting (2.48), (2.49), (2.50), (2.51) from (2.1), (2.2), (2.3) we get the error estimate

$$(2.55) \qquad \left\|\binom{y-y_T}{z-z_T}\right\|_{[1,T]} \le \text{const}\left\|S\binom{y(T,\varepsilon)}{z(T,\varepsilon)} - \gamma(T)\right\|.$$

Inserting (2.44), (2.45) into the right hand side of (2.55) (when using (2.52), (2.53)) gives the uniform estimate

$$(2.56) \qquad \left\|\binom{y-y_T}{z-z_T}\right\|_{[1,T]} \le \text{const}(\|E_2(H_1E_2^{-1}(g(\cdot,0) - C(0)A^{1}(0)f(\cdot,0))(t) - z(\infty,0)\| + O(\varepsilon)).$$

Convergence follows because (2.28)(c).

Since the solutions of the reduced problem $(\varepsilon = 0)$ (2.48), (2.49) do not generally fulfill (2.51) one has to expect a boundary layer at $t = T$ whose height can be estimated by the right hand side of (2.56).

Estimates of the order of convergence of the first term of the right hand side of (2.56) depending on the decay of f,g as $t \to \infty$ are given in Markowich (1980a), (1980b).

Therefore under the given assumptions the asymptotic boundary condition (2.51) can be consructed with respect to the reduced $(\varepsilon = 0)$ infinite problem.

3. Variable Coefficient Problems.

We consider the problem

$$\left.\begin{aligned} &(3.1) \qquad \varepsilon y' = t^{\alpha}A(t,\varepsilon)y + t^{\alpha}B(t,\varepsilon)z + t^{\alpha}f(t,\varepsilon) \\ &(3.2) \qquad z' = t^{\alpha}C(t,\varepsilon)y + t^{\alpha}D(t,\varepsilon)z + t^{\alpha}g(t,\varepsilon) \end{aligned}\right\} \quad \begin{aligned} &1 \le t < \infty \\ &\alpha > -1 \end{aligned}$$

$$(3.3) \qquad F(\varepsilon)\begin{pmatrix} y(1,\varepsilon) \\ z(1,\varepsilon) \end{pmatrix} = \beta(\varepsilon)$$

$$(3.4) \qquad \begin{pmatrix} y \\ z \end{pmatrix} \in C([1,\infty])$$

where the dimensions are as in Chapter 1 and assume that

$$(3.5) \qquad A,B,C,D,f,g \in C([1,\infty] \times [0,\varepsilon_0]);\ F,\beta \in C([0,\varepsilon_0])$$

holds for some $\varepsilon_0 > 0$ and that F,β,A,B,C,D,f,g are uniformly Lipschitz continuous at $\varepsilon = 0$.

Moreover we assume that the eigenvalues $\lambda(t)$ of $A(t,0)$ split up into two groups such that

$$(3.6) \qquad \operatorname{Re}\lambda_1(t) \ge c_+,\cdots,\operatorname{Re}\lambda_{r_+}(t) \ge c_+,\ c_+ > 0,\quad t \ge 1$$

$$(3.7) \qquad \operatorname{Re}\lambda_{r_++1}(t) \le -c_-,\cdots,\operatorname{Re}\lambda_n(t) \le -c_-,\ c_- > 0,\ t \ge 1\ (n - r_+ = r_-)$$

holds (eigenvalues are counted accoring to algebraic multiplicities) and that there is a transformation to block form

$$A(t,0) = E(t)J(t)E^{-1}(t),\ J(t) = \begin{array}{c} \left[\begin{array}{c|c} J_+(t) & 0 \\ \hline 0 & J_-(t) \end{array}\right] \end{array} \begin{array}{l} \}r_+ \\ \}r_- \end{array}$$

such that the eigenvalues of $J_+(t)\,(J_-(t)$ are $\lambda_1(t),\cdots,\lambda_{r_+}(t)$ $(\lambda_{r_++1}(t),\cdots,\lambda_n(t))$ and

$$(3.9) \qquad \|E\|^{\alpha}_{[1,\infty]} + \|E^{-1}\|^{\alpha}_{[1,\infty]} \le \text{const.}$$

Under the assumptions (3.6), (3.7) and additional smoothness assumption on A this transformation matrix E exists at least locally (everywhere in $[1,\infty]$), but the assumption of the global existence is much more restrictive (see O'Malley (1979)). At first we investigate

$$(3.10) \qquad \varepsilon y' = t^{\alpha}A(t,\varepsilon)y + t^{\alpha}h(t,\varepsilon), \; y \in C([1,\infty])$$

and substitute

$$(3.11) \qquad y = E(t)x$$

obtaining

$$(3.12) \qquad \varepsilon x' = t^{\alpha}J(t)x + t^{\alpha}(E^{-1}(t)(A(t,\varepsilon) - A(t,0))E(t) - \\ -\varepsilon t^{-\alpha}E^{-1}(t)E'(t))x + t^{\alpha}E^{-1}(t)h(t,\varepsilon)$$

$$(3.13) \qquad x \in C([1,\infty]).$$

Using a perturbation approach we first solve

$$(3.14) \qquad \varepsilon u' = t^{\alpha}J(t)u + t^{\alpha}d(t,\varepsilon), \quad u \in C([1,\infty]).$$

According to (3.8) the system (3.14) splits up into

$$(3.14)(a) \qquad \varepsilon u_+' = t^{\alpha}J_+(t)u_+ + t^{\alpha}d_+(t,\varepsilon), \quad u_+ \in C([1,\infty])$$

$$(3.14)(b) \qquad \varepsilon u_-' = t^{\alpha}J_-(t)u_- + t^{\alpha}d_-(t,\varepsilon), \quad u_- \in C([1,\infty]).$$

At first we analyse (3.14a), which we rewrite as

$$(3.15)(a) \qquad \varepsilon u_+' = t^{\alpha}J_+(\infty)u_+ + t^{\alpha}(J_+(t) - J_+(\infty))u_+ + t^{\alpha}d_+(t,\varepsilon)$$

$$(3.15)(b) \qquad u_+ \in C([1,\infty]).$$

We regard (3.15) as an inhomogeneous constant coefficient problem with the fundamental matrix

$$(3.16) \qquad \psi_+(t,\delta,\varepsilon) = \exp\left(\frac{J_+(\infty)}{\varepsilon(\alpha+1)}(t^{\alpha+1} - \delta^{\alpha+1})\right), \qquad \delta > 1$$

and with solution operator

$$(3.17) \qquad (G^+_{\varepsilon,\delta}d_+(\cdot,\varepsilon))(t) = \frac{1}{\varepsilon}\int_{\infty}^{t}\psi_+(t,\delta,\varepsilon)\psi_+^{-1}(s,\delta,\varepsilon)s^{\alpha}d_+(s,\varepsilon)ds, \qquad t > \delta.$$

The solution of (3.15) is

$$u_+ = G^+_{\varepsilon,\delta}(J_+(\cdot) - J_+(\infty))u_+ + G^+_{\varepsilon,\delta}d_+(\cdot,\varepsilon). \tag{3.18}$$

Since (2.39) holds and $J_+(t) \to J_+(\infty)$ the operator $I - G^+_{\varepsilon,\delta}(J_+(\cdot) - J_+(\infty))$ is invertible on $C([\delta,\infty])$ for δ sufficiently large. We obtain

$$u_+(t) = (\underbrace{(I - G^+_{\varepsilon,\delta}(J_+(\cdot) - J_+(\infty)))^{-1}G^+_{\varepsilon,\delta}d_+(\cdot,\varepsilon))(t)}_{(\theta^+_\varepsilon d_+(\cdot,\varepsilon))(t)}, \quad t > \delta. \tag{3.19}$$

To get a solution on $[1,\infty]$ we solve the terminal value problem

$$\varepsilon\tilde{u}'_+ = t^\alpha J_+(t)\tilde{u}_+ + t^\alpha d_+(t,\varepsilon), \quad 1 \le t \le \delta \tag{3.20}$$

$$\tilde{u}_+(\delta) = u_+(\delta). \tag{3.21}$$

and set $(\theta^+_\varepsilon d_+(\cdot,\varepsilon))(t) := \tilde{u}_+(t)$ for $t < \delta$. θ^+_ε is an operator on $[1,\infty]$ and since the eigenvalues of $J_+(t)$ have strictly positive real part

$$\|\theta^+_\varepsilon\|_{[1,\infty]} \le \text{const} \tag{3.22}$$

holds and because of (2.34d), (2.35)

$$\begin{aligned}(\theta^+_\varepsilon d_+(\cdot,\varepsilon))(t) &= \sum_{i=0}^{\infty} ((G^+_{\varepsilon,\delta}(J_+(\cdot) - J_+(\infty)))^i G^+_{\varepsilon,\delta}d_+(\cdot,\varepsilon))(t) = \\ &= -(J_+(t))^{-1}d_+(t,\varepsilon) + O(\varepsilon\|d_+(\cdot,\varepsilon)\|^\alpha_{[\delta,\infty]})\end{aligned} \tag{3.23}$$

holds for $t > \delta$. By continuation (3.23) holds for $t > 1$ (See Ringhofer (1981)).

We rewrite(3.14)(b) analoguously

(3.24)(a) $\quad \varepsilon u'_- = t^\alpha J_-(\infty)u_- + t^\alpha(J_-(t) - J_-(\infty))u_- + t^\alpha d_-(t,\varepsilon)$

(3.24)(b) $\quad u_- \in C([1,\infty])$

and define the fundamental matrix

$$\psi_-(t,\delta,\varepsilon) = \exp\left(\frac{J_-(\infty)}{\varepsilon(\alpha+1)}(t^{\alpha+1} - \delta^{\alpha+1})\right), \quad \delta > 1 \tag{3.25}$$

and solution operator

$$(G^-_{\varepsilon,\delta}d_-(\cdot,\varepsilon))(t) = \frac{1}{\varepsilon}\int_\delta^t \psi_-(t,\delta,\varepsilon)\psi_-^{-1}(s,\delta,\varepsilon)s^\alpha d_-(s,\varepsilon)ds, \quad t > \delta \tag{3.26}$$

such that the general solution of (3.24) is

$$u_- = (I - G^-_{\varepsilon,\delta}(J_-(\cdot) - J_-(\infty)))^{-1}\psi_-(\cdot,\delta,\varepsilon)\bar{\zeta} +$$

(3.27)

$$+ (I - G^-_{\varepsilon,\delta}(J_-(\cdot) - J_-(\infty)))^{-1}G_{\varepsilon,\delta}d_-(\cdot,\varepsilon)$$

for $\bar{\zeta} \in C^{r_-}$ and $t \geq \delta$. We call the first term on the right hand side of (3.27) $\bar{\psi}_-(t,\delta,\varepsilon)\bar{\zeta}$ and the second $\bar{u}_{p_-}(t,\varepsilon)$. Obviously

(3.28) $$\bar{\psi}_-(\delta,\delta,\varepsilon) = I_{r_-}, \quad \bar{\psi}_-(\infty,\delta,\varepsilon) = 0, \quad \bar{u}_{p_-}(\delta,\varepsilon) = 0$$

hold. $\bar{\psi}_-$ has a boundary layer at δ. The homogenous problem (3.14)(b) has a fundamental matrix $\tilde{\psi}_-(t,\varepsilon)$ such that

(3.29) (a) $\tilde{\psi}_-(1,\varepsilon) = I_{r_-}$, (b) $\|\tilde{\psi}_-(t,\varepsilon)\| \leq c_1\exp\left(\frac{-c_2}{\varepsilon(\alpha+1)}(t^{\alpha+1}-1)\right)$

holds for $t \in [1,\delta]$ where $c_1,c_2 > 0$ (see Ringhofer (1981) and under more general assumptions O'Malley (1978)). We set

(3.30) $$\hat{\psi}_-(t,\varepsilon) = \bar{\psi}_-(t,\delta,\varepsilon)\tilde{\psi}_-(\delta,\varepsilon), \qquad t \geq 1.$$

Since $\hat{\psi}_-(\delta,\varepsilon) = \tilde{\psi}_-(\delta,\varepsilon)$ we obtain $\hat{\psi}_- \equiv \tilde{\psi}_-$ and the boundary layer has been shifted from δ to 1. On $[1 + O(\varepsilon|\ln\varepsilon|),\infty]$ the matrix $\hat{\psi}_-$ is smooth. Another particular solution is

(3.31) $$\tilde{u}_{p_-}(t,\varepsilon) = \frac{1}{\varepsilon}\int_1^t \tilde{\psi}_-(t,\varepsilon)\tilde{\psi}_-^{-1}(s,\varepsilon)s^\alpha d_-(s,\varepsilon)ds, \quad t \in [1,\delta].$$

Since $\|\tilde{\psi}_-(t,\varepsilon)\tilde{\psi}_-^{-1}(s,\varepsilon)\| \leq c_3\exp\left(\frac{-c_2}{\varepsilon(\alpha+1)}(t^{\alpha+1}-s^{\alpha+1})\right)$ holds on $[1,\delta]$ we derive

(3.32) $$\|\tilde{u}_{p_-}(\cdot,\varepsilon)\|_{[1,\delta]} \leq \text{const}\|d_+(\cdot,\varepsilon)\|_{[1,\delta]}.$$

Setting

(3.33) $$\hat{u}_{p_-}(t,\delta) = (\theta^-_\varepsilon d_-(\cdot,\varepsilon))(t) := \bar{\psi}_-(t,\delta,\varepsilon)\tilde{u}_{p_-}(\delta,\varepsilon) + \bar{u}_{p_-}(t,\varepsilon)$$

we obtain $\hat{u}_{p_-} \equiv \tilde{u}_{p_-}$ and

(3.34) $$\|\theta^-_\varepsilon\|_{[1,\infty]} \leq \text{const}$$

because on $[1,\delta]$ we use (3.32) and on $[\delta,\infty]$ we use estimate (3.33) and (3.27). As general solution of (3.14)(b) we take

$$(3.35)\qquad u_-(t,\varepsilon) = \hat{\psi}_-(t,\varepsilon)\zeta + (\theta_\varepsilon^- d_-(\cdot,\varepsilon))(t), \quad t \geq 1$$

and we find

$$(3.36)\qquad \begin{aligned} u_-(t,\varepsilon) = \hat{\psi}_-(t,\varepsilon)(\zeta + J_-^{-1}(1)d_-(1,\varepsilon)) - J_-^{-1}(t)d_-(t,\varepsilon) + \\ + O(\varepsilon\|d_-(\cdot,\varepsilon)\|^\alpha_{[1,\infty]}) \end{aligned}$$

uniformly on $[1,\infty]$.

Setting $\theta_\varepsilon = \begin{pmatrix}\theta_\varepsilon^+\\ \theta_\varepsilon^-\end{pmatrix}$ we write the solution of (3.12), (3.13) as

$$(3.37)\qquad x = \begin{bmatrix} 0 \\ \hat{\psi}_-(\cdot,\varepsilon)\end{bmatrix}\zeta + \theta_\varepsilon(E^{-1}(A(\cdot,\varepsilon) - A(\cdot,0))E - \varepsilon\tilde{E})x + \theta_\varepsilon E^{-1}h(\cdot,\varepsilon)$$

where $\tilde{E}(t) = t^{-\alpha}E^{-1}(t)E'(t)$ has been set. (3.5), (3.9) guarantee that $\tilde{A}(t,\varepsilon) = E^{-1}(t)(A(t,\varepsilon) - A(t,0))E - \varepsilon\tilde{E}(t) \to 0$ as $\varepsilon \to 0$ uniformly on $[1,\infty]$. Therefore $(I - \theta_\varepsilon\tilde{A}(\cdot,\varepsilon))^{-1}$ exists on $C([1,\infty])$ and is bounded *uniformly in* ε *such that*

$$(3.38)\qquad \begin{aligned} x(t,\varepsilon) = \left((I - \theta_\varepsilon\tilde{A}(\cdot,\varepsilon))^{-1}\begin{bmatrix} 0 \\ \hat{\psi}_-(\cdot,\varepsilon)\end{bmatrix}\right)(t)\zeta + \\ + ((I - \theta_\varepsilon\tilde{A}(\cdot,\varepsilon))^{-1}\theta_\varepsilon E^{-1}h(\cdot,\varepsilon))(t), \quad t \geq 1 \end{aligned}$$

holds for $\zeta \in C^{r_-}$.

$$(3.39)\qquad y(t,\varepsilon) = \sum(t,\varepsilon)\begin{bmatrix} 0 \\ I_{r_-}\end{bmatrix}\rho - A^{-1}(t,0)h(t,\varepsilon) + O(\varepsilon\|h(\cdot,\varepsilon)\|^\alpha_{[1,\infty]}), \quad t \geq 1$$

holds where $\sum(t,\varepsilon)\begin{bmatrix} 0 \\ I_{r_-}\end{bmatrix}$ is the boundary layer term (at $t = 1$) fulfilling the estimate (3.29)(b) and $\rho \in C^{r_-}$. This is proven by using the series expansion of (3.38).

Returning to the coupled problem (3.1), (3.2), (3.4) we assume that

$$(3.40)\qquad A,B,C,D,f,g \in C^1_\alpha([1,\infty]) \text{ uniformly in } \varepsilon.$$

From (3.39) we get for fixed $z \in C^1_\alpha([1,\infty])$

$$y(t,\varepsilon) = \sum(t,\varepsilon)\begin{bmatrix}0\\ I_{r_-}\end{bmatrix}\rho - A^{-1}(t,0)(B(t,0)z(t,\varepsilon) + f(t,0)) + \tag{3.41}$$

$$+ \varepsilon(L_\varepsilon^{(1)}z)(t) + \varepsilon(L_\varepsilon^{(2)}f)(t)$$

where $L_\varepsilon^{(i)} = C^1_\alpha([1,\infty]) \to C([1,\infty])$ and $\|L_\varepsilon^{(i)}\|^\alpha_{[1,\infty]} < \text{const}, \; i > 1,2.$
Inserting (3.41) into (3.2) gives

$$z' = t^\alpha(D(t,0) - C(t,0)A^{-1}(t,0)B(t,0))z + t^\alpha\varepsilon(L_\varepsilon^{(3)}z)(t) +$$

$$+ t^\alpha(L_\varepsilon^{(4)}f)(t) + t^\alpha(C(t,\varepsilon)\sum(t,\varepsilon)\begin{bmatrix}0\\ I_{r_-}\end{bmatrix}\rho + g(t,\varepsilon)), \tag{3.42}$$

$$z \in C([1,\infty]).$$

Again the operators $L_\varepsilon^{(j)} : C^1_\alpha([1,\infty]) \to C([1,\infty]), \;\; j = 3,4$ are uniformly (in ε) bounded.

Setting $\bar{D}(t,\varepsilon) = D(t,\varepsilon) - C(t,\varepsilon)A^{-1}(t,\varepsilon)B(t,\varepsilon)$ and assuming that (2.14), (2.18b) holds for $\bar{D}(\infty,0)$ we can solve

$$z' = t^{\bar\alpha}\bar{D}(t,0)z + t^{\bar\alpha}\bar{g}(t,\varepsilon), \quad z \in C^1_\alpha([1,\infty]) \tag{3.43}$$

by using the theory developed by de Hoog and Weiss (1980a,b). We obtain for $\xi \in C^{\tilde{r}_-}$

$$z = (I-E_2H_\delta E_2^{-1}(\bar{D}(\cdot,0) - \bar{D}(\infty,0))E_2)^{-1}\phi(\cdot,\delta)\begin{bmatrix}0\\ I_{\tilde{r}_-}\end{bmatrix}\xi + \tag{3.44}$$

$$+(I-E_2H_\delta E_2^{-1}(\bar{D}(\cdot,0) - \bar{D}(\infty,0))E_2)^{-1}E_2H_\delta E_2^{-1}\bar{g}(\cdot,\varepsilon), \qquad t > \delta$$

where $H_\delta, \phi(t,\delta)$ are defined in (2.26), (2.24) resp. and δ is sufficiently large. The right hand side of(3.44) can be continued to $[1,\delta]$ and we obtain

$$z(t,\varepsilon) = \hat{\phi}(t)\begin{bmatrix}0\\ I_{\tilde{r}_-}\end{bmatrix}\xi + (\Gamma\bar{g}(\cdot,\varepsilon))(t), \quad t > 1 \tag{3.45}$$

where $\Gamma : C([1,\infty]) \to C^1_\alpha([1,\infty])$ and $\|\Gamma\|_{[1,\infty]} \le$ const. Applying this to (3.42) gives

$$z(t,\varepsilon) = \hat{\phi}(t)\begin{bmatrix} 0 \\ I_{\tilde{r}_-} \end{bmatrix}\xi + \varepsilon(\Gamma L_\varepsilon^{(3)} z)(t) + (\Gamma L_\varepsilon^{(4)} f)(t) +$$

(3.46)

$$+ (\Gamma g(\cdot,\varepsilon))(t) + (\Gamma(C(\cdot,\varepsilon)\textstyle\sum(\cdot,\varepsilon\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix})(t)\rho$$

$\Gamma L_\varepsilon^{(3)} : C^1_\alpha([1,\infty]) \to C^1_\alpha([1,\infty])$ and $\|\Gamma L_\varepsilon^{(3)}\|^\alpha_{[1,\infty]} \le$ const. Therefore $(I - \varepsilon\Gamma L_\varepsilon^{(3)})^{-1}$ exist as operators on $C^1_\alpha([1,\infty])$ for ε sufficiently small and

$$z(t,\varepsilon) = \hat{\phi}(t)\begin{bmatrix} 0 \\ I_{\tilde{r}_-} \end{bmatrix}\xi + (\Gamma(g(\cdot,0) - C(\cdot,0)A^{-1}(\cdot,0)f(\cdot,0)))(t) +$$

(3.47)

$$+ (\Gamma C(\cdot,\varepsilon)\textstyle\sum(\cdot,\varepsilon)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix})(t)\rho + O(\varepsilon).$$

Using the exponential decay of $\sum(t,\varepsilon)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix}$ and the definition of H_δ it is easy to show that

$$\|\Gamma C(\cdot,\varepsilon)\textstyle\sum(\cdot,\varepsilon)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix}\|_{[1,\infty]} = O(\varepsilon). \tag{3.48}$$

So we obtain

<u>Theorem 3.1.</u> Let the given assumptions on A,B,C,D,f,g hold and assume that the $(r_- \times \tilde{r}_-) \times (r_- \times \tilde{r}_-)$-matrix

$$F(0)\begin{bmatrix} E(1)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix} & E(1)D_+E(1)^{-1}A^{-1}(0)B(0)\hat{\phi}(1)\begin{bmatrix} 0 \\ I_{\tilde{r}_-} \end{bmatrix} \\ 0 & \hat{\phi}(1)\begin{bmatrix} 0 \\ I_{\tilde{r}_-} \end{bmatrix} \end{bmatrix}$$

in nonsingular ($F(\varepsilon)$ is a $(r_- + \tilde{r}_-) \times (n+m)$-matrix). Then the boundary value problem (3.1), (3.2), (3.3), (3.4) has for sufficiently small ε and for all $\beta(\varepsilon) \in R^{r_- + \tilde{r}_-}$, $f,g \in C^1_\alpha([1,\infty])$ (uniformly in ε) a unique solution y,z. The continuity statements of Theorem 2.1 hold and

$$z(t,\varepsilon) = \hat{\phi}(t)\begin{bmatrix} 0 \\ I_{\tilde{r}_-} \end{bmatrix}\xi + (\Gamma(g(\cdot,0) - C(\cdot,0)A^{-1}(\cdot,0)f(\cdot,0)))(t) + \tag{3.49}$$
$$+ O(\varepsilon),\ \xi \in C^{\tilde{r}_-}$$

$$y(t,\varepsilon) = \sum(t,\varepsilon)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix}\rho - A^{-1}(t,0)B(t,0)z(t,0) + f(t,0)) + \tag{3.50}$$
$$+ O(\varepsilon),\quad \gamma \in C^{r_-}$$

hold uniformly in $[1,\infty]$. $y(\infty,\varepsilon)$, $z(\infty,\varepsilon)$ are as in (2.46), (2.47) when $A(0)$, $B(0)$, $C(0)$, $D(0)$ are substituted by $A(\infty,0)$, $B(\infty,0)$, $C(\infty,0)$, $D(\infty,0)$.

The 'finite' problem is

$$\left.\begin{aligned} &\varepsilon y_T' = t^\alpha A(t,\varepsilon)y_T + t^\alpha B(t,\varepsilon)z_T + t^\alpha f(t,\varepsilon) && (3.51)\\ &z_T' = t^\alpha C(t,\varepsilon)y_T + t^\alpha D(t,\varepsilon)z_T + t^\alpha g(t,\varepsilon) && (3.52)\end{aligned}\right\}\ 1 \leq t \leq T$$

$$F(\varepsilon)\binom{y_T(1,\varepsilon)}{z_T(1,\varepsilon)} = \beta(\varepsilon) \tag{3.53}$$

$$S(T,\varepsilon)\binom{y_T(T,\varepsilon)}{z_T(T,\varepsilon)} = \gamma(T,\varepsilon) \tag{3.54}$$

where $\gamma(T,\varepsilon) \in R^{r_+ + \tilde{r}_+}$, $S(t,\varepsilon)$ is a $(r_+ + \tilde{r}_-) \times (n+m)$-matrix. We assume that (2.18)(a) holds for $A(\infty,0)$. Then by proceeding as de Hoog and Weiss (1980a) did, we find that we can use (2.52), (2.53) in order to set up the asymptotic boundary condition (3.54). (We do not have to know $E(t)$ explicitely since it can be chosen such that $E(t) \to E_1$ as $t \to \infty$). The convergence estimate (2.56) holds if we add $O(\exp(-\frac{\rho}{(\alpha+1)}T^{\alpha+1}))$, $\rho > 0$, which is an estimate for the order of decay of $\hat{\phi}(t)$, to the right hand side.

4. Quasiliner Problems.

We investigate

$$\left.\begin{aligned} &(4.1)\qquad \varepsilon y' = t^{\alpha}A(z,t)y + t^{\alpha}f(z,y,t,\varepsilon) \\ &(4.2)\qquad z' = t^{\alpha}g(z,y,t,\varepsilon) \end{aligned}\right\} \begin{array}{l} \alpha > -1 \\ 1 \leq t < \infty \end{array}$$

$$F(\varepsilon)\begin{pmatrix} y(1,\varepsilon) \\ z(1,\varepsilon) \end{pmatrix} = \beta(\varepsilon) \tag{4.3}$$

$$\begin{pmatrix} y \\ z \end{pmatrix} \in C([1,\infty]) \tag{4.4}$$

where $A(z,t)$ is an $n\times n$-matrix, f an n-vector, g an m-vector and we assume that the Problem (4.1), (4.2) is quasilinear:

$$\frac{\partial f}{\partial y} = O(\varepsilon) \tag{4.5}$$

for $t \in [1,\infty]$, $\varepsilon \in [0,\varepsilon_0]$ and y,z in compact sets. We get immediately

$$f(z,y,t,0) = f(z,0,t,0). \tag{4.6}$$

We now assume that $F(\varepsilon)$ is a $k \times (n+m)$-matrix (k will be specified later) and

$$f,g \in C^2(\mathbb{R}^{m+n} \times [1,\infty] \times [0,\varepsilon_0]) \cap C^1_{\alpha}([1,\infty]) \tag{4.7}$$

Now we proceed as Ringhofer (1981) did. We split $F(0)$ into

$$F(0)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = F_z(0)\xi_2 + F_y(0)\xi_1,\ \xi_1 \in \mathbb{R}^n,\ \xi_2 \in \mathbb{R}^m \tag{4.8}$$

and we assume that there is an integer $r_+ \leq n$ such that

$$F_y(0)\xi_1 = F_{y_+}(0)\xi_1^+ + F_{y_-}(0)\xi_1^- \tag{4.9}$$

where

$$\xi_1 = \begin{pmatrix} \xi_1^+ \\ \xi_1^- \end{pmatrix},\quad \xi_1^+ \in \mathbb{R}^{r_+},\ \xi_1^- \in \mathbb{R}^{r_-},\quad r_- = n-r_+ \tag{4.10}$$

and the $k \times r_-$ matrix $F_{y_-}(0)$ ($k \geq r_-$ is assumed) has maximal rank r_-

Therefore, there is a $k \times k$ matrix Z such that

$$(4.11) \qquad ZF_{y_-}(0) = \underbrace{\begin{bmatrix} V \\ 0 \end{bmatrix}}_{r_-} \begin{matrix} \}r_- \\ \}k-r_- \end{matrix}, \quad Z = \underbrace{\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}}_{k} \begin{matrix} \}r_- \\ \}k-r_- \end{matrix}$$

holds where V is nonsingular.

The main assumption is the following. The reduced problem

$$(4.12) \qquad z' = t^{\alpha} g(y,z,t,0), \quad 1 \le t < \infty$$

$$(4.13) \qquad 0 = A(z,t)y + f(z,0,t,0), \quad 1 \le t < \infty$$

$$(4.14) \qquad Z_2 F(0)\binom{y(1)}{z(1)} = Z_2 \beta(0)$$

$$(4.15) \qquad \binom{y}{z} \in C([1,\infty])$$

has an isolated solution (see Keller (1975)) $y = \bar{y}$, $z = \bar{z}$ and

$$(4.16) \qquad A(z,t) = \left[\begin{array}{c|c} A_+(z,t) & 0 \\ \hline \underbrace{0}_{r_+} & \underbrace{A_-(z,t)}_{r_-} \end{array}\right] \begin{matrix} \}r_+ \\ \}r_- \end{matrix}$$

holds in $C_{\varphi} = \{(z,t) \,|\, \|z-\bar{z}(t)\| \le \varphi, \ t \in [1,\infty]\}$, $\varphi > 0$, where the eigenvalues $\lambda_+(z,t)$ of $A_+(z,t)$ and $\lambda_-(z,t)$ of $A_-(z,t)$ fulfill

$$(4.17) \qquad \mathrm{Re}\,\lambda_+(z,t) \ge c_+ > 0\,, \quad (z,t) \in C_{\varphi}$$

$$(4.18) \qquad \mathrm{Re}\,\lambda_-(z,t) \le -c_- < 0, \quad (z,t) \in C_{\varphi}.$$

This guarantees that (4.13) can be solved for z locally around $\bar{z}$ and

$$(4.19) \qquad \bar{y} = \bar{y}(\bar{z},t) = -A^{-1}(\bar{z},t)f(\bar{z},0,t,0)$$

holds. We now assume that the matrix

$$(4.20) \qquad \bar{D} = \frac{\partial g}{\partial z}(\bar{z}(\infty),\bar{z}(\infty),\infty,0) + \frac{\partial g}{\partial y}(\bar{z}(\infty),\bar{y}(\infty),\infty,0)\frac{\partial \bar{y}}{\partial \bar{z}}(\bar{z}(\infty),\infty).$$

(A^{-1} is supposed to be smooth) fulfills

$$(4.21) \qquad \bar{D} = EJE^{-1}, \quad J = \begin{array}{c} \left[\begin{array}{c|c} J^+ & 0 \\ \hline 0 & J^- \end{array}\right] \begin{array}{l} \}r_+ \\ \}\tilde{r}_- \end{array} \\ \begin{array}{cc} \tilde{r}_+ & \tilde{r}_- \end{array} \end{array}$$

where the eigenvalues of $J^+(J^-)$ have positive (negative) real parts.

Therefore we assume that $Z_2F(0)$ is a matrix $\tilde{r}_- \times (n+m)$ matrix, $Z_2\beta(0) \in R^{r_-}$ and $k = r_- + \tilde{r}_-$ such that (4.12), (4.13), (4.14), (4.15) is well posed with respect to the number of 'finite' boundary conditions (see Markowich (1980a), de Hoog and Weiss (1980a,b)). Obviously $z_\infty = \bar{z}(\infty)$, $y_\infty = \bar{y}(\infty)$ are solutions of

$$(4.22) \qquad \text{(a)} \quad 0 = g(y_\infty,z_\infty,\infty,0), \quad \text{(b)} \quad 0 = A(z_\infty,\infty)y_\infty + f(z_\infty,0,\infty,0)$$

and we assume that z_∞, y_∞ are isolated and that $f(y_\infty,z_\infty,t,0) \equiv 0$, $g(y_\infty,z_\infty,t,0) \equiv 0$, $t > \delta > 1$ holds. Therefore $\bar{D}$ as in (4.20) can be calculated a priori at these roots.

Let $\psi(t,\varepsilon)$ denote the fundamental matrix of

$$(4.23) \qquad \varepsilon v' = t^\alpha A(\bar{z}(t),t)v, \quad \psi(1,\varepsilon) = I.$$

We only state the existence result since the proof goes along the lines of the proof given in Ringhofer (1981) for finite-interval problems using the linear theory developed in chapters 2,3 of this paper.

<u>Theorem 4.1.</u> Let $F(\varepsilon) \in C([0,\varepsilon_0])$ be a $(r_-+\tilde{r}_-) \times (n+m)$ matrix. Under the given assumption the problems (4.1), (4.2), (4.3), (4.4) has a locally unique solution y,z for ε sufficiently small such that

$$y(t,\varepsilon) = \psi(t,\varepsilon)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix}\zeta + \bar{y}(t) + O(\varepsilon)$$

$$z(t,\varepsilon) = \bar{z}(t) + O(\varepsilon)$$

for some $\zeta \in C^{r_-}$ holds uniformly in $[1,\infty]$.

From chapter 3 we conclude that

$$\left\|\psi(t,\varepsilon)\begin{bmatrix}0\\ I_{r_-}\end{bmatrix}\right\| \leq \text{const.}\exp\left(\frac{-c}{\varepsilon(\alpha+1)}(t^{\alpha+1}-1)\right), \quad c > 0$$

holds. t-asymptotics for $\bar{z}(t),\bar{y}(t)$ can be obtained from Markowich (1980a):

$$\bar{z}(t) = \bar{z}(\infty) + E\phi(t)\begin{bmatrix}0\\ I_{\tilde{r}_-}\end{bmatrix}\xi + 0\left(\left\|\phi(t)\begin{bmatrix}0\\ I_{\tilde{r}_-}\end{bmatrix}\right\|^2\right) \tag{4.25}$$

for $\xi \in C^{\tilde{r}_-}$ where

$$\phi(t) = \exp\left(\frac{J}{\alpha+1}(t^{\alpha+1}-1)\right) \tag{4.26}$$

holds. From (4.19) we get

$$\bar{y}(t) = \bar{y}(\infty) + \frac{\partial\bar{y}}{\partial\bar{z}}(\bar{z}(\infty),\infty)E\phi(t)\begin{bmatrix}0\\ I_{\tilde{r}_-}\end{bmatrix}\xi + 0\left(\left\|\phi(t)\begin{bmatrix}0\\ I_{\tilde{r}_-}\end{bmatrix}\right\|^2\right). \tag{4.27}$$

The approximating 'finite' problems are

$$\left.\begin{aligned} &\varepsilon y_T' = t^\alpha A(z_T,t)y_T + t^\alpha f(z_T,y_T,t,\varepsilon) && (4.28)\\ &z_T' = t^\alpha g(y_T,z_T,t,\varepsilon) && (4.29)\end{aligned}\right\} \quad 1 \leq t \leq T$$

$$F(\varepsilon)\begin{pmatrix}y_T(1,\varepsilon)\\ z_T(1,\varepsilon)\end{pmatrix} = \beta(\varepsilon) \tag{4.30}$$

$$S(T,\varepsilon)\begin{pmatrix}y_T(T,\varepsilon)\\ z_T(T,\varepsilon)\end{pmatrix} = \gamma(T,\varepsilon) \tag{4.31}$$

where $S(T,\varepsilon)$ is a $(r_+ + \tilde{r}_+) \times (n+m)$ matrix and $\gamma(T,\varepsilon) \in R^{r_+ + \tilde{r}_+}$. We choose

$$(4.32) \qquad S \equiv S(T,\varepsilon) = \begin{bmatrix} [I_{r_+},0] & -[I_{r_+},0]\frac{\partial \bar{y}}{\partial \bar{z}}(\bar{z}(\infty),\infty) \\ 0 & \begin{bmatrix} I_{\tilde{r}_+},0 \end{bmatrix} E^{-1} \end{bmatrix}$$

and

$$(4.33) \qquad \gamma \equiv \gamma(T,\varepsilon) = S\begin{pmatrix} \bar{y}(\infty) \\ \bar{z}(\infty) \end{pmatrix}.$$

Then we obtain

$$(4.34) \qquad \left\| S\begin{pmatrix} y(T,\varepsilon) \\ z(T,\varepsilon) \end{pmatrix} - \gamma \right\| = O\left(\left\| \phi(t)\begin{bmatrix} 0 \\ I_{\tilde{r}_-} \end{bmatrix} \right\|^2 \right) + O(\varepsilon)$$

and by using the linear stability result (2.54) we get by proceeding as de Hoog and Weiss (1980a) did

$$(4.35) \qquad \left\| \begin{pmatrix} y_T - y \\ z_T - z \end{pmatrix} \right\|_{[1,T]} \le \text{const} \left\| S\begin{pmatrix} y(T,\varepsilon) \\ z(T,\varepsilon) \end{pmatrix} - \gamma \right\|$$

for the *locally unique solution* y_T, z_T of (4.28), (4.29), (4.30), (4.31) such that (4.34) constitutes the convergence estimate.

As in the linear case this asymptotic boundary condition only depends on the reduced 'infinite' problem.

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This paper deals with systems of singularly perturbed ordinary differential equations posed as boundary value problems on an infinite interval. The system is assumed to consist of singularly perturbed (fast) components and unperturbed (slow) components and to have a singularity of the second kind at ∞. Under the assumption that there is no turning point we derive uniform asymptotic expansions (as the perturbation parameter tends to zero) for the fast and slow components uniformly on the whole infinite line. The second goal of the paper is to derive convergence estimates for the

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20. Abstract (continued)

solutions of 'finite' singular perturbation problems obtained by cutting the infinite interval at a finite (far out) point and by substituting appropriate additional boundary conditions at the far end. Using a suitable choice for these boundary conditions the order of convergence is shown to depend only on the decay property of the infinite solution.