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DISCRETIZATION ERRORS IN THE FAR FIELD CONDITIONS FOR OSCILLATORY SUBSONIC FLOWS

K. G. GUDERLEY

University of Dayton Research Institute
300 College Park Avenue
Dayton, OH 45469

December 1981
Final Report for Period June 1981 - December 1981

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This technical report has been reviewed and is approved for publication.

Charles L. Keller

CHARLES L. KELLER
Mathematician
Aeroelastic Group
Analysis & Optimization Branch

Fred A. Picchioni

FREDERICK A. PICCHIONI, LtCol, USAF
Analysis & Optimization Branch
Structures & Dynamics Division

FOR THE COMMANDER

Ralph L. Kuster

RALPH L. KUSTER, Jr., Col, USAF
Chief, Structures & Dynamics Division

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In their complete form, far field conditions are given by an infinite number of global linear equations between the potential and its normal derivative at the far boundary of the computed part of a flow field. The report illustrates by means of an example the effects of a discretization carried out in the partial differential equation for different formulations of the far field conditions. If one considers the potential and its normal derivative at all

points of the far boundary as infinite vectors, then the far field conditions are established by ~~means~~ of the infinite matrices. The appearance of these matrices depends upon the choice of the "test" functions used in ~~the~~ formulating ~~of~~ the far field conditions, but the relations obtained for different formulations arise from each other by premultiplication with some matrix. In a numerical approach an approximation of the potential and its normal derivative in terms of a finite number of parameters must be used, ~~in other words~~ the vector is restricted to a finite dimensional subspace. The matrices encountered in ~~the~~ formulating ~~of~~ the far field conditions depend upon the choice of the subspace and of the the relative eigenvectors and eigenvalues of the two matrices. For vectors in an infinite vector space they are independent of the choice of the test functions. The effect of the discretization can be recognized by the deviation of the eigenvectors and eigenvalues from the ideal case. This idea is carried out for different choices of the systems of test functions and of the representation for the potential.

PREFACE

This is the final Technical Report to appear under Grant AFOSR-78-3524, entitled "Mathematical Questions Related to the Computation of Compressible Flow Fields", to the Universtiy of Dayton for the Applied Mathematics Group, Analysis and Optimization Branch, Structures and Dynamics Division, FDL, AFWAL under Project 2304, Task 2304N1 and Program Element 61102F. The work was performed during the period June through December 1981. Dr. Karl G. Guderley, of the University of Dayton Research Institute was Principal Investigator. The numerical work has been carried out very competently by Mr. Keith Miller. Dr. Charles L. Keller, AFWAL/FIBRC (513) 255-5350, Wright-Patterson AFB, OH, 45433 was program manager.

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SECTION 1
INTRODUCTION

In Reference 1, far field conditions for small periodic oscillations superimposed on a subsonic flow have been derived. They express the effect of the distant field on the portion of the flow field that is to be treated numerically. One obtains linear relations in which the potential and its normal derivative at all points of the far boundary of the computed part of the flow field appear simultaneously. We speak of global conditions since all boundary points are involved. Approximate local far field conditions have been derived by Bayliss, Gunzberger and Turker (Reference 2).

The present report clarifies by means of an example some of the questions which arise in the application of these conditions; of particular interest are the errors introduced by the discretizations which are necessary in a numerical treatment. The example is rather simple; it can be treated analytically. The analytical properties will be helpful in the discussion but, since one wants to imitate more realistic problems, they will not be used in a numerical approach. Actually, the numerical solutions play a very minor role. The problem has been programmed and solved for different boundary conditions, but only to check the feasibility of the approach, not to provide a production routine.

A comparison of the results so obtained with exact solutions will not be carried out because it does not allow us to separate different error sources (errors due to the discretization in the differential equations, in the far field conditions, and in satisfactory boundary conditions at the profile). Instead, we shall examine how the discretization affects different eigenfunctions.

In realistic problems the differential equation governing unsteady perturbations is not identical with the one for perturbations in a parallel subsonic flow, in particular not in the vicinity of the profile. This does not limit the scope of the present study because in the distant field, the latter gives an acceptable approximation.

SECTION II
SOME PRELIMINARY OBSERVATIONS

We consider the unsteady two dimensional field generated by a plate which extends along the x axis of a Cartesian system of coordinates x,y from x = -1 to x = +1. Along the plate the velocity normal to it, that is the normal derivative of the oscillating part of the potential ϕ is prescribed. For an infinitely thin oscillating plane the value of ϕ_y at its upper and lower side are, of course, the same, but we shall allow for a more general situation. The problem has a number of symmetries which allow one to separate it into smaller problems; in the subproblem treated numerically here the potential ϕ is symmetric with respect to the x axis. The components of the perturbation velocity then have opposite signs at the upper and lower side of the plate. To complete the formulation of the boundary value problem one must prescribe at a distance the "far field" conditions under study in the present report.

The equations for the distant field can ultimately be reduced to the Helmholtz equation

$$\phi_{xx} + \phi_{yy} + \mu^2\phi = 0 \tag{1}$$

It arises from the equation of two dimensional acoustics

$$\bar{\phi}_{xx} + \bar{\phi}_{yy} - \bar{\phi}_{tt} = 0 \tag{2}$$

by setting

$$\bar{\phi}(x,y,t) = \phi(x,y)\ \exp(i\mu t) \tag{3}$$

In this formulation it is assumed that the time has been scaled so that the velocity of sound is 1. The time for one period of the oscillatory motion is then $T = 2\pi/\mu$; this formula gives also the wave length (because of the choice of the velocity of sound).

To get some appreciation for the magnitude of μ we mention that its value is π for a wave with a length equal to the width of the plate (which is 2 in the present example).

If the plate is embedded in a subsonic parallel flow, one must make several transformations in order to arrive at the Helmholtz equation (see Ref. 1). If in such a flow the plate oscillates with a circular frequency, ν, then one obtains (after the transformation to the Helmholtz equation)

$$\mu = \frac{\nu a L}{a^2 - U^2} = \frac{\nu L}{a(1-M^2)} \tag{4}$$

where U is the free stream velocity, "a" is the free stream velocity of sound, $M = U/a$ is the free stream Mach number, and L is some characteristic length (here the half chord of the plate).

During one period $T = 2\pi/\nu$ a perturbation travels strictly upstream or strictly downstream by the respective wave lengths, via $(2\pi/\nu)a(1-M)$ and $(2\pi/\nu)a(1+M)$.

Frequently a reduced frequency, here denoted by μ', given by

$$\mu' = \nu\, L/a \tag{5}$$

is introduced. Then

$$\mu = \mu' \frac{M}{1-M^2} \tag{6}$$

The frequencies ν are determined by the physical situation under investigation.

The transformation to the Helmholtz equation generates a wave pattern in which the wave length is the same in all directions. For a given ν one has the following expressions for the wave length

Helmholtz equation $2\pi/\mu = (2\pi/\nu L)a(1-M^2)$

waves traveling upstream in a parallel flow $(2\pi/\nu L)a(1-M)=(2\pi/\nu L)a(1-M^2)(1+M)^{-1}$

waves traveling downstream in a parallel flow $(2\pi/\nu L)a(1+M)=(2\pi/\nu L)a(1-M^2)(1-M)^{-1}$

The wave length for the Helmholtz equation is the harmonic mean between the wave lengths for upstream and downstream traveling waves.

The fact that after the transformation to the Helmholtz equation the wave length is larger than for waves traveling upstream in the original field is of practical interest. The grid points of a finite difference net must be close enough to reproduce the waviness of the unsteady field. After the transformation to the Helmholtz equation a grid which is coarser than that of the original problem is admissible. Of course, for small values of ν and Mach numbers not too close to one this is less important, for the wave lengths are large to begin with and the solution resembles in essence that of the Laplace equation. But, under different circumstances, it may be advantageous to carry out such a transformation even though the differential equation for the vicinity of the profile differs from the Helmholtz equation.

SECTION III
TRANSFORMATIONS

In a numerical approach one must provide a net in which the discretization is carried out. One might use a rectangular net in the physical plane. An elimination strategy for this purpose is described in Reference 3. In Reference 1 a different net obtained by a conformal mapping has been suggested. For practical purposes the first net is probably preferable because it retains the orientation imposed by the free stream direction. The net of Reference 1, used here, makes it possible to study the problem in greater detail. The general conclusions do not depend upon the choice of the net.

The starting point is the Helmholtz equation, Eq. (1). Let

$$x + iy = z \tag{6}$$

$$\xi + i\eta = \zeta \tag{7}$$

and

$$z = g(\zeta)$$

let

$$g' = dg/d\zeta . \tag{8}$$

where g is an analytic function. Then one obtains

$$\phi_{\xi\xi} + \phi_{\eta\eta} + \mu^2 |g'|^2 \phi = 0 \tag{9}$$

We choose specifically the familiar mapping

$$g(\zeta) = \cosh \zeta \tag{10}$$

Then

$$\begin{aligned} x &= \mathrm{Re}\, g = \cosh \xi \cos \eta \\ y &= \mathrm{Im}\, g = \sinh \xi \sin \eta \end{aligned} \tag{11}$$

for $\zeta = 0$ one obtains $y = 0$ and x varies from -1 to $+1$ and back as η varies from 0 to 2π. This is the map of the plate surface. Both, the lines $\eta = 0$ and $\eta = 2\pi$ map into the x axis from $x = 1$ to $x = \infty$, the line $\eta = \pi$ maps into the x axis from $x = -1$ to $-\infty$. The whole flow field is mapped into a strip of width 2π which extends from $\xi = 0$ to $\xi = \infty$.

One obtains, by eliminating η

$$\frac{x^2}{\cosh^2 \xi} + \frac{y^2}{\sinh^2 \xi} = 1$$

and by eliminating ξ

$$\frac{x^2}{\cos^2 \eta} - \frac{y^2}{\sin^2 \eta} = 1$$

The lines ξ = const and η = const are, respectively, ellipses and hyperbolae with foci at $x = +1$ and $x = -1$.

A number of formulae are listed for future use. One has

$$g' = \sinh(\zeta) \tag{12}$$

$$|g'|^2 = (\sinh \zeta)(\sinh^* \zeta)$$

where the star denotes the conjugate complex. Writing the hyperbolic functions in terms of exponential functions and carrying out the multiplication one finds

$$|g'|^2 = 1/2\,(\cosh 2\xi - \cos 2\eta) \tag{13}$$

We also introduce polar coordinates r, θ in the physical plane

$$g(\zeta) = x + iy = r \exp i\theta \tag{14}$$

Then

$$r = |g(\zeta)| = |g(\zeta)g^*(\zeta)|^{1/2}$$

and ultimately,

$$r = [1/2(\cosh(2\xi) + \cos(2\eta))]^{1/2} \tag{15}$$

For ξ large, one has

$$\cosh 2\xi \sim 1/2 \exp (2\xi)$$

and therefore

$$r \sim 1/2 \exp \xi$$

one observes that

$$\exp(i\theta) = \frac{x+iy}{r} = \frac{g(\zeta)}{r} = \frac{g(\zeta)}{|g(\zeta)|} \tag{16}$$

Hence, asymptotically for large values of ξ

$$\exp(i\theta) \sim \exp(i\eta) \tag{17}$$

We note that, because of Eq. (16)

$$\begin{aligned} \cos(n\theta) &= \mathrm{Re}(\exp(i\theta))^n = \mathrm{Re}[(g(\zeta)/r)^n] \\ \sin(n\theta) &= \mathrm{Im}(\exp(i\theta))^n = \mathrm{Im}[(g(\zeta)/r)^n] \end{aligned} \tag{18}$$

These expressions are readily evaluated by complex arithmetic. Considering x and y as functions of ξ and η, one obtains

$$\begin{aligned} \partial x/\partial\xi &= \mathrm{Re}\ g' \\ \partial x/\partial\eta &= -\mathrm{Im}\ g' \\ \partial y/\partial\xi &= \mathrm{Im}\ g' \\ \partial y/\partial\eta &= \mathrm{Re}\ g' \end{aligned} \tag{19}$$

Furthermore, if ξ and η are considered as functions of x and y,

$$\begin{aligned}
\partial\xi/\partial x &= |g'|^{-2}\ \mathrm{Re}\ g' \\
\partial\xi/\partial y &= |g'|^{-2}\ \mathrm{Im}\ g' \\
\partial\eta/\partial x &= -|g'|^{-2}\ \mathrm{Im}\ g' \\
\partial\eta/\partial y &= |g'|^{-2}\ \mathrm{Re}\ g'
\end{aligned} \tag{20}$$

Since

$$\log g = \log r + i\theta$$

one obtains in analogy to Eq. (19)

$$\begin{aligned}
\partial \log r/\partial\xi &= \mathrm{Re}(g'/g) \\
\partial \log r/\partial\eta &= -\mathrm{Im}(g'/g) \\
\partial\theta/\partial\xi &= \mathrm{Im}(g'/g) \\
\partial\theta/\partial\eta &= \mathrm{Re}(g'/g)
\end{aligned} \tag{21}$$

By substituting Eq. (14) into

$$\frac{g'}{g} = \frac{\mathrm{Re}\ g' + i\ \mathrm{Im}\ g'}{\mathrm{Re}\ g + i\ \mathrm{Im}\ g}$$

one finds

$$\begin{aligned}
g'/g = (1/r)[(\mathrm{Re}(g')\cos\theta &+ (\mathrm{Im}(g'))\sin\theta \\
&+ i((\mathrm{Im}(g')\cos\theta - (\mathrm{Re}(g')\sin\theta)]
\end{aligned} \tag{22}$$

One has, specifically

$$\frac{g'}{g} = \frac{\sinh \zeta}{\cosh \zeta}$$

Introducing

$$\exp \xi = u \tag{23}$$

and separating the real and imaginary parts one obtains

$$\frac{g'}{g} = \frac{u^2 - u^{-2} + 2i \sin 2\eta}{u^2 + u^{-2} + 2 \cos 2\eta} \tag{24}$$

Hence, asymptotically, for large u

$$\begin{aligned} \mathrm{Re}(g'/g) &= 1 \\ \mathrm{Im}(g'/g) &= 2u^{-2} \sin 2\eta \end{aligned} \tag{25}$$

Substituting Eq. (13) into Eq. (9), one obtains

$$\phi_{\xi\xi} + \phi_{\eta\eta} + (\mu^2/2)\ (\cosh 2\xi - \cos 2\eta)\phi = 0 \tag{26}$$

Replacing ξ by u, (Eq. (23)), one obtains a form that is more convenient for asymptotic considerations and in addition gives a suitable mesh for numerical work

$$\phi_{uu} + u^{-1}\phi_u + u^{-2}\phi_{\eta\eta} + \mu^2[(1+u^{-4})/4 - (u^{-2}/2)\cos(2\eta)]\phi = 0 \tag{27}$$

SECTION IV
PARTICULAR SOLUTIONS

The far field conditions (for the Helmholtz equation), Eq. (57) of Ref. 1, specialized to the case of plane flows is given by

$$\int_C [\omega(x,y)\phi_n(x,y) - \omega_n(x,y)\phi(x,y)]ds = 0 \tag{28}$$

Here C denotes the far boundary of the computed part of the flow field. The subscript n denotes the derivative normal to this boundary, ds is the line element, ϕ, defined in Eq. (3), is the potential expressing harmonic perturbations. In the distant field ϕ satisfies Eq. (1), in the near field, it will in general satisfy a more complicated differential equation. The functions ω have the character of test functions, they satisfy Eq. (1) at least in the distant and usually also in most of the near field. In addition, we define the test functions used in the present report. They must satisfy the far field conditions. One set is already given in Reference 1.

$$\omega = \chi_m = H_m^{(2)}(\mu r) \begin{cases} \cos m\theta \\ \sin m\theta \end{cases} \tag{29}$$

The second set is obtained by a product hypothesis in the ξ,η plane, namely

$$\omega = \psi_k(\xi,\eta) = f_k(\xi)g_k(\eta) \tag{30}$$

Introducing a separation constant λ_k one then obtains

$$\frac{d^2f_k}{d\xi^2} + (-\lambda_k + \frac{\mu^2}{2}\cosh(2\xi))f_k = 0 \tag{31}$$

$$\frac{d^2g_k}{d\eta^2} + (\lambda_k - \frac{\mu^2}{2}\cos(2\eta))g_k = 0 \tag{32}$$

The second of these equations (with a different normalization of the constant μ^2) is the equation for Mathieu functions, the first one is that for radial Mathieu functions. One might refer to the literature for further developments, but for the present purposes it is nearly as convenient to derive the necessary equations directly and to generate the information needed by the computer.

The expressions (29) and (30) can, of course, also be used to represent the functions ϕ in the distant field.

The periodicity conditions for ϕ carry over to the functions g_k. One must, therefore, determine the separation constants λ_k in such a manner that

$$g_k(0) = g_k(2\pi)$$

and

$$dg_k/d\eta\Big|_{\eta=0} = dg_k/d\eta\Big|_{\eta=2\pi} \tag{33}$$

The operator of Eq. (32), together with the periodicity conditions (33), is self adjoint. The eigenvalues λ_k are real, the eigenfunctions are real and orthogonal to each other. The symmetries of the problems suggest the following hypotheses

$$
\begin{aligned}
g_k^{(1)} &= a_{1,k}^{(1)}\, 2^{-1/2} + \sum_{\ell=2}^{\infty} a_{\ell,k}^{(1)} \cos((2\ell-2)\eta) \\
g_k^{(2)} &= \sum_{\ell=1}^{\infty} a_{\ell,k}^{(2)} \cos(2\ell-1)\eta \\
g_k^{(3)} &= \sum_{\ell=1}^{\infty} a_{\ell,k}^{(3)} \sin(2\ell\eta) \\
g_k^{(4)} &= \sum_{\ell=1}^{\infty} a_{\ell,k}^{(4)} \sin((2\ell-1)\eta)
\end{aligned}
\tag{34}
$$

The individual functions occurring in the right-hand sides of Eq. (34), including the constant term in the first expression, have the same normalization constant π over the interval $0 < \eta \leq 2\pi$. The expressions Eqs. (34) are substituted into Eq. (32). Collecting the individual Fourier components, one obtains four separate systems of equations for the constant $a^{(i)}_{\ell,k}$, $i = 1,\ldots 4$. We combine the constants $a^{(i)}_{\ell,k}$ (for fixed values of i and k) into a vector $\vec{a}^{(i)}_k$ (whose ℓ^{th} component is $a^{(i)}_{\ell,k}$). Then these problems can be written in the form

$$M^{(i)}\,\vec{a}^{(i)}_k - \lambda^{(i)}_k\,\vec{a}^{(i)}_k = 0 \qquad (35)$$

the matrices $M^{(i)}$ are given by

$$M^{(i)} = D^{(i)} + (\mu^2/4)M^{(i,1)}$$

where the $D^{(i)}$'s are diagonal matrices with

$$D^{(1)}_{nn} = (2n-2)^2 \quad , \qquad n = 1,2,\ldots$$

$$D^{(2)}_{11} = 2$$

$$D^{(2)}_{nn} = (2n-1)^2 \quad , \qquad n = 2,3,\ldots$$

$$D^{(3)}_{nn} = (2n)^2 \quad , \qquad n = 1,2,3,\ldots$$

$$D^{(4)}_{11} = 0$$

$$D^{(4)}_{nn} = (2n-1)^2 \quad , \qquad n = 2,3,\ldots$$

The matrices $M^{(i,1)}$ are symmetric. One has

$$M^{(i,1)}_{12} = M_{21} = 2^{1/2}$$

$$M^{(i,1)}_{n,n+1} = M^{(i,1)}_{n+1,n} = 1 \quad , \qquad n = 2,3,\ldots \qquad (36)$$

$$M^{(i,1)}_{n,n+1} = M^{(i,1)}_{n+1,n} = 1 \quad , \qquad n=1,2,\ldots, \; i = 2,3,4,\ldots$$

(36) continued

all other elements of the matrices $M^{(i,1)}$ are zero.

Let $A^{(i)}$ be a matrix whose k^{th} column is given by $\vec{a}_k^{(i)}$ and let $\Lambda^{(i)}$ be a diagonal matrix with the k^{th} element $\lambda_k^{(i)}$. Then one can write

$$M^{(i)}A^{(i)} - A^{(i)}\Lambda^{(i)} = 0$$

If an eigenvalue routine for matrices is available, (best a routine for banded matrices), then numerical solutions to the problem (Eq. (35)) are readily obtained by truncating the infinite matrix. In the present context, only a limited number, n, of eigenvalues and eigenvectors, $a_k^{(i)}$ is used. The infinite matrix is truncated to an m by m matrix where m is chosen large enough so that the coefficients $a^{(i)}_{m,k}$ $(k = 1,2,\ldots n)$ is smaller than an assigned small number ϵ, while the largest coefficient is normalized to 1, say. In our computations, carried out with $n = 5$ and $\epsilon = 10^{-5}$, it was sufficient to take $m = 9$ if $\mu < 4$.

The Mathieu functions g_k are nearly monotonic (not more than one minimum of the absolute value and no maxima) in the region where $\lambda_k - (\mu^2/4) \cos 2\eta < 0$; they are oscillatory where this quantity is larger than zero. If μ^2 is not small, one has a nearly monotonic region for the first eigenvalues in the vicinity of $\eta = 0$ and $\eta = \pi$. In the vicinity of $\eta = \pi/2$ and $\eta = 3\pi/2$ the functions are always oscillatory. The oscillatory regions grow as λ is increased.

For the separation constants $\lambda_k^{(i)}$ obtained from the eigenvalue problem one must solve Eq. (31) for the radial Mathieu functions. For ξ sufficiently large these functions are always oscillatory, because contribution $(\mu^2/2) \cosh (2\xi)$ makes the coefficient of f positive. The particular solutions which represent outgoing waves are found by studying the asymptotic behavior for large values of ξ. It is practical to replace r by u, (Eq. (23)). Then one obtains from Eq. (31).

$$\frac{d^2 f_k^{(i)}}{du^2} + \frac{1}{u}\frac{df_k^{(i)}}{du} + \left(\frac{\mu^2}{4} - \frac{\lambda_k^{(i)}}{u^2} + \frac{\mu^2}{4u^4}\right) f_k^{(i)} = 0 \qquad (37)$$

The form of the dominant part of the coefficient of $f_k^{(i)}$ suggests a hypothesis

$$f_k^{(i)}(u) = \tilde{f}_k^{(i)}(u)\exp(-i(\mu/2)u) \qquad (38)$$

Outgoing waves are obtained by choosing here the negative sign in argument of the exponential function, because the argument of the exponential function in the hypothesis (3) has the positive sign.

Then one obtains

$$\frac{d^2 \tilde{f}_k^{(i)}}{du^2} + \left(\frac{1}{u} - i\mu\right)\frac{d\tilde{f}_k^{(i)}}{du} + \left[-\frac{i\mu}{2u} - \frac{\lambda_k}{u^2} + \frac{\mu^2}{4u^4}\right] \tilde{f}_k^{(i)} = 0 \qquad (39)$$

In subsequent formulae the indices k and i will be omitted. A formal development

$$\tilde{f} = u^{-1/2} \sum_0^\infty c_n u^{-n} \qquad (40)$$

leads to the recurrence relation (obtained by collecting terms with power $u^{-(1/2) - (n-1)}$),

$$i\mu n\, c_n = c_{n-1}(\lambda - (1/2 - n)^2) - (\mu^2/4)c_{n-3} \qquad (41)$$

According to the hypothesis (Eq. (40)) all coefficients with negative subscripts n are zero. The last equation is therefore satisfied for n = 0. (This shows that the factor $u^{-1/2}$ in Eq. (40) is correct.) The coefficient c_0 is taken to be 1. From there on, one proceeds to larger values of n. The series is semiconvergent. If it is terminated after a finite number of terms, then the error can be expected to be of the order of the first neglected term.

This determines how many terms and which minimum value of u one must take in order to obtain an assigned accuracy. This procedure works well provided that the value of μ is not too small.

For a value of u chosen according to this discussion one computes f and df/du. With these initial values one then integrates Eq. (37) in the direction toward smaller values of u. For larger values of u, the values of f and df/du can be computed directly from the asymptotic representation derived here. Notice that f_k and df_k/du are complex.

Eq. (37) is linear, it therefore can be reduced to a first order differential equation. One sets

$$p(u) = (df/du)/f(u)$$

or (42)

$$df/du = p(u)\ f(u)$$

Hence

$$d^2f/du^2 = (dp/du)f(u) + p(u)(df/du)$$

$$= [(dp/du) + p^2(u)]f(u)$$

One then obtains from Eq. (37), the following Ricatti equation

$$\frac{dp}{du} + p^2 + \frac{1}{u}\,p + \left(\frac{\mu^2}{4} - \frac{\lambda}{u^2} + \frac{\mu^2}{4u^4}\right) = 0 \tag{43}$$

Once p(u) has been found one obtains f(u) by a quadrature

$$\log f = \int_1^u p\,(v)\,dv + \text{const} \tag{44}$$

This procedure may be numerically advantageous especially if only (df/du)/f(u) is needed, for the function p is likely to be smoother than the function f. But this is merely a technicality.

In the present problem the particular solutions in terms of Mathieu functions are natural because a series in these particular solutions (with coefficients determined by the boundary conditions at the plate) converges throughout the flow field. One thus obtains a representation of the flow field and a formulation of the far field conditions with which other approximations can be compared. To a large extent this observation is valid also for problems in which the equation obtained by transforming the basic equation differs somewhat from the Helmholtz equation. The use of a representation of the field in terms of Mathieu functions $g_k^{(i)}(\eta)$ need not be restricted to theoretical discussions. For practical applications one must, of course, first provide representations of the functions $g_k^{(i)}(\eta)$ and of $f_k^{(i)}(u)/df_k^{(i)}/du)$ for the value of u at which the far field conditions are applied. This can be done by means of the relations derived above.

In the region $r > 1$ the particular solutions $\psi_k^{(i)}$, (Eq. (30)) can be expressed in terms of the particular solution $\chi(r,\theta)$, Eq. (29). In principle one will evaluate a particular solution $\psi_k^{(i)}$ along some curve r = const ($r > 1$), and then represent it in terms of the functions χ_ℓ by means of a Fourier analysis with respect to θ. The result has the form

$$\psi_k^{(i)} = \sum \beta_{\ell,k}^{(i)} \chi_\ell \tag{45}$$

Certain coefficients will be zero because of symmetry properties. The coefficients $\beta_{\ell,k}$ are constant. The values of the constants (not needed in the future discussions) are readily obtained if one carries out the above matching procedure along a line of u = const in the limit $u \to \infty$. Consider, for instance $\psi_k^{(i)}$. One has, according to the first of Eqs. (34), using the representation of f_k given by Eqs. (38) and (40)

$$\begin{aligned}\psi_k^{(1)} &= g_k^{(1)}(\eta)\ f_k^{(1)}(u)\\ &= [a_{1,k}^{(1)}\ 2^{-1/2} + \sum_{\ell=2}^{\infty} a_{\ell,k}^{(1)} \cos((2\ell-2)\eta)]u^{-1/2}\exp(-i(\mu/2)u)\end{aligned}$$

We write Eq. (45) in a form where the notation for the coefficients β is analogous to that for the coefficient in "a" in the last equation

$$\psi_k^{(1)} = \beta_{1,k}^{(1)} \; H_o^{(2)} \; (\mu r) + \sum_{\ell=2}^{\infty} \; \beta_{\ell,k}^{(1)} \; H_{2\ell-2}^{(2)}(\mu r)\cos((2\ell-2)\theta)$$

One has for u large

$$r \sim u/2, \quad \theta \sim \eta$$

and

$$H_k^{(2)} \; (\mu r) \sim \exp[i(\pi/2)(k + \tfrac{1}{2})]\,(2/\pi\mu r)^{1/2}\exp(-i\mu r)$$

One then obtains

$$\beta_{1,k}^{(1)} = a_{1,k}^{(1)} \; 2^{-1/2}(1/2)(\mu\pi)^{1/2} \; \exp(-i(\pi/4)$$

$$\beta_{1,k}^{(1)} = a_{\ell,k}^{(1)} \; (1/2)(\mu\pi)^{1/2} \; \exp[-i(\pi/2)(2\ell - 2 + \tfrac{1}{2})] \tag{46}$$

SECTION V

EVALUATION OF THE FAR FIELD CONDITIONS

Our discussions will be based on a discretization of the differential equation in the u-η system. Accordingly, one must express the far field conditions (Eq. (28)) in terms of these variables. One has

$$\phi_n = \text{grad}\phi \cdot \vec{e}_n = \phi_x \cos(n,x) + \phi_y \sin(n,x)$$

and

$$ds = -dx \sin(n,x) + dy \cos(n,x)$$

$$\phi_n\, ds = \phi_x dy - \phi_y dx \qquad (47)$$

where $\vec{e}_n$ is the unit vector normal to the contour C; (n,x) denotes the angle between the normal and the x-axis, and dx and dy are the components of a line element along the curve C, which is now given by an ellipse u = const. Then, from Eq. (19)

$$\begin{aligned} dx &= -\text{Im}(g')d\eta \\ dy &= \text{Re}(g')d\eta \end{aligned} \qquad (48)$$

$$\begin{aligned} \phi_x &= \phi_\xi\, \xi_x + \phi_\eta\, \eta_x \\ \phi_y &= \phi_\xi\, \xi_y + \phi_\eta\, \eta_y \end{aligned} \qquad (49)$$

Substituting Eq. (48) and (49) into Eq. (47) and using Eqs. (20) and (23), one finds that along a line u = const

$$\phi_n ds = \phi_\xi d\eta = u\phi_u\, d\eta \qquad (50)$$

Analogously,

$$\omega_n\, ds = u\omega_u\, d\eta$$

If both ϕ and ω are considered as functions of u and η, then one obtains as far field conditions

$$\int_0^{2\pi} (\omega\phi_u - \omega_u\phi)\,d\eta = 0 \tag{51}$$

Substituting here the expressions ψ_k, Eq. (30), one obtains

$$f_k(u) \int_0^{2\pi} g_k(\eta)\,\phi_u(u,\eta)\,d\eta - f_k'(u) \int_0^{2\pi} g_k(\eta)\,\phi(u,\eta)\,d\eta = 0 \tag{52}$$

where the integration is to be carried out along the line u = const which gives the contour C. The values of $f_k(u)$, $f_k'(u)$ and the functions $g_k(\eta)$ are considered as known.

In an alternative derivation of the last equation one would carry out a Fourier decomposition of $\phi_u(u,\eta)$ and $\phi(u,\eta)$ (u = const) in terms of the system of orthogonal functions $g_k(\eta)$ and postulate that each of the components represent an outgoing wave. This derivation may appear simpler, but it is applicable only if the contour C is a line u = const while in principle the formulation (28) is valid for any contour.

Further formulae are needed, if one chooses for the test functions ω in Eq. (28) the expressions χ_m defined in Eq. (29)

$$\chi_m = H_m^{(2)}(\mu r)\,\cos(m\theta)$$

We assume that a routine for generating Bessel functions of real argument is available. Let $Y_m(z)$ denote Bessel functions of the second kind. One has

$$H_m^{(2)}(z) = J_m(z) - i\,Y_m(z) \tag{53}$$

We note the following relations

$$H_m^{(2)}(z)' = dH_m^{(2)}/dz = 1/2\ H_{m-1}^{(2)}(z) - 1/2\ H_{m+1}^{(2)}(z) \tag{54}$$

$$H_{m-1}^{(2)}(z) + H_{m+1}^{(2)}(z) = (2m/z) H_m^{(2)}(z) \tag{55}$$

In evaluating

$$\int_0^{2\pi} \chi_m \phi_n \, ds$$

one uses, again, Eq. (50) to express ϕ_n ds. The radius r and the angle θ, which are needed in the evaluation of χ_m are expressed in terms of η (and u) by Eqs. (15) and (16), the trigonometric functions of θ ultimately by Eq. (18). It is best first to compute

$$\zeta = \xi + i\eta = \log u + i\eta$$

and then to proceed with complex arithmetic (along the curve C for which the evaluation is carried out, one has u = const.)

In addition, one must evaluate $(d\chi_m/du)ds$. One has

$$\frac{d}{dn}(\chi_m) ds = \frac{\partial}{\partial r}(\chi_m) r \, d\theta - \frac{\partial}{\partial \theta}(\chi_m) \frac{1}{r} dr$$

With Eq. (21) one then obtains for a line u = const

$$\frac{d}{dn}(\chi_m) ds = \left\{ \frac{\partial}{\partial r}(\chi_m) r \, \mathrm{Re}\left(\frac{g'}{g}\right) + \frac{\partial}{\partial \theta}(\chi_m) \, \mathrm{Im}\left(\frac{g'}{g}\right) \right\} d\eta$$

Hence, after substitution of the specific expressions (29)

$$\frac{d}{dn}(\chi_m) ds = \left\{ r \, H_m^{(2)} \cos(m\theta) \mathrm{Re}\left(\frac{g'}{g}\right) - m \, H_m^{(2)} \sin(m\theta) \mathrm{Im}\left(\frac{g'}{g}\right) \right\} d\eta \tag{56}$$

An alternative, more symmetric form, is obtained by using Eqs. (54), (55), and (22)

$$\frac{d}{dn}(\chi_m)ds = (\mu/2)\left\{H^{(2)}_{m-1}(\mu r)[\cos((m-1)\theta)Re(g') - \sin((m-1)\theta)Im(g')]\right.$$

$$\left. -H^{(2)}_{m+1}(\mu r)[\cos((m+1)\theta)Re(g') + \sin((m+1)\theta)Im(g')]\right\} d\theta \tag{57}$$

Now the integral Eq.(51), with ω given by χ_m can be evaluated. One needs for this purpose Eqs. (50) and (56) or (57).

A formula for the approximate far field conditions of Bayliss, Gunzberger and Turkel is obtained by the following considerations. The fact that particular solutions representing outgoing waves in terms of Mathieu functions have the asymptotic form for large values of u

$$\psi_k(u,\eta) = g_k(\eta)\exp(-i(\mu/2)u)[u^{-1/2} + c_{1,k}\, u^{-3/2} + \dots]$$

(see Eqs. (38) and (40)) implies that ϕ is asymptotically given by

$$\phi = \exp(-i(\mu/2)u)[u^{-1/2}\, h_1(\eta) + u^{-3/2}\, h_2(\eta) + \dots] \tag{58}$$

where $h_1(\eta)$ is arbitrary and function h_i, $i > 1$ are expressed in terms of functions h_j and their derivatives with subscripts $j < i$. The operation

$$\left(\frac{\partial}{\partial u} + \frac{5}{2u} + \frac{i\mu}{2}\right)\left(\frac{\partial}{\partial u} + \frac{1}{2u} + \frac{i\mu}{2}\right)$$

applied to the first two terms in Eq. (58) gives zero for every value of η. Writing this operation in detail, one obtains

$$\phi_{uu} + \phi_u\left(\frac{3}{u} + i\mu\right) + \phi\left(-\frac{\mu^2}{4} + \frac{3}{4u^2} + \frac{3}{2}\frac{i\mu}{u}\right) = 0 \tag{59}$$

This equation is satisfied for any function ϕ which has the form of the first two terms in Eq. (58). The term $\partial^2\phi/\partial u^2$, which does not fit the concept of a boundary condition for a second order partial differential equation is eliminated by means of the governing Eq. (27). Then one obtains

$$\phi_u\left(\frac{2}{u} + i\mu\right) - \frac{1}{u^2}\phi_{\eta\eta} + \phi\left[\frac{3}{4u^2} + \frac{\mu^2}{2u^2}\cos 2\eta + \frac{3i\mu}{2u} - \frac{\mu^2}{2} - \frac{\mu^2}{4u^4}\right] = 0 \tag{60}$$

This condition is applied along a line u = const for $0 \leq \eta \leq 2\pi$. The presence of $\phi_{\eta\eta}$ can be taken as an indication that this formulation, although it has local character, is an approximation to conditions of a global nature. If one forms analogous conditions on the basis of additional terms in Eq. (58), then one encounters higher derivatives with respect to η.

SECTION VI

RELATION BETWEEN DIFFERENT FAR FIELD CONDITIONS

It is convenient to symbolize the function $\phi(\eta)$, $\phi_u(\eta)$, $\psi(\eta)$, $\psi_u(\eta)$, $\chi(\eta)$, $\chi_u(\eta)$ and cos $m\eta$ (or sin $m\eta$), which are actually elements of a function space, by vectors. The integrations needed in the far field conditions are then considered as scalar products, which will be written as products of a row and a column matrix. One of the far field conditions then appears in the form

$$\left[\leftarrow \omega(u,\eta) \rightarrow\right] \begin{bmatrix} \uparrow \\ \phi_u(u,\eta) \\ \downarrow \end{bmatrix} - \left[\leftarrow \omega_u(u,\eta) \rightarrow\right] \begin{bmatrix} \uparrow \\ \phi(u,\eta) \\ \downarrow \end{bmatrix} = 0$$

Here one must substitute for $\omega(\eta)$ all functions of either the set $\psi_k(\eta)$ or the set $\chi_k(\eta)$. Combining symbolically these row vectors into matrices, one then obtains

$$\begin{bmatrix} \leftarrow & \cdots & \rightarrow \\ \leftarrow & \psi_k(u,\eta) & \rightarrow \\ \leftarrow & \cdots & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \phi_u(u,\eta) \\ \downarrow \end{bmatrix} - \begin{bmatrix} \leftarrow & \cdots & \rightarrow \\ \leftarrow & \psi_{k,u}(u,\eta) & \rightarrow \\ \leftarrow & \cdots & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \phi(u,\eta) \\ \downarrow \end{bmatrix} = 0 \tag{61}$$

and

$$\begin{bmatrix} \leftarrow & \cdots & \rightarrow \\ \leftarrow & \chi_k(u,\eta) & \rightarrow \\ \leftarrow & \cdots & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \phi_u(u,\eta) \\ \downarrow \end{bmatrix} - \begin{bmatrix} \leftarrow & \cdots & \rightarrow \\ \leftarrow & \chi_{k,u}(u,\eta) & \rightarrow \\ \leftarrow & \cdots & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \phi(u,\eta) \\ \downarrow \end{bmatrix} = 0 \tag{62}$$

In all of these equations u is constant. Because of the special form of the functions ψ_k (Eq. (30)), one can rewrite Eq. (61):

$$D_1 \; B \begin{bmatrix} \uparrow \\ \phi_u(u,\eta) \\ \downarrow \end{bmatrix} - D_2 \; B \begin{bmatrix} \uparrow \\ \phi(u,\eta) \\ \downarrow \end{bmatrix} = 0 \tag{63}$$

where D_1 and D_2 are diagonal matrices in which the elements $D_{1,kk}$ and $D_{2,kk}$ are respectively given by $f_k(u)$ and $f_k'(u)$ and where

$$B = \begin{bmatrix} \leftarrow & \cdot & \cdot & \cdot & \rightarrow \\ \leftarrow & & g_k(\eta) & & \rightarrow \\ \leftarrow & \cdot & \cdot & \cdot & \rightarrow \end{bmatrix} \tag{64}$$

Because of the orthogonality of the functions $g_k(\eta)$ one finds that the eigenvectors $p_k(\eta)$ of the eigenvalue problem

$$D_1 \; B \begin{bmatrix} \uparrow \\ p_k(\eta) \\ \downarrow \end{bmatrix} - \lambda_k D_2 B \begin{bmatrix} \uparrow \\ p_k(\eta) \\ \downarrow \end{bmatrix} = 0 \tag{65}$$

are given by

$$p_k(\eta) = g_k(\eta) \tag{66}$$

and that the eigenvalues are

$$\lambda_k = \frac{D_{1,kk}}{D_{2,kk}} = \frac{f_k(u)}{f_k'(u)} \tag{67}$$

The relation between the formulations Eq. (61) and (62) is established on the basis of Eq. (4[illegible]) which is now written in the form

$$\begin{bmatrix} \cdot \\ \text{[illegible]}_k(u,\text{[illegible]}) \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \text{[illegible]}(u,\text{[illegible]}) \\ \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \text{[illegible]}_k \\ \cdot \end{bmatrix}$$

where [illegible]$_k$ is a vector in the ordinary sense, for which the [illegible]th component is given by [illegible]$_{[illegible],k}$. Hence

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \text{[illegible]}_{k,u}(u,\text{[illegible]}) & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} & & \\ & \text{[illegible]}_{\ell,k} & \\ & & \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \text{[illegible]}_{,u}(u,\text{[illegible]}) & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{68}$$

where β denotes a constant matrix in the usual sense for which the element with subscripts k and ℓ is given by $\beta_{k,\ell}$. One recognizes, by substituting the last relation into Eq. (61), that Eq. (62) arises by premultiplying Eq. (61) by the matrix $[\beta]$. The matrix $[\beta]$ is well behaved. The relative eigenvectors and eigenvalues of the two matrices occurring in Eqs. (61) and (62) are therefore the same. After a discretization this result will hold only approximately, the deviations from the ideal case will serve to characterize the effect of the discretization.

In discussing the approximate far field conditions of Bayliss, Gunzburger, and Turkel, Eq. (60), it suffices if one considers one component of the development of $\phi(u,\eta)$ and $\phi_u(u,\eta)$ along a line u = const with respect to the functions $g_k(\eta)$. We assume accordingly that

$$\phi(u,\eta) = q_k(u)\ g_k(\eta)$$

then (69)

$$\phi_u(u,\eta) = q_k'(u)\ g_k(\eta)$$

The function $q_k(u)$ replaces the function $f_k(u)$ in Eq. (30). Then one obtains from the approximate far field conditions Eq. (60)

$$\frac{dq_k}{du}\ g_k\left(\frac{2}{u} + i\mu\right) + q_k\left(-\frac{1}{u^2}\frac{d^2g_k}{d\eta^2} + \left(\frac{3}{4u^2} + \frac{\mu^2}{2u^2}\cos 2\eta + \frac{3i\mu}{2u} - \frac{\mu^2}{2} - \frac{\mu^2}{4u^4}\right)g_k\right) = 0$$

and by substituting Eq. (32)

$$\frac{dq_k}{du}\left(\frac{2}{u} + i\mu\right) + q_k\left(\frac{\lambda_k}{u^2} + \frac{3}{4u^2} + \frac{3}{2}\frac{i\mu}{u} - \frac{\mu^2}{2} - \frac{\mu^2}{4u^4}\right) = 0 \qquad (70)$$

The same relation is obtained if one assumes that q_k has the form

$$q_k(u) = (\alpha_1\, u^{-1/2} + \alpha_2\, u^{-3/2})\ \exp(-i(\mu/2)u)$$

where α_1 and α_2 are arbitrary constants. Any expression of this form satisfies the second order differential equation

$$\left(\frac{d}{du} + \frac{5}{2u} + \frac{i\mu}{2}\right)\left(\frac{\partial}{\partial u} + \frac{1}{2u} + \frac{i\mu}{2}\right)q_k(u) = 0$$

In the derivation of Eq. (60), the condition has been imposed that $\phi(u,\eta)$ (here given by Eq. (69)) satisfy the partial differential equation (27) at the value of u which gives the curve C. This leads to Eq. (37) with $f_k(u)$ replaced by $q_k(u)$. Eq. (70) is then obtained by eliminating from the last expression $d^2q_k(u)/du^2$ by means of Eq. (37).

The eigenfunctions pertaining to the approximate conditions Eq. (60) are again the functions $g_k(\eta)$, but the eigenvalues $f_k(u)/f_k'(u)$ are replaced by the expression obtained from Eq. (70). This equation arises by assuming that the function $g_k(u)$ which replaces $f_k(u)$ has the correct asymptotic form, and satisfies the differential equation for f for the chosen finite value of u.

SECTION VII
DISCRETIZATION

So far the far field conditions have been formulated as if the functions ϕ involved can be represented exactly, and, at least in principle, it has been assumed that the set of test functions to be used in the far field conditions is infinite. Actually, the functions ϕ must be approximated by means of a finite number of expressions and the set of test functions for the far field conditions can only be finite.

An ideal case arises if ϕ satisfies the Helmholtz equation even in the computed part of the flow field. Of course, one would then proceed analytically, but this case serves well in comparing different formulations. If one represents the solution ϕ by a superposition of particular solutions ϕ_k (defined in Eq. (30)), the Helmholtz equation and the far field conditions are satisfied. To satisfy the boundary conditions at the plate one must use an infinite series in these particular solutions. This series converges throughout the flow field. In practice, one will truncate the series. The only error which arises here is caused by the approximation to the boundary conditions at the plate.

In cases of practical interest, the equation obtained by transforming the original problem differs from the Helmholtz equation, particularly in the vicinity of the plate. (It is of course assumed that at great distances the transformed equation is very close to the Helmholtz equation). The transformation of the flow field to a half infinite strip in the ξ,η plane may nevertheless be practical, as the transformation provides a suitable mesh in the physical plane. The fact that in this report the analysis is carried out in the ξ,η (or rather in the u,η) system ought to be considered as incidental; it facilitates the integrations necessary in the formulation of the far field conditions.

In the u,η-plane, ϕ is periodic along lines u = const. For a function which, in addition, is analytic along such lines, a representation in terms of analytic periodic functions has excellent convergence properties. (The coefficients a_n (sav) of a Fourier series decrease for sufficiently large n faster than any negative power of n.) One such representation is by means of a truncated series in Mathieu functions

$$\phi(u,\eta) = \sum q_k(u)\ g_k(\eta) \tag{71}$$

where, so far, the coefficients q_k are unknown functions of u. If the problem is governed by the Helmholtz equation, then this hypothesis leads back to a representation of ϕ by a superposition of functions ψ_k. The only errors arise because general boundary conditions at the plate cannot be satisfied exactly by a truncated series.

For a more general differential equation one is led to a system of second order ordinary differential equations for the functions $q_k(u)$. At large distances the functions $q_k(u)$ will have the same behavior as the functions $f_k(u)$, which is determined by Eq. (37). For u large, one has approximately

$$f_k = u^{-1/2} \exp(-i(\nu/2)u)$$

This means that the relative magnitude of these functions for different values of k remains the same (unless k is very large, then the asymptotic development is not applicable). Therefore, the number of functions $g_k(\eta)$ in the representation of ϕ cannot be reduced as one proceeds from intermediate to large values of u. Such a reduction would be possible for the Laplace equation. More functions $g_k(\eta)$ may be needed close to the plate in order to express local effects of the boundary conditions.

To reduce from a certain value of u on the number of functions q_k, one will set functions $g_k(u)$ for the expressions that are to be disregarded, equal to zero. This implies that at this value of u one introduces the condition $q_k = 0$ for those values of q_k. The functions $q_k(u)$ resemble the functions $f_k(u)$ and those functions are nearly monotonic, for $\mu^2 u^2/4 < \lambda_k$, for larger values of u, they are oscillatory. If one imposes the condition $q_k(u) = 0$ for a value of u where the function $f_k(u)$ is oscillatory, then there is a possibility that large errors are introduced because of a resonance phenomenon. To avoid this, one will reduce the number of functions $g_k(\eta)$ used for the representation of ϕ only at values of u for which, for the value of k in question, f_k is nonoscillatory.

If ϕ is represented at each value of u in terms of Mathieu functions $g_k(\eta)$, then one must provide a representation for these functions, obtained for instance by the procedure shown in Section IV. This may well be a worthwhile approach. In this case, no approximations are encountered in the far field conditions.

The periodicity of ϕ with respect to η is alternatively expressed by the use of a truncated Fourier series

$$\phi(u,\eta) = \sum q_k(u)\ h_k(\eta) \qquad (72)$$

where

$$h_0(\eta) = 2^{-1/2}$$

$$h_{2k}(\eta) = \cos(k\eta), \quad h_{2k-1}(\eta) = \sin(k\eta); \quad k=1,\ldots 2$$

In the hypothesis (72) a certain minimum number of functions $g_k(\eta)$ are needed in order to give a sufficiently accurate representation of ϕ for large values of u. We have also shown that the functions $g_k(\eta)$ can be represented by a Fourier series in η. In this sense the hypothesis Eqs. (71) and (72) are related to each other. All Fourier components (implicitly) present in Eq. (71) must also be present in Eq. (72) in order to give a desired accuracy for large values of u. This means that the maximum value of k occurring in Eq. (72) must exceed that occurring in Eq. (71). The number of additional terms depends upon the value of μ, it increases with μ. For the detailed discussion of this report we have used the function h defined in Eq. (72). Sometimes, however, one might find it desirable to use different functions for $h_k(\eta)$. The good convergence properties of the Fourier series hold only if the functions to be represented are analytic along a line u = const. If they have discontinuities (for instance in higher derivative) or even if they are analytic and strongly peaked (that is, if in the complex η-plane poles lie close to the real η axis) then the convergence will be poor. For subsonic flows the governing differential equation has analytic coefficients and ϕ will be analytic except at the plate where the boundary conditions may introduce singularities. For flows with an embedded supersonic region even the coefficients of the differential equation need not be analytic (especially toward the end of the supersonic region). Under these circumstances one will prefer representations which are more suitable to reflect local properties of ϕ, for instance finite difference or finite element approaches. This expresses itself in the choice of the functions $h_k(\eta)$. For the value of u, at which the far field conditions are applied, one deals, of course, with analytic functions. Adopting the notation of the preceeding section, we write the expression Eq. (72) in the form

$$\begin{bmatrix} \uparrow \\ \phi(u,\eta) \\ \downarrow \end{bmatrix} = \begin{bmatrix} \vdots & & \uparrow & & \vdots \\ \cdot & \cdots & h_\ell(\eta) & \cdots & \cdot \\ \vdots & & \downarrow & & \vdots \end{bmatrix} \begin{bmatrix} \uparrow \\ q(u) \\ \downarrow \end{bmatrix} \tag{73}$$

where the number of functions $h_\ell(\eta)$ and of components of the vector $\vec{q}(u)$ is finite. The expression Eq. (71) arises by replacing in this equation $h_k(\eta)$ by $g_k(\eta)$.

Let us first study Eq. (61); in which the far field conditions are expressed by means of Mathieu function. They are written in the form of Eq. (63).

$$D_1 B \begin{bmatrix} \uparrow & & \uparrow \\ \cdots h_\ell(\eta) \cdots \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \vec{q}\,'(u) \\ \downarrow \end{bmatrix} - D_2 B \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \cdots h_\ell(\eta) \cdots \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \vec{q}(u) \\ \downarrow \end{bmatrix} = 0$$

The number of equations expressed by this formula equals the number of rows in the matrices D_1B and D_2B. This number must be chosen equal to the number of components in the vector $\vec{q}(u)$. Let this number be given by n_1. We substitute into this equation the expression (64) for B. Let

$$\begin{array}{c} \uparrow \\ n_1 \text{ rows} \\ \downarrow \end{array} \begin{bmatrix} \leftarrow \cdot \cdot \cdot \cdot \rightarrow \\ \leftarrow q_k(\eta) \rightarrow \\ \leftarrow \cdot \cdot \cdot \cdot \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \cdots h_\ell(\eta) \cdots \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \bar{B}_1 \qquad (74)$$

Let $\bar{D}_1$ and $\bar{D}_2$ be the matrices and D_1 and D_2 truncated in the same manner. Then one has

$$\bar{D}_1 \; \bar{B}_1 \begin{bmatrix} \uparrow \\ \vec{q}\,'(u) \\ \downarrow \end{bmatrix} - \bar{D}_2 \; \bar{B}_1 \begin{bmatrix} \uparrow \\ q(u) \\ \downarrow \end{bmatrix} = 0 \qquad (75)$$

In the preceding section we had characterized the far field conditions by the relative eigenvectors and eigenvalues of the two governing matrices (here $\bar{D}_1 \; \bar{B}_1$ and $\bar{D}_2 \; \bar{B}_1$). Accordingly, we consider the problem

$$\bar{D}_1 \; \bar{B}_1 \; \vec{V}_\ell - \nu_\ell \; \bar{D}_2 \; \bar{B}_1 \; \vec{V}_\ell = 0 \qquad (76)$$

where ν_ℓ is the ℓ^{th} eigenvalue and $\vec{V}_\ell$ the ℓ^{th} eigenvector. Obviously,

$$\vec{V}_\ell = \bar{B}_1^{-1} \; \vec{e}_\ell \qquad (77)$$

where $\vec{e}_\ell$ is the unit vector in the direction of the ℓ^{th} component of $\vec{V}$ (i.e., the k^{th} component of $\vec{e}_\ell$ is given by $\delta_{\ell,k}$). Then

$$\nu_\ell = \bar{D}_{1,\ell\ell} / \bar{D}_{2,\ell\ell} \qquad (78)$$

This result holds whether $\bar{B}_1$ is an infinite or a finite matrix. If $h_k(\eta) = g_k(\eta)$, then because of the orthogonality of the $g_k(\eta)$ the matrix $\bar{B}_1$ is the unit matrix (provided that the functions $g_k(\eta)$ are normalized). Then

$$\vec{V}_\ell = \vec{e}_\ell$$

The function $\phi(u,\eta)$ pertaining to this eigenvector is given by

$$\begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ \vdots & & & & \vdots \\ \cdots & & g_k(\eta) & & \cdots \\ \vdots & & & & \vdots \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \vec{e}_\ell \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow \\ g_\ell(\eta) \\ \downarrow \end{bmatrix}$$

This is the result obtained in Eq. (66). It holds whether the matrix $\bar{B}_1$ is truncated or not. In this case no error will arise in the far field conditions.

Assume next that the functions $h_\ell(\eta)$ are the terms of a Fourier series. Let B_1 be the matrix $\bar{B}_1$ without truncation. Then one finds in the j^{th} row of the matrix B_1 the Fourier coefficients of g_j. The eigenfunctions are given, as before, by the ℓ^{th} column of B_1^{-1}. For the nontruncated matrix

$$B_1^{-1} = B_1^{T}$$

(where T stands for transpose). This is seen as follows. The matrix

$$\begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ \vdots & & & & \vdots \\ \cdots & & h_j(\eta) & & \cdots \\ \vdots & & & & \vdots \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix}$$

maps an element of the vector space (in which $\vec{V}$ lies) into a function space spanned by the individual functions $h_j(\eta)$. Specifically, it generates the functions which has the components of $\vec{V}$ as coefficients of the Fourier decomposition. On the other hand

$$\begin{bmatrix} \leftarrow \cdots & \cdots \rightarrow \\ \leftarrow \cdots & h_j(\eta) \rightarrow \\ \leftarrow \cdots & \cdots \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ p(\eta) \\ \downarrow \end{bmatrix}$$

yields the coefficients of the Fourier decomposition of $p(\eta)$ because the $h_j(\eta)$ are assumed to belong to an orthogonal set. It follows from this description that

$$\begin{bmatrix} \leftarrow \cdots & \cdots \rightarrow \\ \leftarrow \cdot h_j(\eta) & \cdot \rightarrow \\ \leftarrow \cdots & \cdots \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \cdots & h_j(\eta) & \cdots \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \vec{V} \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow \\ \vec{V} \\ \downarrow \end{bmatrix}$$

and on the other hand that

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vdots \;\cdot\; \cdot & h_j(\eta) & \cdot \;\cdot\; \vdots \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \cdots & \rightarrow \\ \leftarrow & h_j(\eta) & \rightarrow \\ \leftarrow & \cdots & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \vec{p}(\eta) \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow \\ \vec{p}(\eta) \\ \downarrow \end{bmatrix}$$

The same holds, of course, for any other orthonormal set $q_k(\eta)$. It then follows, because of Eq. (74), that

$$B_1^T B_1 = \begin{bmatrix} \leftarrow & \cdots\cdots & \rightarrow \\ \leftarrow & h_\ell(\eta) & \rightarrow \\ \leftarrow & \cdots\cdots & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vdots \;\cdot\; \cdot & g_k(\eta) & \cdot \;\cdot\; \vdots \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \cdots & \rightarrow \\ \leftarrow & g_k(\eta) & \rightarrow \\ \leftarrow & \cdots & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vdots \;\cdot\; \cdot & h_j(\eta) & \cdot \;\cdot\; \vdots \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

is the identity operator. Hence, the results announced above

$$B_1^T = B_1^{-1} \tag{79}$$

We mentioned that the j^{th} row of B_1 gives the components of a decomposition of g_j in terms of the orthogonal set $h(\eta)$. According to Eq. (77), the eigenvector $\vec{V}_j$ is given by the j^{th} column of $B_1^{-1} = B_1^T$. This means that for nontruncated matrices

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vdots \;\cdot\; \cdot & h_k(\eta) & \cdot \;\cdot\; \vdots \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \vec{V}_j \\ \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow \\ g_j(\eta) \\ \downarrow \end{bmatrix}$$

In the representation of the flow field discussed first, [illegible](u,η) is expressed for each value of u as a linear combination of Mathieu functions $g_\ell(\eta)$. The eigenvector $\vec{V}_\ell$ is then given by the unit vector $\vec{e}_\ell$ and the function represented by it is immediately given by $g_\ell(\eta)$. If $\phi(u,\eta)$ is represented for each value of u by a Fourier series, then the eigenvectors $\vec{V}_j$ has as components the Fourier coefficients of the function $g_j(\eta)$. It follows that for far field conditions in the form (63) (i.e., far field conditions which use Mathieu functions as test functions) the eigenvectors $\vec{V}_\ell$ obtained with these two choices of representations for $\phi(u,\eta)$ represent the same function $g_\ell(\eta)$. This holds only for nontruncated matrices, for Eq. (79) is subject to this restriction.

Eq. (79) is based upon the observation that $B_1^T B_1$ is the identity operator. For completeness, the meaning of the operator $\bar{B}_1^T B_1$ (where $\bar{B}_1$ denotes the truncated matrix) is explored in the Appendix.

In practical computations one need not evaluate the eigenvectors $\vec{V}_\ell$. They are introduced here to describe the working of the far field conditions. One has according to Eq. (77)

$$\vec{V}_\ell = \bar{B}_1^{-1} \vec{e}_\ell \tag{80}$$

Let $\tilde{g}_\ell$ be the function that is represented by $\vec{V}_\ell$.

$$\tilde{g}_\ell(\eta) = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ & h_k(\eta) & \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \bar{B}_1^{-1} \end{bmatrix} \vec{e}_\ell$$

The function $\phi(u,\eta)$ can be expressed as a superposition of these eigenfunctions

$$\phi(u,\eta) = \sum \tilde{q}_\ell(u)\ \tilde{g}_\ell(\eta) \quad ; \quad \phi_u(u,\eta) = \sum \tilde{q}_\ell'(u)\tilde{g}_\ell(\eta)$$

Then one has for the value of u which gives the curve C

$$q_\ell(u)/q'_\ell(u) = \nu_\ell$$

where according to Eqs. (78) and (67)

$$\nu_\ell = D_{1,\ell\ell}/D_{2,\ell\ell} = f_k(u)/f'_k(u) \tag{81}$$

For truncated matrices the functions $q_k(v)$ are in general not identical with the functions $q_k(v)$ for nontruncated matrices.

For the practical application of the far field conditions (75) one must form the matrix $\bar{B}_1$. (In Eq. (80) its inverse is encountered.) We observed above that the j^{th} row of B_1 consists of the Fourier coefficients of the function $q_j(v)$. In $\bar{B}_1$ this set of coefficients is truncated.

The coefficients of a Fourier development of the functions $q_j(v)$ have been used previously to provide a representation of these Mathieu functions. They arise by solving the eigenvalue problem Eq. (35) for the infinite matrices $M^{(i)}$. In practice these matrices are truncated but only to the extent that all Fourier coefficients of those functions $q_j(v)$ which are used in the computations are found with a prescribed accuracy. Actually, a more drastic truncation is admissible in the present context. We observed above that the number of terms occurring in the sum of Eq. (72) is larger if one represents ϕ by a Fourier development than if one uses Mathieu functions $q_k(v)$. The last few eigenvectors are inaccurate in any case because of the truncation of $\bar{B}_1$. This can be tolerated because the number of functions $h_k(v)$ (of Fourier components) is supposed to be large enough, so that in the (conceptual) decomposition of ϕ the last few eigenvectors occur only with very small coefficients.

The accuracy of the formulation of the far field conditions need not exceed the accuracy with which the partial differential equation is satisfied. If one approximates the function $\phi(u,v)$ by a truncated series of the form (71) then the resulting system of ordinary differential equations is simply given by Eq. (31) with f_k replaced by q_k.

If one represents $\phi(u,\eta)$ in the form (72) (in which the functions $h_k(\eta)$ are the terms of a Fourier series) then one finds by substitution into the partial differential Eq. (27)

$$\vec{q}'' + u^{-1}\vec{q}' + (\omega^2/4)(1+u^{-4})\vec{q} - u^{-2}M\vec{q} = 0 \tag{82}$$

here $\vec{q}(u)$ is a vector valued function whose k^{th} component is given by the function $q_k(u)$ occurring in Eq. (72). The matrix M is given by the matrices $M^{(i)}$ defined in Eq. (36). Some modification of the indices is needed because the Eqs. (36) take the symmetries of the problem into account.

Eq. (82) holds, in the first place, for the infinite system. But, because we use in Eq. (72) a Fourier series truncated to n_1 terms, one must truncate also the matrix M to a size n_1 by n_1. (This was the form in which the problem has originally been programmed.)

The matrix $\bar{B}_1$ encountered in the far field conditions (when one uses Mathieu functions as test functions and a truncated Fourier development for the representation of ϕ), contains in its j^{th} row the truncated set of Fourier coefficients for the Mathieu function $g_j(\eta)$. They are computed by solving the eigenvalue problem for the infinite matrices $M^{(i)}$, Eqs. (36). This has been stated before.

It is consistent with the approximation used in solving the partial differential equation if one approximates the rows of the matrix $\bar{B}_1$ by the components of the eigenvectors of the matrices $M^{(i)}$ truncated to the dimension n_1 by n_1.

The eigenvalue problem Eq. (35) yields also the eigenvalues λ_k which are used in Eq. (31) to determine f_k/f_k' (Eq. 81) for the value of u which represents the outer boundary C. The detailed procedure is described in conjunction with Eqs. (37) through (41). If one truncates the matrices $M^{(i)}$ to n_1 by n_1 matrices, then there will be an error in the later eigenvalues λ_k and in the values f_k/f_k'. Such an error is admissible.

The error encountered in this procedure can be visualized if one computes the angle between the approximate and the exact eigenvectors $q_k(\eta)$ and the relative error in the eigenvalues. The functions $q_k(\eta)$ are given by their Fourier coefficients, here combined into vectors. Consider two vectors $\vec{V}_1$ and $\vec{V}_2$ with components a_i and b_i. The scalar product is defined by

$$[\vec{V}_1, \vec{V}_2] = \sum_i a_i b_i \tag{83}$$

The angle is then defined by

$$\cos \beta = \frac{[\vec{V}_1, \vec{V}_2]}{[\vec{V}_1, \vec{V}_1]^{1/2} [\vec{V}_2, \vec{V}_2]^{1/2}} \tag{84}$$

In the present case the vector $\vec{V}_2$ which represents $q_k(\eta)$ has only n_1 components. Accordingly, one has $a_i = 0$, $i > n_1$. The vector $\vec{V}_1$ which represent $g_k(\eta)$ has an infinite number of components, but the Fourier series converges very well. The relative error in the eigenvalues is given by

$$\frac{|\lambda_1 - \lambda_2|}{|\lambda_1|}$$

The approximate far field conditions of Bayliss, Gunzburger and Turkel Eq. (60) arise from the assumption that ϕ has the form of Eq. (58). This leads to Eq. (59). In this equation ϕ_{uu} is eliminated by means of the original partial differential equation. Of course, the requirement that the partial differential equation be satisfied is needed only for the value of u which corresponds to the outer boundary C. The approximation to the condition (60) for cases where the potential is approximated by the truncated Fourier series Eq. (72) is best found by retracing this derivation. Eq. (59) then assumes the form

$$\dot{q}''+\dot{q}'(3u^{-1}+i\omega)+\dot{q}(-\omega^2/4+(3/4)u^{-2}+(3/2)i\omega u^{-1})=0, \qquad (85)$$

From this equation $\dot{q}''$ is eliminated by means of Eq. (82), which is an expression of the partial differential equation. One obtains

$$\dot{q}^{1}(2u^{-1}+i\omega)+\dot{q}[-(\omega^2/2)(1+u^{-4}/2)+(3/4)u^{-2}+(3/2)i\omega u^{-1}]+u^{-2}M\dot{q}=0$$

where one must substitute the value of u for the contour C. This is the form of the Bayliss, Gunzburger, Turkel condition as it is used in the practical work.

In this equation the truncated matrix M appears again. For an analysis of the effect of this boundary condition (in conjunction with a representation of the flow field by a truncated Fourier series in η) we assume that $\dot{q}$ is represented by a linear combination of the pertinent eigenvectors $\vec{V}_\ell$ (which in turn give representations for the eigenfunctions $q_\ell(\eta)$. Let the coefficient be $\bar{q}_\ell(u)$. Because of the definition of $\vec{V}_\ell$ one has

$$\bar{M}\,\vec{V}_\ell = \lambda_\ell\,\vec{V}_\ell$$

where $\bar{M}$ denotes the truncated matrix M and λ_ℓ is the eigenvalue which belongs to $q_\ell(\eta)$. Therefore from Eq. (85)

$$q_\ell'(2u^{-1}+i\omega)+q_\ell[-(\omega^2/2)(1+u^{-4}/2)+(3/4)u^{-2}+(3/2)i\omega u^{-1}+\lambda_\ell u^{-2})=0 \qquad (86)$$

The application of the Bayliss, Gunzburger, Turkel condition thus has the following effect. For the value of u corresponding to the curve C one decomposes $\dot{q}(u,\eta)$ and $\dot{q}_u(u,\eta)$ in terms of the approximate eigenfunctions $\bar{q}_\ell(\eta)$ and then postulates that the ratio of the coefficients $q_\ell'(u)/q_\ell(u)$ assumes the value computed from Eq. (86). The error encountered in this procedure is due to the deviation of the approximate eigenfunctions $\bar{q}_\ell(\eta)$ from the exact eigenfunctions $q_\ell(\eta)$ and the error in the eigenvalues $f_\ell(u)/f_\ell'(u)$. This error is characterized by

$$\frac{|f_\ell(u)/f_\ell'(u) - \tilde{q}_\ell(u)/\dot{q}_\ell(u)|}{|f_\ell(u)/f_\ell'(u)|} \tag{87}$$

This expression must be evaluated for the values of ℓ used in the representation of the solution and for the value of u which gives the boundary C.

Finally, we study the effect of a truncation if one uses as test functions the expressions χ defined in Eq. (29). They are a product of Hankel functions in r and trigonometric functions in θ.

It was shown in Section VI that the formulation of the far field conditions in terms of test functions originates from a formulation in terms of test functions ψ by premultiplication with a matrix (Eqs. (61), (62), and (68)). We found, furthermore, that if one equation of the system Eq. (62) is satisfied for some value of u, it will be satisfied for any other value of u.

Nevertheless, one may have reservations about applying far field conditions in terms of functions χ if u is small. Along each line u = const the function $\chi(u,\eta)$ is analytic, and, therefore, a representation in terms of a Fourier series in η is possible. The coefficient of $\chi_j(u,\eta)$ appear in the j^{th} row of the matrix

$$\begin{bmatrix} \leftarrow & & \rightarrow \\ \cdots & \chi_j(u,\eta) & \cdots \\ \leftarrow & & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ & h_j(u) & \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

But the functions χ are singular at r = 0; one has $\chi_m \sim r^{-m} \cos m\theta$. Along a line u = const one has small values of r for $\theta = \pi/2$ and $\theta = 3\pi/2$, and there the functions χ have peaks, particularly pronounced for values of u close to 1. Then a considerable number of Fourier coefficients will be needed to give an acceptable approximation to χ. Because of the truncation of the Fourier series to n_1 terms the coefficients which are not negligible will be disregarded.

Another argument seems to support these reservations. The representation of a test function ψ in terms of function χ converges only if $r > 1$. The smallest value of r along a line u = const is reached at the y axis. There $r = y = \sinh \xi = \frac{1}{2}\,(u - \frac{1}{u})$. For $u < 1 + \sqrt{2}$, there exist points along a line u = const where the development of the functions ψ_j in terms of χ_j does not converge. (The reason is again the behavior of the functions χ_j.) One may then have doubt whether the formulation of the far field conditions in terms of the test function χ_k can replace the formulation in terms of test functions ψ_k. Of course, the underlying theory shows that the functions χ_k are legitimate test functions.

Substituting the hypothesis (72) (written in the form of Eq. (73)) into Eq. (62), one obtains, if one uses nontruncated expressions

$$B_2(u) \begin{bmatrix} \uparrow \\ \vec{q}(u) \\ \downarrow \end{bmatrix} - B_3(u) \begin{bmatrix} \vec{q}\,'(u) \end{bmatrix} = 0 \qquad (88)$$

where

$$B_2 = \begin{bmatrix} \leftarrow \cdots \chi_k(u,\eta) \rightarrow & \begin{matrix} \uparrow \\ h(\eta) \\ \downarrow \end{matrix} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vdots & \ldots h(\eta) \ldots & \vdots \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \leftarrow \cdots \chi_{k,u}(u,\eta) \cdots \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vdots & \ldots h(\eta) \ldots & \vdots \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

It follows from Eq. (68) and the definition (30) for ψ_k that Eq. (75) arises from the last equation by premultiplication with the matrix $[\beta]$. We had found that the j^{th} eigenvector of the matrix D_1B relative to the matrix D_2B is given by a vector whose components are the coefficients of the development of g_j with respect to the system of functions h_j, and that the j^{th} eigenvalue is given by $f_j(u)/f_j'(u)$. The eigenvalues and eigenvectors remain unchanged by the premultiplication by $[(\beta)]^{-1}$. These are, therefore, also the eigenvalues and eigenvectors of the matrix B_2 relative to B_3. Using the notation of Eq. (35) (but without the superscript i, which is introduced to distinguish between cases of different symmetries), one, therefore, has

$$D_1 B_1 \vec{V}_k - \frac{f_k'(u)}{f_k(u)} D_2 B_1 \vec{V}_k = 0$$

and, consequently, also

$$B_2 \vec{V}_k - \frac{f_k'(u)}{f_k(u)} B_3 \vec{V}_k = 0$$

We assert that for the first few eigenvectors this equation is rather well satisfied, even if the matrices B_2 and B_3 and the vectors $\vec{V}_k$ are truncated. This assertion is based on the fact that the truncation of the first few eigenvectors changes these eigenvectors only by a very small amount, because the components lost by the truncation are close to zero. The truncation of B_2 and B_3 to a finite number n_1 of rows means simply, that one considers only the first n_1 equations, rather than all of them. The truncation to a finite number n_1 of columns might appear more serious because the elements that are omitted are not small. However, in the original infinite matrix they are multiplied by the components of the eigenvector which are small, and which are now eliminated because of the truncation. Incidentally, the eigenvectors $\vec{V}_k$ with the largest eigenvalues will satisfy the truncated equations only poorly, because then the components lost by truncation are large. We emphasized above that the number n_1, which determines the size of the matrix, must be larger than the number of eigenfunctions which are important to represent the solution because n_1 represents the functions $g_k(\eta)$, which are important for the solution with sufficient accuracy. In other words, for $n > n_1$, the components $\vec{V}_{k,n}$ with $n > n_1$ of the vector $\vec{a}_k$ must be negligible for those vectors $\vec{V}_k$ which are important in the solution ψ. We, therefore, expect that a truncation of the matrices B_2 and B_3 is permissible even at fairly small values of u. However, it should be understood, that because of the peaks which occur in the function $\rho(u,\eta)$ for u close to 1, it is mandatory that the integrations needed in the determination of the matrix elements of B_2 and B_3 be carried out with sufficient precision.

To determine quantitatively to what extent these observations are correct, one forms the eigenvalues and eigenvectors of the truncated Matrix, B_2 relative to B_3. The elements of the matrices are complex because the functions ϕ_k are complex. For a comparison of the eigenvectors, one uses the Hermitian scalar product. If

$$V_1 = \begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \end{bmatrix}, \quad V_2 = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \end{bmatrix}$$

then,

$$[\vec{V}_1, \vec{V}_2] = \sum_i a_i b_i^* \tag{89}$$

where the asterisk denotes the conjugate complex. Then we set

$$\cos \phi = \frac{|[\vec{V}_1, \vec{V}_2]|}{[\vec{V}_1, \vec{V}_1]^{1/2}\,[\vec{V}_2, \vec{V}_2]^{1/2}} \tag{90}$$

It can be shown that $\phi = 0$ only if $V_1 = \text{const}\ V_2$. For the relative error in the eigenvalues, we form

$$\frac{\left|\lambda_k - \dfrac{f_k}{f_k'}\right|}{\left|\dfrac{f_k}{f_k'}\right|} \tag{91}$$

SECTION VIII
NUMERICAL RESULTS

In this section results which are of interest for practical work are collected.

Figure 1 shows a number of curves u = const. They can be considered as ideal boundaries, for they take the fact into account that the profile (which is the inner boundary) lies at or close to the x axis. After the transformation of the flow field into a strip of the ξ,η plane one obtains a particularly simple representation of the potential which holds throughout the flow field and displays clearly the dominant effects at large values of u (or ξ). This holds for the underlying steady flow field, as well as for superimposed harmonic perturbations. Deviations from these boundaries which may be desirable because of practical considerations are admissible. In fact, the far field conditions studied here hold for rather arbitrary boundaries. Which of these boundaries one chooses depends upon the character of the partial differential equation. In deriving the far field conditions the assumption is made, that in the field outside of this boundary one deals, after the necessary transformations have been carried out, with the Helmholtz equation.

Part of the transformations which lead to the Helmholtz equation is a Prandtl-Glauert transformation in the physical plane. The ellipses in the physical plane which by this transformation are mapped into the curve $u = 5$ are shown for different Mach numbers in Figure 2.

The constant μ which appears in the Helmholtz equation depends upon the reduced frequency $\mu' = \omega L/u$. The curves of Figure 3 show μ versus the Mach number for different values of μ'. This graph allows one to determine the range of values of μ which is of practical importance.

It was mentioned that an iterative solution of the Helmholtz equation is possible (but perhaps not practical) up to a frequency for which a solution with the homogeneous boundary condition $\phi_y = 0$ at the plate $\phi = 0$ at the far boundary exists. Figure 4 shows the pertinent values of $1/\mu$ versus u. (The frequency limitation becomes more stringent as the value of u for the outer boundary is increased.)

The remaining data refer to the far field conditions directly. Let us first repeat the leading ideas. The far field conditions are represented by an infinite system of linear equations. Two different forms of this system have been considered, namely the representation of the far field conditions in terms of products of ordinary and radial Mathieu functions in the η, ξ system, and the representation in terms of products of Hankel functions of r and trigonometric functions of θ. These two representations are equivalent to each other. In numerical work only a finite number of equations of these systems can be used. The equations retained are, of course, individually correct; the approximation lies in the fact that only a limited number of these conditions are satisfied. The far field conditions do not introduce errors, but they will admit certain errors.

The partial differential equation can be reduced to an infinite system of ordinary differential equations. Different forms of the system may arise if different representations are chosen. We have studied a representation in terms of Mathieu functions of η and radial Mathieu functions of ξ alternatively in terms of a development in trigonometric functions with respect to η with coefficients that depend upon u. Here errors arise by the truncation of the system. We have studied how the truncation of the system representing the partial differential equation interacts with the truncation of the system which gives the far field conditions.

The representation of the flow field in terms of Mathieu functions converges for the Helmholtz equation throughout the field. The only effect of the truncation of the system is seen in the boundary conditions at the plate which in general are satisfied only approximately. The representation in terms of Mathieu function clearly expresses the directional character of the radiation (provided that the inner boundary is given by a slit along the x axis). On account of these observations this representation of the flow field can be regarded as best from a theoretical point of view. If one uses this representation in a practical computation then the flow field at a distance can be approximated with a minimum number of terms. If one applies to such a representation a formulation of the far field conditions in terms of Mathieu function, then the far field conditions are perfectly satisfied.

For a procedure of this kind one needs, of course, a representation of the Mathieu functions. Table 1 gives for $\mu = 0.5$, $\mu = 2.0$ and $\mu = 3.5$, the Fourier coefficient of the development of $g_k^{(1)}(\eta)$ defined in Eq. (34), for the five lowest eigenvalues. Let us consider the coefficients $a_{\ell,k}$ as elements of a matrix, then one has for $\mu = 0$ only terms in the main diagonal (for one deals with the Laplace equation). As μ increases, the important terms are found in the vicinity of the main diagonal. The "spreading out" of the Fourier coefficients does not increase as one proceeds to Mathieu functions with larger eigenvalues. The presence of off-diagonal terms shows the directional effect of the radiation. It is present even for the lowest eigenvalue, but the effect is pronounced only for larger values of μ. (For the values of μ considered, it is fairly weak.) The eigenvalues are shown in Table 1 underneath in an extra row. They must be used to compute the values of $f_k(u)/f_k'(u)$.

The counterpart to a representation of the flow field and of the far field conditions in terms of Mathieu functions is a formulation in the physical plane in terms of Hankel functions of the radius r and trigonometric functions of the angle θ. Such a representation would be practical, if the inner boundary is given by a circle. If the inner boundary is a plate, it introduces an undesirable singularity at the origin. For a combination of a a representation of the flow field in terms of trigonometric functions of θ (this is the important part) and of the far field conditions in terms of Hankel trigonometric functions, the far field conditions would be perfectly satisfied.

We return to the formulation of the problem in the u, η plane and consider the representation of the flow field in terms of a development of ψ in terms of a truncated Fourier series in η with coefficients that depend upon u. Such a representation can express a solution in terms of Mathieu function which we consider as optimal only approximately. This fact is reflected in the expressions for the flow field that are compatible with the far field conditions. In the above analysis we have adopted the following characterization of the far field conditions. The functions ψ and ψ_u are represented as linear combinations of the relative eigenfunctions of the matrices B_1 and B_2, the ratio of the coefficients of corresponding terms in the linear combinations for ψ and ψ_u is then given by the relative eigenvalues. For the nontruncated system (in any representation) the eigenfunctions are Mathieu functions of η the relative eigenvalues are given by the ratio $f(u)/f'(u)$ obtained from the radial Mathieu functions. Table 2 refers to a representation of the flow field in terms of trigonometric functions in η while the far field conditions are expressed by means of Mathieu functions. (We have restricted ourselves to solutions which are symmetric with respect to the x and the y axis and then restricted ourselves to the first five terms, the last term represents eight cosine waves over the whole contour, u = const.). While the Fourier expansion of the Mathieu function is obtained by the eigenvectors or an infinite matrix one now deals with the eigenvectors of the same matrix truncated to the size given by the number of terms in the flow field. The error in the functions admitted arises because the eigenvectors and eigenvalues are not the same in the two cases.

Table 2 gives for the $\mu = 0.5$, $\mu = 2.0$ and $\mu = 3.5$, the angle between the exact and the approximate eigenvectors and the discrepancy in the eigenvalues. One cannot expect perfect agreement for later eigenfunctions, because the truncation of the Fourier series suppresses Fourier terms which are important in the exact eigenfunctions. The table shows that the first few eigenvectors and eigenvalues are indeed very close to the exact values. The errors increase with μ. The eigenvalues found here are needed in order to determine from Eq. (37) the ratio $f_k/(u)/f_k'(u)$ for the value of u chosen for the outer boundary. It would be equally justified if one computes the values of $f_k(u)/f_k'(u)$ for the infinite matrices. Then no error in the eigenvalues would be encountered.

If one uses the same representation for the flow field and expresses the far field conditions by a truncated system of Hankel functions in r and trigonometric functions in θ one will obtain slightly different results, because of the difference in the far field conditions. Table 3 shows for different values of μ and of u the angle of corresponding eigenvectors for the truncated and nontruncated matrices B_2 and B_3 and the error in the eigenvalues. In this case the evaluation of the matrix elements is critical. The matrix elements are integrals over periodic analytic functions of η. In this case the trapezoidal rule gives excellent results, but the integrands are rather peaked for small values of u and therefore the integration interval must be chosen small enough. Tables 3 have been computed for an interval of integration $\pi/32$ but actually for the values of u considered the results are the same if one uses the interval $\pi/16$. Serious discrepancies arise however for an interval $\pi/8$. The shortest wave length in the representation of ζ by a Fourier series in η truncated to 5 terms is $\pi/4$. An interval $\pi/8$ would give just 2 points to a full wave. For small values of u the peakedness of the integrands is increased because of the character of the Hankel functions, and then smaller integration intervals will probably be necessary. However, $u = 4$ is probably small enough for all practical purposes (see Fig. 1). The deviations from the ideal case are significant only for the highest eigenvalue.

Table 4 shows for $\mu = 3.5$ the angle between the eigenvectors for a flow field represented by means of trigonometric functions in u with the far field conditions expressed either by Mathieu functions (ordinary and radial) or by Hankel functions in r and trigonometric functions in θ. The angles are extremely small; for smaller values of μ, they are even smaller). This shows that the deviation of the eigenvectors from their ideal values shown in Tables 2 and 3 is primarily due to the discretization of the partial differential equation and not to the form of the far field conditions.

The approximate far field conditions of Bayliss Gunzburger and Turkel are mainly of interest if one represents the flow field in terms of a truncated Fourier development. We found that the eigenfunction decomposition which one finds for these boundary conditions is the same as if one imposes the far field conditions in the form of a truncated system of Mathieu functions and these errors are already shown in Table 2. The main error occurs because of the falsification of the ratio $f_k(u)/f_k'(u)$. Table 5 shows this error for different eigenfunctions, different values of μ and different values of u. As expected the error is small for sufficiently large u and μ. Assume for instance that errors in the fourth and fifth eigenvectors are unessential, because their contribution to the solution is small and that a falsification of f/f' for the third eigenvector by .05 is admissible, then boundary conditions cannot be used for $\mu = .5$ and u as large as 10, they can be applied for $\mu = 2$ down to $u = 5$ and for $\mu = 3.5$ down to $u = 3$.

SECTION IX
GENERAL OBSERVATIONS

One will ask to what extent the results of the present analysis can be carried over to more realistic problems.

The transformations which in the present case lead to the Helmholtz equation can be useful for higher values of μ, because it reduces the waviness due to waves that travel upstream.

The choice of the boundary of the computed flow field as a line u = const arises because of the shape of the inner boundary (here the plate). It can be expected to be advantageous for all two-dimensional airfoil problems, but some modifications are admissible.

The far field (especially its directional characteristics) is best represented in terms of Mathieu functions. This is a consequence of the shape of the inner boundary.

The idea of representing ϕ in terms of Mathieu functions throughout the flow field requires, of course, that one employs coordinates (or a mesh) given by the u, η-system. The idea of conformal mapping has been used even in the transonic case, therefore, a precedent to this procedure exists. The use of a Mathieu function development (as well as Fourier series development in η) may offer difficulties in the vicinity of the air foil, where the function ϕ may be rather peaked (because the character of the boundary conditions, and also because of the discontinuities which may be encountered in a transonic flow field).

The representatives of far field conditions in terms of Mathieu functions (even if the boundary is not exactly a line u = const) is quite feasible (it gives best accuracy with a minimum number of terms).

The representation of the flow field in terms of a Fourier development in η requires a few more terms than a representation in terms of Mathieu functions, if one wants to capture the directional characteristics of the far field equally well. We mentioned above that in the representations of the field close to the profile both forms have the same limitations. One expects that other representations of the flow field will have the same requirements for the representation of the distant field as a Fourier representation. They may have advantages in the vicinity of the profile.

Errors in satisfying at the far field conditions originate mainly by the truncation needed in the partial differential equation. Whether one represents the far field conditions by means of Mathieu functions or by Hankel functions in r and trigonometric functions θ has practically no effect, if the outer edge of the computed parts of the profile is given by a line $u > 4$.

The approximate far field conditions of Bayliss, Gunzburger, and Turkel can be applied only if μ and u are sufficiently large.

APPENDIX

According to Eq. (71)

$$\bar{B}_1 = \left\updownarrow n_1 \right. \begin{bmatrix} \leftarrow & \cdots & \rightarrow \\ \leftarrow & g_k(\eta) & \rightarrow \\ \leftarrow & \cdots & \rightarrow \end{bmatrix} \underbrace{\begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ \vdots & \cdots & h_j(\eta) & \cdots & \vdots \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix}}_{\longleftarrow \; n_1 \; \longrightarrow}$$

Let S_h and S_g the function subspaces spanned respectively by the function $h_j(\eta)$, and $g_k(\eta)$, $j = 1 \ldots n_1$, $k = 1 \ldots n_1$. The matrix

$$\begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ \vdots & \cdots & h_j(\eta) & \cdots & \vdots \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix}$$

maps a point of the n_1-dimensional $\vec{V}$ space into an element of the subspace S_h. The matrix

$$\begin{bmatrix} \leftarrow & \cdots & \rightarrow \\ \leftarrow & h_k(\eta) & \rightarrow \\ \leftarrow & \cdots & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ \vdots & \cdots & h_j(\eta) & \cdots & \vdots \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix}$$

is the identity mapping in the n_1-dimensional $\vec{V}$ space. The matrix

$$\begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ \vdots & \cdots & h_k(\eta) & \cdots & \vdots \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & \cdots & \rightarrow \\ \leftarrow & \cdots h_j(\eta) \cdots & \rightarrow \\ \leftarrow & \cdots & \rightarrow \end{bmatrix}$$

gives the orthogonal projection of a general element of the function space into the space S_h. Using Eq. (79) one readily demonstrates that it has the property of a projection operator

$$P^2 = P$$

Corresponding results hold for the set of orthonormal functions $g_k(\eta)$. Now we consider the sequence of mappings given by

$$B^T B\,\vec{V} = \begin{bmatrix} \leftarrow & h_j(\eta) & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \cdots\, g_k(\eta) \,\cdots \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & g_k(\eta) & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \cdots\, h_j(\eta) \,\cdots \\ \downarrow \end{bmatrix} \vec{V}$$

$$= \begin{bmatrix} \leftarrow & h_j(\eta) & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \cdots\, h_j(\eta) \,\cdots \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & h_j(\eta) & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \cdots\, g_k(\eta) \,\cdots \\ \downarrow \end{bmatrix}$$

$$\begin{bmatrix} \leftarrow & g_k(\eta) & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \cdots\, h_j(\eta) \,\cdots \\ \downarrow \end{bmatrix} \vec{V}$$

One starts, of course, on the right.

$$\begin{bmatrix} \uparrow \\ \cdots\, h_j(\eta) \,\cdots \\ \downarrow \end{bmatrix} \vec{V}$$

gives an element of the subspace S_h. The premultiplication by

$$\begin{bmatrix} \uparrow \\ \cdots\, g_k \,\cdots \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & g_k(\eta) & \rightarrow \end{bmatrix}$$

projects this element into the subspace S_g. The subsequent premultiplication by

$$\begin{bmatrix} \uparrow \\ \cdots\, h_j(\cdot) \,\cdots \\ \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & h_j(\eta) & \rightarrow \end{bmatrix}$$

gives a projection from S_g into S_h. The premultiplication by

$$\begin{bmatrix} \leftarrow & h_j(\cdot) & \rightarrow \end{bmatrix}$$

generates the Fourier coefficients. They form an element of the V space. The projections occurring here are orthogonal $B^T B$ will be an identity mapping, only if the subspaces S_h and S_g are the same.

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LEGENDA TO FIGURES

Figure 1 Curves u = const in the x,y plane after a Prandtl-Glauert Coordinate Distortion has been carried out.

Figure 2 Lines $u = 5$ in the x,y plane for different Mach numbers, if the Prandtl-Glauert Coordinate Distortion is not carried out. Such curves constitute ideal outer boundaries for a computed flow field.

Figure 3 Values of μ for different reduced frequencies μ' as function of the Mach number.

Figure 4 Values of u for which, at a given value $(1/\mu)$ an iterative procedure will theoretically converge (the admissible region has been shaded). It is assumed that in the iterative procedure the values of Φ at the outer boundary will be recomputed in each iteration step and then kept fixed during the flow field computation. The theoretical limit arises for a chosen value of u at the lowest frequency μ for which a standing wave flow with $\Phi = 0$ at the outer boundary and $\Phi_\eta = 0$ at the inner boundary can arise.

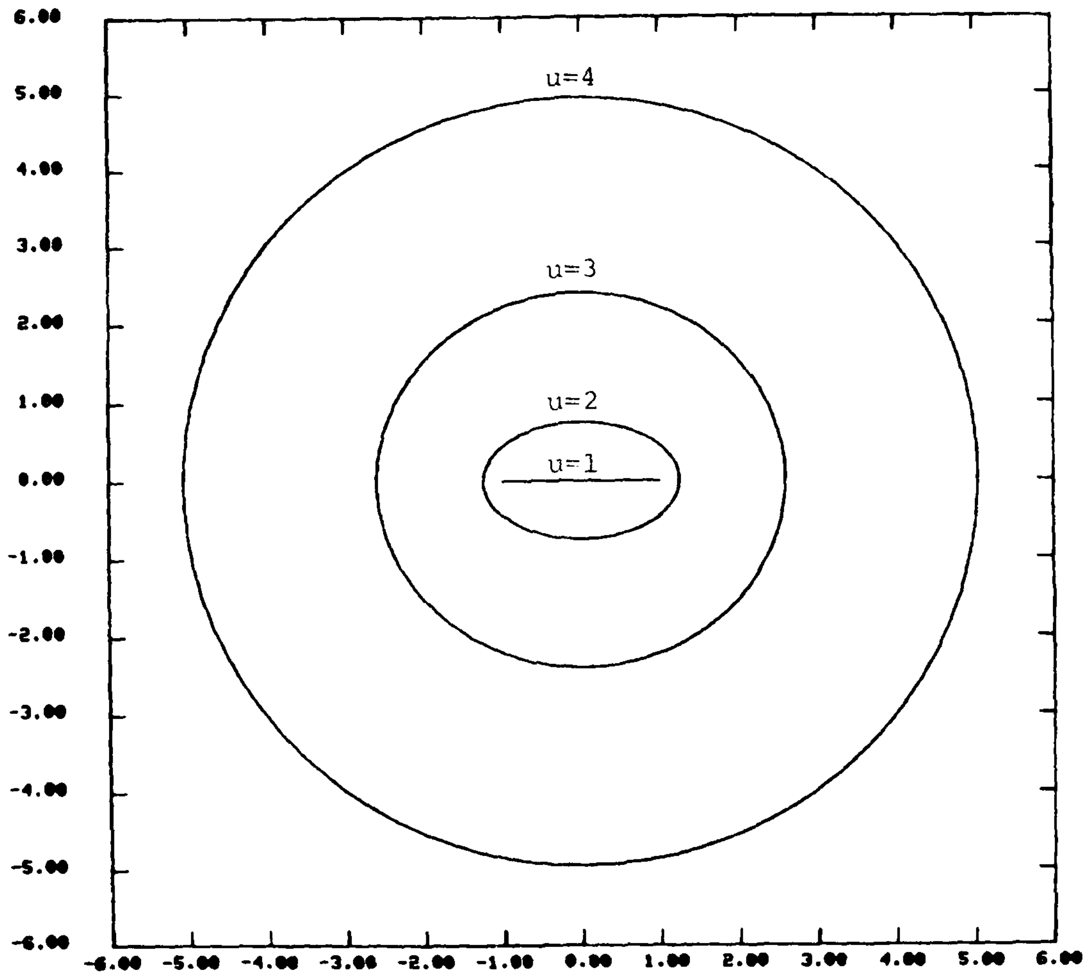

Figure 1. Constant u Contours in x, y Plane With Prandtl-Glauert Coordinate Distortion.

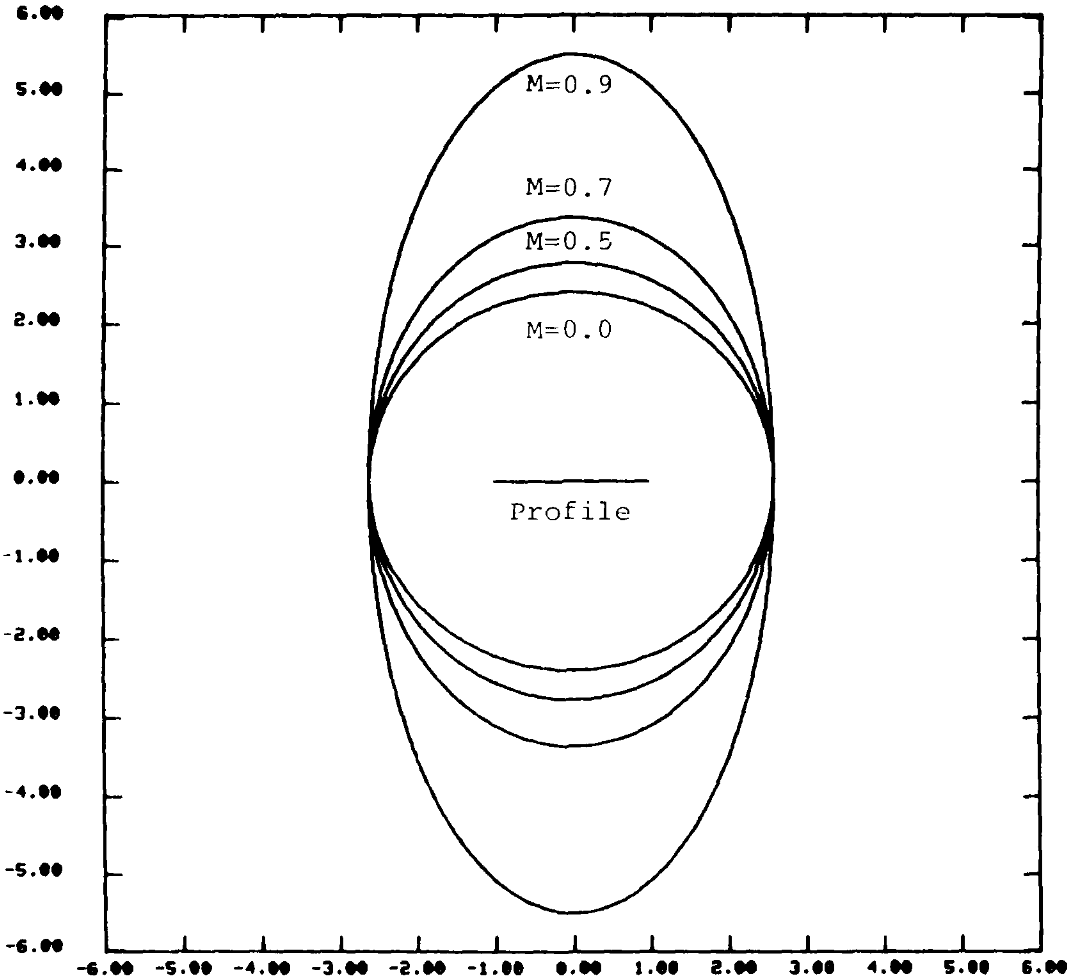

Figure 2. Mach Number Contours for u = 5 in x, y Plane Without Prandtl-Glauert Coordinate Distortion.

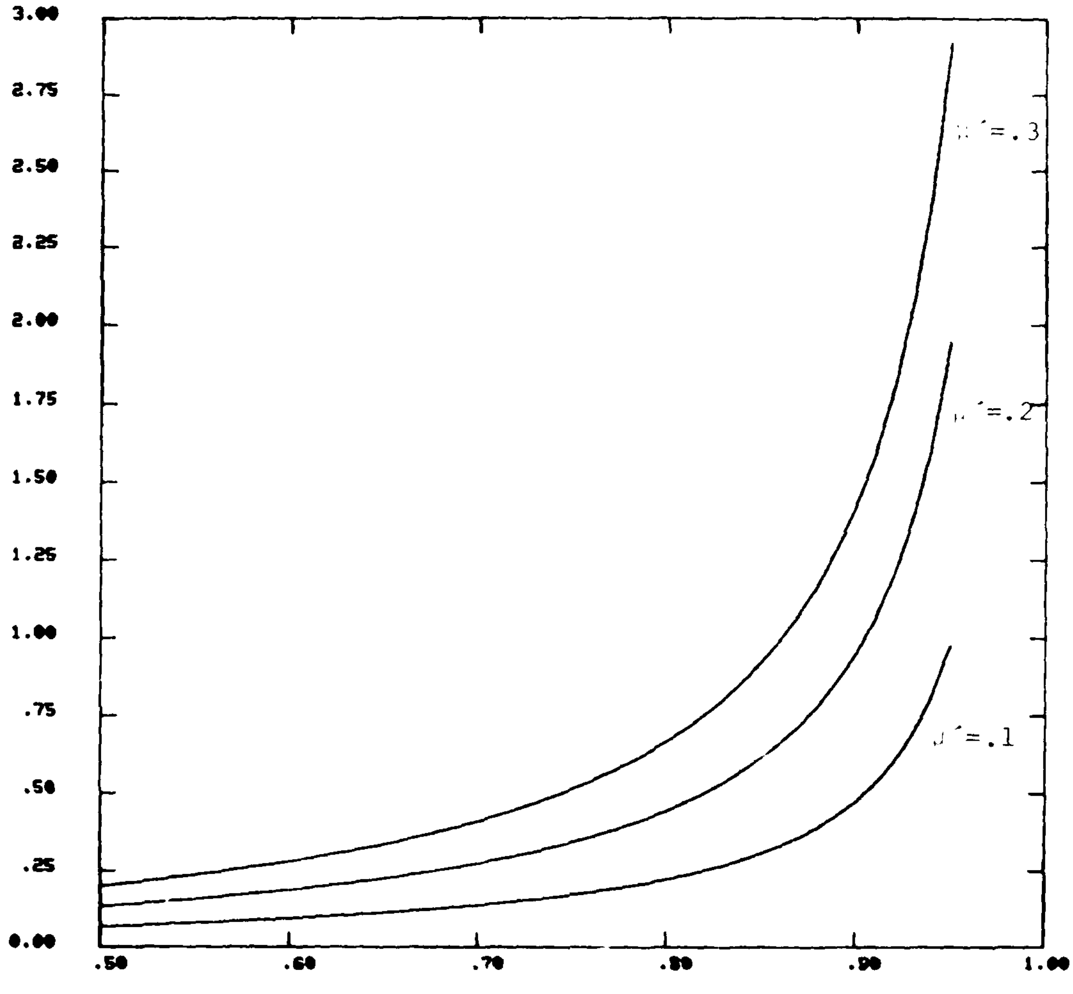

Figure 3. μ as a Function of Mach Number for Different Reduced Frequencies μ´.

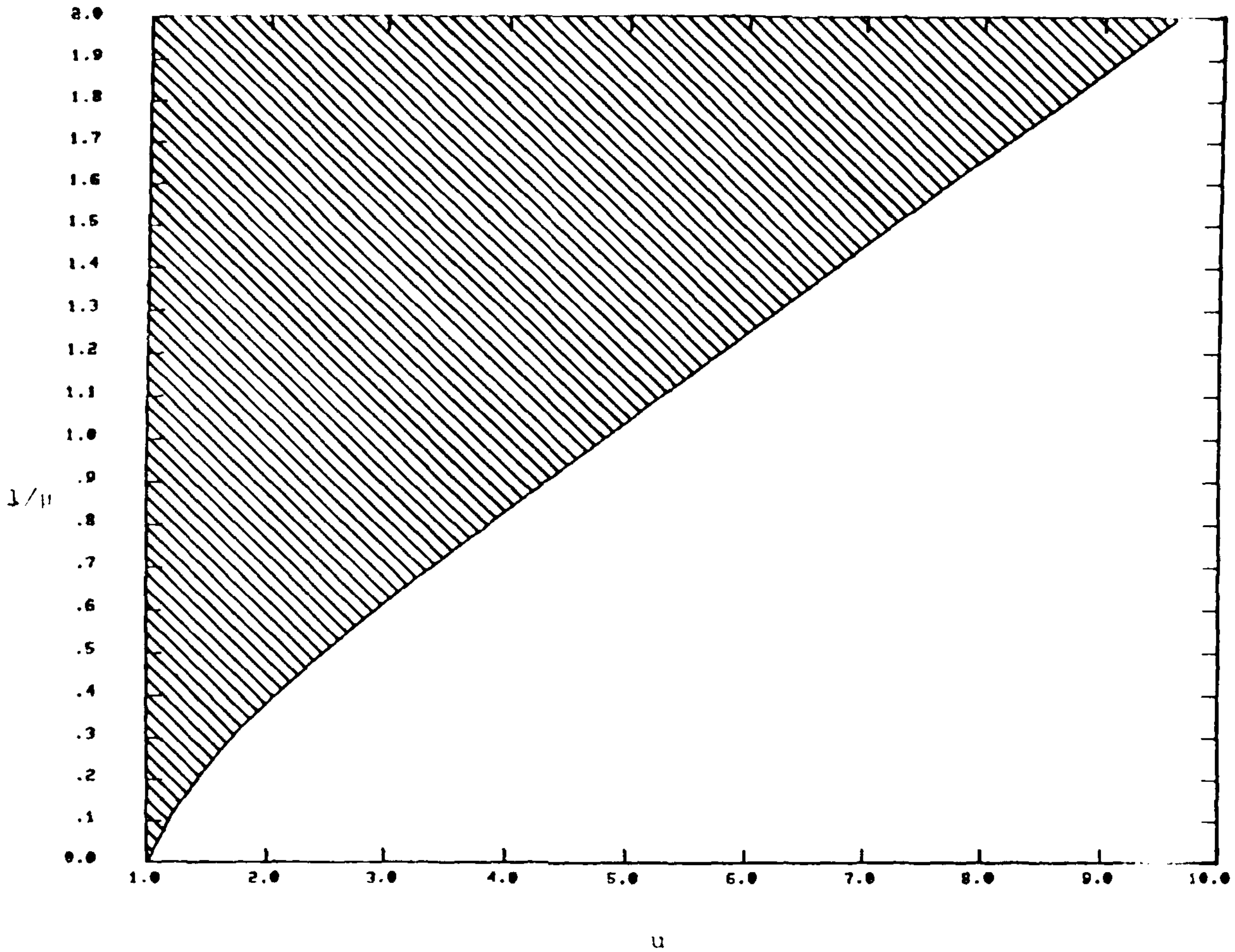

Figure 4. Region of Convergence for Iterative Procedure.

LEGENDA TO TABLES

Table 1 Fourier Coefficients of five Mathieu Functions which are symmetric to the x and y Axes and corresponding Eigenvalues.

Table 2 A characterization of the Far Field Errors for a Flow Field expressed by Trigonometric Functions in [illegible] and Far Field Conditions formulated by means of Mathieu Functions.

The table shows the angle between the relative eigenvector of the two matrices which appear in the far field conditions computed once for a perfect representation of the flow field and a second time for a Fourier series truncated to five terms. In practice these eigenvectors are computed from the matrix which give a Fourier representation for Mathieu functions. The infinite matrix has been approxmated by a nine by nine matrix, the approximate eigenvectors arise from a matrix truncated to five by five. Also given are the relative errors in the eigenvalues for these matrices. These eigenvalues serve to determine the value of $f(u)/f'(u)$ from Eq. (35). The values $f(u)/f'(u)$ are the eigenvalues which appear in the characterization of the far field conditions.

Table 3 A Characterization of the Far Field Errors for a Flow Field expressed by Trigonometric Functions in [illegible] and Far Field Conditions given by Hankel Functions in the Radius and Trigonometric Functions in the Angle in the Physical Plane.

The table shows the angle between the relative eigenvectors of the two matrices which appear in the far field conditions computed once for the perfect representation of the flow and a second time if the Fourier series is truncated to five terms. The error in the eigenvalues refers to the matrix which appears in the far field conditions.

Table 4 Angle between the Relative Eigenvalues between the Two Matrices which appear in the Far Field Conditions for a Flow Field represented by a Truncated Series in Trigonometric Functions in [illegible]. The far field conditions are expressed once in terms of Mathieu functions and a second time in terms of Hankel functions in the radius and trigonometric functions in the angle.

Table 5 Relative Error in the Eigenvalues for the conditions of Bayliss, Gungburger and Turkel, for a Flow Field Represented by Trigonometric Functions in [illegible]. The Error in the Eigenfunctions is the Same as in Figure 2.

TABLE 1*

*See Legenda on page 59.

TABLE 2*

N	MU = .5 ANGLE(RADIANS)	ERROR OF EIGENVALUES	MU = 2.0 ANGLE(RADIANS)	ERROR OF EIGENVALUES	MU = 3.5 ANGLE(RADIANS)	ERROR OF EIGENVALUES
1	.00000	.00000	.00000	.00000	.00001	.00000
2	.00000	.00000	.00000	.00000	.00004	.00000
3	.00000	.00000	.00001	.00000	.00035	.00000
4	.00000	.00000	.00056	.00000	.00524	.00005
5	.00174	.00000	.02777	.00043	.08488	.00404

*See Legenda on page 59.

TABLE 3A*

MU = .5

U	EIGENVECTORS #	ANGLE(RADIANS)	EIGENVALUES RELATIVE ERROR
25.0	1	.00000	.00000
	2	.00000	.00000
	3	.00000	.00000
	4	.00000	.00000
	5	.00174	.00007
10.0	1	.00000	.00000
	2	.00000	.00003
	3	.00000	.00003
	4	.00000	.00001
	5	.00174	.00005
8.0	1	.00000	.00001
	2	.00000	.00003
	3	.00000	.00002
	4	.00000	.00001
	5	.00174	.00007
6.0	1	.00000	.00003
	2	.00000	.00004
	3	.00000	.00002
	4	.00000	.00002
	5	.00174	.00012
4.0	1	.00000	.00007
	2	.00000	.00004
	3	.00000	.00002
	4	.00000	.00002
	5	.00175	.00023

*See Legenda on page 59.

TABLE 3b*

MU 2.0

U	EIGENVECTORS #	ANGLE (RADIANS)	EIGENVALUES RELATIVE ERROR
25.0	1	.00000	.00000
	2	.00000	.00000
	3	.00001	.00000
	4	.00056	.00000
	5	.02777	.00002
10.0	1	.00000	.00001
	2	.00000	.00002
	3	.00001	.00000
	4	.00056	.00001
	5	.02777	.00005
8.0	1	.00000	.00000
	2	.00000	.00005
	3	.00001	.00001
	4	.00056	.00002
	5	.02777	.00051
6.0	1	.00000	.00005
	2	.00000	.00006
	3	.00001	.00001
	4	.00058	.00003
	5	.02777	.00291
4.0	1	.00000	.00010
	2	.00000	.00002
	3	.00001	.00003
	4	.00072	.00002
	5	.02781	.00425

*See Legenda on page 59.

TABLE 3c*

NU 3.5

U	EIGENVECTORS #	ANGLE(RADIANS)	EIGENVALUES RELATIVE ERROR
15.0	1	.00001	.00000
	2	.00004	.00000
	3	.00035	.00000
	4	.00525	.00000
	5	.08488	.00007
10.0	1	.00001	.00015
	2	.00005	.00007
	3	.00036	.00010
	4	.00530	.00000
	5	.08487	.00029
8.0	1	.00001	.00008
	2	.00005	.00002
	3	.00036	.00009
	4	.00534	.00000
	5	.08487	.00039
6.0	1	.00001	.00003
	2	.00005	.00007
	3	.00036	.00002
	4	.00545	.00002
	5	.08487	.00037
4.0	1	.00001	.00008
	2	.00005	.00004
	3	.00040	.00007
	4	.00625	.00013
	5	.08491	.01856

*See Legenda on page 59.

TABLE 4*

MU = 3.5

U	EIGENVECTORS #	ANGLE (RADIANS)
25.0	1	.00000
	2	.00000
	3	.00002
	4	.00041
	5	.00041
10.0	1	.00000
	2	.00001
	3	.00006
	4	.00106
	5	.00097
8.0	1	.00000
	2	.00001
	3	.00007
	4	.00136
	5	.00118
6.0	1	.00000
	2	.00001
	3	.00010
	4	.00195
	5	.00152
4.0	1	.00001
	2	.00003
	3	.00021
	4	.00395
	5	.00280

*See Legenda on page 59.

TABLE 5*

RELATIVE ERROR OF EZE' FOR MU = .5

	#1	#2	#3	#4	#5
15.0	.000377	.003762	.243427	1.445522	1.211586
14.0	.000487	.004995	.333113	1.321456	[illegible]
13.0	.000597	.006723	.453250	1.195566	1.054694
12.0	[illegible]	.009184	.591454	1.086398	.996267
11.0	.000850	[illegible]	.704230	.996438	.942693
10.0	.001060	.018161	[illegible]	.923463	.902153
9.0	.001430	[illegible]	[illegible]	.864492	.873265
8.0	.002100	.039153	[illegible]	.816891	.844969
7.0	.003223	[illegible]	.657408	.778551	.821453
6.0	[illegible]	[illegible]	.619933	.747846	.802088
5.0	[illegible]	[illegible]	[illegible]	.725544	.786393
4.0	[illegible]	.128730	.553907	.704720	.773993
3.0	[illegible]	[illegible]	[illegible]	.690693	.764609
2.0	[illegible]	[illegible]	.532188	.680957	.758030
1.0	[illegible]	[illegible]	.528835	.675031	.754050

RELATIVE ERROR OF EZE' FOR MU = 2.0

	#1	#2	#3	#4	#5
10.0	[illegible]	.000107	.003012	.020615	.093333
9.0	[illegible]	.000163	.004697	.033804	.166162
8.0	[illegible]	[illegible]	.007764	.059910	.328823
7.0	[illegible]	.000440	.013853	.117977	.750167
6.0	[illegible]	[illegible]	.027410	.269262	1.918954
5.0	[illegible]	.001637	.062857	.745000	2.043040
4.0	[illegible]	[illegible]	.178946	1.512912	1.318163
3.0	[illegible]	[illegible]	.585987	1.087264	.996526
2.0	[illegible]	[illegible]	[illegible]	.816660	.844814
1.0	[illegible]	[illegible]	.548897	.700454	.772193

RELATIVE ERROR OF EZE' FOR MU = 3.5

	#1	#2	#3	#4	#5
10.0	[illegible]	.000030	.000312	.001790	.006410
9.0	[illegible]	.000045	.000480	.002793	[illegible]
8.0	[illegible]	.000072	[illegible]	.004626	[illegible]
7.0	[illegible]	.000121	.001342	.008280	[illegible]
6.0	[illegible]	[illegible]	[illegible]	.016526	[illegible]
5.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

*See Legenda on page 59.

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