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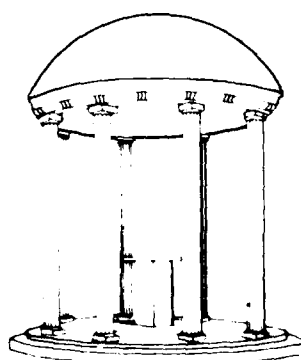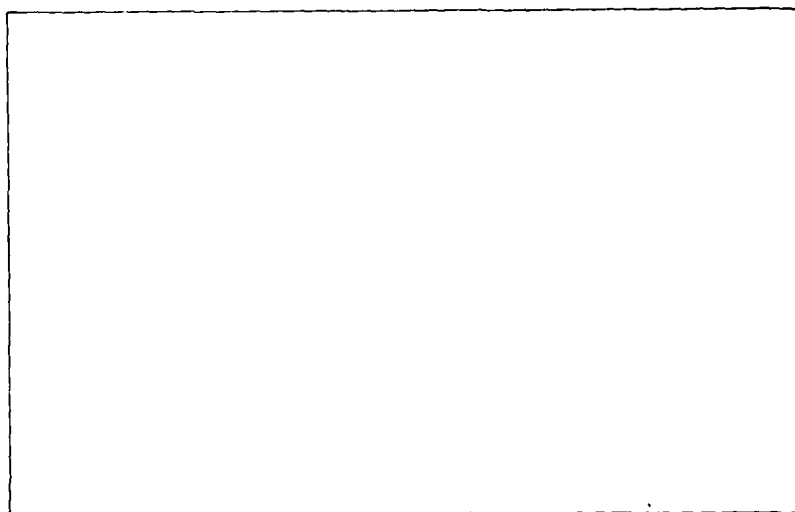
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CHOOSING SINGLE-ITEM SERVICE OBJECTIVES IN
A MULTI-ITEM BASE-STOCK INVENTORY SYSTEM

Technical Report #20

J. Christopher Mitchell

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper considers a multi-item, multi-period base-stock inventory system. The model differs from standard treatments in that shortage costs are replaced by stockout probability constraints to be satisfied in each period. The value of such a model is that it is often easier to express service objectives in terms of stockout probability constraints than it is to specify shortage costs. Specifically, system service is defined in

terms of a weighted average of single-item stockout probabilities. An optimal policy minimizes system costs while satisfying a constraint on system service. Necessary and sufficient conditions for a policy to be optimal are derived for the base-stock system, and a computationally efficient algorithm to find such a policy is developed for the special case of exponential demands and zero leadtimes. It is also shown by means of an example that operating costs can be reduced significantly when this model is used rather than the simpler uniform service model often used by managers.

FOREWORD

As part of the on-going research program in "Decision Control Models in Operations Research," Mr. J. Christopher Mitchell examines the problem of minimizing costs in a multi-item system by an appropriate choice of single-item service objectives. Mr. Mitchell investigates the properties of optimal policies and, in the special case of exponentially distributed demand, develops an efficient algorithm for computing optimal policies. He shows that this approach can lead to significant savings over the uniform service approach that is often used in applied settings.

This report is somewhat of a departure from earlier reports in that it is the first to consider a system of interdependent items. Other related reports dealing with the research program are listed on the following pages.

Harvey M. Wagner
Principal Investigator

Richard Ehrhardt
Co-Principal Investigator

MacCormick, A. (1974), <u>Statistical Problems in Inventory Control</u>, ONR and ARO Technical Report 2, December 1974, School of Organization and Management, Yale University, 244 pp.

Estey, A. S. and R. L. Kaufman (1975), <u>Multi-Item Inventory System Policies Using Statistical Estimates: Negative Binomial Demands (Variance/Mean = 9)</u>, ONR and ARO Technical Report 3, September 1975, School of Organization and Management, Yale University, 85 pp.

Ehrhardt, R. (1975), <u>Variance Reduction Techniques for an Inventory Simulation</u>, ONR and ARO Technical Report 4, September 1975, School of Organization and Management, Yale University, 24 pp.

Kaufman, R. (1976), <u>Computer Programs for (s,S) Policies Under Independent or Filtered Demands</u>, ONR and ARO Technical Report 5, School of Organization and Management, Yale University, 65 pp.

Kaufman, R. and J. Klincewicz (1976), <u>Multi-Item Inventory System Policies Using Statistical Estimates: Sporadic Demands (Variance/Mean = 9)</u>, ONR and ARO Technical Report 6, School of Organization and Management, Yale University, 58 pp.

Ehrhardt, R. (1976), <u>The Power Approximation: Inventory Policies Based on Limited Demand Information</u>, ONR and ARO Technical Report 7, June 1976, School of Organization and Management, Yale University, 106 pp.

Klincewicz, J. G. (1976), <u>Biased Variance Estimators for Statistical Inventory Policies</u>, ONR and ARO Technical Report 8, August 1976, School of Organization and Management, Yale University, 24 pp.

Klincewicz, J. G. (1976), <u>Inventory Control Using Statistical Estimates: The Power Approximation and Sporadic Demands (Variance/ Mean = 9)</u>, ONR and ARO Technical Report 9, November 1976, School of Organization and Management, Yale University, 52 pp.

Klincewicz, J. R. (1976), <u>The Power Approximation: Control of Multi-Item Inventory Systems with Constant Standard-Deviation-To-Mean Ratio for Demand</u>, ONR and ARO Technical Report 10, November 1976, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 47 pp.

Kaufman, R. L. (1977), <u>(s,S) Inventory Policies in a Nonstationary Demand Environment</u>, ONR and ARO Technical Report 11, April 1977, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 155 pp.

Ehrhardt, R. (1977), Operating Characteristic Approximations for the Analysis of (s,S) Inventory Systems, ONR and ARO Technical Report 12, April 1977, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 109 pp.

Schultz, C. R., R. Ehrhardt, and A. MacCormick (1977), Forecasting Operating Characteristics of (s,S) Inventory Systems, ONR and ARO Technical Report 13, December 1977, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 47 pp.

Schultz, C. R. (1979), (s,S) Inventory Policies for a Wholesale Warehouse Inventory System, ONR Technical Report 14, April 1979, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 75 pp.

Schultz, C. R. (1980), Wholesale Warehouse Inventory Control with Statistical Demand Information, ONR Technical Report 15, December 1980, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 74 pp.

Ehrhardt, R. and G. Kastner (1980), An Empirical Comparison of Two Approximately Optimal (s,S) Inventory Policies, Technical Report 16, December 1980, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 22 pp.

Ehrhardt, R. (1980), (s,S) Policies for a Dynamic Inventory Model with Stochastic Lead Times, Technical Report 17, December 1980, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 20 pp.

Mosier, C. (1981), Revised (s,S) Power Approximation, Technical Report 18, February 1981, School of Business Administration, University of North Carolina at Chapel Hill, 18 pp.

Blazer, D. and M. McClelland (1981), An Inventory Model for Special Handling of Extreme Value Demands, Technical Report 19, December 1981, School of Business Administration, University of North Carolina at Chapel Hill, 10 pp.

CHOOSING SINGLE-ITEM SERVICE OBJECTIVES IN

A MULTI-ITEM BASE-STOCK INVENTORY SYSTEM

J. Christopher Mitchell*

- Abstract -

This paper considers a multi-item, multi-period base-stock inventory system. The model differs from standard treatments in that shortage costs are replaced by stockout probability constraints to be satisfied in each period. The value of such a model is that it is often easier to express service objectives in terms of stockout probability constraints than it is to specify shortage costs. Specifically, system service is defined in terms of a weighted average of single-item stockout probabilities. An optimal policy minimizes system costs while satisfying a constraint on system service. Necessary and sufficient conditions for a policy to be optimal are derived for the base-stock system, and a computationally efficient algorithm to find such a policy is developed for the special case of exponential demands and zero lead-times. It is also shown by means of an example that operating costs can be reduced significantly when this model is used rather than the simpler uniform service model often used by managers.

*Curriculum in Operations Research and Systems Analysis, The University of North Carolina at Chapel Hill

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1. INTRODUCTION

We consider a simple multi-item inventory system and study the problem of specifying system-wide service objectives. Most theoretical models considered in the literature assume that holding or shortage costs are applied to any excess inventory or unsatisfied demand, respectively. A major difficulty in applying such models in practice is in the specification of shortage costs. Frequently, the manager will set shortage cost parameters based on an objective of satisfying demand with at least some minimum probability. Such an approach is often preferred because subjective factors can more easily be expressed as a probability of satisfying demand than as a cost for each shortage incurred.

A significant shortcoming of this approach is that it usually entails the setting of one system-wide probability of demand satisfaction that is then applied uniformly to all items in the system. In this paper we raise the issue of specifying different service objectives for individual items while still satisfying some given system-wide objective. The value of this approach is that system costs can be reduced below those of a method that requires identical service for all items.

Specifically, we consider a base-stock inventory model, a multi-item, multi-period inventory problem with stochastic demands, fixed leadtimes, and stationary ordering, holding, and shortage costs. For each item i there is an ordering cost c_i per unit ordered, incurred upon delivery $L_i \geq 0$ periods after order placement. There is also a holding cost h_i per unit of inventory on hand, incurred at the end of each period. The demand realized for item

i in period n is D_{in}, with absolutely continuous cdf Φ_i and density ϕ_i. We assume that $D_{in} \geq 0$ with probability 1, and $\Phi_i(0) = 0$. The mean of Φ_i is μ_i, and all demands D_{in} are mutually independent. The initial inventory level for item i is u_i, and the single period discount factor is β, where $0 < \beta \leq 1$. There are N items and T (possibly infinite) periods in which decisions must be made. When T is finite, there are also final salvage periods $T + L_i + 1$ for $i = 1,\ldots,N$.

A solution to the single-product version of this problem is given by Veinott [2, pp. 754-757] when a unit shortage cost p_i, incurred at the end of each period, is specified for each item i. Let ϕ_i^* denote the (L_i+1)-fold convolution of ϕ_i with itself. There is a myopic optimal policy $x = (x_1,\ldots,x_N)$ given by

$$\phi_i^*(x_i) = \frac{p_i - (1-\beta)c_i}{p_i + h_i},$$ where x_i is the base-stock level for item i in all periods. Thus the policy requires that whenever the inventory position (stock on hand plus stock on order) for item i falls below x_i, an order is placed to increase it up to x_i. The probability of meeting the demand for any period for item i is thus $\dfrac{p_i - (1-\beta)c_i}{p_i + h_i}$. Typically, a manager wishes to meet all demands in all periods with a probability of at least α, for some prescribed $0 < \alpha < 1$, and hence sets each shortage cost p_i such that $\dfrac{p_i - (1-\beta)c_i}{p_i + h_i} = \alpha$.

This paper will discuss an alternative base-stock inventory model, one without shortage costs. Instead of these costs, we require that some specified set of service-level constraints be

satisfied in each period. Except for the most simple constraints, it is considerably more difficult to find an optimal policy when service-level constraints, rather than shortage costs, are used.

We formulate this model as a constrained nonlinear program (NLP). We discuss two types of service-level constraints. The first is the one frequently used by managers, namely, requiring that all demands in all periods be met with a probability of at least α. The second is more general than the first. It requires that all demands for item i in all periods be met with a proba- bility of at least α_i, for $i = 1,...,N$, where the α_i are not necessarily all the same. We also require a system-wide constraint on the α_i in each period. Each item is controlled by a base-stock policy that assures demand satisfaction with probability α_i for item i. Moreover, a solution $x = (x_1,...,x_N)$ of this constrained NLP is equivalent to specifying shortage costs p_i for each item i so that $\dfrac{p_i - (1-\beta)c_i}{p_i + h_i} = \alpha_i$. Although we assume that the demand distributions are absolutely continuous, we note that if the demand distributions are discrete a deterministic policy may not be opti- mal [2, p. 757]. Nonetheless, we would recommend that research be focused on deterministic base-stock policies, since these are more appropriate in practice.

For our constrained NLP we derive equations representing first- order necessary and second-order sufficient conditions, the solution to which is an optimal policy. However, it is not clear how to solve these equations efficiently for a general demand distribution.

For exponential demand distributions we derive a computationally efficient algorithm to solve these equations. We then consider an example in which we compare the total system costs when each of the two types of service-level constraints is used. We show that there are inventory systems of practical interest for which there is a substantial cost savings when using the general constraint rather than the simpler one often used by managers.

2. THE GENERAL INVENTORY MODEL

Recall that for our alternative base-stock model we have for each item $i = 1,\ldots,N$ and each period $n = 1,\ldots,T+L_i$, the following: ordering cost c_i, holding cost h_i, fixed leadtime $L_i \geq 0$, demands D_{in} with continuous cdf Φ_i, density ϕ_i and mean μ_i, initial inventory u_i, and a single-period discount factor β with $0 < \beta \leq 1$. Each item i is controlled by a myopic base-stock policy x_i, and $x = (x_1,\ldots,x_N)$. We seek to minimize the total expected discounted cost over the entire horizon subject to a system-wide service constraint being satisfied in every period.

Let $C_{ni}(x_i)$ be the expected discounted cost for item i in period $n + L_i$. If the leadtime is greater than 0, costs in periods $1,\ldots,L_i$ will be independent of x_i. Now for $n = 1,\ldots,T$ and $i = 1,\ldots,N$,

$$C_{ni}(x_i) = \beta^{n+L_i-1}\left[h_i E(Y_{n+L_i,i}) + c_i E(R_{ni})\right], \qquad (1)$$

where Y_{mi} is the inventory on hand for item i at the end of period m, and R_{ni} is the order size for item i placed in period n (and hence received and paid for in period $n + L_i$).

Since x_i is inventory on hand plus on order in each period,

$$Y_{n+L_i,i} = \left(x_i - \sum_{j=n}^{n+L_i} D_{ij}\right)^+ ,$$

where $(y)^+ = \max\{0,y\}$. And since the policy is a base-stock policy,

$$R_{in} = \begin{cases} x_i - u_i & n = 1 \\ D_{in} & n = 2,\ldots,T \end{cases} .$$

Note that the value of R_{i1} implies that if the initial inventory u_i exceeds the base-stock level x_i, the excess is salvaged at the unit ordering cost c_i.

Recall that ϕ_i^* is the (L_i+1)-fold convolution of Φ_i with itself. Let ϕ_i^* be its density. Then since the D_{ij} are iid with cdf Φ_i for $j = n,\ldots,n+L_i$, we have

$$E(Y_{n+L_i,i}) = E\left(x_i - \sum_{j=n}^{n+L_i} D_{ij}\right)^+$$

$$= \int_0^{x_i} (x_i-t)d\phi_i^*(t) .$$

Since $\phi_i(0) = 0$ for all i, integration by parts yields

$$E(Y_{n+L_i,i}) = \int_0^{x_i} \phi_i^*(t)dt \quad \text{for} \quad i = 1,\ldots,N , \qquad n = 1,\ldots,T . \qquad (2)$$

We also have

$$E(R_{ni}) = \begin{cases} x_i - u_i & n = 1 \\ u_i & n = 2,\ldots,T \end{cases} \qquad (3)$$

We assume a salvage operation for item i in period $T + L_i + 1$, where any excess stock is sold at its unit ordering cost, and any unfilled demand is satisfied immediately. Hence, the salvage cost is given by

$$-\beta^{T+L_i} c_i \left(x_i - \sum_{n=T}^{T+L_i} D_{in} \right),$$

which has expected value

$$C_{T+1,i}(x_i) = -\beta^{T+L_i} c_i [x_i - (L_i+1)u_i] \, .$$

Our introduction of salvage cost follows the development in [2], where it is on artifice to guarantee a myopic optimal policy. When the horizon T is long, the salvage terms are an acceptable approximation. For the infinite horizon case, which is considered at the end of Section 3, the artifice evidently vanishes.

Let $\hat{C}(x) = \sum_{n=1}^{T+1} \sum_{i=1}^{N} C_{ni}(x_i)$, the total discounted expected cost excluding the policy-independent costs incurred in periods $1,\ldots,L$, where $L = \min\{L_i : i = 1,\ldots,N\}$. Also let

$$H(x) = \sum_{i=1}^{N} \beta^{L_i} h_i \int_{0}^{x_i} \phi_i^*(t)dt$$

and

$$h(x) = \sum_{i=1}^{N} \beta^{L_i} c_i x_i .$$

Then (1), (2), (3), and (4) imply that

$$\hat{C}(x) = \sum_{n=1}^{T} \sum_{i=1}^{N} \beta^{n+L_i-1} h_i \int_0^{x_i} \phi_i^*(t)dt$$

$$+ \sum_{i=1}^{N} \left[\beta^{L_i} c_i (x_i - u_i) + \sum_{n=2}^{T} \beta^{n+L_i-1} c_i u_i \right]$$

$$- \sum_{i=1}^{N} \beta^{T+L_i} c_i [x_i - (L_i+1)u_i] .$$

After some algebraic simplifications, for $\beta < 1$ we can write

$$\hat{C}(x) = C(x) + \left(\frac{\beta - \beta^T}{1-\beta} \right) \sum_{i=1}^{N} \beta^{L_i} c_i u_i + \beta^T \sum_{i=1}^{N} \beta^{L_i} c_i (L_i+1) u_i$$

$$- \sum_{i=1}^{N} \beta^{L_i} c_i u_i , \tag{5}$$

where $C(x) = \frac{1-\beta^T}{1-\beta} [H(x) + (1-\beta)h(x)] .$ $\qquad$ (6)

Note that $\hat{C}(x) - C(x)$ are policy-independent costs.

In summary, we have that the total expected discounted cost excluding all policy-independent costs is

$$C(x) = \frac{1-\beta^T}{1-\beta} [H(x) + c(x)] , \tag{7a}$$

$$\text{where} \quad H(x) = \sum_{i=1}^{N} \left(\beta^{L_i} h_i \right) \int_0^{x_i} \phi_i^*(t) dt \tag{7b}$$

$$\text{and} \quad c(x) = (1-\beta)h(x) = \sum_{i=1}^{N} (1-\beta)(\beta^{L_i} c_i) x_i \ . \tag{7c}$$

We are interested in minimizing $C(x)$ subject to a service constraint on the probability of satisfying demand. We consider two types of service constraints. The first type requires that the demand for every item in every period be satisfied with a probability of at least α, where $0 < \alpha < 1$. We call this service requirement Independent and Identical Service (IIS). Thus, there will be N service constraints

$$\phi_i^*(x_i) \geq \alpha \quad \text{for} \quad i = 1,\ldots,N \ , \tag{8}$$

which must be satisfied in every period. IIS is an approach frequently used in practice. Indeed, if we make the reasonable assumption that for some myopic optimal policy $x = (x_1,\ldots,x_N)$ these constraints are tight, finding such a policy is equivalent to specifying the shortage costs p_i in Veinott's model [2] described in the Introduction by

$$\frac{p_i - (1-\beta)c_i}{p_i + h_i} = \alpha, \quad \text{for} \quad i = 1,\ldots,N \ .$$

The second type of service constraint is a generalization of IIS and is called General Service (GS). For this service the system of N constraints on the probability of satisfying demand is

replaced by a single constraint requiring that a weighted average of the probabilities be at least α. The value of this approach is that varying service levels can be applied to the items se as to reap the greatest benefits in system-wide costs. We use the constraint

$$\sum_{i=1}^{N} k_i [\phi_i^*(x_i) - \alpha] \geq 0 , \qquad (9)$$

where $k_1, \ldots, k_N > 0$ are specified weights. In the following sections we often set $k_i = \mu_i$.

We will require that constraint (9) be satisfied in every period. This does not destroy the myopic character of the optimal solution.

We minimize the cost function given by (7a), (7b), and (7c) subject to the constraint (9). In summary, we have formulated the following optimization problem:

$$\left. \begin{array}{l} \text{minimize } C(x) = \dfrac{(1-\beta^T)}{1-\beta} [H(x) + c(x)] \\[3ex] \text{subject to } g(x) = \displaystyle\sum_{i=1}^{N} k_i [\phi_i^*(x_i) - \alpha] \geq 0 , \\[3ex] \text{where } H(x) = \displaystyle\sum_{i=1}^{N} \beta^{L_i} h_i \int_0^{x_i} \phi_i^*(t)\,dt \\[3ex] \text{and } c(x) = \displaystyle\sum_{i=1}^{N} (1-\beta)\beta^{L_i} c_i x_i \end{array} \right\} \qquad (10)$$

We mention again that a myopic optimal policy $x = (x_1, \ldots, x_N)$ found to be the solution of this NLP is equivalent to specifying

shortage costs p_i as $\dfrac{p_i - (1-\beta)c_i}{p_i + h_i} = \phi_i^*(x_i)$ in Veinott's model

[2] discussed in the Introduction. Also, the probability α_i that the demand for item i be met in any period is

$$\alpha_i = \frac{p_i - (1-\beta)c_i}{p_i + h_i}.$$

3. ANALYSIS

3.1 First- and Second-Order Optimality Conditions

We use NLP techniques to analyze (10). Specifically, we derive first-order necessary and second-order sufficient conditions for x to be a solution of (10). We also investigate when this local optimum can be known to be a global optimum.

It is clear that if (10) has a solution x, it has one with $g(x) = 0$, and so we assume that $g(x) = 0$ at optimality. Since there is only one constraint $g(x) \geq 0$, and g is differentiable everywhere on $(0, +\infty)$, a first-order necessary condition for x to solve (10) is that there exist $\lambda \geq 0$ such that

$$\nabla C(x) - \lambda \nabla g(x) = 0 ,$$

$$\text{where} \quad g(x) = 0 ,$$

(11)

as proved in [1, p. 41]. Now

$$\frac{\partial}{\partial x_i} C(x) = \frac{(1-\beta^T)}{1-\beta} \left[\beta^{L_i} h_i \phi_i^*(x_i) + (1-\beta)\beta^{L_i} c_i \right]$$

and

$$\frac{\partial}{\partial x_i} g(x) = k_i \phi_i^*(x_i) \ .$$

Hence (11) is equivalent to

$$\lambda^* = \left(\frac{1-\beta}{1-\beta^T}\right)\lambda = \frac{\beta^{L_i}[h_i \phi_i^*(x_i) + (1-\beta)c_i]}{k_i \phi_i^*(x_i)}$$

being independent of i for all $i = 1,\ldots,N$.

Suppose that at optimality ϕ_i^* and ϕ_i^* can be written in the following form for $i = 1,\ldots,N$:

$$\left. \begin{aligned} \phi_i^*(x_i) &= v_i \alpha, \quad \text{some} \quad 0 < v_i < 1/\alpha \\[2ex] \phi_i^*(x_i) &= \frac{\beta^{L_i}\left[h_i v_i \alpha + (1-\beta)c_i\right]}{\lambda^* k_i} , \quad \text{some} \quad \lambda^* > 0 \ . \end{aligned} \right\} \tag{12}$$

Since λ^* is independent of i, (12) together with $g(x) = 0$, which is

$$\sum_{i=1}^{N} k_i[v_i - 1] = 0 \ ,$$

satisfy the necessary condition (11). We summarize these results in the following theorem.

Theorem 1: A first-order necessary condition for x to solve (10) is that there exist $v_1,\ldots,v_N$, with $0 < v_i < 1/\alpha$, and $\lambda^* > 0$, such that for $i = 1,\ldots,N$,

(a) $\qquad \phi_i^*(x_i) = v_i \alpha$

(b) $\qquad \phi_i^*(x_i) = \dfrac{\beta^{L_i}\left[h_i v_i \alpha + (1-\beta)c_i\right]}{\lambda^* k_i}$ $\qquad \left.\vphantom{\begin{array}{c}a\\b\\c\\d\\e\end{array}}\right\}$ (13)

(c) $\qquad \displaystyle\sum_{i=1}^{N} k_i[v_i - 1] = 0 .$

Given that the first-order condition (11) holds, we proceed to derive a second-order sufficient condition that x solve (10). As proved in [1, p. 48], a second-order condition is the following:

For all $z \in \mathbf{R}^N$ with $z \neq 0$,

$$z^T[\nabla^2 C(x) - \lambda \nabla^2 g(x)]z > 0 , \qquad (14)$$

where $\lambda > 0$ is the same parameter as given in (11).

Now

$$\nabla^2 C(x) = \frac{(1-\beta^T)}{1-\beta}\begin{bmatrix} \beta^{L_1}h_1\phi_1^*(x_1) & & 0 \\ & \ddots & \\ 0 & & \beta^{L_N}h_N\phi_N^*(x_N) \end{bmatrix} , \qquad (15)$$

and

$$\nabla^2 g(x) = \begin{bmatrix} k_1\phi_1^{*\prime}(x_1) & & 0 \\ & \ddots & \\ 0 & & k_N\phi_N^{*\prime}(x_N) \end{bmatrix} , \qquad (16)$$

and so (14) is equivalent to

$$\frac{(1-\beta^T)}{1-\beta} \sum_{i=1}^{N} \left[\beta^{L_i} h_i \phi_i^*(x_i) - \lambda^* k_i \phi_i^{*\prime}(x_i) \right] z_i^2 > 0 ,$$

because $\lambda = \dfrac{1-\beta^T}{1-\beta} \lambda^*$.

Since it is sufficient that each term in the summand be positive, we have established the following theorem:

Theorem 2. Given that (11) holds for x, a second-order sufficient condition for x to solve (10) is that for $i = 1,\dots,N$,

$$\frac{d}{dx_i} \phi_i^*(x_i) < \beta^{L_i} \frac{h_i}{\lambda^* k_i} \phi_i^*(x_i) . \tag{17}$$

Note that (17) is trivially satisfied if ϕ_i^* is positive and nonincreasing at x_i.

3.2 Undiscounted and Infinite Horizon Models

We briefly consider what changes are necessary to model the systems that are undiscounted and/or have an infinite planning horizon. It turns out that although the formulation (10) must be modified somewhat, the first- and second-order conditions (13) and (17) remain intact.

The undiscounted finite horizon model is obtained by setting $\beta = 1$. The only change necessary in the formulation occurs with respect to (7a), where it is assumed $\beta < 1$. However, when $\beta = 1$, (7a) becomes

$$C(x) = T[H(x) + c(x)] .$$

Moreover, when $\beta = 1$, (7b) and (7c) become

$$H(x) = \sum_{i=1}^{N} h_i \int_0^{x_i} \phi_i^*(t)$$

and $c(x) = 0$, respectively.

Therefore the undiscounted finite horizon model analogous to (10) is

$$
\left.
\begin{aligned}
&\text{minimize} \quad C(x) = TH(x) \\
&\text{subject to} \quad g(x) = \sum_{i=1}^{N} k_i [\phi_i^*(x_i) - \alpha] \geq 0 , \\
&\text{where} \quad H(x) = \sum_{i=1}^{N} h_i \int_0^{x_i} \phi_i^*(t)dt .
\end{aligned}
\right\}
\qquad (18)
$$

It is straightforward to verify that the first-order necessary and second-order sufficient conditions for optimality are still (13) and (17), respectively, with $\beta = 1$ inserted in those formulas.

The undiscounted infinite horizon model is obtained by setting $\beta = 1$ and $T = +\infty$. It is clear from (18) that the total expected undiscounted cost is infinite, and so we must use an alternative optimality criterion. We seek to minimize the average expected cost per period. Thus this is the previous model (18) with $T = 1$, and so the optimality conditions (13) and (17) hold with $\beta = 1$ and $T = 1$ (not $T = +\infty$) inserted in those formulas.

The discounted infinite horizon model is obtained by setting $T = +\infty$ (and $\beta < 1$). In the formulation of the model (10) the

only dependence on T is found in the objective function, which becomes

$$C(x) = \frac{1}{1-\beta}[H(x) + c(x)]$$

when $T = +\infty$. In order to guarantee that the total expected discounted cost is finite, we must verify that the policy-independent terms in (5) are finite for $T = +\infty$. The only dependence on T are the terms

$$\frac{\beta-\beta^T}{1-\beta} \quad \text{and} \quad \beta^T ,$$

and for $\beta < 1$,

$$\lim_{T\to\infty} \frac{\beta-\beta^T}{1-\beta} = \frac{\beta}{1-\beta} < +\infty \quad \text{and} \quad \lim_{T\to\infty} \beta^T = 0 .$$

Thus the total expected discounted cost is finite, and the analog to the model (10) for the discounted infinite horizon model is

$$\left.\begin{aligned}
&\text{minimize} \quad C(x) = \frac{H(x) + c(x)}{1 - \beta}\\[2mm]
&\text{subject to} \quad g(x) = \sum_{i=1}^{N} k_i[\phi_i^*(x_i) - \alpha] \geq 0 ,\\[2mm]
&\text{where} \quad H(x) = \sum_{i=1}^{N} \beta^{L_i} h_i \int_0^{x_i} \phi_i^*(t)dt\\[2mm]
&\text{and} \quad c(x) = \sum_{i=1}^{N} (1-\beta)\beta^{L_i} c_i x_i .
\end{aligned}\right\} \quad (19)$$

Again, the first- and second-order optimality conditions (13)
and (17) remain unchanged, as is easily shown.

3.3 Global Optima

We now seek to establish conditions that guarantee that a
local optimum to (10) is in fact a global optimum. In general, a
local optimum need not be a global optimum. However, if (10) is a
convex NLP, i.e., if on their respective domains $C(x)$ is a convex
function and $g(x)$ is a concave function, then any local optimum
is also a global optimum, as proved in [1, p. 96]. Now (15) implies
that $C(x)$ is always convex since $\nabla^2 C(x)$ is always positive def-
inite. However, (16) implies that $g(x)$ is concave if and only if
ϕ_i^* is nonincreasing at x_i for all $i = 1,\ldots,N$ (since this is
equivalent to $\nabla^2 g(x)$ being negative semi-definite at x). Recall
that this is almost the same condition that guaranteed that the
second-order sufficient condition (17) be satisfied.

In our examples, we consider the exponential and uniform dis-
tributions. Since their density functions are always nonincreasing
on their supports, it will be sufficient to search for local optima
using (13) and (17), for they will also be global optima.

4. EXPONENTIAL DEMAND

We consider the special case where $L_i = 0$ for $i = 1,\ldots,N$,
and all demands are exponentially distributed. Using the first-
and second-order conditions we derive a computationally efficient
algorithm to compute an optimal base-stock policy x.

Since the exponential density is positive and decreasing on its support, the sufficient condition (17) is trivially satisfied for all $x \geq 0$. Thus we need only search for $x \geq 0$ which satisfies the necessary condition (13). Because $L_i = 0$,

$$\Phi_i^*(x) = \Phi_i(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{(-x/\mu_i)} & x \geq 0 \end{cases},$$

and

$$\phi_i^*(x) = \phi_i(x) = \begin{cases} 0 & x < 0 \\ \dfrac{1}{\mu_i} e^{(-x/\mu_i)} & x \geq 0 \end{cases}.$$

So when $x \geq 0$, (a) and (b) of (13) are equivalent to

$$v_i = \frac{\lambda^* k_i - \mu_i(1-\beta)c_i}{\alpha(\lambda^* k_i + \mu_i h_i)} . \tag{20}$$

This, together with (c) of (13), is equivalent to

$$\sum_{i=1}^{N} k_i \left[\frac{\lambda^* k_i - \mu_i(1-\beta)c_i}{\alpha(\lambda^* k_i + \mu_i h_i)} - 1 \right] = 0 . \tag{21}$$

Now $\Phi_i(x_i) = v_i \alpha$ if and only if

$$x_i = -\mu_i \log (1 - v_i \alpha) , \tag{22}$$

and so if we can solve (21) for $\lambda^* > 0$, (20) gives us v_i and (22) gives us x_i. We have the following theorem:

<u>Theorem 2</u>: There is a unique $\lambda^* > 0$ that solves (21).

<u>Proof</u>: Let

$$f(\lambda) = \sum_{i=1}^{N} k_i \left[\frac{\lambda k_i - \mu_i(1-\beta)c_i}{\alpha(\lambda k_i + \mu_i h_i)} - 1 \right] .$$

Then $f(0) < 0$ and $f(+\infty) > 0$. Since f is continuous on $(0,+\infty)$, the Intermediate Value Theorem implies that there exists $\lambda^* \in (0,+\infty)$ such that $f(\lambda^*) = 0$. It is straightforward to verify that $f'(\lambda) > 0$ on $(0,+\infty)$, and so Rolle's Theorem implies that λ^* is unique. $\square$

Thus we have the following algorithm to compute x when $L_i = 0$ and the demands are exponentially distributed with means μ_i:

$$
\left.
\begin{array}{l}
\text{(1)} \quad \text{Solve} \ \displaystyle\sum_{i=1}^{N} k_i \left[\frac{\lambda k_i - \mu_i(1-\beta)c_i}{\alpha(\lambda k_i + \mu_i h_i)} - 1 \right] = 0 \\[4pt]
\qquad \ \ \text{for} \ \lambda^* > 0 . \\[18pt]
\text{(2)} \quad \text{Set} \ v_i = \dfrac{\lambda^* k_i - \mu_i(1-\beta)c_i}{\alpha(\lambda^* k_i + \mu_i h_i)} \ \text{for} \ i = 1,\ldots,N. \\[18pt]
\text{(3)} \quad \text{Set} \ x_i = -\mu_i \ \log \ (1- v_i\alpha) \ \text{for} \ i = 1,\ldots,N.
\end{array}
\right\} \quad (23)
$$

Step (1) can be done, for example, using a bisection method. Even for very large N, this is not computationally very difficult.

Recall that in the Introduction we discussed two possible service constraints we might impose on the inventory system. The first, which we call Independent and Identical Service (IIS),

requires that we use the N constraints given in (8) for each
period. The second service constraint, which we call General
Service (GS), requires that we use the single constraint (9) in
each period.

If we use the IIS constraint, the fact that the constraints
are tight at optimality implies that the optimal value for x can
be found by solving the system of equations

$$\Phi_i(x_i) = \alpha, \quad i = 1,\ldots,N ,$$

for $x = (x_1,\ldots,x_N)$. Hence, the optimal base-stock policy is given
by

$$x_i = -\mu_i \log(1-\alpha) . \tag{24}$$

If we use the GS constraint, the optimal base-stock policy x
must be found using the algorithm given in (23). Hence, the impact
of imposing the GS constraint as opposed to the IIS constraint is
to replace $\mu_i \log(1-\alpha)$ in (24) with $\mu_i \log(1-v_i\alpha)$, where v_i
is algorithmically computed.

It is clear that the GS constraint is less restrictive than
the IIS constraint, and hence, the optimal objective value of (7a)
subject to the GS constraint is no larger than that subject to the
IIS constraint. However, it is a good deal more work to find the
base-stock policy in the GS case than in the IIS case. One may
wonder for what system parameters, if any, the extra work and ex-
pense involved in using (23) is justified. In other words, when,
if ever, is the cost function (7) evaluated at the solution (23)
of (10) significantly smaller than when evaluated at (24)? We
explore this question with an illustrative two-item system.

We consider a single period, two-item system that is undiscounted. It is straightforward to verify that the objective function (10) is

$$C(x) = \sum_{i=1}^{2} h_i [x_i - \mu_i \phi_i(x_i)] . \tag{25}$$

We denote the optimal base-stock policy for the IIS and GS models by $x^I = (x_1^I,\ldots,x_N^I)$ and $x^G = (x_1^G,\ldots,x_N^G)$, respectively. Then (24) and (23) imply, respectively, that for $i = 1,\ldots,N$,

$$\left.\begin{array}{l} x_i^I = -\mu_i \log (1-\alpha) \\[2em] x_i^G = -\mu_i \log (1-v_i\alpha) \end{array}\right\}, \tag{26}$$

where the v_i are found by algorithm (23).

Let D be the percent decrease in minimum system cost when the GS constraint, rather than the IIS constraint, is used. Then,

$$\begin{aligned} D &= D(\mu_1,\mu_2,h_1,h_2,) \\[1em] &= 100\left[\frac{C(x^I) - C(x^G)}{C(x^I)}\right] \\[1em] &= 100\left[1 - \frac{C(x^G)}{C(x^I)}\right], \end{aligned}$$

and (25) and (26) imply that

$$D = 100 \left[1 - \frac{[\alpha + \log(1-\alpha)] \sum_{i=1}^{2} h_i \mu_i}{\sum_{i=1}^{2} [v_i \alpha + \log(1-v_i \alpha)] h_i \mu_i} \right]. \qquad (27)$$

In order to investigate the sensitivity of D to the parameters μ_i, h_i and α, we show that for fixed α, D is a function only of μ_2/μ_1 and h_2/h_1. We then graph D evaluated at pairs $(\mu_2/\mu_1, h_2/h_1)$, and see for what pairs D is large.

Theorem 3: For fixed α, D as given in (27) is a function only of μ_2/μ_1 and h_2/h_1.

Proof: The proof is in the Appendix. □

In Figures 1 and 2 we have plotted the level curves of D for $\alpha = .8$ and $\alpha = .9$, respectively. We used the IMSL routine ZBRENT to compute Step (1) of algorithm (23).

We consider $D = 5\%$ to be a significant decrease in optimal cost. In Figures 1 and 2 we have plotted the level curves corresponding to $D = 5$. We call the two regions in each figure for which $D \geq 5$ critical regions. Whenever a pair $(\mu_2/\mu_1, h_2/h_1)$ falls within a critical region, a significant decrease in optimal cost results from using the GS rather than the IIS constraint. The symmetry observable in the graphs follows from the easily verifiable fact that

$$D\left(\frac{\mu_1}{\mu_2}, \frac{h_1}{h_2}\right) = D\left(\frac{\mu_2}{\mu_1}, \frac{h_2}{h_1}\right).$$

FIGURE 1

Critical Regions of Decreased Cost

$$[\alpha = .8, \quad D = D(\mu,h)]$$

$h = \dfrac{h_2}{h_1}$

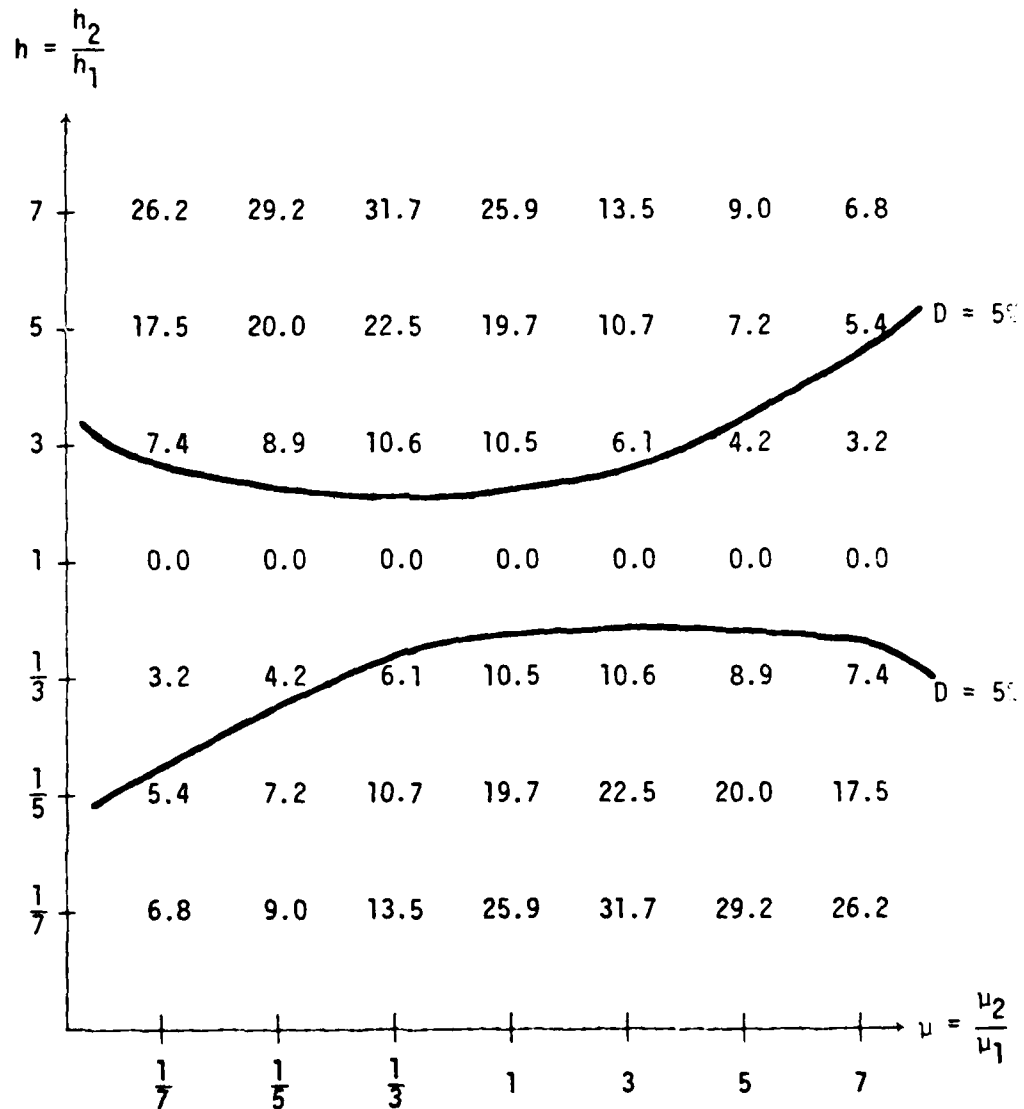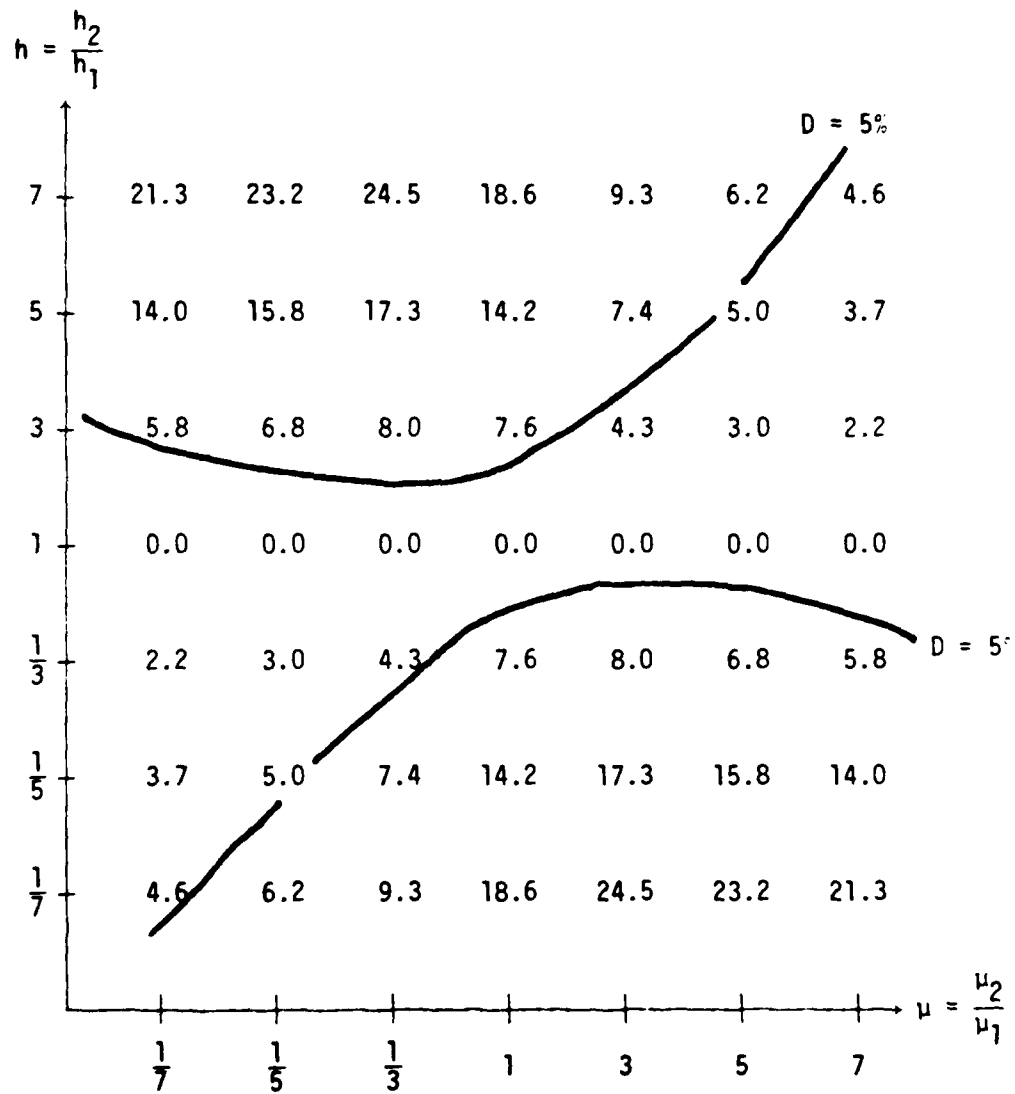|	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{3}$	1	3	5	7	
7	26.2	29.2	31.7	25.9	13.5	9.0	6.8	
5	17.5	20.0	22.5	19.7	10.7	7.2	5.4	$D = 5$
3	7.4	8.9	10.6	10.5	6.1	4.2	3.2	
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
$\frac{1}{3}$	3.2	4.2	6.1	10.5	10.6	8.9	7.4	$D = 5$
$\frac{1}{5}$	5.4	7.2	10.7	19.7	22.5	20.0	17.5	
$\frac{1}{7}$	6.8	9.0	13.5	25.9	31.7	29.2	26.2	

$\mu = \dfrac{\mu_2}{\mu_1}$

FIGURE 2

Critical Regions of Decreased Cost

$$[\alpha = .9, \; D = D(\mu, h)]$$

Note that the largest values of D are found in the regions where h_2/h_1 and μ_1/μ_2 are large, and where h_1/h_2 and μ_2/μ_1 are large. A system for which this is typical is one in which some items have a very low expected demand and a very high holding cost, or vice versa [high holding cost ≈ high unit purchase cost (≈ expensive items)]. Many inventory systems of practical interest contain such items. For such systems one can decrease the total system cost substantially by using the base-stock policy given by (23) rather than (24). Finally, we note that the critical region is larger when $\alpha = .8$ than when $\alpha = .9$.

5. CONCLUSIONS AND EXTENSIONS

We have formulated a constrainted NLP to solve a base-stock model with a service-level constraint rather than shortage costs. This model has the advantage that a manager can more easily specify service objectives than shortage costs. We derived first-order necessary and second-order sufficient conditions for an optimal base-stock policy. For the special case of zero lead-time and exponentially distributed demands, we derived a computationally efficient algorithm to find an optimal base-stock policy. We showed that for some inventory systems there is a substantial cost savings when using this algorithm instead of the independent and identical service approach that is often used in practice.

5.1 Uniformly Distributed Demands

One can do analysis similar to that in Section 4 for the special case when $L_i = 0$ and demands are distributed uniformly.

Given that the demand distribution for item i is uniform on

(a_i, b_i), so $\mu_i = \dfrac{a_i + b_i}{2}$ and the variance is $\sigma_i^2 = \dfrac{(b_i - a_i)^2}{12}$,

we have the following algorithm, analogous to (23), to determine

our optimal base-stock policy x.

$$
\left.
\begin{array}{ll}
(1) & \text{Solve } \displaystyle\sum_{i=1}^{N} k_i\left[\dfrac{\lambda h_i}{h_i \sigma_i} - 1\right] = 0 \quad \text{for } \lambda^* > 0. \\[3em]
(2) & \text{Set } v_i = \dfrac{\lambda^* k_i}{h_i \sigma_i} \quad \text{for } i = 1,\ldots,N. \\[3em]
(3) & \text{Set } x_i = a_i + \sqrt{12}\,\sigma_i v_i \alpha, \quad \text{for } i = 1,\ldots,N.
\end{array}
\right\} \quad (28)
$$

Analysis similar to that done for exponential demands could

be done here. We note that although the uniform density is a two-

parameter density, only the variance parameter is used to deter-

mine the v_i in algorithm (28). The reason for this is that the

mean parameter is a "shift parameter," that is, its only effect on

the density or cdf is to move it right or left. It is easily seen

from (13) that a shift parameter never affects the value of v_i

for any distribution. Of course, the value of x_i is shifted as

necessary.

5.2 Possible Directions of Research

There are obvious difficulties involved in solving the equations

for the first-order necessary condition (13) if the cdf ϕ_i^* cannot

be written in closed form. A direction of research is to examine

numerical methods for solving (13) for, say, the normal or gamma

distributions. A basic difficulty is involved in approximating the cdf so that the equations (13) can be solved, and so that the solution obtained is a good approximation of the true policy x. ͵

According to Veinott [2, pp. 754-757], the generalized base-stock model with shortage costs still has, for some nonstationary parameters, a myopic base-stock policy that is optimal. A simple model of this sort is one in which all costs are stationary, but the demand distribution means vary in time. In order to guarantee that a myopic base-stock policy is optimal, it is necessary that, for each item in the system, the means are nondecreasing in time. This, however, is not strong enough to guarantee that a myopic base-stock policy be optimal when GS service constraints, rather than shortage costs, are used. Thus, another direction of research is to investigate optimal policies for this model. Since they will not be myopic, this seems a very difficult problem.

A more promising direction is to introduce a set-up cost in the model. The infinite horizon case using the stationary demand distributions can be investigated. This model seems similar to the infinite horizon case for (10), and so results similar to those in this paper may be forthcoming. In particular, results for the exponential demand distributions may be readily derived.

APPENDIX

Theorem 3: For fixed α, D as given in (27) is a function only of $\frac{\mu_2}{\mu_1}$ and $\frac{h_2}{h_1}$.

Proof: We note that for $a, b, \hat{a}, \hat{b} > 0$, $\frac{a}{b} = \frac{\hat{a}}{\hat{b}}$ if and only if there is some $\gamma > 0$ such that $\hat{a} = \gamma a$ and $\hat{b} = \gamma b$. Hence, we must show that if $\hat{\mu}_1 = \gamma\mu_1$, $\hat{\mu}_2 = \gamma\mu_2$, $\hat{h}_1 = \delta h_1$, and $\hat{h}_2 = \delta h_2$, then $D(\hat{\mu}_1, \hat{\mu}_2, \hat{h}_1, \hat{h}_2) = D(\mu_1, \mu_2, h_1, h_2)$. Let $D = D(\mu_1, \mu_2, h_1, h_2)$ and $\hat{D} = D(\hat{\mu}_1, \hat{\mu}_2, \hat{h}_1, \hat{h}_2)$. Also let v_i and $\hat{v}_i$ be given by (23) for the parameters μ_i, h_i and $\hat{\mu}_i, \hat{h}_i$, respectively. Then (27) implies

$$D = 100\left[1 - \frac{[\alpha + \log(1-\alpha)]\sum_{i=1}^{2} h_i\mu_i}{\sum_{i=1}^{2}[v_i\alpha + \log(1-v_i\alpha)]h_i\mu_i}\right]$$

and

$$\hat{D} = 100\left[1 - \frac{[\alpha + \log(1-\alpha)]\sum_{i=1}^{2} \delta h_i\gamma\mu_i}{\sum_{i=1}^{2}[\hat{v}_i\alpha + \log(1-\hat{v}_i\alpha)]\delta h_i\gamma\mu_i}\right],$$

since $\hat{\mu}_i = \gamma\mu_i$ and $\hat{h}_i = \delta h_i$. Simplifying yields

$$\hat{D} = 100\left[1 - \frac{[\alpha + \log(1-\alpha)]\sum_{i=1}^{2} h_i \mu_i}{\sum_{i=1}^{2} [\hat{v}_i \alpha + \log(1-\hat{v}_i \alpha)]h_i \mu_i}\right] .$$

Evidently it is sufficient to show that $\hat{v}_i = v_i$ for $i = 1,2$. Since $\beta = 1$, (23) implies that

$$v_i = \frac{\lambda^* k_i}{\alpha(\lambda^* k_i + \mu_i h_i)} ,$$

where $\lambda^* > 0$ is the unique solution to

$$\sum_{i=1}^{2} k_i\left[\frac{\lambda k_i}{\alpha(\lambda k_i + \mu_i h_i)} - 1\right] = 0 . \qquad (29)$$

Similarly,

$$\hat{v}_i = \frac{\hat{\lambda}^* k_i}{\alpha(\hat{\lambda}^* k_i + \delta\sigma\mu_i h_i)} ,$$

where $\hat{\lambda}^* > 0$ is the unique solution to

$$\sum_{i=1}^{2} k_i\left[\frac{\hat{\lambda} k_i}{\alpha(\hat{\lambda} k_i + \delta\gamma\mu_i h_i)} - 1\right] = 0 .$$

We can write this equation as

$$\sum_{i=1}^{2} k_i \left[\frac{\left(\frac{\hat{\lambda}}{\delta\gamma}\right)k_i}{\alpha\left[\left(\frac{\hat{\lambda}}{\delta\gamma}\right)k_i + \mu_i h_i\right]} - 1 \right] = 0 ,$$

and so since λ^* is the unique solution to (29), we have

$$\frac{\hat{\lambda}^*}{\delta\gamma} = \lambda^* .$$

Thus,

$$\hat{v}_i = \frac{\lambda^* \delta\gamma k_i}{\alpha(\lambda^* \delta\gamma k_i + \delta\gamma\mu_i h_i)} = v_i ,$$

as needed. □

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