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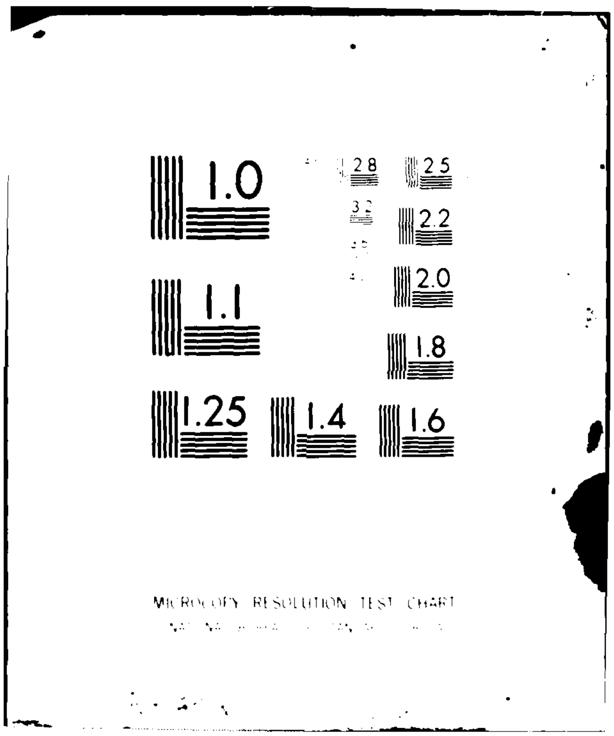
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GLOBAL GEOPOTENTIAL MODELLING FROM SATELLITE-TO-SATELLITE TRACKING

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October, 1981

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The error analysis of the global modelling of the geopotential has been carried out up to degree and order 331 of the spherical harmonic expansion, for data from a low-low satellite-to-satellite tracking (SST) mission. The sphericity and the rotation of the Earth have been considered, as well as the discrete nature of the data, assumed to consist of time averages of the measured range-rate sampled at regular intervals. The expansion of the potential has been truncated at degree n = 331, because little information on higher degrees is likely to be present in the data. Two theories have been used: that of least

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) squares adjustment, and that of least squares collocation; above degree $n = 200$ the accuracies predicted according to collocation are significantly better than those according to least squares. In this report there is also a discussion on how to process SST data to obtain very high resolution models of the gravitational field. Descriptions and listings of computer programs are included.

To reduce the computer time and storage needed to set up and to invert the normal matrix, a somewhat simplified orbital geometry and an approximate model of the data have been adopted; no orbit determination errors have been considered. Some arguments are given to justify these shortcuts, which may not affect seriously the validity of the results. An extention of the theory to non-polar orbits is given.

The main results according to collocation can be summed up as follows: if the two satellites move in much the same polar, circular orbit at a height of 160 km and at a distance of 300 km from each other; if the accuracy of the averaged range rate is $\sqrt{2} \times 10^{-6} \mathrm{m\ s}^{-1}$, the averaging interval is 4 s, and sampling takes place every 4 s; if residual data are used, with respect to a reference model of specified accuracy, complete to degree and order 20, then:

(1) the relative error in the estimated potential coefficients could be better than 1% up to degree $n = 130$, than 10% up to $n = 210$, and than 50% up to $n = 270$;

(2) the accuracy of point geoid undulation implied by the coefficients could be better than 0.05 mm rms in the band from 3000 km to 40030 km (total error in this band), and better than 10 cm rms in the band from 140 km to 3000 km (also total error).

Foreword

This report was prepared by Dr. Oscar L. Colombo. Much of the research described in this report was carried out while Dr. Colombo was a Post Doctoral Research at The Ohio State University where the studies were supported under Air Force Contract No. F19628-79-C-0027, The Ohio State University Research Foundation Project 711664. The contract covering this research is administered by the Air Force Geophysics Laboratory, Hanscom Air Force Base, Massachusetts, with Mr. George Hadgigeorge, Contract Monitor. The actual writing of this report was carried out by Dr. Colombo at the Geodetic Institute of the University of Stuttgart in the Federal Republic of Germany.

Acknowledgements

I wish to thank Professor Rapp, Project Supervisor, for providing the encouragement and the support that allowed me to begin and to carry out this research. Thanks also to Reiner Rummel, who is partly responsible for my interest in this problem, and who provided useful comments on parts of the manuscript; to Carl Wagner, for some interesting correspondence on SST; and to Susan Carroll, who did the typing.

This report was written during the first months of a visit to the Geodetic Institute of the University of Stuttgart, in the Federal Republic of Germany. This visit was sponsored by the Alexander von Humboldt Foundation. To the staff of this institution, and to Professor Erik Grafarend, whose liberality allowed me to finish this work, my most sincere appreciation.

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1. Introduction

More than a decade ago, in 1968, Muller and Sjogren published their paper on lunar mass concentrations, or "mascons". In it they announced one of the most important conclusions about the internal structure of a member of the Solar System ever reached from the analysis of gravitation alone. With remarkable insight, they used the numerically differentiated range-rate tracking data of the lunar orbiters, deployed as part of the Apollo program, as "direct observations" of the acceleration of gravity along the line of sight between the tracking station on Earth and the spacecrafts circling the Moon. A simple plot of this information showed strong and roughly circular anomalies over the flatlands, or maria, suggesting the presence of large bodies of abnormal density buried in the lunar crust.

The idea that high resolution information on a gravitational field could be extracted by a simple analysis from differentiated range-rate data was seen, naturally enough, as having potential value for the study of our own planet. One more or less obvious adaptation of the idea was having two satellites: one in a very high orbit tracking the second one on a low orbit, much as the "high" station on Earth had tracked the "low" orbiters near the Moon. A different satellite-satellite tracking (SST) configuration, later to be known as the "low-low" approach (the first being the "high-low", of course), was proposed by Wolff in 1969: he thought that the relative velocity along the line of sight between two bodies following as close as possible the same near circular, polar orbit was, as a first approximation, the difference in anomalous potential between the two. As the Earth rotates, this pair would have eventually covered the whole planet with observations of gravity, all taken with the same "instrument", creating a global data set of extraordinary density and of uniform quality. From such data one should have been able to recover an almost as detailed, and far more complete, picture of the field than from terrestrial data alone. Though this idea suffered from some theoretical and practical problems, its appeal to the imagination was such that it inspired much research in spite of its shortcomings.

Among the first to help clarify the theory behind SST was Schwarz (1970), who proposed a rigorous mathematical formulation. Many studies followed to find out the best satellite arrangement, how the data could be processed, and what accuracy could be expected from the results. To all this one must add the work done on the development of suitable hardware, also along several lines.

In the mid-1970's, during the Apollo-Soyusz and the Geos-3 missions, "high-low" range-rate tracking data were gathered using the ATS-6 satellite, in geostationary orbit, as the high spacecraft, because of its steerable radar antenna. During the same period, the idea of a dedicated gravitational satellite mission begun to develop. Both NASA and its European counterpart ESA drew up preliminary plans for a Gravity Satellite (GRAVSAT), and for a Space Laser Low Orbit Mission (SLALOM), respectively. Of the two, SLALOM is the most concretely defined at present (CSTG Bulletin No.2, 1980). It involves the simultaneous tracking by laser interferometry from the Space Laboratory, carried on the Space Shuttle, of two reflecting spheres to determine their relative velocities with respect to the shuttle and to each other. Recovery of gravity information is to be limited to a region over the Eastern Mediterranean, and to a period of some seven days. This experiment is expected to be carried out during this decade. GRAVSAT, on the other hand, is a global concept that may take one of several possible configurations based on the SST principle or, alternatively, resort to an orbiting gradiometer. At present it appears likely that the SST idea will be tried first, in a GRAVSAT-A mission in the late 1980's, while the gradiometer, whose development into a practical instrument for this purpose is still at the "breadboard" stage, may be used in a GRAVSAT-B mission sometime in the 1990's. Several types of gradiometers have been considered. A promising design seems to be a supercooled instrument now under development at the University of Maryland, which may achieve a sensitivity of better than 0.001 E (Paik, 1981).

Of the various implementations of the "low-low" principle, the most likely to be adopted appears to be the DISCOS system of two drag-compensated satellites, capable of remaining in orbit for up to six months so close to the Earth that, without the periodical use of small rocket engines, their orbits would decay very quickly due to the atmospheric resistance. Two proof-masses, one inside each craft, will be kept in permanent free fall by the compensating mechanism, so only gravitational forces act upon them. Their relative line of sight velocity will be measured by an extremely accurate radar interferometric technique developed at the Applied Physics Laboratory of Johns Hopkins University (Pisacane and Yionoulis, 1980). Accuracies of better than $10^{-6} ms^{-1}$ are expected, provided that no serious problems due to ionospheric propagation are encountered.

Apart from the "high-low" and the "low-low" configurations, an intermediate "butterfly" arrangement, where two satellites follow different elliptical orbits, has been considered as well.

The SST data collected during the middle of the last decade were analysed in different ways. Among others, Kahn et al. (1978) tried to recover gravity anomalies using the "numerical" method of satellite geodesy, with the aid of the computer programme GEODYN; Hajela (1978) followed, instead, the original idea of treating the differentiated range-rates as gravity observations, and used least squares collocation as the processing technique. Marsh and Marsh (1978) attempted what was, in essence, an experiment like that of Muller and Sjogren with Earth data, to detect crustal and upper mantle structures. All these studies have been restricted to local areas, as no complete global set of SST has been obtained yet.

The work that has been done on the error analysis of the recovery of values of mean gravity anomalies, mean and point geoidal undulations, etc., can be divided, broadly, along two main lines: the "numerical" method of satellite geodesy, and the "direct" approach that goes back to Muller and Sjogren: using the differentiated range-rate values as observations. As an example of the first line, one can mention the work of Douglas et al. (1980), as one of the most recent. Of the second, the author is best familiar with the work of Hajela (1974), Rummel et al. (1976), Krynski (1978), Rapp and Hajela (1979), and Rummel (1980). All of these have considered local gravity field model improvements relying on the theory of least squares collocation. The ultimate accuracy of global recovery given a certain quality of data has been investigated, among others, by Breakwell (1979), who used a flat-earth approximation, while Jekeli and Rapp (1980) have employed a spherical, non-rotating Earth approximation.

While all the work mentioned above has been in progress, numerous meetings, involving government agencies and members of the scientific community that are potential users of GRAVSAT data, have taken place. Of several reports on these activities, there are two of particular importance that describe the objectives of a GRAVSAT mission and also define the required accuracies and other specifications for the results so they can be used meaningfully for geodetic and geophysical purposes. One is the report of a special panel of the Committee on Geodesy of the US National Academy of Sciences (1979); the other is by the GRAVSAT Users Working Group (1980). The main requirements defined so far are:
a) For geological and geophysical applications, gravity information should be resolved at the Earth's surface through wavelengths of 100 km and to an accuracy of 2.5 to 10 mgals;
b) Oceanographers need geoid heights accurate to some 10 cm in the band from 100 km to 3000 km.

The first specification is relevant to the study of the structure of the crust and the upper mantle. The second one is associated with the analysis of the instantaneous and average shape of the sea surface in studies that use such data as satellite altimetry. The difference between the sea surface and a horizontal surface can reveal many as yet unknown aspects of surface currents, tides, and transients of various kinds, particularly in regions of the oceans that are not sufficiently accessible by other means. Also of consequence to oceanography, as well as to many practical aspects of geodesy, such as satellite positioning techniques, is the obtention of a model for the gravity field that permits the calculation of very precise spacecraft orbits, so this is another important objective of GRAVSAT. Among the latest global studies those by Breakwell, and by Jekeli and Rapp, already mentioned, suggest that the DISCOS system could achieve both the quality and quantity of data needed to meet goals (a) and (b).

The present report describes the theory behind a global error analysis of a low-low mission of the DISCOS type, and gives the corresponding results. The latter are in broad agreement with some of the previous studies, notably Breakwell's and Jekeli and Rapp's. The approach taken has been the "direct" one of considering the differentiated range-rate signal as equal to the component of the gravitational line of sight acceleration, in an inertial frame, along the line of sight direction. This is only an approximation, but it simplifies greatly the mathematical treatment, and previous reports by Hajela (1978) and by Rummel (1980), already mentioned, indicate that it may be in good agreement with reality at short spatial wavelengths, both for the "high-low" and the "low-low" configurations, respectively. This model of the line of sight signal has been modified somewhat here by substracting a term that depends only on radial distance to the geocenter to eliminate some unrealistic long-wave phenomena related to the large zero harmonic and to the other even zonals of the geopotential.

This work is concerned primarily with the optimal recovery of a geopotential model in the form of a spherical harmonic expansion of the potential of such a high degree and order (331) that its truncation error at satellite altitude (160 km) is nearly negligible, particularly in the presence of noise.(1) It differs from previous studies of this sort in that it considers a spherical, rotating Earth, and discrete data consisting of time-averages of the instaneous range rate. Observation equations are obtained under the simplifying assumptions that the orbits are perfectly circular, the satellites separation is constant, and that the orbits repeat themselves exactly every 179 days, i.e., the length of the whole mission. From these equations a normal matrix is formed that is then inverted taking advantage of its block-diagonal structure. Two adjustment techniques are considered and implemented: least squares adjustment, and least squares collocation. An argument based on an infinite series of successive approximations is used to show how the results obtained under such simplifying assumptions can be, in fact, close indicators of the maximum amount of information than can be extracted from a real, three-dimensional distribution of SST data. While orbital errors are not included in the analysis, it turns out that the effect of such errors on the estimated parameters is very small, although this particular result depends strongly on the model adopted for the line of sight signal.

The last section considers how the principles used in this study can be applied to the actual processing of SST data in order to obtain a very high resolution spherical harmonic model of the gravity field. Such models have important mathematical advantages, and there is no great problem in using them if this is done with adequate techniques (see Colombo (1981), for instance). Global data reduction is a very important problem, to which not enough attention has been paid so far.

The potential impact of SST and of satellite gradiometry on the future of geodesy and of geophysics appears great. It is a difficult and beautiful task, the work of many, through many years, to develop what begun as a simple and bright idea for studying the Moon into a tool for increasing our understanding of our own planet and, eventually, of the rest of the Solar System. If we are successful, this task could have a deeper and more lasting effect on pure and applied Earth sciences than any other ever attempted in geodesy before.

---

(1) In this "band-limited" situation, the optimal estimates of any other field-related quantities (undulations, gravity anomalies, etc.) can be obtained directly from the optimal estimates of the potential coefficients (see, for instance, Colombo (1981), paragraph 2.18). The same is true of the accuracies of those quantities.

1.1 The Low-low Configuration

Figure 1.1 shows two drag-compensated satellites S_1 and S_2 that have been placed almost in the same near-circular polar orbit so that, in inertial space, the plane of the common orbit contains the Earth's mean axis of rotation, which is aligned in the picture with the x_3 axis. The geocentric angle separating both satellites is $\psi = 2\sin^{-1}(\frac{\rho}{2R})$ radians, where ρ is the length of the segment of line of sight between S_1 and S_2. The positions of the spacecrafts in the system of geocentric inertial coordinates x_1, x_2, x_3 are represented by the vectors $\underline{x}_1$ and $\underline{x}_2$, respectively. The line of sight vector $\underline{x}_{12} = \underline{x}_1 - \underline{x}_2$ is oriented, according to the picture, from South to North in the ascending passes. The sense of the orbit (prograde or retrograde) is not important. The mean orbital radius is R.

Both satellites turn round the Earth with approximately the same angular velocity $\omega = \sqrt{GMR^{-3}}$ while the Earth itself turns on its axis with mean angular velocity Ω. The points directly below each satellite describe groundtracks that envelope the whole surface of the planet as the mission progresses. Both satellites have the same instantaneous longitude, but their groundtracks are not identical: they are always ψ rad apart in the S-N direction and $\Omega\omega^{-1}\psi$ rad in the E-W direction. In what follows, the word "groundtrack", unless further explanation is given, should refer to that of the point midways between the satellites, along the line of sight, $P = \underline{x}_1 + \frac{1}{2}\underline{x}_{12}$.

The measuring device detects the relative velocity along the line of sight between two proof-masses, one inside each satellite, which are kept in permanent free-fall round the Earth by the drag-compensating mechanism. This mechanism compensates not only for drag, but for all other non-gravitational influences of consequence as well. The relative line of sight velocity is

$$v_{12} = \dot{\underline{x}}_{12}^T \underline{e}_{12} \tag{1.1}$$

where

$$\dot{\underline{x}}_{12} = \dot{\underline{x}}_1 - \dot{\underline{x}}_2$$

is the relative inertial velocity, $\dot{\underline{x}}_1$ and $\dot{\underline{x}}_2$ being the absolute ones, while

$$\underline{e}_{12} = -\rho^{-1}\underline{x}_{12} \tag{1.2}$$

is the unit vector along the line between S_1 and S_2, and

$$\rho = \|\underline{x}_{12}\| = \sqrt{(x_{11}-x_{12})^2 + (x_{21}-x_{22})^2 + (x_{31}-x_{32})^2} \tag{1.3}$$

is the euclidean norm of $\underline{x}_{12}$, i.e., the distance between the two spacecrafts. The observed values of v_{12} are averaged over Δa seconds and then transmitted to earth-side stations every Δt seconds, where $\Delta a < \Delta t$. These averages constitute the SST data to be studied here, and the expression for any one of them is

$$\bar{v}_{12}(t) = \int_{t-\Delta a}^{t} v_{12}(\tau)\, d\tau \tag{1.4}$$

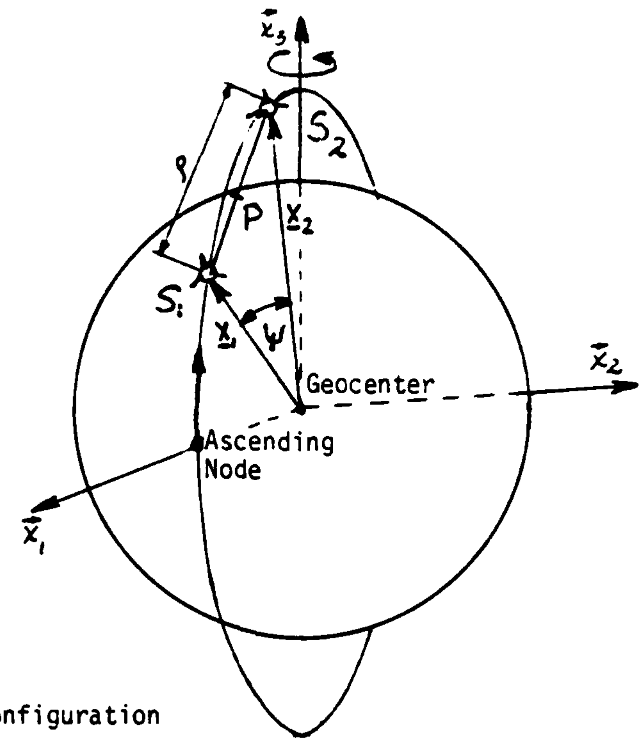

Figure 1.1:
The Low-Low Configuration

1.2 The Band-limited Assumption

The line of sight velocity v_{12} reflects small changes in the velocities of both satellites about their common average $\upsilon = R\omega$. These changes are brought about by the anomalous gravitational field, which is the difference between the true field and that of a homogeneous geocentric sphere of the same mass as the Earth and radius smaller than R .

The actual gravitational potential V can be represented in geocentric spherical coordinates (radial distance r , latitude ϕ , longitude λ) by a spherical harmonic expansion

$$V(r,\phi,\lambda) = \frac{GM}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^n \bar{P}_{nm}(\sin\phi)\,[\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda] \quad (1.5.a)$$

where $\bar{P}_{nm}(\sin\phi)$ is the fully normalized associated Legendre function of the first kind, and $\bar{C}_{nm}$, $\bar{S}_{nm}$ are fully normalized spherical harmonic coefficients. The following alternative notation will be used, wherever possible, in this work:

$$V(r,\phi,\lambda) = \frac{GM}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\alpha=0}^{1}\left(\frac{a}{r}\right)^n \bar{C}^{\alpha}_{nm}\,\bar{Y}^{\alpha}_{nm}(\phi,\lambda) \quad (1.5b)$$

with

$$\bar{C}^{\alpha}_{nm}\,\bar{Y}^{\alpha}_{nm}(\phi,\lambda) = \bar{P}_{nm}(\sin\phi)\begin{cases}\bar{C}_{nm}\cos m\lambda & \alpha = 0\\ \bar{S}_{nm}\sin m\lambda & \alpha = 1\end{cases}$$

M is the mass of the Earth, G the universal constant of gravitation, "a" is the mean Earth radius, "n" indicates the degree, and "m" the order of each term in the expansion.

The three terms with $n=1$ are zero, because here the geocenter coincides with the origin on coordinates; the zero harmonic equals $\frac{GM}{r}$, so $\bar{C}_{00}=1$. The anomalous potential is, therefore,

$$R(r,\phi,\lambda) = V(r\phi,\lambda) - \frac{GM}{r} \tag{1.6}$$

The disturbing potential T, the modelling of which is the concern of this report, is the difference between the true potential V and some reference model potential U of the form

$$U(r,\phi,\lambda) = \frac{GM}{r} \sum_{n=0}^{NM} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} \bar{C}^{\alpha\,(M)}_{nm} \bar{Y}^{\alpha}_{nm}(\phi,\lambda) \tag{1.7}$$

where NM is a relatively small integer (20 or 30). The objective of this study is finding out the accuracy with which a model of T of the same form as U, but truncated at a much higher order N, could be recovered from low-low SST data.

As explained later in this section, the time derivative of the line of sight velocity can be approximated by the component of the inertial acceleration aimed along the line of sight, minus a term independent from ϕ and λ. This acceleration is a linear combination of the three accelerations

$$a_r(r,\phi,\lambda) = \frac{\partial V}{\partial r} = -\frac{GM}{r^2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} \left(\frac{a}{r}\right)^n \bar{C}^{\alpha}_{nm} \bar{Y}^{\alpha}_{nm}(\phi,\lambda) \tag{1.8,a}$$

$$a_\phi(r,\phi,\lambda) = \frac{1}{r}\frac{\partial V}{\partial \phi} = \frac{GM}{r^2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} \left(\frac{a}{r}\right)^n \bar{C}^{\alpha}_{nm} \frac{\partial}{\partial \phi} \bar{Y}^{\alpha}_{nm} \tag{1.8,b}$$

$$a_\lambda(r,\phi,\lambda) = \frac{1}{r\cos\phi}\frac{\partial V}{\partial \lambda} = \frac{GM}{r^2\cos\phi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{\alpha=0}^{1} \left(\frac{a}{r}\right)^n \bar{C}^{\alpha}_{nm} \frac{\partial}{\partial \lambda} \bar{Y}^{\alpha}_{nm} \tag{1.8,c}$$

where

$$\frac{\partial}{\partial \phi} \bar{Y}^{\alpha}_{nm} = \frac{d}{d\phi} \bar{P}_{nm}(\sin\phi) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} m\lambda$$

and

$$\frac{\partial}{\partial \lambda} \bar{Y}^{\alpha}_{nm} = m\, \bar{P}_{nm}(\sin\phi) \begin{Bmatrix} -\sin \\ \cos \end{Bmatrix} m\lambda$$

Because all these expansions converge outside the Earth's bounding sphere, the general terms of all of them should tend to zero with n tending to infinity, so the size of the harmonics should, in general, decrease as n increases. This decay must be accentuated by the factors $(\frac{a}{r})^n$, wherefore the field should become smoother with altitude, as the higher frequency terms vanish faster than the rest with increasing r. At satellite height $h=R-a$ this smoothing should mean that, above a certain degree N, all terms in the expansion can be neglected. Consequently, the field can be regarded as band-limited, with terms restricted to degrees in the band $0 \leq n \leq N$. As the error analysis method presented in section 2 requires computer time in proportion to N^4, it is important to find a realistic value for N that is also as low as possible. The reasoning that follows attempts to provide a guide for such a choice using the decay in the spectrum of the line of sight inertial acceleration as a criterion.

To simplify matters, Earth rotation can be ignored, the orbit can be assumed to be perfectly circular, and the geocentric separation ψ to be constant. Under these conditions, all field dependent functions are periodical, the line of sight acceleration among them, and can be represented by Fourier series such as

$$a_{12}(t) = \sum_{k=0}^{\infty} a_k \cos k\,\omega t + b_k \sin k\,\omega t$$

As the Earth does not rotate in this case, one can choose an arbitrary system of geocentric coordinates $r'=r$, ϕ', λ' where the "equator" coincides with the plane of the orbit, and then make the substitution

$$\lambda = [\omega t]_{\text{MODULE } 2\pi}$$

so that the Fourier series becomes

$$a_{12}(\lambda) = \sum_{k=0}^{\infty} a_k \cos k\lambda + b_k \sin k\lambda$$

To find out the coefficients of this series, consider first the inertial line of sight acceleration in this system of coordinates. For the first satellite, let

$$a_1 = \begin{pmatrix} a_r\ (R,\ \phi' = 0,\ \lambda') \\ a\phi\ (R,\ \phi' = 0,\ \lambda') \\ a\lambda\ (R,\ \phi' = 0,\ \lambda') \end{pmatrix}^T \underline{e}_{12} \equiv \underline{a}_1^T\ \underline{e}_{12}$$

be the projection of its inertial acceleration along that line, and let

$$a_2 = \underline{a}_2^T\ \underline{e}_{12}$$

be the corresponding projection for the second satellite. Then the line of sight acceleration is

$$\begin{aligned} a_{12} &= a_1 - a_2 \\ &= \underline{e}_{12}^T(a_r(S_1)\ \underline{r}_0^{(S_1)} + a_{\phi'}\ (S_1)\ \underline{\phi}_0^{(S_1)} + a_{\lambda'}\ (S_1)\ \underline{\lambda}_0^{(S_1)}) \\ &\quad - \underline{e}_{12}^T(a_r(S_2)\ \underline{r}_0^{(S_2)} + a_{\phi'}\ (S_2)\ \underline{\phi}_0^{(S_2)} + a_{\lambda'}\ (S_2)\ \underline{\lambda}_0^{(S_2)}) \end{aligned} \qquad (1.9)$$

where $r^{(P)}$, $\underline{\phi}^{(P)}$, and $\underline{\lambda}^{(P)}$ are the unit vectors pointing upwards,"S-N", and "W-E" at the general point $P(r,\phi',\lambda')$. Because the circular orbit now lies on the equatorial plane of the rotated system of coordinates, we have

$$\underline{e}_{12}^T\underline{\phi}_0^{(S_1)} = \underline{e}_{12}^T\underline{\phi}_0^{(S_2)} = 0 \qquad (1.10,a)$$

From Figure 1.2 it is easy to see that

$$\underline{e}_{12}^T\underline{r}_0^{(S_1)} = -\underline{e}_{12}^T\underline{r}_0^{(S_2)} = -\sin\frac{\psi}{2} \qquad (1.10,b)$$

and

$$\underline{e}_{12}^T\underline{\lambda}_0^{(S_1)} = \underline{e}_{12}^T\underline{\lambda}_0^{(S_2)} = \cos\frac{\psi}{2} \qquad (1.10,c)$$

Calling the longitudes of S_1 and S_2 λ' and $\lambda' + \psi$, respectively,

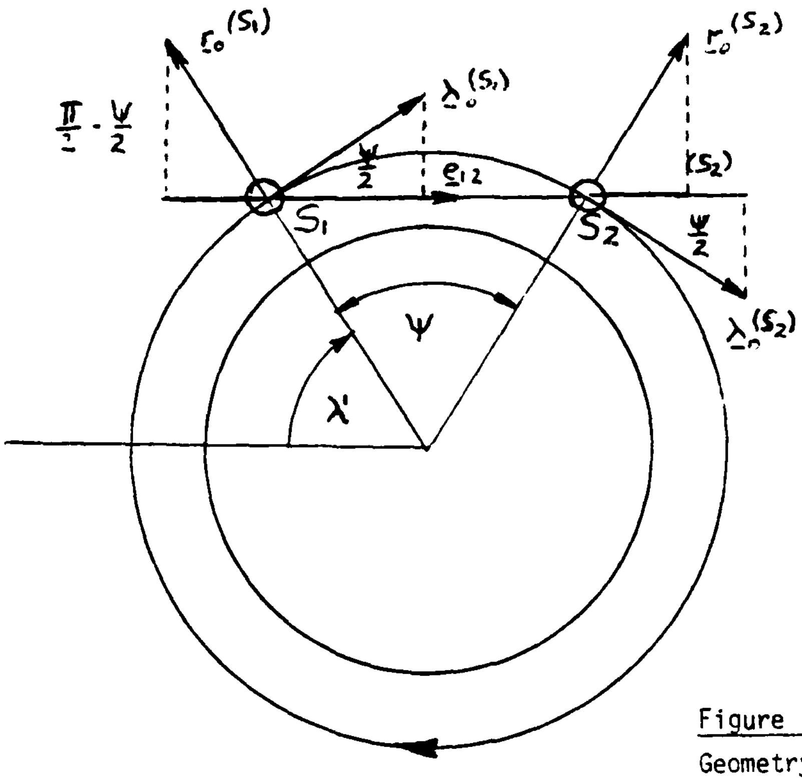

Figure 1.2:

Geometry of the Satellite Pair

replacing (1.8, a-c) and (1.10, a-c) in (1.9), rearranging terms (which is valid, because all the expansions converge uniformly outside the bounding sphere), and making use of simple trigonometric identities,

$$a_{12}(R,\phi'=0,\lambda') = \frac{GM}{R^2}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\bar{P}_{nm}(0)\left(\frac{a}{r}\right)^n \Big[(n+1)\bar{C}_{nm}\sin\frac{\psi}{2} + m\bar{S}_{nm}\cos\frac{\psi}{2}\Big]\cos n\lambda'$$

$$+ \Big[(n+1)\bar{S}_{nm}\sin\frac{\psi}{2} - m\bar{C}_{nm}\cos\frac{\psi}{2}\Big]\sin m\lambda' - \Big[-(n+1)\bar{C}_{nm}\sin\frac{\psi}{2} + m\bar{S}_{nm}\cos\frac{\psi}{2}\Big]$$

$$\cdot \cos m(\lambda'+\psi)$$

$$- \Big[-(n+1)\bar{S}_{nm}\sin\frac{\psi}{2} - m\bar{C}_{nm}\cos\frac{\psi}{2}\Big]\sin m(\lambda'+\psi) \qquad (1.11)$$

Therefore, the coefficients of the Fourier series for a_{12} are

$$a_m = \frac{GM}{R^2}\sum_{n=m}^{\infty}\bar{P}_{nm}(0)\left(\frac{a}{r}\right)^n \Big\{\bar{C}_{nm}\Big[(n+1)\sin\frac{\psi}{2}(1+\cos m\psi) +$$

$$+ m\cos\frac{\psi}{2}\sin m\psi\Big] + \bar{S}_{nm}\Big[m\cos\frac{\psi}{2}(1-\cos m\psi) + (n+1)\sin\frac{\psi}{2}\sin m\psi\Big]\Big\} \qquad (1.12,a)$$

and

$$b_m = \frac{GM}{R^2}\sum_{n=m}^{\infty}\bar{P}_{nm}(0)\left(\frac{a}{r}\right)^n \Big\{\bar{C}_{nm}\Big[-(n+1)\sin\frac{\psi}{2} - m\cos\frac{\psi}{2}(1-\cos m\psi)\Big] +$$

$$+ \bar{S}_{nm}\Big[(n+1)\sin\frac{\psi}{2}(1+\cos m\psi) + m\cos\frac{\psi}{2}\sin m\psi\Big]\Big\} \qquad (1.12,b)$$

where "m" has replaced the original subscript "k" for obvious reasons. The time integral $\int a_{12}(t)\, dt$ has a term of the form $a_0 t$ so it cannot be identical to v_{12} because the latter is periodical under the current assumptions. This problem can be solved by removing the constant term from the Fourier series of a_{12}, so the time derivative of v_{12} would be (approximately)

$$\begin{aligned} \hat{a}_{12} &= a_{12} - a_0 \\ &\cong \dot{v}_{12} \end{aligned} \tag{1.13}$$

and

$$v_{12} = \int_0^\lambda \hat{a}_{12}\,(R,0,\lambda')\,\frac{d\lambda'}{\omega} = \int_0^{t=\frac{\lambda}{\omega}} \hat{a}_{12}\,(\omega t')\,dt'$$

$$= \sum_{m=1}^{\infty} a_m \frac{\sin\ \omega t}{m\,\omega} - b_m \frac{\cos m\,\omega t}{m\,\omega} + \sum_{m=1}^{\infty} \frac{b_m}{m\,\omega}$$

If (1.13) can be accepted as a valid approximation, then

$$\frac{1}{T_s}\int_0^{T_s} \hat{a}_{12}^2\, dt = \tfrac{1}{2}\sum_{m=1}^{\infty} a_m^2 + b_m^2 \qquad \left(\text{where } T_s = \frac{2\pi}{\omega}\right)$$

$$= \sum_{m=1}^{\infty} P_m \tag{1.14}$$

where $P_m = \frac{1}{2}(a_m^2+b_m^2)$ is the power at the mth frequency. Consider now the averaging operator $M\{\}$, which will be encountered again in Section 2 in connection with the use of least squares collocation as an alternative to the usual least squares adjustment for obtaining a field model. This operator represents an average over all rotations; here it can be seen as averaging over all possible circular orbits round a non-rotating Earth. As shown, for instance, in (Colombo, 1981, Section 2),

$$M\{\bar{C}_{nm}^2\} = M\{\bar{S}_{nm}^2\} = \frac{\sigma_n^2}{2n+1} \tag{1.15,a}$$

$$M\{\bar{C}_{nm}\,\bar{S}_{kq}\} = 0 \qquad \text{for all integer } n, m, k, \text{ and } q \tag{1.15,b}$$

$$M\{\bar{C}_{nm}\,\bar{C}_{kq}\} = M\{\bar{S}_{nm}\,\bar{S}_{kq}\} = 0 \qquad \text{for } n \neq k,\ m \neq q \tag{1.15,c}$$

where

$$\sigma_n^2 = \sum_{m=0}^{n} \bar{C}_{nm}^2 + \bar{S}_{nm}^2 \tag{1.16}$$

is usually called the "nth degree variance" of the potential coefficients. The average orbital power per wavelength is, then,

$$\hat{P}_m = M\{P_m\} = \tfrac{1}{2}M\{a_m^2 + b_m^2\} \tag{1.17,a}$$

for $\hat{a}_{12}$, and

$$S_m = \frac{1}{\omega^2 m^2}\hat{P}_m \tag{1.17,b}$$

for v_{12}. Squaring (1.12, a-b) and modifying the resulting expansion according to (1.15,a-c), (1.17,b) finally becomes

$$S_m = P_{rm}\,(1+\cos m\psi) + P_{tm}(1-\cos m\psi) + 2P_{rtm}\,\sin m\psi \qquad (1.18)$$

where

$$P_{rm} = \frac{G^2\,M^2}{2R^4\omega^2 m^2}\sum_{n=m}^{\infty}\frac{\sigma_n^2}{2n+1}\left(\frac{a}{R}\right)^{2n}(n+1)^2\,(1-\cos\psi)\,\bar{P}^2_{nm}\,(0) \qquad (1.19,a)$$

is the contribution from the radial acceleration,

$$P_{tm} = \frac{G^2\,M^2}{2R^4\omega^2 m^2}\sum_{n=m}^{\infty}\frac{\sigma_n^2}{2n+1}\left(\frac{a}{R}\right)^{2n} m^2(1+\cos\psi)\,\bar{P}^2_{nm}\,(0) \qquad (1.19,b)$$

the corresponding contribution from the along-track acceleration, and

$$P_{rtm} = \frac{G^2\,M^2}{2R^4\omega^2 m^2}\sum_{n=m}^{\infty}\frac{\sigma_n^2}{2n+1}\left(\frac{a}{R}\right)^{2n} m(n+1)\,\sin\psi\,\bar{P}^2_{nm}(0) \qquad (1.19,c)$$

is the average crosspectral power of the two.

At a height $R-a = 160$ km, the angular frequency of the satellites is $\omega = \sqrt{\frac{GM}{R}} = 1.196 \times 10^{-3}$ rad. s^{-1}. If the separation between the satellites is $\rho = 300$ km, so $\psi = 0.04594$ rad. , and if the σ_n are those used in the error analysis of Section 3, empirically derived from terrestrial and satellite data, then the S_m are as listed in the table below:

Table 1.1

Spectral R.M.S. $(S_m)^{\frac{1}{2}}$ of the Line of Sight Velocity, and $(\hat{P}_m)^{\frac{1}{2}}$ (acceleration)

spatial frequency m (cycles per rev.)	$(S_m)^{\frac{1}{2}}$ (cm s^{-1})	$(\hat{P}_m)^{\frac{1}{2}}$ (mgal)
0	0.	---
1	.335	.410
2	48.854	119.813
3	.348	1.279
4	.205	1.005
5	.170	1.044
10	.678 x 10^{-1}	.833
20	.268 "	.656
30	.192 "	.706
40	.130 "	.637
50	.932 x 10^{-2}	.572
100	.855 x 10^{-3}	.105
200	.394 x 10^{-4}	.967 x 10^{-2}
300	.107 x 10^{-5}	.394 x 10^{-3}
400	.207 x 10^{-7}	.101 x 10^{-4}
700	.817 x 10^{-11}	.701 x 10^{-8}
1000	.544 x 10^{-14}	.660 x 10^{-11}

Total rms of signal above

m = 0	48.86 cm s^{-1}	120.32 mgal
20	0.06 cm s^{-1}	4.44 mgal

To obtain the values listed above, the series in (1.19,a-c) were truncated at n=1100, though much the same results were obtained with n=500 (up to m=300), which suggests a strongly band-limited nature for $\hat{a}_{12}$ and v_{12}. The preponderance of the term with $m=2$ is due to the large second zonal and is related to the Earth oblatness. The signal in the data consists of time averages of v_{12}, so its spectrum should be smoother than the entries in the table suggest. For the purpose of this discussion, such a refinement is not necessary, and the effect of time-averaging will not be considered until Section 2.

With Δt=4 seconds, some $\frac{2\pi}{\Delta t\omega} \approx 1427$ samples of v_{12} are taken during each revolution of the satellites. Accordingly, the "Nyquist frequency" of the data is N_y=713 . It is clear from the table that the power above this frequency is negligible, so that aliasing problems related to the sampling rate are likely to be insignificant. The question of choosing N , the highest degree in a band-limited model of the potential, so that this model can be regarded as realistic, is not an easy one, except for the fact that N does not have to be larger than 713. For this study the value of N=331 has been chosen purely on the basis that this number appeared to be, on preliminary estimates, the largest N compatible with the computing resources available to the author. The results in Section 3 show recovery errors of more than 80% of the actual values for the coefficients of harmonic n=330 and, for a number of reasons discussed in Section 4, these estimates are rather optimistic at the upper end of the spectrum. So, perhaps, N=331 is truly close to the upper limit of resolution for global estimation procedures of the kind considered here (least squares and collocation).

1.3 An Approximate Model for the Line of Sight Velocity

To estimate the accuracy with which the spherical harmonic coefficients of the geopotential can be recovered from SST data, one needs a model that relates the information in these data to those coefficients. A rigorous approach involves the solution of many variational differential equations for the two satellites, which could be done, in an average sense, by the "analytical" method so well described in Kaula's "Satellite Geodesy" (1966), and in an instantaneous sense, by the "numerical" approach, an example of which is the theory behind the famous "GEODYN" computer programme (Martin et al., 1970). Regrettably, except for some major break-through in computer science, the exact application of either technique to the global study of SST data from low-orbiting satellites does not seem feasible. The problem is the sheer size of the spherical harmonic model needed to represent this data realistically, with N > 300, as suggested in the previous paragraph, and some 4×10^6 observations over a six month mission. The largest models obtained to date (for instance GEM 9, by Lerch et al., 1977) by either technique from satellite tracking data alone have not exceeded N=30, and have already involved very lengthy operations. This does not mean that these methods have no role in the analysis of SST data: the "numerical", at any rate, has been used already for the recovery of gravity anomalies from

Apollo-Soyuzs data (Kahn, 1979), and for the error analysis of a GRAVSAT mission (Douglas et al., 1980), but all this has been done on a local basis, and the purpose of this report is to look at the problem globally.

If a rigorous approach is not practicable, then one must seek some reasonable approximation that makes the task easier. In this work the author has followed the example of previous studies (Hajela, 1978; Rummel 1980), assuming that the time-derivative of the line of sight velocity can be approximated to a sufficient extent by the line of sight component of the inertial acceleration. As explained in the previous paragraph, the inertial acceleration due to gravitation has an average component along that line that is not zero, due to the powerful zero harmonic GMr^{-2} and, to a much lesser extent, to the even zonals (consider (1.12,a) with $m=0$). A non-zero mean acceleration would bring the two satellites (in this case) together, contradicting the assumption that they can follow (with the right initial conditions) the same orbit with the same mean angular velocity. The effect of the zero harmonic alone, for a height of 160 km and $\rho=300$ km, is

$$a_0 = 2 \sin\left(\frac{\psi}{2}\right) \frac{GM}{R^2} \cong 43 \text{ gal}.$$

and the interpretation of this is plain enough: in the field of a central point mass two objects initially at rest would fall along the lines joining their initial positions to the attracting mass and, because their motions converge at this point, they would be moving closer to each other as well. Clearly, this is not the case when the two bodies turn in the same circular orbit, where the relative velocity and its derivative are always zero. The approximation, it can be argued, can be much better for higher frequency effects, i.e., the departures of both velocity and acceleration from their values for a central mass field caused by the mass anomalies. The question is too complex to be settled by a simple argument, so the conclusions arrived at by previous authors and by this one, too, for that matter, are of necessity no more than educated guesses of a provisional nature. In the discussion that follows, I start by deriving rigorously the relationship between velocity and acceleration, and then introduce "order of magnitude" estimates for some of the factors involved.

To get directly to the results of interest, one should consider the residual line of sight velocity

$$\delta v_{12} = v_{12} - v_{12}^{(c)} \tag{1.20}$$

where $v_{12}^{(c)}$ is the velocity $(\underline{x}_1^{(c)} - \underline{x}_2^{(c)})^T \underline{e}_{12}^{(c)}$ computed by integrating the equations of motion numerically with the reference potential model U: this can be called the reference velocity. The residual velocity is what is likely to constitute the data in a real life situation, and has been taken for such in the error analysis of Section 3. Removing the reference velocity (at least in the ideal case where the model truly represents the field up to degree and order NM) has the advantage of taking away the very large effect of the second zonal whose presence is undesirable from a numerical point of view. As the computed orbit is likely to reflect very closely the effect of the attraction of the Sun, the Moon, and of the major planets, as well as the indirect effect of the tidal deformation of the Earth, the residual data are likely to be freer from these unwanted gravitational signals than the original data, resulting in less interference with the desired information. Non-gravitational effects are removed, presumably, by the drag-compensating mechanism .

The time-derivative of the residual line of sight velocity is

$$\delta\dot{v}_{12} = \frac{d}{dt}(\dot{\underline{x}}_{12}^T \underline{e}_{12} - \dot{\underline{x}}_{12}^{(c)T} \underline{e}_{12}^{(c)}) = \ddot{\underline{x}}_{12}^T \underline{e}_{12} - \ddot{\underline{x}}_{12}^{(c)T} \underline{e}_{12}^{(c)} + \dot{\underline{x}}_{12}^T \dot{\underline{e}}_{12} - \dot{\underline{x}}_{12}^{(c)T} \dot{\underline{e}}_{12}^{(c)}$$

$$= \ddot{\underline{x}}_{12}^T \underline{e}_{12} - \ddot{\underline{x}}_{12}^{(c)T} \underline{e}_{12}^{(c)} + \dot{\underline{x}}_{12}^T (\dot{\underline{x}}_{12} - \dot{\underline{x}}_{12}^T \underline{e}_{12} \underline{e}_{12}) \rho^{-1} - \dot{\underline{x}}_{12}^{(c)T} (\dot{\underline{x}}_{12}^{(c)} - \dot{\underline{x}}_{12}^{(c)T} \underline{e}_{12}^{(c)} \underline{e}_{12}^{(c)}) \rho^{-1(c)}$$

$$\cong [\ddot{\underline{x}}_{12} - \ddot{x}_{12}^{(c)}]^T \underline{e}_{12} + \rho^{-1} [\dot{\underline{x}}_{12}^T \dot{\underline{x}}_{12} - \dot{\underline{x}}_{12}^{(c)T} \underline{x}_{12}^{(c)} + (\dot{\underline{x}}_{12}^{(c)T} \underline{e}_{12})^2 - (\dot{\underline{x}}_{12}^T \underline{e}_{12})^2] \tag{1.21}$$

assuming that the computed orbit differs from the true orbit by only a few meters, as it is believed to be the case when using contemporary tracking data and field models, so $\underline{e}_{12}^{(c)} \cong \underline{e}_{12}$ and $\rho^{(c)} \cong \rho$. The first term in (1.21) is the residual inertial line of sight acceleration, so the second term corresponds to the discrepance between $\delta\dot{v}_{12}$ and $\delta a_{12} = a_{12} - a_{12}^{(c)}$. This second term can be written

$$\varepsilon = \rho^{-1}[(\dot{\underline{x}}_{12}^T \underline{e}_n)^2 - (\dot{\underline{x}}_{12}^{(c)T} \underline{e}_n)^2] \tag{1.22}$$

where $\underline{e}_n$ is the unit vector pointing outward along the normal to the line of sight. Therefore

$$\delta\dot{v}_{12} = \delta a_{12} + \varepsilon$$

$$= \delta a_{12} + \rho^{-1} \delta(\dot{\underline{x}}_{12}^T \underline{e}_n)^2 \tag{1.23}$$

So far the rigorous analysis. How large can ε be? Taking the mean value of ε over all rotations

$$M\{\varepsilon\} = M\{\delta(\dot{\underline{x}}_{12}^T \underline{e}_n)^2\} = \rho^{-1}(\sum_{m=1}^{\infty} S_m^n - \sum_{m=1}^{\infty} S_m^{n(c)}) \tag{1.24}$$

where S_m^n and $S_m^{n(c)}$ are the average power at m cycles per revolution in $\dot{x}_{12}^T e_n$ and $\dot{x}_{12}^{(c)T} e_n$, respectively. These are, probably, of the same order of magnitude as $\dot{x}_{12}^T e_{12}$ and $\dot{x}_{12}^{(c)T} e_{12}$. As the model is restricted to spacial frequencies of NM cycles per revolution or less, it should be $S_m^{n(c)}=0$ if $m > NM$. Replacing S_m^n with the S_m in (1.24), for $m > NM$ the summation should add up to

$$M_{\{\varepsilon\}}^{m>NM} \cong \rho^{-1} \sum_{m=NM+1}^{\infty} S_m \tag{1.25}$$

where $\overset{m>NM}{\varepsilon}$ symbolizes the contribution to ε from frequencies above NM . Comparing the <u>mean</u> value for the mth component of ε to the root mean square value for the corresponding component of δa_{12} , or $P_m^{\frac{1}{2}}$,

$$\frac{M\{\varepsilon_m\}}{\hat{P}_m^{\frac{1}{2}}} = \rho^{-1}\frac{S_m}{P_m^{\frac{1}{2}}} = \rho^{-1}\,\frac{\hat{P}_m^{\frac{1}{2}}}{m^2\omega^2} \cong 3\frac{\hat{P}_m^{\frac{1}{2}}}{m^2} \qquad (1.26)$$

if ρ=300 km and $\omega \cong 1.2 \times 10^{-3}$. In spite of the presence of $\hat{P}_m^{\frac{1}{2}}$, this result is, in fact, dimensionless. From Table 1.1 follows that for NM=20, for instance, the right hand side of the expression above is $5. \times 10^{-3}$, Rummel (1981) has found, by a similar reasoning, that the ratio of $M\{\delta\dot{x}_{12}^2\}$ to the total rms of δa_{12} (with NM =20) may be less than 1.5×10^{-7}, which may be in agreement with expression (1.26): the relative error decreases with frequency because of the m^{-2} factor and of the fast decay of $\hat{P}_m^{\frac{1}{2}}$ with m , so the total ratio should be less than the partial ratio at m=20 . Clearly, there are several assumptions involved in these results which have no obvious justification, except that they are not too unreasonable. As the question of "reasonable" or "unreasonable" is a tricky one, this author would conclude that, at this point, the omens look favorable for the model of the line of sight signal proposed here, but detailed studies should be done to verify this question further. It is not just a matter of deciding how to conduct an error analysis of a SST mission, but also whether it may be possible to find an algorithm for reducing the data from such a mission which, along the lines explained in section 5, could be capable of resolving potential coefficients to a much higher degree and order than it is possible at present with existing techniques. One way of conducting a more conclusive study of this matter may be to compute simulated orbits using a field with a broad spectrum, such as that of the point mass model used by Wagner and Colombo (1979), making this model rotate like the Earth, and to compute also the orbits corresponding to a "model potential" consisting of the first NM harmonics of the point mass field. From both sets of orbits one can obtain all the information needed to calculate the two terms in (1.23) and to compare them with each other in various ways, thus throwing a much brighter light on this whole subject. The field of the point mass model referred to above has, roughly, the same power spectrum as that of the Earth, but it is very easy to compute because it consists only of 200 mascons.

Not only the issue of the accuracy of the model proposed here for the line of sight signal can be clarified by numerical studies, but perhaps better models may emerge that share with the present one all the practical advantages and that are also truer to the real situation.

The model chosen for $\dot{v}_{12}$ in this work is not a_{12} but, as explained in the previous paragraph, $\hat{a}_{12} = a_{12} - a_0$, where a_0 is a term independent of ϕ and λ , about which more will be said in section 2, and equal to the contribution to a_{12} of the zero harmonic and of the zero frequency terms in the Fourier expansions of the even zonals. It could be argued that the difference between this and the model used by previous workers (who like Rummel, have chosen $\dot{v}_{12} = a_{12}$) is trivial, because a_0 must be almost entirely removed when the reference velocity is substracted from the data. This is true only if the orbit along which the reference velocity is computed coincides with the true orbit. In general, this is not the case, as orbital errors are present in the reference orbit. Because of these errors, the "removal" of the zero harmonic and even zonal effects accomplished by substracting the reference velocity is not as thorough as if such effects had never been included in the definition of the time-derivative of v_{12} , as proposed here. If left in, the very large pseudo-effect of the zero harmonic would give the impression that orbital errors have an influence on the estimated potential coefficients that is more important than with the present model. Therefore, the choice between the two models is not at all trivial. This is another aspect that could be clarified by careful numerical simulations.

2. The Mathematical Model

This section presents the observation and normal equations for the adjustment of spherical harmonic coefficients of the Earth's gravitation potential from low-low tracking data. These equations are derived here under some simplifying assumptions. The admissibility of such simplifications is discussed elsewhere, particularly in sections 1 and 4.

2.1 Simplifying Assumptions

Computations involving spherical harmonic expansions can be speeded up greatly if there are regularities in the distribution of the data. While such regularities may not occur in reality, actual and ideal distributions may be close enough to each other to ensure that the results obtained for the ideal situation are also applicable to the real one. The assumptions in question are:

1) both satellites describe coplanar, circular, polar orbits with the same geocentric radius R and with the same mean angular velocity ω ;

2) the plane of the orbits is fixed in inertial space, and fluctuations in the geocentric angle ψ between the two satellites are disregarded;

3) the Earth's angular velocity vector is fixed in inertial space, its magnitude is constant and equal to Ω , and its direction coincides with that of the figure axis;

4) the mission lasts an integer number of days N_D ; ω and Ω are commensurable, so the groundtracks of the satellites repeat themselves (for the first time) after N_D days; the total number of revolutions of the mid-point between satellites, N_r , is prime with respect to N_D ;

5) there is perfect compensation for ionospheric propagation, for radar pointing errors, and for all non-gravitational forces such as drag and solar pressure; the attractions of the Sun, Moon and major planets have been accounted for exactly when computing the satellite reference orbits and velocities (the data consists of residual velocities, i.e., differences between measured and reference values);

6) data are sampled at constant intervals of Δt seconds without interruptions during the whole mission; there is an exact number N_p of sampling intervals in the total time $T = N_D \times 24 \times 3600$ s , and $N_p = \frac{T}{\Delta t}$ is an even number;

7) the sampled values consist of the residual line of sight velocities averaged over Δa seconds, where $\Delta a \leq \Delta t$;

8) all data errors are uncorrelated, have zero mean, and the same standard deviation;

9) the line of sight inertial acceleration differs from the time derivative of the line of sight velocity only by a function of r (constant for a circular orbit);

10) at satellite altitude, the detectable gravitational signal is confined to harmonic terms of degree no larger than $N = 331$; the highest frequency in such terms is less than half of the sampling frequency $\frac{1}{\Delta t}$ c/s ;

11) The only source of uncertainty is the presence of errors in the measured line of sight velocities.

Assumptions (5). (6), (7), and (8) by and large state what a perfectly successful mission should produce in terms of data; assumptions (9) and (10) have been explained already in section 1; assumption (11) is basically sound if assumption (9) is admissible, because as mentioned in paragraph (1.3) and further argued in section 4, (9) implies that the coupling between orbit determination and field modelling is weak, so orbital errors, provided they do not exceed a few meters, have little effect on the recovered potential coefficients. Assumptions (1) through (4) define a simplified geometry, the validity of which is treated in detail in section 4.

2.2 The Extended Legendre Function

The equation

$$L_{nm}(\phi) = \frac{(-1)^m(2n!)}{2^n n!(n-m)!} \cos^m\phi \, [\sin^{n-m}\phi - \frac{(n-m)(n-m-1)}{2(2n-1)} \sin^{n-m-2}\phi$$

$$+ \frac{(n-m)\dots(n-m-3)}{2.4(2n-1)(2n-3)} \sin^{n-m-4}\phi \; . \; . \; . \;] \qquad (2.1)$$

defines the function L_{nm} of ϕ that has the following properties:

(a) $L_{nm}(\phi) = P_{nm}(\sin\phi)$ and $\frac{d^h}{d\phi^h} L_{nm}(\phi) = \frac{d^h}{d\phi^h} P_{nm}(\sin\phi)$ if $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$

(b) $L_{nm}(\phi) = L_{nm}(-\phi)$ if $n - m$ is even (2.2,a)

$L_{nm}(\phi) = -L_{nm}(-\phi)$ if $n - m$ is odd (2.2,b)

$L_{nm}(\phi) = L_{nm}(\pi-\phi)$ if m is even (2.2,c)

$L_{nm}(\phi) = -L_{nm}(\pi-\phi)$ if m is odd (2.2,d)

in the interval $0 \leq \phi \leq 2\pi$.

From (a) and (b) one can infer that
(1) $L_{nm}(\phi)$ is even if $n-m$ is even, odd if $n-m$ is odd;
(2) $L_{nm}(\phi)$ has a finite Fourier expansion where the highest term is the nth harmonic and only sines or cosines are present, depending on the parity of $(n-m)$:

$$L_{nm}(\phi) = \sum_{p=0}^{n} h_p^{nm} \cos p\phi \quad \text{if } n - m \text{ is even} \qquad (2.3,a)$$

$$L_{nm}(\phi) = \sum_{p=0}^{n} h_p^{nm} \sin p\phi \quad \text{if } n - m \text{ is odd;} \qquad (2.3,b)$$

(3) $L_{nm}(\phi)$ has half wave symmetry ($L_{nm}(\phi)=L_{nm}(\pi-\phi)$) if m is even, and half wave antisymmetry ($L_{nm}(\phi)=-L_{nm}(\pi-\phi)$) if m is odd. Sums of sines or of cosines which have such symmetries can contain only even or odd harmonics, so the Fourier expansion of L_{nm} must have only even terms if n is even, and only odd terms if n is odd. As a consequence, the Fourier coefficients

h_p^{nm} with p of opposite parity from n are all zero;
(4) $L_{nm}(\phi)$ is continuous and infinitely differentiable in $0 \le \phi < 2\pi$ and takes, together with all its derivatives, the same values as P_{nm} and all its derivatives in the interval $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$.

The function L_{nm} is the analytical continuation of P_{nm} in the interval $0 \le \phi \le 2\pi$, and can be called for this reason the extended Legendre function of the first kind, degree n and other m in $0 \le \phi \le 2\pi$. Multiplying L_{nm} by the same normalizing factor as P_{nm} one obtains the fully normalized extended Legendre function

$$\bar{L}_{nm}(\phi) = \begin{cases} \sqrt{2n+1} \; L_{nm}(\phi) & \text{if } m = 0 \\ \sqrt{2(2n+1)\frac{(n-m)!}{(n+m)!}} \; L_{nm}(\phi) & \text{otherwise} \end{cases} \tag{2.4}$$

Consider now the spherical harmonic of degree n and order m

$$\bar{Y}^{\alpha}_{nm}(\phi,\lambda) = \bar{P}_{nm}(\sin\phi) \left\{ {\cos \atop \sin} \right\} m\lambda \tag{2.5}$$

The maximum circle containing both poles and the point $(\phi=0,\lambda)$ on the equator consists of two meridians, of longitudes λ and $\lambda+\pi$, respectively. Along this circle, points can be ordered according to a parameter ϕ' in the interval $0 \le \phi' \le 2\pi$, which increases continuously from $(\phi'=0,\lambda)$ towards the N pole and beyond. The geocentric latitude ϕ , on the other hand, first increases towards the pole like ϕ' , but on crossing the pole begins to decrease again as it approaches the equator at $(\phi=0,\lambda+\pi)$, and it is negative in the southern hemisphere. The harmonic $\bar{Y}^{\alpha}_{nm}$ is a continuous function of ϕ' along the meridional circle, the same as its derivatives, and it follows from (2.5) and (2.2,c-d) that

$$\begin{aligned} \bar{Y}^{\alpha}_{nm}(\phi,\lambda+\pi) &= \bar{P}_{nm}(\sin\phi) \left\{ {\cos \atop \sin} \right\} m(\lambda+\pi) \\ &= (-1)^m \bar{P}_{nm}(\sin\phi) \left\{ {\cos \atop \sin} \right\} m\lambda \\ &= \bar{L}_{nm}(\pi-\phi) \left\{ {\cos \atop \sin} \right\} m\lambda \\ &= \bar{L}_{nm}(\phi') \left\{ {\cos \atop \sin} \right\} m\lambda = \bar{Y}_{nm}(\phi',\lambda) \end{aligned}$$

This relationship shows that by using $\bar{L}_{nm}$ and ϕ' instead of $\bar{P}_{nm}$ and ϕ one can formulate $\bar{Y}^{\alpha}_{nm}$ avoiding the complications that would arise otherwise because of the discontinuity in the value of the longitude at the poles. This is not a real discontinuity in the function $\bar{Y}^{\alpha}_{nm}$, but merely a consequence of the way in which longitudes are defined.

Replacing $L_{nm}^{(\phi')}$ with its Fourier expansion (2.3,a-b) the harmonic becomes

$$\bar{Y}^{\alpha}_{nm}(\phi',\lambda) = \begin{cases} \sum_{p=0}^{n} \bar{h}_p^{nm} \cos p\phi' \left\{ {\cos \atop \sin} \right\} m\lambda & \text{if } n-m \text{ is even} \\ \sum_{p=0}^{n} \bar{h}_p^{nm} \sin p\phi' \left\{ {\cos \atop \sin} \right\} m\lambda & \text{if } n-m \text{ is odd} \end{cases} \tag{2.7}$$

where

$$\bar{h}_p^{nm} = \begin{cases} \sqrt{2n+1} & h_p^{nm} \quad \text{if} \quad m = 0 \\ \sqrt{2(2n+1)\frac{(n-m)!}{(n+m)!}} & h_p^{nm} \quad \text{otherwise} \end{cases}$$

and there are only even Fourier terms if n is even, and odd terms if n is odd.

2.3 Time Series Expression of the Inertial Line of Sight Acceleration

Reasoning as in paragraph (1.2), but considering the line of sight oriented along a meridian (polar orbit) instead of along the equator, one gets the following expression for the line of sight relative inertial acceleration

$$\begin{aligned} a_{12} &= a_1 - a_2 \\ &= \underline{e}_{12}^T \, (\underline{r}_0^{(S_1)} \, a_r \, (S_1) + \underline{\phi}_0^{(S_1)} \, a_\phi (S_1) - \underline{r}_0^{(S_2)} \, a_r (S_2) + \underline{\phi}_0^{(S_2)} \, a_\phi (S_2) \\ &= -(a_r(S_1) + a_r(S_2)) \sin\frac{\psi}{2} + (a_\phi(S_1) - a_\phi(S_2)) \cos\frac{\psi}{2} \end{aligned} \tag{2.8}$$

where terms containing the "across track" acceleration a_λ have been dropped because $\underline{\lambda}_0$ is always normal to the orbital plane and to the line of sight. Writing the radial and "along track" components a_r and a_ϕ according to (1.8,a-b), replacing $\bar{P}_{nm}$ with $\bar{L}_{nm}$, and choosing the mid-point coordinates $\phi' = \frac{\phi_1' + \phi_2'}{2}$, $\lambda = \frac{\lambda_1 + \lambda_2}{2}$ as independent variables,

$$\begin{aligned} a_{12}(R,\phi',\lambda) = \frac{GM}{R^2} \sum_{n=0}^{N} \sum_{m=0}^{n} \Big\{ &(n+1)[\bar{L}_{nm}(\phi'-\tfrac{\psi}{2}) + \bar{L}_{nm}(\phi'+\tfrac{\psi}{2})] \sin\tfrac{\psi}{2} \\ &+ \frac{d}{d\phi'} [\bar{L}_{nm}(\phi'-\tfrac{\psi}{2}) - \bar{L}_{nm}(\phi'+\tfrac{\psi}{2})] \cos\tfrac{\psi}{2} \Big\} (\tfrac{a}{R})^n [C_{nm} \cos m\lambda \\ &+ S_{nm} \sin m\lambda] \end{aligned} \tag{2.9}$$

where N is the smallest degree such that $\bar{C}_{nm}^{\alpha} = 0$, $n > N$, according to the band-limited assumption.

Since $$\bar{L}_{nm} \, (\phi' + \beta) = \sum_{p=0}^{n} \bar{h}_p^{nm} \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} p(\phi' + \beta)$$

($\cos p(\phi'+\beta)$ if $n-m$ is even, $\sin p(\phi+\beta)$ if $n-m$ is odd), it follows from elementary trigonometric relationships that

$$\bar{L}_{nm} \, (\phi'+\beta) = \sum_{p=0}^{n} \bar{h}_p^{nm} \left\{ \begin{matrix} \cos p\phi' \cos p\beta - \sin p\phi' \sin p\beta \\ \sin p\phi' \cos p\beta + \cos p\phi' \sin p\beta \end{matrix} \right\}$$

so

$$\bar{L}_{nm}(\phi'-\tfrac{\psi}{2}) + \bar{L}_{nm}(\phi'+\tfrac{\psi}{2}) = 2 \sum_{p=0}^{n} \bar{h}_p^{nm} \cos p\tfrac{\psi}{2} \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} p\phi'$$

while

$$\mathfrak{L}_{nm}(\phi'-\frac{\psi}{2})-\mathfrak{L}_{nm}(\phi'+\frac{\psi}{2})] = 2\sum_{p=0}^{n}\bar{h}_p^{nm}\sin p\frac{\psi}{2}\left\{\begin{matrix}\sin\\-\cos\end{matrix}\right\} p\phi'$$

and therefore

$$\frac{d}{d\phi'}[\mathfrak{L}_{nm}(\phi'-\frac{\psi}{2})-\mathfrak{L}_{nm}(\phi'+\frac{\psi}{2})] = 2\sum_{p=0}^{n}\bar{h}_p^{nm}\, p\sin p\frac{\psi}{2}\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\} p\phi' \qquad (2.10,b)$$

From expressions (2.9) and (2.10,a-b) follows that

$$a_{12}(R,\phi',\lambda) = \frac{GM}{R^2}\sum_{n=0}^{N}\sum_{m=0}^{n}\Big[\sum_{p=0}^{n}2\bar{h}_p^{nm}(n+1)\cos p\frac{\psi}{2}\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\} p\phi'$$

$$+\sum_{p=0}^{n}2\bar{h}_p^{nm}p\sin p\frac{\psi}{2}\cos\frac{\psi}{2}\left\{\begin{matrix}\cos\\\sin\end{matrix}\right\} p\phi'\Big](\frac{a}{R})^n[\bar{C}_{nm}\cos m\lambda+\bar{S}_{nm}\sin m\lambda] \qquad (2.11)$$

is the relationship between the value of the line of sight relative inertial acceleration for the two satellies and the spherical harmonic coefficients of the gravitational potential, the $\bar{C}^{\alpha}_{nm}$. Carrying out various multiplications indicated in (2.11), and making use of the trigonometric equations

$$2\cos p\phi'\cos m\lambda = \cos(p\phi'+m\lambda)+\cos(p\phi'-m\lambda)$$
$$2\cos p\phi'\sin m\lambda = \sin(p\phi'+m\lambda)-\sin(p\phi'-m\lambda)$$
$$2\sin p\phi'\cos m\lambda = \sin(p\phi'+m\lambda)+\sin(p\phi'-m\lambda)$$
$$2\sin p\phi'\sin m\lambda = -\cos(p\phi'+m\lambda)+\cos(p\phi'-m\lambda)$$

leads to the expression

$$a_{12}(R,\phi',\lambda) = \frac{GM}{R}\sum_{n=0}^{N}\sum_{m=0}^{n}[C_{nm}(\frac{a}{R})^n\sum_{p=0}^{n}a_p^{nm}\left\{\begin{matrix}\cos(p\phi'+m\lambda)+\cos(p\phi'-m\lambda)\\\sin(p\phi'+m\lambda)+\sin(p\phi'-m\lambda)\end{matrix}\right\}$$

$$+\bar{S}_{nm}(\frac{a}{R})^n\sum_{p=0}^{n}a_p^{nm}\left\{\begin{matrix}\sin(p\phi'+m\lambda)-\sin(p\phi'-m\lambda)\\-\cos(p\phi'+m\lambda)+\cos(p\phi'-m\lambda)\end{matrix}\right\} \qquad (2.12)$$

where

$$a_p^{nm} = \bar{h}_p^{nm}[(n+1)\cos p\frac{\psi}{2}\sin\frac{\psi}{2}+p\sin p\frac{\psi}{2}\cos\frac{\psi}{2}] \qquad (2.13)$$

so

$a_p^{nm} = 0$ if n and p have different parities (par. 2.2, remark (3)).

Introducing time as the independent variable through the formulas

$$\phi' = [\omega t]_{mod\ 2\pi} \qquad (2.14,a)$$

and

$$\lambda = -[\Omega t]_{mod\ 2\pi} \qquad (2.14,b)$$

and introducing the scaled harmonic coefficients

$$\tilde{C}_{nm} = \bar{C}_{nm}(\frac{a}{R})^n\frac{GM}{R^2} \qquad (2.15,a)$$

$$\tilde{S}_{nm} = \bar{S}_{nm}(\frac{a}{R})^n\frac{GM}{R^2} \qquad (2.15,b)$$

expression (2.13) becomes

$$a_{12}(t) = \sum_{n=0}^{N} \sum_{m=0}^{n} \tilde{C}_{nm} \sum_{p=0}^{n} a_p^{nm} \left\{ \begin{matrix} \cos(p\omega + m\Omega)t + \cos(p\omega - m\Omega)t \\ \sin(p\omega + m\Omega)t + \sin(p\omega - m\Omega)t \end{matrix} \right\}$$

$$+ \tilde{S}_{nm} \sum_{p=0}^{n} a_p^{nm} \left\{ \begin{matrix} -\sin(p\omega + m\Omega)t + \sin(p\omega - m\Omega)t \\ \cos(p\omega + m\Omega)t - \cos(p\omega - m\Omega)t \end{matrix} \right\} \tag{2.16}$$

This last formula shows a_{12} as a time series representable by a finite Fourier series where the angular frequencies present have all possible values $p\omega \pm m\Omega$, $0 \leq p, m \leq N$, and the Fourier coefficients are sums of terms of the type $\tilde{C}_{nm} a_p^{nm}$ and $\tilde{S}_{nm} a_p^{nm}$, respectively. The time origin chosen should not influence the outcome of this analysis. The choice implied by (2.14,a-b) corresponds to $t = 0$ when the mid-point between the satellites is directly over the equatorial point (0,0).

2.4 The Correction Term

As explained in paragraph (1.2), the model for the time derivative of the relative line of sight velocity used in this study is not the relative inertial line of sight acceleration a_{12}, but this acceleration minus a time-invariant term a_0 (expression (1.13)).or modified acceleration $\hat{a}_{12}$. According to (2.16), this time-invariant part can only be due to terms where $p\omega \pm m\Omega = 0$. As indicated later in paragraph (2.6), one of the assumptions made in this study imply that $p\omega \pm m\Omega \neq 0$ if $0 < m \leq N$ so this leaves only the case $p=0$, $m=0$, corresponding to the even zonals, $n=0$ in particular. Therefore, the corrected acceleration is

$$\hat{a}_{12}(t) = a_{12}(t) - a_0$$

where

$$a_0 = \sum_{n=0}^{N} a_0^{n0} \tilde{C}_{n0} = \sum_{n=0}^{N} a_0^{n0} \bar{C}_{n0} \left(\frac{a}{R}\right)^n \frac{GM}{R^2} \tag{2.17}$$

is a term that depends only on R, as expected.

2.5 The Observation Equation

One of the assumptions made in paragraph (2.1) was that the orbits are periodic with a period T equal to the length of the whole mission. Consequently, the various angular frequencies $(p\omega \pm m\Omega)$ present in the right hand side of (2.16) are harmonics of fundamental frequency $\omega_0 = \frac{2\pi}{T}$. The modified line of sight relative inertial acceleration $\hat{a}_{12}$ is regarded here as the true time derivative of the relative line of sight velocity, in accordance with expression (1.13). Therefore

$$v_{12}(t) = \int_0^t \hat{a}_{12}(t)\, dt + v_{12}(0) \tag{2.18}$$

Because of the assumption of orbital periodicity, replacing the integrand in (2.18) with the difference between the right hand side of (2.16) and a_0 according to (2.17), and integrating this difference term by term, one gets the following expression, where no zero frequency term is present:

$$v_{12}(t) = \sum_{n=2}^{N} \sum_{m=0}^{n} \tilde{C}_{nm} \sum_{p=z}^{n} a_p^{nm} \begin{Bmatrix} \frac{\sin(p\omega + m\Omega)t}{p\omega + m\Omega} + \frac{\sin(p\omega - m\Omega)t}{p\omega - m\Omega} \\ -\frac{\cos(p\omega + m\Omega)t}{p\omega + m\Omega} - \frac{\cos(p\omega - m\Omega)t}{p\omega - m\Omega} \end{Bmatrix}$$

$$+ \tilde{S}_{nm} \sum_{p=z}^{n} a_p^{nm} \begin{Bmatrix} \frac{\cos(p\omega + m\Omega)t}{p\omega + m\Omega} - \frac{\cos(p\omega - m\Omega)t}{p\omega - m\Omega} \\ \frac{\sin(p\omega + m\Omega)t}{p\omega + m\Omega} - \frac{\sin(p\omega - m\Omega)t}{p\omega - m\Omega} \end{Bmatrix} \quad (2.19)$$

where

$$z = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases} \quad (2.20)$$

and where $\tilde{C}_{10}$, $\tilde{C}_{11}$, $\tilde{S}_{11}$ have been dropped as unknowns, because the origin of coordinates coincides with the geocenter.

The actual observations consist of time averages $\bar{v}_{12}$ of the measured relative line of sight velocity. This averages are taken over identical intervals of length Δa seconds, spaced Δt seconds apart. The <u>signal</u> in the observations is, then, the line of sight velocity averaged over Δa :

$$\bar{v}_{12}(t) = \frac{1}{\Delta a} \int_{t-\Delta a}^{t} v_{12}(\tau)\, d\tau$$

$$= \frac{1}{\Delta a} \sum_{n=2}^{N} \sum_{m=0}^{n} \tilde{C}_{nm} \sum_{p=z}^{n} a_p^{nm} \begin{Bmatrix} \frac{S((p\omega + m\Omega)t, \Delta a)}{(p\omega + m\Omega)^2} + \frac{S((p\omega - m\Omega)t, \Delta a)}{(p\omega - m\Omega)^2} \\ -\frac{C((p\omega + m\Omega)t, \Delta a)}{(p\omega + m\Omega)^2} - \frac{C((p\omega - m\Omega)t, \Delta a)}{(p\omega - m\Omega)^2} \end{Bmatrix}$$

$$+ \tilde{S}_{nm} \sum_{p=z}^{n} a_p^{nm} \begin{Bmatrix} \frac{C((p\omega + m\Omega)t, \Delta a)}{(p\omega + m\Omega)^2} - \frac{C((p\omega - m\Omega)t, \Delta a)}{(p\omega - m\Omega)^2} \\ \frac{S((p\omega + m\Omega)t, \Delta a)}{(p\omega + m\Omega)^2} - \frac{S((p\omega - m\Omega)t, \Delta a)}{(p\omega - m\Omega)^2} \end{Bmatrix} \quad (2.21)$$

where

$$C((p\omega \pm m\Omega)t, \Delta a) = \cos(p\omega \pm m\Omega)t \sin(p\omega \pm m\Omega)\,\Delta a$$
$$+ \sin(p\omega \pm m\Omega)t\,(1 - \cos(p\omega \pm m\Omega)\,\Delta a) \quad (2.22,a)$$

and

$$S((p\omega \pm m\Omega)t, \Delta a) = \sin(p\omega \pm m\Omega)t \sin(p\omega \pm m\Omega)\,\Delta a$$
$$- \cos(p\omega \pm m\Omega)t\,(1 - \cos(p\omega \pm m\Omega)\,\Delta a) \quad (2.22,b)$$

The <u>observed value</u> at t_i consists of $\bar{v}_{12}$ plus the <u>measurement noise average</u> over $t_i - \Delta a \le t \le t_i$, or n_i :

$$\bar{v}_{12}(t_i) + n_i = v_{12}(t_i)(\text{obs}) \quad (2.23)$$

Rearranging the order of summation with respect to n and m in (2.19) and replacing the result in (2.23) one arrives to the <u>observation equation</u> as it will be used in the error analysis

$$\frac{1}{\Delta a} \sum_{m=0}^{N} \sum_{n=\max(m,2)}^{N} \tilde{C}_{nm} \sum_{p=z}^{n} a_p^{nm} \begin{Bmatrix} \frac{S((p\omega + m\Omega)t_i, \Delta a)}{(p\omega + m\Omega)^2} + \frac{S((p\omega - m\Omega)t_i, \Delta a)}{(p\omega - m\Omega)^2} \\ \frac{-C((p\omega + m\Omega)t_i, \Delta a)}{(p\omega + m\Omega)^2} - \frac{C((p\omega - m\Omega)t_i, \Delta a)}{(p\omega - m\Omega)^2} \end{Bmatrix}$$

$$
+ \tilde{S}_{nm} \sum_{p=z}^{n} a_p^{nm} \left\{ \begin{array}{l} \dfrac{C((p\omega + m\Omega)t_i,\Delta a)}{(p\omega + m\Omega)^2} - \dfrac{C((\omega - m\Omega)t_i,\Delta a)}{(p\omega - m\Omega)^2} \\ \dfrac{S((p\omega + m\Omega)t_i,\Delta a)}{(p\omega + m\Omega)^2} - \dfrac{S((p\omega - m\Omega)t_i,\Delta a)}{(p\omega - m\Omega)^2} \end{array} \right\} = \tilde{v}^{(t_i)}_{(obs)} + r_i \tag{2.24}
$$

where max(m,2) indicates the largest of m and 2, and where r_i is the residual or difference between the measurement and the value calculated by replacing the $\tilde{C}^{\alpha}_{nm}$ with numbers in the left hand side. When these numbers are the true values of the unknown $\tilde{C}^{\alpha}_{nm}$ and the model is perfect (as it is supposed to be in the case here), then

$$
r_i = - n_i
$$

(all this is in keeping with established practice in geodetic literature). Consider the following vector notation

$$
\underline{\tilde{v}}_{12(obs)} = [\tilde{v}^{(0)}_{12(obs)} \; v^{(\Delta t)}_{12(obs)} \cdots v^{(i\Delta t)}_{12(obs)} \cdots v^{(T-\Delta t = (N_p-1)\Delta t)}_{12(obs)}]^T \tag{2.25,a}
$$

$$
\underline{r} = [r_0 \;\; r_1 \;\; \ldots \; r_i \; \ldots \; r_{Np-1}]^T \tag{2.25,b}
$$

$$
\underline{c}_m = [\tilde{C}_{mm} \; \tilde{C}_{(m+1)m} \cdots \tilde{C}_{Nm}]^T \tag{2.25,c}
$$

$$
\underline{s}_m = [\tilde{S}_{mm} \; \tilde{S}_{(m+1)m} \cdots \tilde{S}_{Nm}]^T \tag{2.25,d}
$$

$$
\underline{c} = [\underline{c}_0^T \; \underline{c}_1^T \; \underline{s}_1^T \; \underline{c}_2^T \; \underline{s}_2^T \cdots \underline{c}_m^T \; \underline{s}_m^T \cdots \underline{c}_N^T \; \underline{s}_N^T]^T \tag{2.25,e}
$$

where $\underline{\tilde{v}}_{12(obs)}$ and $\underline{r}$ are both N_p - vectors and $\underline{c}$ is a N_c - vector, N_p being the total number of observations (some 3.9×10^6 measurements over six months if $\Delta t = 4$) and $N_c = (N+1)^2 - 3$, the number of coefficients in the band[1] $2 \leq n \leq N$. The set of all observation equations, or system of observation equations, in matrix notation, is

$$
A\underline{c} = \underline{\tilde{v}}_{12(obs)} + \underline{r} \tag{2.26}
$$

where A is the $N_p \times N_c$ matrix of the observation equations. The unknowns, according to (2.24), are the scaled potential coefficients $\tilde{C}^{\alpha}_{nm}$, instead of the actual coefficients $\bar{C}^{\alpha}_{nm}$, which are the ones desired. Once the $\tilde{C}_{nm}$ are known, however, the $\bar{C}^{\alpha}_{nm}$ can be obtained by a trivial operation based on (2.15,a-b). The same is true of the standard deviations, which are the quantities relevant to this error analysis.

The observation equation (2.24) presents the relationship between data and unknowns in time. The corresponding relationship in space can be obtained directly from (2.24) by replacing ϕ' and λ for t in accordance to (2.14,a-b). The temporal representation, however, makes the overall treatment of what is to follow simpler, and is adopted here for this reason.

---

[1] Coefficients with $n > NM$ are actual field coefficients; with $n \leq NM$ (NM being the highest degree in the reference field model mentioned in paragraph (1.2)) the $\bar{C}^{\alpha}_{nm}$ correspond to residual coefficients: the differences between the actual coefficients and the coefficients of the model, if the data consists of residual velocities, as assumed in section 3.

2.6 The Condition that N_r and N_D be Relative Primes

If Earth rotation and satellites revolution are congruent over the length of the mission, the total number of revolutions N_r and the total number of days N_D are both positive integers. According to condition (4) in paragraph 2.1 they are also relative primes, i.e., without common factors other than the unity. As shown here this property rules out certain relationships between the frequencies $p\omega \pm m\Omega$ in (2.24), the absence of which simplifies the mathematical treatment of the error analysis, as it will be explained in paragraph 2.7. Taking m in the interval $-N \leq m \leq N$, $p\omega \pm m\Omega$ can be written more simply as $p\omega + m\Omega$. The relationships of interest have the form

$$p\omega + m\Omega = j\omega + q\Omega \tag{2.27}$$

where either $p \neq j$, $m \neq q$, or both. The question as to whether such relationships are possible can be answered in two parts:

(a) Case where $m \neq q$:

If (2.27) is possible, then

$$\frac{m-q}{j-p} = \frac{\omega}{\Omega} = \frac{\omega\,\omega_0^{-1}}{\Omega\,\omega_0^{-1}} = \frac{N_r}{N_D} \tag{2.28}$$

where ω_0 is the fundamental angular frequency of the mission: $\omega_0 = \frac{2\pi}{T}$. As both N_r and N_D are positive it follows that $\frac{|m-q|}{|j-p|} = \frac{N_r}{N_D}$ so

$$|m-q|\, N_D = |j-p|\, N_r$$

As $|m-q|\, N_D$ and $|j-p|\, N_r$ are both positive integers and N_r and N_D have no common factors, it must be $|m-q| = N_r K$ for some $K \geq 1$, so

$$|m-q| \geq N_r \tag{2.29,a}$$

As $-N \leq m \leq N$ and $-N \leq q \leq N$, it follows that, under the band-limited assumption,

$$|m-q| \leq 2N \tag{2.29,b}$$

As a consequence of (2.29,a-b)

$$N_r \leq 2N \tag{2.29,c}$$

if (2.28) is true. In the present case, the number of revolutions over six months with the satellites at a height of 160 km exceeds 2900, while the maximum degree n of the harmonics in the band considered here is, according to paragraph (1.2), $N = 331$. This means that $2N < N_r$, contradicting (2.9,c) and thus, (2.28). Therefore, for the given satellite height and field bandwith, (2.27) is impossible if $m \neq q$.

(b) $m = q$:

In this case (2.28) is never true, because the first member is always zero except in the trivial case $p = j$, where it is indeterminate. Therefore

(2.27) is also impossible when $m = q$ and , thus, whenever either $p \neq j$ or $m \neq q$ apply.

An immediate consequence of the impossibility of (2.27) under the assumptions of paragraph 2.1 is that

$$p\omega \pm m\Omega = 0 \tag{2.30}$$

is also impossible, except in the trivial case $p = 0$, $m = 0$, which has been excluded by the way in which the modified line of sight acceleration is defined (see expressions (1.13) and (2.17)). Since the rate of precession of the node of a polar orbit is nil, condition (2.30) corresponds to what is known in satellite geodesy as a resonant orbit. So the assumption that N_r and N_D are relative primes excludes resonances. This means that the orbit is not allowed to repeat itself after a number of revolutions that is a submultiple of N_r , so no sub-cycles occur within the grand cycle whose period is the length of the whole mission, i.e., T seconds or N_D days. Because of this, the sampling of the gravity field over the face of the Earth is the most even that is possible, as with resonances many arcs would be superimposed, so the "footprint" of the mid-point between the satellites would cover the ground rather coarsely. Without resonances, the arcs are all different and, thus, better spread out.

In reality, resonance is most unlikely to occur exactly: the actual situation is much too complicated. How realistic is, even so, the assumption that N_r and N_D are relative primes? While perfect congruence (both numbers integers) is also quite unlikely, the situation need not be too different from that implied by the assumption. Choosing $N_D = 179$ (a prime number) and $N_r = 2933$ (another prime), both are then relative primes. For the satellites to orbit, at 160 km, 2933 times around the Earth in precisely 179 days, a change in ω of only -4 parts per thousand from its actual value is needed or, equivalently, an increase in Ω of 4 p.p. thousand.

2.7 The Scalar Product of Two Columns of the A Matrix

If the N_p - vectors $\underline{a}^{\alpha}_{nm}$ and $\underline{a}^{\beta}_{kq}$ are the columns in the matrix of observation equations A of (2.26) corresponding to the scaled coefficients $\tilde{C}^{\alpha}_{nm}$ and $\tilde{C}^{\beta}_{kq}$, respectively, then their scalar product

$$p^{\alpha\beta}_{nmkq} = (\underline{a}^{\alpha}_{nm})^T \underline{a}^{\beta}_{kq} = \sum_{i=0}^{N_p-1} a^{\alpha}_{nm}(t_i)\, a^{\beta}_{kq}(t_i) \tag{2.31}$$

should depend, according to (2.24), on what the mission parameters Δa , Δt , and ψ are, and also on the values of sums of products of cosines and sines of the various arguments $(p\omega \pm m\Omega)t_i$, where the discrete values t_i cover the whole length of the mission. The products of the columns of A have to be obtained as part of the formation of the normal matrix of the adjustment, the inverse of which provides the a posteriori accuracies that are the main objective of the error analysis. Sums of products of sines and cosines sampled at regular intervals are strongly influenced by the relationship between the sampling frequency (inverse of the sampling interval Δt) and the highest frequency in the arguments of the trigonometric functions involved. In particular, it is important to know whether the highest frequency is below the Nyquist frequency (one half the sampling frequency) or not, in order to choose the most convenient treatment for those sums of trigonometric products. In the case at hand $\Delta t = 4$ s, so the Nyquist frequency is

$$N_y = \frac{1}{2\Delta t} \text{ cycles/s}$$
$$= 0.125 \text{ c/s}$$

while the highest frequency, corresponding to the term $(N\omega + N\Omega)t$ is, for $N = 331$.

$$f_{max} = \frac{331}{2\pi}(\omega + \Omega) = 0.066 \text{ c/s}$$

Therefore,

$$f_{max} < N_y \tag{2.32}$$

or the highest frequency is less than the Nyquist frequency. Under this condition, the following formulas apply:

$$\sum_{i=0}^{N_{p-1}} \cos(p\omega \pm m\Omega)t_i \cos(j\omega \pm q\Omega)t_i = \sum_{i=0}^{N_{p-1}} \sin(p\omega \pm m\Omega)t_i \sin(j\omega \pm q\Omega)t_i$$
$$= \begin{cases} \frac{N_p}{2} & \text{if } (p\omega \pm \Omega) = (j\omega \pm q\Omega) \\ 0 & \text{otherwise} \end{cases} \tag{2.33,a}$$

$p = j = m = q = 0$ being excluded, and N_p being even;

$$\sum_{i=0}^{N_{p-1}} \cos(p\omega \pm m\Omega)t_i \sin(j\omega \pm q\Omega)t_i = 0 \quad \text{always} \tag{2.33,b}$$

From these formulas, and from expressions (2.22,a-b), one gets

$$\sum_{i=0}^{N_{p-1}} C((p\omega \pm m\Omega)t_i,\Delta a)\, C((j\omega \pm q\Omega)t_i,\Delta a)$$
$$= \sum_{i=0}^{N_{p-1}} S((p\omega \pm m\Omega)t_i,\Delta a) S((j\omega \pm q\Omega)t_i,\Delta a) = \begin{cases} N_p(1-\cos(p\omega \pm m\Omega)\Delta a) & \text{if } p\omega \pm m\Omega = j\omega \pm q\Omega \\ 0 & \text{otherwise} \end{cases} \tag{2.34,a}$$

$p = j = m = q = 0$ being excluded, and N_p being even;

$$\sum_{i=0}^{N_{p-1}} C((p\omega \pm m\Omega)t_i),\Delta a)\, S((j\omega \pm q\Omega)t_i),\Delta a) = 0 \quad \text{always} \tag{2.34,b}$$

As a result of expressions (2.34,a-b) above, of the fact that $a_p^{nm} = 0$ if p has different parity from n, and of the fact that $p\omega \pm m\Omega \neq j\omega \pm q\Omega$ if at least $p \neq j$, as explained in the previous paragraph, replacing $a_{nm}^{\alpha}(t_i)$ and $a_{kq}^{\beta}(t_i)$ in (2.31) by their respective expressions according to (2.24), i.e.,

$$a_{nm}^{\alpha(t_i)} = \frac{1}{\Delta a} \sum_{p=z}^{n} a_p^{nm} \ \{\ \}^{\alpha}$$

where $\{\ \}^{\alpha}$ is the first or the second bracketin (2.24), depending on α, and similarly for $a_{kq}^{\beta}(t_i)$, one gets

$$p_{nmkq} = \begin{cases} 0 \ \text{ if } \ \alpha \neq \beta\ , \ m \neq q\ , \ n \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix} \text{ and } k \begin{Bmatrix}\text{odd}\\ \text{even}\end{Bmatrix} \\ \dfrac{N_p}{\Delta a} \displaystyle\sum_{p=z}^{\min(n,k)} a_p^{nm} a_p^{km} \left[\frac{(1-\cos(p\omega+m\Omega)\,\Delta a)}{(p\omega + m\Omega)^4} + \frac{(1-\cos(p\omega-m\Omega)\,\Delta a)}{(p\omega - m\Omega)^4}\right] \\ \min(n,k) \text{ being the smallest of } n \text{ and } k\ . \end{cases} \tag{2.35}$$

as the expression for the scalar product, which shows that under the assumptions the product is zero in the majority of cases. This results in many elements in the normal matrix being zero as well, which is a major advantage when setting up this matrix, as only the relatively few non-zero elements have to be computed. Moreover, as shown in paragraph (2.9), a suitable ordering of the unknowns groups this non-zero elements in a block-diagonal structure, so inverting the very large normal matrix reduces itself to inverting the much smaller diagonal blocks, a crucial fact as far as the feasibility of obtaining the a posteriori covariance matrix is concerned. Finally, expression (2.35) shows that the non-zero elements can be computed from the values of the a_p^{nm}, which are the Fourier coefficients of the $\bar{L}_{nm}$ multiplied by scale factors (expression (2.13)). Because the a_p^{nm} are zero when p and n have different parities, because $\bar{L}_{nm}$ are expansions of sines only or of cosines only, and because n cannot be larger than N = 331, the total number of terms in the summation of (2.35) is much less than that of the terms in the summation of (2.31), resulting in considerable economies when computing the non-zero elements of the normal matrix.

The Fourier coefficients can be obtained by computing $\bar{L}_{nm}(\phi')$ at regular intervals $\Delta\phi' = \frac{2\pi}{2n}$ (the highest frequency in $\bar{L}_{nm}(\phi')$ is that of $\cos n\phi'$ or $\sin n\phi'$) and then carrying out a numerical Fourier analysis, or discrete Fourier transform, of these values. This is done in the program of appendix B by means of a mixed-radix Fast Fourier Transform algorithm chosen on grounds of efficiency. Finding the required value of $\bar{L}_{nm}$ in the interval $0 \leq \phi' \leq 2\pi$ is simplified by the use of expressions (2.2,a-c) and of the relationship $\bar{L}_{nm}(\phi') = \bar{P}_{nm}(\phi')$ if $0 \leq \phi' \leq \frac{\pi}{2}$, which reduces most of the effort to that of computing the values of $\bar{P}_{nm}$ at regular intervals in $0 \leq \phi' \leq \frac{\pi}{2}$.

2.8 Least Squares Adjustment

Consider the quadratic form

$$Q = \underline{r}^T P \underline{r} \tag{2.36}$$

where P is a symmetric $N_p \times N_p$ matrix and where $\underline{r}$ is the vector of residuals

$$\underline{r} = A\underline{c} - \bar{\underline{v}}_{12}(\text{obs}) \tag{2.37}$$

according to (2.24), (2.25,b) and (2.26). The form (2.36) is a function of $\underline{c}$, the vector of potential coefficients, through (2.37). The vector $\hat{\underline{c}}$ that minimizes the quadratic form must satisfy the condition

$$\left(\frac{\partial Q}{\partial \underline{c}}\right)^T = (A^T PA)\,\hat{\underline{c}} - A^T P \underline{\bar{v}}_{12\,(obs)} = \underline{0} \tag{2.38}$$

(where $\underline{0}$ is a null Nc - vector) so

$$\hat{\underline{c}} = (A^T PA)^{-1}\, A^T P \underline{\bar{v}}_{12\,(obs)} \tag{2.39}$$

and the form has a minimum at $\hat{\underline{c}}$ provided P *is a positive matrix.* Expression (2.39) above can be written

$$\hat{\underline{c}} = F_p\, \underline{\bar{v}}_{12\,(obs)} \tag{2.40}$$

where F_p is the optimal estimator matrix corresponding to the weights matrix P. In general $\hat{\underline{c}} \neq \underline{c}$, because, even if the signal in the data $(\underline{\bar{v}}_{12})$ and the unknown parameters $(\underline{c})$ are related exactly to each other by the matrix equations $\underline{\bar{v}}_{12} = A\underline{c}$, the data $\underline{\bar{v}}_{12\,(obs)}$ contains noise $\underline{n}$ in addition to the signal. The noise $\underline{n}$ propagates into the estimate

$$\hat{\underline{c}} = F_p\, \underline{\bar{v}}_{12\,(obs)} = F_p\,(\underline{\bar{v}}_{12} + \underline{n}) = F_p\, \underline{\bar{v}}_{12} + F_p\, \underline{n} \tag{2.41}$$

resulting in a difference between $\hat{\underline{c}}$ and $\underline{c}$, or error,

$$\hat{\underline{c}} - F_p\, \underline{\bar{v}}_{12} = F_p\, \underline{n} = -\underline{e}_n \tag{2.42}$$

The variance-covariance matrix of these errors in $\hat{\underline{c}}$, or a posteriori random errors, is

$$E_n = E\,\{\underline{e}\ \ \underline{e}^T\} \tag{2.43}$$

where $E\{\ \}$ is the usual mathematical expectation operator of statistics. The diagonal elements of the error matrix are the a posteriori variances of the estimated coefficients, or formal accuracies of the adjustment. If

$$P^{-1} = D = E\,\{\underline{n}\ \underline{n}^T\} \tag{2.44}$$

where D is the $N_p \times N_p$ symmetrical and positive matrix corresponding to the data errors and known as the a priori variance-covariance matrix, then

$$\begin{aligned} E_n &= E\,\{F_D\, \underline{n}\, (F_D\, \underline{n})^T\} = E\,\{F_D\, \underline{n}\, \underline{n}^T\, F_D^T\} \\ &= F_D\, E\,\{\underline{n}\, \underline{n}^T\}\, F_D^T = F_D\, D\, F_D^T \\ &= (A^T D^{-1} A)^{-1} A^T D^{-1} D\, D^{-1} A\, (A^T D^{-1} A)^{-1} \\ &= (A^T D^{-1} A)^{-1} = G^{-1} \end{aligned} \tag{2.45}$$

where $F_D = (A^T D^{-1} A)^{-1} A^T D^{-1}$

Matrix

$$G = A^T D^{-1} A \tag{2.46}$$

is known as the normal matrix, because (2.38) can be written

$$G \, \hat{\underline{c}} = \underline{b} \tag{2.47,a}$$

with

$$\underline{b} = A^T D^{-1} \, \bar{\underline{v}}_{12\,(obs)} \tag{2.47,b}$$

and (2.47,a) is the system of the normal equations. Therefore, when $P = D^{-1}$, the a posteriori variance-covariance matrix is identical to the inverse of the normal matrix. In particular, when all data errors are uncorrelated $(E\{n_i \, n_j\} = 0)$ and all have the same standard deviation σ, then

$$D = \sigma^2 I$$

where I is the unit $N_p \times N_p$ matrix, so

$$G = A^T \sigma^{-2} I A = \sigma^{-2} A^T A \tag{2.48}$$

and the elements of the normal matrix have the form

$$g^{\alpha\beta}_{nmkq} = \sigma^{-2} (\underline{a}^{\alpha}_{nm})^T \, \underline{a}^{\beta}_{kq} = \sigma^{-2} \, p^{\alpha\beta}_{nmkq} \tag{2.49}$$

where, in the case under study, $p^{\alpha\beta}_{nmkq}$ can be calculated from (2.35). Among the important properties of the least squares estimate $\hat{\underline{c}}$ is that of being a minimum variance estimate when $P = D^{-1}$, which means that the diagonal elements of G^{-1} (and, then, their sum or trace $t_r\{E_n\}$) are minimized. If the probability distribution of the error is gaussian, then the estimate $\hat{\underline{c}}$ is best in the sense that the a posteriori variances are the smallest for all estimates, linear or nonlinear. Moreover, the most likely value of $\hat{\underline{c}}$ coincides with $\underline{c}$, so $\hat{\underline{c}}$ is also the maximum likelihood estimate. When the model $\bar{\underline{v}}_{12} = A\underline{c}$ is perfect, as assumed here, and $E\{\underline{n}\} = \underline{0}$, then

$$\begin{aligned} E\{\hat{\underline{c}}\} &= (A^T D^{-1} A)^{-1} A^T D^{-1} (E\{\bar{\underline{v}}_{12}\} + E\{\underline{n}\}) = (A^T D^{-1} A)^{-1} A^T D^{-1} A \, \underline{c} \\ &= \underline{c} \end{aligned} \tag{2.50}$$

For this reason, the estimate $\hat{\underline{c}}$ is called unbiased. Expression (2.50) further indicates that, in the absence of noise, the estimates are identical to the true values of the unknowns.

The rather impressive list of properties of the least squares estimator (2.39) with $P = D^{-1}$, together with the relative simplicity of the theory and rather straightforward nature of the calculations involved, have made this type of estimator the most widely used in geodesy as well as in many other branches of technology and of natural science. In reality, models are never exact, statistics never truly gaussian (this would include cases where the error is extremely large, but such occurrences are usually edited-out during pre-processing), or even truly known, so the practical implications of the theoretical properties of the estimator are somewhat obscure. What is clear is that (2.39) with $P = D^{-1}$ is a "sensible" way of processing data, as it gives larger weights to the better measurements (those with smaller variances), and less weight to the worst (larger variances), at least when D is diagonal. If, as in the present case, all measurements are assumed to be equally good (same σ), then they are all weighted equally. From the point of view of the error analysis which is the purpose of this study, the determination of the formal accuracies requires creating and inverting the normal matrix G .

2.9 The Structure of the Normal Matrix

One of the assumptions made in paragraph (2.1) was that the errors in the data, the elements n_i of $\underline{n}$, were random, uncorrelated, and had all the same standard deviation σ . Consequently, the elements of the normal matrix G can be calculated from (2.49) and (2.35) which, combined, give

$$g^{\alpha\beta}_{nmkq} = \begin{cases} 0 \quad \text{if} \quad m \neq q \ , \ \alpha \neq \beta, \ n \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix} \text{ and } k \begin{Bmatrix}\text{odd}\\ \text{even}\end{Bmatrix} \\ \sigma^{-2} \frac{N_p}{\Delta a^2} \sum\limits_{p=z}^{\mathrm{Min}(n,k)} a_p^{nm} \, a_p^{km} \left[\frac{(1-\cos(p\omega + m\Omega)\Delta a)}{(p\omega + m\Omega)^4} + \frac{(1-\cos(p\omega - m\Omega)\Delta a)}{(p\omega - m\Omega)^4} \right] \end{cases} \tag{2.51}$$

As already pointed out (par. (2.7)), there are many elements in G that are zero and it is possible to arrange the unknown $\tilde{C}^{\alpha}_{nm}$ so that G exhibits a very convenient structure. Consider the ordering given in expressions (2.25,c-e) to the elements of $\underline{c}$, where

$$\underline{c} = [\underline{c}_0^T \ \underline{c}_1^T \ \underline{s}_1^T \ \underline{c}_2^T \ \underline{s}_2^T \ \ldots \ \underline{c}_m^T \ \underline{s}_m^T \ \ldots \ \underline{c}_N^T \ \underline{s}_N^T \ ]^T$$

$$\underline{c}_m = [\tilde{C}_{mm} \ \tilde{C}_{(m+1)m} \ \tilde{C}_{(m+2)m} \ \ldots \ \tilde{C}_{Nm}]^T$$

$$\underline{s}_m = [\tilde{S}_{mm} \ \tilde{S}_{(m+1)m} \ \ldots \ \tilde{S}_{Nm}]^T$$

and now separate each $\underline{c}_m$ and $\underline{s}_m$ in two halves $\underline{c}_m^{\gamma=0}$, $\underline{c}_m^{\gamma=1}$, and $\underline{s}_m^{\gamma=0}$, $\underline{s}_m^{\gamma=1}$, so $\underline{c}_m^{\gamma=0}$, $\underline{s}_m^{\gamma=0}$ contain only $\tilde{C}^{\alpha}_{nm}$ where n is even, $\underline{c}_m^{\gamma=1}$, $\underline{s}_m^{\gamma=1}$ only $\tilde{C}^{\alpha}_{nm}$ where n is odd. With the unknowns arranged in this way, the $\underline{c}_m$, s_m contain coefficients of the same order m , the C_{nm} "cosine" coefficients and the $\tilde{S}_{nm}$ "sine" coefficients, and each of these are split further according to parity. If all this is done, then G can be partitioned, as shown in figure 2.1, into blocks G_{mq} . Those along the main diagonal, or G_{mm} , correspond to unknowns of equal order m , and are further partitioned each into four blocks, according to α and β, and each of these four blocks into another four, according to the parities of n and k . All G_{mq} , except for those along the main diagonal, are zero matrices according to (2.51). In the diagonal G_{mm} only the sub-blocks where $\alpha = \beta$ and n and k have the same parity (dashed in the picture) can contain nonzero elements. This sub-blocks are diagonal sub-blocks, so the whole normal matrix is zero except for those diagonal sub-blocks: G is block diagonal.

More concretely, G contains 4x(N+1)-4 diagonal blocks that are not null matrices. According to (2.51), the elements of "cosine" blocks ($\alpha = \beta = 0$) are identical to those of "sine" blocks ($\alpha = \beta = 1$) in corresponding positions. This means that only the 2x(N+1)-1 "odd n" and "even n"

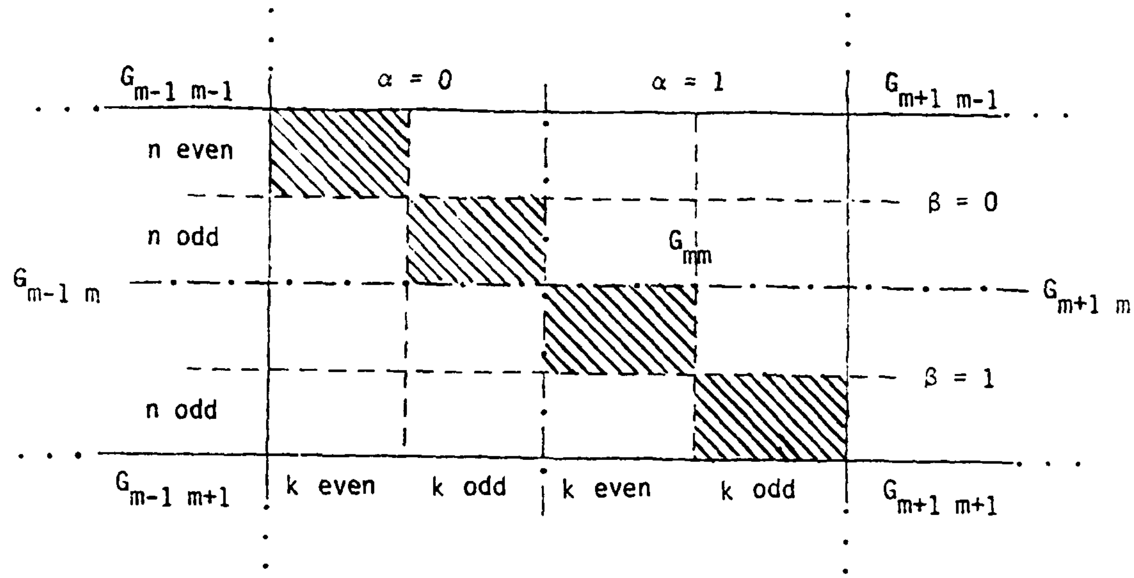

Figure 2.1: structure of the normal matrix

blocks have to be formed when setting up G, and only they have to be inverted when inverting G. The smallest blocks to be inverted have only one element and correspond to $m = N$; the two largest blocks have $\frac{1}{4}(N+1)^2$ elements each and correspond to $m = 0$ (zonals) with n even and odd, respectively. Altogether, the $2x(N+1)-1$ blocks contain in the order of $\frac{1}{24}(N+1)^3$ different elements, as the blocks are also symmetrical. In the case of a general symmetrical matrix the total number of different elements to be computed is, approximately, $\frac{1}{2}(\text{dimension})^2$, or $\frac{1}{2}(N+1)^4$ in the case at hand. The reduction in calculations when setting up G is, therefore, of the order of $\frac{1}{12}(N+1)$. Furthermore, expression (2.13) shows that the a_p^{nm} can be computed from the scaled Fourier coefficients $\bar{h}_p^{nm}$ of the $\bar{L}_{nm}$. As many $\bar{h}_p^{nm}$ are zero, as pointed out in paragraph (2.7), the number of operations using (2.51) rather than the general expression (2.49) is reduced by at least an order of magnitude. All these savings in computing make the setting up of such an enormous matrix (332^2 elements) quite feasible, although certainly not trivial when one adds to it the effort needed to obtain the values of $\bar{L}_{nm}$ from which the $\bar{h}_p^{nm}$ are then computed by Fourier analysis to give the a_p^{nm} according to (2.13). This whole operation required 50 c.p.u. minutes in the AMDHAL 470v/VI-II computer at O.S.U., using double precision and FORTRAN H extended in the highest optimizing mode. Another advantage of (2.51) is that it permits the setting up of the normal matrix without first having to form the observation equations matrix A, which is truly gigantic ($3.9 \times 10^6 \times 11 \times 10^4$ elements).

The inversion of a general matrix of the size of G, since the number of operations required to invert a $d \times d$ matrix is of the order of d^3, would be impossibly laborious, requiring something like $(N+1)^6$ operations, or thereabouts. As only the small non-zero diagonal blocks have to be inverted, and only half of those are different, the actual number of operations is of the order of $\frac{1}{96}(N+1)^4$, or some $96\,(N+1)^2 = 1.06 \times 10^7$ times less than for a general matrix. Using the same computer, compiler, etc. mentioned above, the inversion of G required some 15 min of central processor unit time. This is still a very large number of calculations. However, because the inversion involves the processing of independent blocks, each relatively small, the rounding errors due to finite register length (64 bits each double word) are confined so they can not affect the results in any appreciable way. Another property of a block-diagonal matrix is that, as the blocks are created and inverted independently, the whole procedure is ideally suited for parallel processing.

The programs used to form and invert G are documented and listed in appendix B .

2.10 The Existence of G^{-1}

Differential measurements, such as the relative line of sight velocity between the satellites, tend to be associated with observation and normal matrices that are rank-defficient, so it is reasonable to wonder whether the inverse of G , so important to an error analysis, does in fact exist at all. As shown here, this is not an entirely idle question, because there are cases where G is singular, though fortunately not with the mission parameters chosen in this study.

Two trivial examples of singular configurations are $\psi = 0$, when the whole matrix A vanishes, and G with it, and $\psi = \pi$, when all the columns of A corresponding to odd harmonic degrees are zero. If A is singular, G is singular too, so G^{-1} does not exist if

$$A\underline{z} = \underline{0}$$

for some

$$\underline{z} \neq \underline{0}$$

Calling Z^{α}_{nm} to the element of $\underline{z}$ corresponding to C^{α}_{nm} in $\underline{c}$, then, according to (2.13), (2.22,a-b), and (2.24), the elements y_i of the vector $y = A\underline{z}$ have the form

$$y_i = \sum_{\beta=0}^{1} \sum_{m=0}^{N} \sum_{p=z}^{N} f^{\beta}_{pm} \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} (p\omega \pm m\Omega)t_i \tag{2.52,a}$$

where

$$f^{\beta}_{pm} = \sum_{\alpha=0}^{1} \sum_{j=m}^{N} a^{jm}_{p} f(\Delta a) Z^{\alpha}_{jm} \tag{2.52,b}$$

with

$$f(\Delta a) = \frac{1}{(p\omega \pm m\Omega)} \left\{ \begin{matrix} \sin(p\omega \pm m\Omega)\Delta a \\ 1 - \cos(p\omega \pm m\Omega)\Delta a \end{matrix} \right\} ;$$

$$a^{jm}_{p} = h^{nm}_{p} [(j+1) \cos p\frac{\psi}{2} \sin\frac{\psi}{2} + p \sin p \frac{\psi}{2} \cos \frac{\psi}{2}] \text{ (expression (2.13))}$$

Expression (2.52.a) is a Fourier expansion with coefficients f^{β}_{pm} , and it can be zero only if all such coefficients vanish, as long as the sampling rate is higher than the highest frequency in the expansion, as it is the case here according to paragraph (2.7). In general, some elements of $\underline{z}$ can be zero, so assume that Z^{α}_{nm} is the element of highest degree and order that is not zero. In such case, $(n\omega + m\Omega)$ is the highest angular frequency in y_i. The only terms of this frequency are $f^1_{nm} \sin(n\omega + m\Omega)$ and $f^0_{nm} \cos(n\omega + m\Omega)$ which cannot cancel each other out. For both to be zero for all t_i must be $f^0_{nm} = f^1_{nm} = 0$. The other relevant factors involved in (2.52,b) are: Z^{α}_{nm}, which is assumed to be not zero; n^{nm}_{R} which is the Fourier coefficient of $[\Lambda^{\pm}_{nm})$ of highest frequency and cannot, for this reason, be zero; $\sin(n\omega + m\Omega)\,\Delta a$ and $1 - \cos(n\omega + m\Omega)\Delta a$ with $\Delta a > 0$, which cannot be simultaneously zero because for this to occur must be $(n\omega + m\Omega)\Delta t \geq (n\omega + m\Omega)\Delta a = k2\pi$, as $\Delta a \leq \Delta t$, but this is precluded by the sampling ratio Δt^{-1} being more than twice the highest frequency $(n\omega + m\Omega)/2\pi$ and, therefore, $(n\omega + m\Omega)\Delta a < (n\omega + m\Omega)\,\Delta t < 2\pi$. The only way in which the terms of frequency $(n\omega + m\Omega)$ can vanish for all t_i, for a given ψ, is that

$$(n+1)\cos n\frac{\psi}{2}\sin\frac{\psi}{2} + n\sin n\frac{\psi}{2}\cos\frac{\psi}{2} = 0 \tag{2.53}$$

for some n in $2 \leq n \leq N$, as $n = 0$ and $n = 1$ have been excluded from the model (paragraphs 2.4 and 2.5). This is a necessary condition for G^{-1} not to exist. Conversely,

$$(n+1)\cos n\frac{\psi}{2}\sin\frac{\psi}{2} + n\sin n\frac{\psi}{2}\cos\frac{\psi}{2} \neq 0 \tag{2.54}$$

for all n in $2 \leq n \leq N$ is a sufficient condition for G^{-1} to exist, or G not to be singular. Fortunately (2.54) is met by all n in $2 \leq n \leq 331$ with the chosen inter-satellite distance of 300 km (ψ approx. 0.046 rad). According to (2.53), the critical values of ψ at which (2.53) is met for some n in the band must be isolated points in $0 \leq \psi \leq 2$. If ψ coincided with some Φ, and this were truly a point of singularity (remember that 2.53) gives only a necessary condition), then one could choose $\psi = \Phi + \delta\psi$, with $\delta\psi$ arbitrarily small, and G^{-1} would still exist. From a practical point of view, G should become increasingly ill-conditioned as $\delta\psi \to 0$ and singularity is approached, so eventually it would be impossible to calculate G^{-1} numerically even when, theoretically, the inverse does exist. For that reason the stability of the numerical inversion of G should be checked by obtaining the differences

$$\partial\underline{z} = \underline{z} - G_c^{-1}G\underline{z} \tag{2.55}$$

where G_c^{-1} is the computed inverse and $\underline{z}$ an arbitrary N_c vector whose components are chosen from a sequence of random rumbers. The stability can be judged from the number of significant figures that $\underline{z}$ and $G_c^{-1}G\,\underline{z}$ have in common.

For small ψ, as it is the case with intersatellite separations, the first term in (2.53) can be much smaller than the second term, so the necessary condition for singularity can be written, after some obvious simplification

$$\sin n\frac{\psi}{2} = 0 \tag{2.56}$$

which is met for those critical values Φ where $n\Phi = 0, 2\pi, 4\pi, \ldots k2\pi, \ldots$ Looking at the derivation of (2.13), on which (2.53) is based, one can see that the term ignored corresponds to the effect of the radial component of the inertial acceleration on the line of sight velocity. In a flat Earth this component would be always perpendicular to the line of sight and have no effect at all, so (2.56) could be applicable to a flat Earth geometry. Breakwell (1979) has used a flat Earth approach, his results showing peaks in the noise to signal ratio, or percentage error, at those spatial wavelengths γ satisfying the condition $\psi = k\gamma$ with k integer. The spatial wavelength on the sphere for a harmonic of degree n is $\frac{2\pi}{n}$; replacing γ with $\frac{2\pi}{n}$ in "Breakwell's condition" $\psi = k\gamma$, one gets (2.56). A singularity in the operator involved. the G matrix for instance, would result in the relative error being infinite at some wavelength, indicating complete loss of information at the corresponding frequencies due to the differencial nature of the measurements. While such singularities were not found in the spherical Earth analysis reported here because of the parameters chosen, there were nevertheless, some very gentle and broad ripples in the relative error as a function of the harmonic degree n, with maxima at those degrees where $n\psi$ was closest to $k2\pi$, as the reader can see by looking carefully at the results listed in appendix C.

2.11 Least Squares Collocation

In general, a linear estimator of $\underline{c}$ from $\underline{\bar{v}}_{12(obs)}$ has the form

$$\underline{\hat{c}} = F\,\underline{\bar{v}}_{12(obs)} = F(\underline{\bar{v}}_{12} + \underline{n}) \tag{2.57}$$

where F is the $N_c \times N_p$ estimator matrix. In the case of least squares adjustment, as the reader may remember, this matrix was called F_p (expression (2.41)). As the least squares estimator of paragraph 2.8 is unbiased when the model $\underline{\bar{v}}_{12} = A\underline{c}$ is perfect, the error in $\underline{\hat{c}}$ is due purely to the data errors:

$$\underline{e}_n = \underline{c} - \underline{\hat{c}} = -F\underline{n} \quad \text{(see (2.42))}$$

The purpose of least squares adjustment is to minimize the covariances of the elements of $\underline{e}_n$ or, equivalently, the trace of the error matrix

$$E_n = E\{\underline{e}_n\,\underline{e}_n^T\} = E\{F\underline{e}_n\,\underline{e}_n^T F^T\} = FE\{\underline{e}_n\,\underline{e}_n\}\,F^T = F\,D\,F^T \tag{2.58}$$

Not all linear estimators are unbiased. In general

$$\underline{e}_b = \underline{c} - F\,\underline{\bar{v}}_{12} \neq 0 \tag{2.59}$$

where $\underline{e}_b$ is the N_c - vector of bias errors in $\underline{\hat{c}}$. The gravity field $(\underline{c}, \underline{\bar{v}}_{12})$ is "deterministic" (albeit unknown), while the noise $\underline{n}$ is "stochastic", and the errors in $\underline{\hat{c}}$ that one and the other give rise to are also "deterministic" and "stochastic", respectively. What this difference boils down to is that the noise can be interpreted mathematically as a random process while the field and the bias cannot. To obtain estimators that minimize the total error (bias plus the effect of the noise) a method known as least squares collocation has been developed (Moritz, 1967, Krarup 1969) that treats both parts of the error in a way that is formally very similar, by using the operator $E\{\ \}$ for the noise, and the operator

$M\{\ \}$ (average over all rotations) for the bias. $E\{\ \}$ is a stochastic operator, while $M\{\ \}$ is a <u>geometrical</u> operator. If ε_b is the bias in the estimate $\hat{\bar{C}}^{\alpha}_{nm}$ and ε_n the effect of the noise, then the error measure or "covariance" to be minimized is

$$\sigma_\varepsilon^2 = M\{\varepsilon_b^2\} + E\{\varepsilon_n^2\}$$

This is a <u>hybrid</u> error measure (geometric plus stochastic), but has the advantage of being <u>quadratic</u> while ε_b and ε_n are <u>linear</u> functions of $\bar{v}_{12}$ and $\underline{n}$, which simplifies the mathematical treatment. The rotations involved in the application of $M\{\ \}$ are those of the system of coordinates and, "attached" to it, as it were, of the pattern in which the data are sampled (i.e., the orbits). After each rotation both measurement sites and coordinates are different, and so are the values measured and the estimates, together with the bias. $M\{\ \}$ averages the error over all such "possible outcomes" (one outcome = one rotation). If the statistics of the noise $\underline{n}$ are gaussian, so are those of ε_n. If $\varepsilon_b = 0$, one can determine the likelihood of $\hat{C}^{\alpha}_{nm}$ being within so many "sigmas" of the true value, if one knows only the covariance and the mean value of the data errors. With the "geometrical" statistics based on $M\{\ \}$ one would need, in general, not only $M\{\varepsilon_b^2\}$, but the higher moments $M\{\varepsilon_b^3\}$, $M\{\varepsilon_b^4\}$, . . . as well, as there is no great reason at present to believe that the gravity field is sufficiently "gaussian" to ignore them. Such moments can be "propagated" (in the sense in which covariances are "propagated") from the corresponding moments of the geopotential. The second moment is the <u>covariance function</u> (the Legendre transform of the spectrum σ_n^2) which depends only on the geocentric angle between two points on the same sphere. Higher moments are functions of many such angles. Determining the second moment or covariance from empirical data is not easy; obtaining the higher moments must be even harder, so probably working with these moments to obtain intervals of confidence is not practicable, at least at present. In practice, the choice of any approximation technique, such as collocation, should depend on how well it works for the sort of problem to be solved with it. In the case of geopotential coefficients determination, this author has conducted several numerical tests (Colombo, 1981) and found out that the square of the actual errors in $\hat{\underline{c}}$ can be very close to their hybrid a posteriori covariances when estimating $\hat{\underline{c}}$ with collocation. The data considered, however, were point and mean gravity anomalies, not SST data. For a discussion of the theory and applications of collocation in geodesy, the reader could see Moritz (1980).

To treat biases and propagated errors in a way that is <u>formally</u> the same, one could define a "variance-covariance" matrix for the biases

$$E_b = M\{\underline{e}_b\, \underline{e}_b^T\} \tag{2.60}$$

As already mentioned in paragraph 1.2, expressions (1.15,a-c),

$$M\{\bar{C}^{\alpha}_{nm}\} = 0\ ;\quad M\{(\bar{C}^{\alpha}_{nm})^2\} = \frac{\sigma_n^2}{2n+1}\ ;\quad M\{\bar{C}^{\alpha}_{nm}\, \bar{C}^{\beta}_{kq}\} = 0 \quad \text{if } \alpha \neq \beta,\ n \neq k, \text{ or}$$

$m \neq q$, so the matrix

$$C = M\{\underline{c}\, \underline{c}^T\} \tag{2.61}$$

is diagonal. Each diagonal element in C equals

$$C_{nmkq}^{\alpha\beta} = \begin{cases} \delta\sigma_n^2\ (2n+1)^{-1} & \text{if } n \le M \\ \sigma^2\ (2n+1)^{-1} & \text{if } n > M \end{cases}$$

where the $\delta\sigma_n^2$ correspond to the discrepancies $\bar{C}_{nm}^{\alpha}-\hat{C}_{nm}^{\alpha}$ between the actual coefficients and those of the reference model (paragraph 1.2). Replacing (2.59) in (2.60) and multiplying out one gets

$$E_b = M\{(\underline{c} - F\,\bar{\underline{v}}_{12})\,(\underline{c} - F\bar{\underline{v}}_{12})^T\} = M\{\underline{c}\,\underline{c}^T\} - 2M\{\underline{c}\ \bar{\underline{v}}_{12}^T\ F^T\} + M\{F\bar{\underline{v}}_{12}\bar{\underline{v}}_{12}^T\ F^T\}$$

$$= C - 2CA^TF^T + FACA^TF^T \tag{2.62}$$

because $\bar{\underline{v}}_{12} = A\underline{c}$. The error measure to be minimized is the trace of the <u>combined error matrix</u>

$$E_T = E_b + E_n = C - 2CA^TF + FACA^TF^T + FDF^T = C - 2CA^TF^T + F(ACA^T + D)\ F^T \tag{2.63}$$

according to (2.58) and (2.62), or $\mathrm{tr}\ \{E_T\}$. The optimal estimator matrix minimizes $t_r\ \{E_T\}$, so it must satisfy the matrix equation

$$\frac{1}{2}\,\frac{\partial\,\mathrm{tr}}{\partial F}\ \{E_T\} = F\ (ACA^T + D) - CA^T = 0 \quad \text{(null matrix)} \tag{2.64}$$

This equation is known as the "normal" equation. The solution is

$$F = CA^T\ (ACA^T + D)^{-1} \tag{2.65}$$

Using the matrix identity

$$CA^T\ (ACA^T + D)^{-1} = (A^TD^{-1}A + C^{-1})\ A^T\ D^{-1} \tag{2.66}$$

(see, for instance, Uotila (1967), equation (29)) the optimal estimator is, finally, according to (2.46), (2.57), (2.65), and (2.66),

$$\hat{\underline{c}} = (A^TD^{-1}A + C^{-1})^{-1}\ A^T\ D^{-1}\ \bar{\underline{v}}_{12(obs)}$$

$$= (G + C^{-1})\ A^TD^{-1}\ \bar{\underline{v}}_{12(obs)} \tag{2.67}$$

which, except for the term C^{-1} , is the same as expression (2.39) (with $P = D^{-1}$) for leastsquares adjustment. If all the degree variances σ_n^2 of the potential are non zero for $2 \le n \le N$, C^{-1} exists, $G + C^{-1}$ is positive definite (sufficient condition for F in (2.65) to minimize $t_r\ \{E_T\}$), and $(G + C^{-1})^{-1}$ exists as well, regardless of whether G is positive definite or singular. This is the case in the present study, so the problem discussed in paragraph 2.10 is not relevant, at least in theory, to least squares collocation. Ill-conditioning in $G + C^{-1}$, as distinct from singularity, is another matter. Fortunately, the stability of the inversion of $G + C^{-1}$ was as good as for G alone. As C^{-1} is diagonal, $G + C^{-1}$ retains the block-diagonal structure of G .

Expression (2.63) can be transformed as follows, when F satisfies (2.64):

$$
\begin{aligned}
E_T &= C - FAC = C - (A^T D^{-1} A + C^{-1})^{-1} A^T D^{-1} AC \\
&= [I - (A^T D^{-1} A + C^{-1})\ A^T D^{-1} A]C \\
&= (A^T D^{-1} A + C^{-1})[A^T D^{-1} A^T C^{-1} - A^T D^{-1} A]C \\
&= (A^T D^{-1} A + C^{-1})^{-1} I \\
&= (G + C^{-1})^{-1} \qquad (2.68)
\end{aligned}
$$

so $(G + C^{-1})^{-1}$ is the "hybrid" variance-covariance matrix of the estimates, corresponding to G^{-1} in least squares, from which it differs only in the term C^{-1}. Expressions such as (2.67) and (2.68) have been used already in satellite geodesy for modelling the geopotential (Lerch et al., 1977).

In general, the bias matrix

$$
\begin{aligned}
E_b &= (A^T D^{-1} A + C^{-1})^{-1} - E_n = (G + C^{-1})^{-1} - (G + C^{-1})^{-1} G (G + C^{-1})^{-1} \\
&= (G + C^{-1})^{-1} C^{-1} (G + C^{-1})^{-1} \qquad (2.69)
\end{aligned}
$$

is not zero, so the optimal estimator of collocation can be a biased estimator. Before the introduction of collocation in geodesy, the deliberate use of biased estimators in this discipline was unusual. In other fields. statistics for instance, biased estimators have had considerable application, for example in so-called "ridge regression" (see Bibby and Toutenberg, 1977) which is formally very similar to least squares collocation. The intentional use of biased estimators is quite common in communications, control engineering and in data processing generally, where the Wiener and Kalman filters have many uses. Both types of filter are, in fact, biased estimators whose mathematical formalism resembles that of collocation. But biased estimators are extensively used in everyday life too: every radio and audio system contains filters that "estimate" the desired component in the input signal (voice, music) by rejecting those frequencies at which unwanted signals, or noise, are most prevalent, and reinforcing those where the desired ones dominate. While the output is made inteligible, the signal itself has been distorted because those frequencies where it overlaps the noise have been smoothed out while the others have been boosted. If no noise were present, the true signal would not be identical to the "estimate" or audible output, the difference being a bias. The example is quite germane, as the amplifiers and filters in question are usually linear, like the estimators of collocation.

Estimating the covariance function or the power spectrum of the gravitational potential from existing data, always incomplete and imperfect, is a problem that entails some deep and difficult theoretical questions (Moritz, 1980). From a practical point of view, one could estimate the degree variances needed for C in (2.67) and (2.68) from the SST data itself by using expressions (1.18) and (1.19,a-c) to set up observation equations relating the average spectral power S_m *(observable, in principle)* to the unknown σ_n^2 $(0 < n \leq N)$, and then adjusting the latter in the manner proposed by Wagner and Colombo (1979) for high-low SST data. The short-arc spectral technique discussed there for circular orbits was later extended by Wagner (1980) to elliptical orbits as well.

In the last analysis, how well the σ_n^2 must be known depends on how sensitive the estimates and the posteriori "hybrid covariances" are to errors in the adopted values of the degree variances. In section 3 there is a comparison of results obtained using two different models for the power spectrum, showing low sensitivity even when the discrepancy between the spectra is considerable, so this may not be a serious problem.

Throughout the discussion one has considered the estimation of the actual potential coefficients $\bar{C}^{\alpha}_{nm}$ instead of the scaled $\tilde{C}^{\alpha}_{nm}$. The only real change needed is in expression (2.61), where the scaled elements of the diagonal of C are

$$\tilde{c}^{\alpha\alpha}_{nmnm} = \frac{(GM)^2 a^{2n}}{R^{2n+4}} \times \begin{cases} \delta\sigma_n^2 \, (2n+1)^{-1} & n \leq M \\ \sigma_n^2 \, (2n+1)^{-1} & n > M \end{cases} \tag{2.70}$$

After inverting the thus modified $(G + C^{-1})$ matrix, the actual a posteriori degree variances

$$\tilde{\sigma}^2\varepsilon_n = \sum_{\alpha=0}^{1} \sum_{m=0}^{n} M\{(\varepsilon\hat{\bar{C}}^{\alpha}_{nm})^2\} + E\{(\varepsilon\hat{\bar{C}}^{\alpha}_{nm})^2\} \ , \quad \text{where} \quad \varepsilon\hat{\bar{C}}^{\alpha}_{nm} = \bar{C}^{\alpha}_{nm} - \hat{\bar{C}}^{\alpha}_{nm} \ ,$$

can be obtained from the scaled a posteriori variances

$$\sigma_n = \sum_{\alpha=0}^{1} \sum_{m=0}^{n} M\{(\varepsilon\tilde{C}^{\alpha}_{nm})^2 + E\,(\varepsilon\hat{\tilde{C}}^{\alpha}_{nm})^2\}$$

with the relationship

$$\sigma^2\varepsilon_n = \frac{R^{2n+4}}{(GM)^2 a^{2n}} \, \tilde{\sigma}^2\varepsilon_n \tag{2.71}$$

2.12 Accuracy of the Computed Geoidal Heights

One of the main geophysical applications of GRAVSAT should be to provide an accurate description of the geoid at sea, as reference surface for oceanographic studies of currents, eddies, etc. The geoid height, or difference in ellipsoidal height between the geoid and the reference ellipsoid, is

$$N(r, \phi, \lambda) = r \sum_{\alpha=0}^{1} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \bar{C}^{\alpha}_{nm} \, \bar{Y}^{\alpha}_{nm} (\phi, \lambda) \left(\frac{a}{r}\right)^n \tag{2.72}$$

assuming that all masses are contained inside the ellipsoid. When this is not the case, and the $\bar{C}^{\alpha}_{nm}$ correspond to the expansion of the geopotential outside the bounding sphere (i.e. those that can be sensed by SST data for $2 \leq n \leq N$), expression (2.72) can be said to correspond to the free air geoid. As the geoid experiences periodical and secular variations, what is discussed here should be regarded as an average geoid for a given period of time (say, during the GRAVSAT mission), but this average can be corrected for the main fluctuations, such as tidal effects, to obtain the instantaneous geoid.

Using the estimated potential coefficients one can compute the first $(N + 1)^2 - 3$ terms in (2.72) (the band-limited assumption does not apply at sea level)

$$\hat{N}(r, \phi, \lambda) = r \sum_{\alpha=0}^{1} \sum_{n=2}^{N} \sum_{m=0}^{n} \hat{C}^{\alpha}_{nm} \bar{Y}^{\alpha}_{nm}(\phi,\lambda) \left(\frac{a}{r}\right)^n \tag{2.73}$$

and the difference between (2.72) and (2.73) is the error in geoid height

$$\delta N(r, \phi, \lambda) = r \sum_{\alpha=0}^{1} \sum_{n=2}^{N} \sum_{m=0}^{n} \epsilon\hat{C}^{\alpha}_{nm} \bar{Y}_{nm} \left(\frac{a}{r}\right)^n + r \sum_{\alpha=0}^{1} \sum_{N+1}^{\infty} \sum_{m=0}^{n} \bar{C}^{\alpha}_{nm} \bar{Y}^{\alpha}_{nm} \left(\frac{a}{r}\right)^n \tag{2.74}$$

The square of this error is

$$\delta N(r,\phi,\lambda)^2 = r^2 \sum_{\alpha=0}^{1} \sum_{\beta=0}^{1} \sum_{n=2}^{\infty} \sum_{k=2}^{\infty} \sum_{m=0}^{n} \sum_{q=0}^{n} \bar{D}^{\alpha}_{nm} \bar{D}^{\beta}_{kq} \bar{Y}_{nm} \bar{Y}_{kq} \left(\frac{a}{r}\right)^{2n},$$

$$\text{where } \bar{D}^{\alpha}_{nm} = \begin{cases} \epsilon\hat{C}^{\alpha}_{nm} & n \le N \\ \bar{C}^{\alpha}_{nm} & n > N \end{cases}$$

If the $\hat{C}^{\alpha}_{nm}$ are obtained by least squares adjustment, the variance of the errors would be

$$E\{\delta N(r, \phi, \lambda)^2\} = \sigma^2 \delta N(r, \phi, \lambda)$$

Keeping r constant and averaging $\sigma^2\delta_N(r, \phi, \lambda)$ over the unit sphere (i.e., with respect to ϕ and λ alone)

$$\sigma^2 \epsilon N(r) = \frac{1}{4\pi} \int_0^{\pi}\int_0^{2\pi} \sigma^2 \delta N(r, \phi, \lambda) \cos\phi \, d\phi \, d\lambda =$$

$$= \frac{r^2}{4\pi} \sum_{\alpha,\beta,n,k,m,q} \left(\frac{a}{r}\right)^{2n} E\{\bar{D}^{\alpha}_{nm} \bar{D}^{\beta}_{kq}\} \int_0^{\pi}\int_0^{2\pi} \bar{Y}^{\alpha}_{nm} \bar{Y}^{\beta}_{kq} \cos\phi \, d\phi \, d\lambda =$$

$$= r^2 \sum_{\alpha nm} \sigma^2 \epsilon^{\alpha\alpha}_{nmnm} \left(\frac{a}{r}\right)^{2n} = r^2 \left(\sum_{n=2}^{N} \sigma^2 \epsilon_n \left(\frac{a}{r}\right)^{2n} + \sum_{n=N+1}^{\infty} \sigma^2{}_n \left(\frac{a}{r}\right)^{2n} \right) \tag{2.75}$$

because of the orthogonality properties of spherical harmonics. In an entirely similar way one can arrive to exactly the same expression for the mean square error when the $\bar{C}^{\alpha}_{nm}$ are estimated by collocation, except that $\sigma^2_{\epsilon N}$ is now a hybrid degree variance in the sense explained in the previous paragraph. As the error depends on r, and r is not constant on the ellipsoid, one could settle for an "average" error by choosing $r = a$, where a is the mean Earth radius, so

$$\sigma^2_{\epsilon N} = a^2 \left(\sum_{m=2}^{N} \sigma^2_{\epsilon_n} + \sum_{n=N+1}^{\infty} \sigma^2{}_n \right) \tag{2.76}$$

which is the expression used in section 3. Notice that the correlations between the errors $\epsilon\hat{C}^{\alpha}_{nm}$, which exist when they have the same order m because $(G + C^{-1})^{-1}$ has the same block-diagonal structure as $(G + C^{-1})$, are eliminated from (2.75) and (2.76) by the orthogonality of the $\bar{Y}^{\alpha}_{nm}(\phi,\lambda)$.

To obtain the accuracy of the geoid height computed with the SST derived model, one has to know the a posteriori accuracies of the coefficients, represented by the $\sigma^2\varepsilon^{\alpha\alpha}_{nmnm}$, and also the actual degree variances of the field for $n > N$. The first we know, to some extent, as formal accuracies derived from the diagonal elements of $(G + C^{-1})^{-1}$; the second are largely unknown, as they correspond to the part of the potential not analyzed yet. Therefore, one must adopt some approximate model for this spectral "tail" between N and ∞ , based largely on indirect evidence, which means that the estimated contribution of this "tail" to ∂N, or truncation error, must depend on rather weak assumptions. A further problem, of more theoretical nature, is that (2.72) is valid only when the potential can be expanded in spherical harmonics. When there are masses above the geoid, as it is the case in reality, the expansion is known to converge only outside the smallest sphere bounding all those masses, or external bounding sphere, and not necessarily at the Earth's surface itself, where it is actually needed. This question is partially answered by a theorem first proposed by Walsh (1927), reported later by Keldych and Lavrentieff (1939), and independently stated by Krarup (1969), according to which there are always spherical harmonic expansions like the one in (2.72) that approximate uniformly the potential, to within any arbitrary accuracy, on and above the Earth's surface. A consequence of this theorem is that the $\bar{C}^{\alpha}_{nm}$ for $n \leq N$, with N finite, in the convergent expansion outside the external bounding sphere (which can be detected by SST) must differ from those in some of the internal approximating expansions also by an arbitrarily small amount. In this sense, the coefficients detected from SST data can be considered to be valid also at the Earth's surface.

The approximating expansions in Walsh's theorem converge uniformly to the functions they approximate above the topography. Their analytical continuations inside the Earth, where they also take definite values, have no physical meaning and are, thus, free from the natural constraints that determine the character of the functions represented (potential, geoid height, the estimation error ∂N according to (2.74)) above the topography or bathymetry (near the sea surface, for example). The free air geoid, say, coincides in the limit with the true geoid where this surface runs above the solid Earth, so it is rather smooth and gentle in such a region, but it can be argued that it may be much rougher in places elsewhere, under the terrain. The mean square value of ∂N in (2.76) corresponds to an average over the sphere of radius $r = a$. This can be interpreted as a spherical approximation to the reference ellipsoid after all masses outside it have been removed by an atmospheric and topographic correction. This is the interpretation made in Section 3, and it involves a spherical approximation error not accounted for in (2.76). One can also regard the average as being done on an actual sphere that dips in and out of the topography and, more importantly, of the equatorial bulge, so a correction for the masses above this sphere is too large to be reliably computed. In this case the average includes the squares of true errors ∂N above the Earth's surface, and of the analytical continuation of ∂N below. As the continuation can have a very different character from that of ∂N in free space, one may wonder just how meaningful is this average, how well does it represent the real errors, particularly where one is most concerned: near the sea surface, for instance. Finally, the presence of sharp features such as ridges, trenches, cordilleras, etc., results usually in strong variations of the geoid which may cause oscillations in the truncated series of the estimated $N(r, \phi, \lambda)$ of the type called "Gibson's phenomenon" in Fourier series, oscillations that may affect the practical use of the global model in areas that are among the most interesting from a geophysical point of view. While all these problems may have simple solutions, none are known to this author yet, so there may be some need for improvement in our present understanding of these rather basic questions, and for a grain of salt when taking some of the results distilled from existing theories. With such reservations in mind, one can regard the undulation estimate of (2.73) as optimal in the sense that if the SST data were used to estimate N directly, instead of finding the $\tilde{C}^{\alpha}_{nm}$ first, the result would be exactly the same. This is so because the field at satellite height has been assumed to be band limited, and in such case any linear combination of the $\tilde{C}^{\alpha}_{nm}$, such as $N(r, \phi, \lambda)$ is according to (2.73), can be optimally estimated by replacing the $\tilde{C}^{\alpha}_{nm}$ with their optimal estimates $\hat{C}^{\alpha}_{nm}$ in the linear combination (see Colombo (1981), paragraph 2.18). The truncated part $(n > N)$ is regarded here as an additional, unknown signal on which there is no information in the satellite data, while the terms with $n \leq N$ constitute the estimated variable.

2.13 The Effect of Some Mission Parameters on Coefficient Accuracy

The diagonal elements of the inverse of the normal matrix, $\sigma^2\varepsilon^{\alpha\alpha}_{nmnm}$ in the sort of notation used for the elements of G in paragraph 2.9, depend on the values of certain mission parameters, such as data accuracy, sampling interval, mission duration, etc., according to simple formulas. Such formulas enable one to recalculate the accuracy of the adjusted coefficients for different values of those parameters without having to repeat the lengthy computations required by the formation and inversion of the normal matrix. This paragraph contains the derivation of some of these convenient formulas, both for least squares adjustment and for least squares collocation.

(a) Least Squares Adjustment:

When $D = \sigma^2 I$, where I is the $N_p \times N_p$ unit matrix, the normal matrix is

$$G = A^T D^{-1} A = \sigma^{-2} A^T I^{-1} A = \sigma^{-2} A^T A$$

and the variance-covariance matrix is

$$G^{-1} = \sigma^2 (A^T A)^{-1}$$

so the diagonal elements of G^{-1} are related to σ by

$$\sigma^2\varepsilon^{\alpha\alpha}_{nmnm}(\sigma) = \left(\frac{\sigma}{\sigma_0}\right)^2 \sigma^2\varepsilon^{\alpha\alpha}_{nmnm}(\sigma_0) \qquad (2.77,a)$$

The elements of G depend on the sampling interval Δt through the number of samples $N_p = \frac{T}{\Delta t}$ (expression (2.51)). Consequently, the elements of G are inversely proportional to Δt, and those of G^{-1} are directly proportional to it, so

$$\sigma^2\varepsilon^{\alpha\alpha}_{nmnm}(\Delta t) = \left(\frac{\Delta t}{\Delta t_0}\right) \sigma^2\varepsilon^{\alpha\alpha}_{nmnm}(\Delta t_0) \qquad (2.77,b)$$

As the elements of G are directly proportional to T, the length of the mission, those of G^{-1} should be inversely proportional to T, <u>provided</u> that $T = k\,T_0$, where k is an integer and T_0 the value originally used when setting up G (i.e., 179 days in the case studied in Section 3). As the orbit has a grand period T_0, extending this period k times results in the orbit being repeated also k times, and a measurement at any given location being taken k well, so the adjustment contains repeated but uncorrelated measurements. Consequently

$$\sigma^2\epsilon^{\alpha\alpha}_{nmnm}(T) = (\frac{T_0}{T})\sigma^2\epsilon^{\alpha\alpha}_{nmnm}(T_0) \tag{2.77,c}$$

provided T/T_0 is integer.

For <u>small</u> changes in height above or below h (the height used to calculate G originally) the changes in satellite angular velocity ω and, consequently, in the shape of the groundtrack of the mid-point, and in the distribution of the measurements along it are small, too, and can be ignored. If ψ is kept constant by changing the separation ρ between satellites so $\sin\frac{\psi}{2} = \frac{\rho}{2R} = \frac{\rho}{2(a+h)}$ is unchanged, then A and G will vary so little as to be considered independent from h. The de-scaled diagonal elements of G^{-1} are then related to h as follows

$$\sigma^2\epsilon^{\alpha\alpha}_{nmnm}(h) = (\frac{a+h}{a+h_0})^{2n}\,\sigma^2\epsilon^{\alpha\alpha}_{nmnm}(h_0) \tag{2.77,d}$$

where a is the mean Earth radius. Expressions (2.77,a-d) can be written together

$$\sigma^2\epsilon^{\alpha\alpha}_{nmnm}(\sigma,\Delta t,T,h) = \frac{\sigma^2\Delta t T_0 (a+h)^{2n}}{\sigma_0^2\Delta t_0 T (a+h_0)^{2n}}\,\sigma^2\epsilon^{\alpha\alpha}_{nmnm}(\sigma_0,\Delta t_0,T_0,h_0) \tag{2.78}$$

The degree variance of the error is $\sigma^2\epsilon_n = \sum_{\alpha=0}^{1}\sum_{m=0}^{n}\sigma^2\epsilon^{\alpha\alpha}_{nmnm}$

so

$$\sigma^2\epsilon_n(\sigma,\Delta t,T,h) = \frac{\sigma^2\Delta t T_0 (a+h)^{2n}}{\sigma_0^2\Delta t_0 T (a+h_0)^{2n}}\,\sigma^2\epsilon_n(\sigma_0,\Delta t_0,T_0,h_0) \tag{2.79}$$

in accordance with (2.78), provided T/T_0 is integer, $|h-h_0|$ is small compared to $R = a+h$, and ψ is kept constant.

(b) <u>Least Squares Collocation</u>:

Rummel et al. (1979) have shown how a singular value decomposition of the normal matrix greatly simplifies the recalculation of the a posteriori variances with different levels of data noise. This idea can be adapted to the global case under consideration. It follows from the previous discussion that

$$G'(\sigma,\Delta t,T) = \frac{\sigma_0^2\,\Delta t_0 T}{\sigma^2\,\Delta t\,T_0}\,G'(\sigma_0,\Delta t_0,T_0) \tag{2.80}$$

where $G'(\sigma_0,\Delta t_0,T_0)$ is the normal matrix G of least squares adjustment pre- and post-multiplied by C^{2} and calculated with the parameter values

σ_0, Δt_0, T_0. If Λ_0 is the diagonal matrix whose diagonal elements λ_i are the eigenvalues of $G'(\sigma_0, \Delta t_0, T_0)$, if all the λ_i are different and nonzero, and if M_0 is the matrix whose columns $\underline{m}_i$ are the corresponding normalized eigenvectors of $G'(\sigma_0, \Delta t_0, T_0)$, so $\underline{m}_i^T \underline{m}_i = 1$, then

$$G'(\sigma, \Delta t, T) = \frac{\sigma_0^2 \Delta t_0 T}{\sigma^2 \Delta t T_0} M_0 \Lambda_0 M_0^T \tag{2.81}$$

It follows that

$$\begin{aligned} C^{\frac{1}{2}}(G(\sigma,\Delta t,T) + C^{-1})C^{\frac{1}{2}} &= G'(\sigma,\Delta t,T) + I \\ &= \frac{\sigma_0^2 \Delta t_0 T}{\sigma^2 \Delta t T_0} M_0 \Lambda M_0 + I \end{aligned}$$

The eigenvalues of $G'(\sigma,\Delta t,T) + I$ are $\frac{\sigma_0^2 \Delta t_0 T}{\sigma^2 \Delta t T_0} \lambda_i + 1$, while the eigenvectors are those of $G'(\sigma,\Delta t,T)$. The eigenvectors of a real symmetric matrix with distinct eigenvalues are orthogonal and equal in number to the dimension of the matrix, so M_0 is a square orthogonal matrix (the columns are orthogonal unit vectors, if one assumes that all the λ_i are different). Consequently, $M_0^{-1} = M_0^T$, so

$$(G'(\sigma, \Delta t, T) + I)^{-1} = M_0\left(\frac{\sigma_0^2 \Delta t_0 T}{\sigma^2 \Delta t T_0} \Lambda_0 + I\right)^{-1} M_0^T \tag{2.82}$$

Therefore, the de-scaled diagonal elements of the inverse of the normal matrix,

$$C^{-\frac{1}{2}}(G'(\sigma, \Delta t, T) + I)^{-1} C^{-\frac{1}{2}} = C^{-\frac{1}{2}} M_0\left(\frac{\sigma_0^2 \Delta t_0 T}{\sigma^2 \Delta t T_0} \Lambda_0 + I\right)^{-1} M_0^T C^{-\frac{1}{2}} \tag{2.83}$$

are

$$\sigma^2 \varepsilon_{nmnm}^{\alpha\alpha} = \frac{2n+1}{\sigma_h^2} \underline{m}_i^T \left(\frac{\sigma_0^2 \Delta t_0 T}{\sigma^2 \Delta t T_0} \Lambda_0 + I\right)^{-1} \underline{m}_i \frac{R^{2n+4}}{(GMa^n)^2} \tag{2.84}$$

where $\sigma^2 \varepsilon_{nmnm}^{\alpha\alpha}$ has the same position in the diagonal of $(G + C^{-1})^{-1}$ as λ_i in the diagonal of Λ_0. The presence of C^{-1} in the expression of the normal matrix seems to preclude a simple relationship between the a posteriori variance and h like that of (2.77,c), because C depends on $R = a+h$, so $G' = C^{\frac{1}{2}} G C^{\frac{1}{2}}$ and Λ_0 are also functions of h (expressions (2.70) and (2.81).

While not as straightforward as (2.78), (2.84) is relatively easy to apply, compared to a full recalculation of the normal and its inverse. To apply this expression one needs to have the eigenvalues and eigenvectors of the normal, instead of its inverse. Obtaining either requires much the same amount of computing.

2.14 The Right Hand Sides of the Normals

What follows, though not applicable to the error analysis that is the main subject of this report, is relevant to the question of data processing discussed in Section 5.

The normal equations of least squares adjustment and of least squares collocation have the same independent terms which, in vector form, can be represented by the N_c - vector $\underline{b}$

$\underline{b} = A^T D^{-1} \underline{\bar{v}}_{12(obs)}$ (expressions (2.47,b) and (2.67))

An individual component of $\underline{b}$, corresponding in position to $\tilde{C}^{\alpha}_{nm}$ in $\underline{c}$, can be written

$$b^{\alpha}_{nm} = (\underline{a}^{\alpha}_{nm})^T D^{-1} \underline{\bar{v}}_{12(obs)} = \frac{1}{\sigma^2} \sum_{i=0}^{N_p} a^{\alpha}_{nm}(t_i)\, \bar{v}_{12}{}^{(t_i)}_{(obs)} \tag{2.85}$$

where $\underline{a}^{\alpha}_{nm}$ is the column vector in A corresponding to the unknown $\tilde{C}^{\alpha}_{nm}$ (paragraph 2.7). According to (2.24), the elements of $\underline{a}^{\alpha}_{nm}$ are

$$a^{0}_{nm}(t_i) = \frac{1}{\Delta a} \sum_{p=z}^{n} a^{nm}_{p} \left\{ \begin{array}{l} \dfrac{S((p\omega+m\Omega)t_i,\Delta a)}{(p\omega+m\Omega)^2} + \dfrac{S((p\omega-m\Omega)t_i,\Delta a)}{(p\omega-m\Omega)^2} \\ -\dfrac{C((p\omega+m\Omega)t_i,\Delta a)}{(p\omega+m\Omega)^2} - \dfrac{C((p\omega-m\Omega)t_i,\Delta a)}{(p\omega-m\Omega)^2} \end{array} \right\} \tag{2.86,a}$$

$$a^{1}_{nm}(t_i) = \frac{1}{\Delta a} \sum_{p=z}^{n} a^{nm}_{p} \left\{ \begin{array}{l} \dfrac{C((p\omega+m\Omega)t_i,\Delta a)}{(p\omega+m\Omega)^2} - \dfrac{S((p\omega-m\Omega)t_i,\Delta a)}{(p\omega-m\Omega)^2} \\ \dfrac{S((p\omega+m\Omega)t_i,\Delta a)}{(p\omega+m\Omega)^2} - \dfrac{S((p\omega-m\Omega)t_i,\Delta a)}{(p\omega-m\Omega)^2} \end{array} \right\} \tag{2.86,b}$$

The $C((p\omega + m\Omega)t_i,\Delta a)$ and $S((p\omega - m\Omega)t_i,\Delta a)$ are orthogonal funcions of t_i, according to (2.34,a-b), so the data $\bar{v}_{12}{}^{(t_i)}_{(obs)}$ can be expanded into a sum of these functions:

$$\bar{v}_{12}{}^{(t_i)}_{(obs)} = \sum_{p=0}^{N_y} \sum_{m=-N_r}^{N_r} \bar{v}^{0}_{pm}\, C((p\omega+m\Omega)t_i,\Delta a) + \bar{v}^{1}_{pm}\, S((p\omega+m\Omega)t_i,\Delta a) \tag{2.87}$$

where $N_y = \frac{\pi}{\omega\Delta t}$ is the Nyquist frequency as defined in paragraph 1.2, while $N_r = \frac{1}{2}\omega\Omega^{-1}N_d$. The $\bar{v}^{1}_{pm}$, $\bar{v}^{0}_{pm}$ coefficients can be obtained from the ordinary Fourier coefficients in $\bar{v}_{12}{}^{(t_i)}_{(obs)} = \sum_{p=0}^{N_y} \sum_{m=-N_r}^{N_r} A_{pm}\cos(p\omega+m\Omega)t_i + B_{pm}\sin(p\omega+m\Omega)t_i$ with the relationships

$$\bar{v}^{0}_{pm} = A_{pm}\sin(p\omega+m\Omega)\Delta a + B_{pm}(1-\cos(p\omega+m\Omega)\Delta a) \tag{2.88,a}$$

$$\bar{v}^{1}_{pm} = A_{pm}(1-\cos(p\omega+m\Omega)\Delta a) + B_{pm}\sin(p\omega+m\Omega)\Delta a \tag{2.88,b}$$

According to (2.34,a-b), (2.85), and (2.86,a-b), the elements of $\underline{b}$ are

$$b^{0}_{nm} = \frac{N_r}{\sigma^2\Delta a} \sum_{p=z}^{n} a^{nm}_{p} \left\{ \begin{array}{l} \dfrac{\bar{v}^{1}_{pm}}{\omega^2_{+}} + \dfrac{\bar{v}^{1}_{p-m}}{\omega^2_{-}} \\ -\dfrac{\bar{v}^{0}_{pm}}{\omega^2_{+}} - \dfrac{\bar{v}^{0}_{p-m}}{\omega^2_{-}} \end{array} \right\} \tag{2.89,a}$$

and

$$b^{1}_{nm} = \frac{N_p}{\sigma^2\Delta a} \sum_{p=z}^{n} a^{nm}_{p} \left\{ \begin{array}{l} \dfrac{\bar{v}^{0}_{pm}}{\omega^2_{+}} - \dfrac{\bar{v}^{0}_{p-m}}{\omega^2_{-}} \\ \dfrac{\bar{v}^{1}_{pm}}{\omega^2_{+}} - \dfrac{\bar{v}^{1}_{p-m}}{\omega^2_{-}} \end{array} \right\} \tag{2.89,b}$$

where $z = \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases}$

(as in (2.20))

where, as before, the upper parts of the curly brackets correspond to (n-m) even, the lower to (n-m) odd, $\omega^2 \pm = \frac{(p\omega \pm m\Omega)^2}{(1-\cos(p\omega \pm m\Omega)\Delta a)}$, and p and n have the same parity.

The Fourier coefficients C_{pm} , S_{pm} can be obtained by means of the Fast Fourier Transform in any of the special forms designed to handle data sets too large to store in the central memory of a computer.

2.15 Oblique Orbits

The error analysis reported in this work is concerned only with polar orbits, which provide the fullest data coverage because they include the polar regions. In the case of oblique orbits, the inclination of the orbital plane with respect to the equator is neither 0° nor 90°. With oblique orbits, even if the other assumptions in paragraph 2.1 are maintained, expression (2.9) for the line of sight inertial acceleration, which is the starting point for deriving the observation equations, has to be modified. Calling , as in Figure 2.2, F to the geocentric angle between the ascending node and a point P along the orbit, and L to the longitude of that node, the band-limited gravitational field potential can be written as follows

$$V(r, F, L) = \frac{GM}{r} \sum_{n=0}^{N} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n \sum_{q=0}^{n} \bar{F}_{nmq}(\iota)\left[\begin{Bmatrix}\bar{C}_{nm}\\ \bar{S}_{nm}\end{Bmatrix} \cos((n-2q)\,F + mL)\right.$$

$$\left. + \begin{Bmatrix}\bar{S}_{nm}\\ \bar{C}_{nm}\end{Bmatrix} \sin((n-2q)\,F + mL)\right] \qquad (2.90)$$

where the upper part of each bracket corresponds to (n-m) even and the lower part to (n-m) odd; ι is the inclination angle between the equator and the orbital plane; F and L are functions of time

$$F = [\omega t]_{\mathrm{mod}_{2\pi}} \qquad (2.91,a)$$

and

$$L = -[(\Omega - \beta(\iota))t]_{\mathrm{mod}_{2\pi}} \qquad (2.91,b)$$

$$= -[\Omega' t]_{\mathrm{mod}_{2\pi}}$$

where

$$\beta(\iota) \simeq -1.35 \times 10^{-6} \cos\iota \quad \mathrm{rad\ s^{-1}} \qquad (2.91,c)$$

is the rate of precession of the node, which is zero for polar orbits ($\iota = \frac{\pi}{2}$); $\bar{F}_{nmq}(\iota)$ is the normalized inclination function

$$\bar{F}_{nmq}(\iota) = \begin{Bmatrix} \sqrt{2n+1} \quad (m=0) \\ \\ \sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}} \end{Bmatrix} \sum_{t=0}^{\min(q,k)} \frac{(2n-2t)!\,(\sin\iota)^{n-m-2t}}{t!\,(n-t)!\,(n-m-2t)!\,2^{2(n-t)}}$$

$$\times \sum_{s=0}^{m} \binom{m}{s} (\cos\iota)^s \sum_{c} \binom{n-m-2t+s}{c} \binom{m-s}{q-t-c} (-1)^{c-k} \qquad (2.92)$$

where k is the integer part of $\frac{n-m}{2}$ and c is summed over all values that make the binomial coefficients nonzero. For the derivation of (2.90) and (2.92), see, for instance, Kaula (1966), Chapter 3 (the notation is somewhat different).

The line of sight between the satellites lies always in the instantaneous orbital plane. The inertial acceleration can be decomposed into three orthogonal components: the radial, the normal to the radial in the instantaneous orbital plane, and the perpendicular to this plane. Of these, only the first two accelerations have nonzero projections on the direction of the line of sight. The radial acceleration is

$$\frac{\partial V}{\partial r} = -\frac{GM}{r^2} \sum_{n=0}^{N} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n (n+1) \sum_{q=0}^{n} F_{nmq}(\iota) \left[\left\{ \begin{matrix} \tilde{C}_{nm} \\ \tilde{S}_{nm} \end{matrix} \right\} \cos((n-2q)F + mL) \right.$$
$$\left. + \left\{ \begin{matrix} \tilde{S}_{nm} \\ \tilde{C}_{nm} \end{matrix} \right\} \sin((n-2q)F + mL) \right] \qquad (2.93,a)$$

and the normal to the radial acceleration (in the instantaneous orbital plane) is

$$\frac{1}{r}\frac{\partial V}{\partial F} = \frac{GM}{r^2} \sum_{n=0}^{N} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n \sum_{q=0}^{n} F_{nmq}(\iota)\,(n-2q) \left[\left\{ \begin{matrix} -\tilde{C}_{nm} \\ \tilde{S}_{nm} \end{matrix} \right\} \sin((n-2q)F + mL) \right.$$
$$\left. + \left\{ \begin{matrix} \tilde{S}_{nm} \\ \tilde{C}_{nm} \end{matrix} \right\} \cos((n-2q)F + mL) \right] \qquad (2.93,b)$$

The coordinate L is common to both satellites, as their orbits are coplanar. Adopting $F = \frac{F_1+F_2}{2}$, the midpoint nodal distance, as the other independent variable ($r = R$ is fixed), the modified inertial line of sight acceleration (inertial accel. minus constant term due to even zonals) is

$$\hat{a}_{12}(R,F,L) = -\left(\frac{\partial V}{\partial r}\left(R,F-\frac{\psi}{2},L\right) + \frac{\partial V}{\partial r}\left(R,F+\frac{\psi}{2},L\right)\right) \sin\frac{\psi}{2}$$
$$+ \frac{1}{R}\left(\frac{\partial V}{\partial F}\left(R,F-\frac{\psi}{2},L\right) - \frac{\partial V}{\partial F}\left(R,F+\frac{\psi}{2},L\right)\right) \cos\frac{\psi}{2} - a_0 \qquad (2.94)$$

Replacing F and L with t as the independent variable in (2.93,a-b), substituting the $\tilde{C}_{nm}$ with the scaled $\tilde{C}^{\alpha}_{nm}$, introducing $p = n-2q$ as subscript, instead of q, eliminating all terms of frequency zero, and replacing the result in (2.94), one can finally arrive to an expression for the observation equation resembling (2.24), after a rather laborious process quite similar to that in paragraph 2.2:

$$\frac{1}{\Delta a} \sum_{m=0}^{N} \sum_{n=\max(m,z)}^{N} \tilde{C}_{nm} \sum_{p=z}^{n} \alpha^{nm}_{\left(\frac{n-p}{2}\right)}(\iota) \left\{ \begin{matrix} S((p\omega + m\Omega')t_i,\Delta a) \\ -C((p\omega + m\Omega')t_i,\Delta a) \end{matrix} \right\} (p\omega + m\Omega')^{-2}$$
$$+ \alpha^{nm}_{\left(\frac{n+p}{2}\right)}(\iota) \left\{ \begin{matrix} S((p\omega - m\Omega')t_i,\Delta a) \\ C((p\omega - m\Omega')t_i,\Delta a) \end{matrix} \right\} (p\omega - m\Omega')^{-2} +$$
$$+ \tilde{S}_{nm} \sum_{p=z}^{n} \alpha^{nm}_{\left(\frac{n-p}{2}\right)}(\iota) \left\{ \begin{matrix} C((p\omega + m\Omega')t_i,\Delta a) \\ S((p\omega + m\Omega')t_i,\Delta a) \end{matrix} \right\} (p\omega + m\Omega')^{-2}$$

$$+ \alpha^{nm}_{(\frac{n\pm p}{2})}(\iota) \left\{ \begin{matrix} -C((p\omega - m\Omega')t_i, \Delta a) \\ S((p\omega - m\Omega')t_i, \Delta a) \end{matrix} \right\} (p\omega - m\Omega')^{-2} = \bar{v}_{12}{}^{(t_i)}_{(obs)} + r_i \qquad (2.95)$$

where

$$\alpha^{nm}_{(\frac{n\pm p}{2})}(\iota) = 2\bar{F}_{nm(\frac{n\pm p}{2})}(\iota) \left[(n+1)\cos p\frac{\psi}{2} \sin\frac{\psi}{2} + p \sin p\frac{\psi}{2} \cos\frac{\psi}{2}\right] \qquad (2.96)$$

While their coefficients are now different in each pair, corresponding pairs of frequencies $p\omega \pm m\Omega$ and $p\omega \pm m\Omega'$, in the same order, appear in (2.24) as they do in (2.95). As a result of this, the same columns in A are orthogonal to each other regardless of the angle ι and, therefore, the same elements in $G = \sigma^{-2}A^TA$ are zero as before, so the normal matrices of least squares adjustment and least squares collocation have the block-diagonal structure shown in Figure 2.1, paragraph 2.9, for oblique as well as for polar orbits. This is true provided that ω and Ω' are congruent, so the total number of satellite revolutions N_r and the total number of apparent turns of the Earth in a node-fixed system of coordinates, during the whole mission, N_D', are relative primes.

For polar orbits, the inclination functions have the property

$$\bar{F}_{nm(\frac{n-p}{2})}\left(\frac{\pi}{2}\right) = (-1)^{n-m}\,\bar{F}_{nm(\frac{n\pm p}{2})}\left(\frac{\pi}{2}\right) \qquad (2.97)$$

($p \neq 0$, p and n with the same parity). Comparing (2.13) and (2.96) one gets, therefore,

$$\bar{h}^{nm}_p = 2\bar{F}_{nm(\frac{n-p}{2})}\left(\frac{\pi}{2}\right) \qquad (2.98)$$

Consequently, the Fourier coefficients $\bar{h}^{nm}_p$ of the $\bar{C}_{nm}$ used in the case of polar orbits can be calculated using the formulas

$$\bar{h}^{nm}_p = \left\{ \begin{matrix} \sqrt{2n+1} \quad (m=0) \\ \sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}} \end{matrix} \right\} \sum_{t=0}^{\min(k,\frac{n-p}{2})} \frac{(2n-2t)!}{t!\,(n-t)!\,(n-m-2t)!\,2^{2(n-t)}}$$

$$\times \sum_c \binom{n-m-2t}{c} \binom{m}{(\frac{n-p}{2})-t-c} (-1)^{c-k} \qquad (2.99)$$

derived from (2.92) with $\iota = \frac{\pi}{2}$. This formula is an alternative to the use of the Fast Fourier Transform as explained in paragraph 2.7.

The similitudes between (2.24) and (2.95) indicate that the computer programs needed to carry out the error analysis of a mission where the orbital plane is inclined with respect to the equator are very similar to those for polar orbits explained and listed in Appendix B. Reasoning once more as in paragraph 2.9, one arrives to an expression for the general element of the G matrix

$$g_{nmkq}^{\alpha\beta} = \begin{cases} 0 \quad \text{if} \quad \alpha \neq \beta \ , \quad m \neq q \ , \quad n \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix} \ \text{and} \ k \begin{Bmatrix}\text{odd}\\ \text{even}\end{Bmatrix} \ , \\ \\ \dfrac{N_p}{\sigma^2 \Delta a} \displaystyle\sum_{p=z}^{\min(n,k)} \quad \alpha^{nm}_{\left(\frac{n-p}{2}\right)}(\iota) \quad \alpha^{km}_{\left(\frac{n-k}{2}\right)}(\iota) \ \dfrac{(-\cos(p\omega + m\Omega')\Delta a)}{(p\omega + m\Omega')^4} \\ \qquad + \alpha^{nm}_{\left(\frac{n+p}{2}\right)}(\iota) \quad \alpha^{km}_{\left(\frac{n+k}{2}\right)}(\iota) \ \dfrac{(1-\cos(p\omega - m\Omega')\Delta a)}{(p\omega - m\Omega')^4} \end{cases} \tag{2.100}$$

that is comparable to (2.51), showing at once *the similitudes and the* differences between the polar orbit and the oblique.

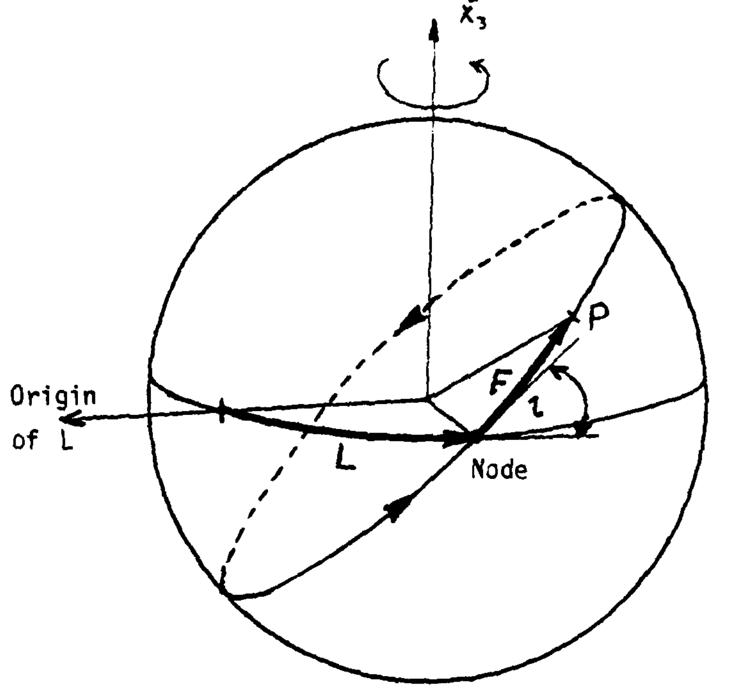

Figure 2.2: Geometry of tne Oblique Orbit

3. Numerical Results

This section presents the results of the error analysis whose theory has been given in sections 1 and 2 . The calculations have been done in accordance to least squares adjustment and to least squares collocation, and both sets of results are shown for comparison.

3.1 Spectral Model and Error Formulas

The degree variance of the error is, according to paragraph 2.13,

$$\sigma^2\varepsilon_n = \sum_{m=0}^{n} \sum_{\alpha=0}^{1} \sigma^2\varepsilon^{\alpha\alpha}_{nmnm} \qquad (3.1)$$

where $\sigma^2\varepsilon^{\alpha\alpha}_{nmnm}$ is the variance of the estimated coefficient $\hat{C}^{\alpha}_{nm}$. The relative error per degree is

$$\rho_n = \frac{\sigma\varepsilon_n}{\sigma_n} \qquad (3.2)$$

where

$$\sigma^2_n = \sum_{m=0}^{n} \sum_{\alpha=0}^{1} (\bar{C}^{\alpha}_{nm})^2 \qquad (3.3)$$

is the degree variance of the n harmonic of the potential. The set of all σ^2_n for $0 \le n < \infty$ is the power spectrum of the potential. This relative error, multiplied by 100, is listed as percentage error per degree in the various tables shown in this section. To calculate (3.2) it is necessary to know the power spectrum, the σ^2_n . The spectrum, like the potential itself, is not entirely known, but there are estimates of the σ^2_n obtained from the analysis of terrestrial and satellite data. For low degree harmonics the σ^2_n can be calculated using (3.3) with the values for the $\bar{C}^{\alpha}_{nm}$ taken from one of the existing spherical harmonic models of the potential. For higher terms, the "tail" beyond the highest degree whose $\bar{C}^{\alpha}_{nm}$ are known, one can choose among the several formulas for σ^2_n as a function of n that are available, each based on a different type of approximation, or on different data. For this study the degree variances up to degree $n = 100$ were taken from a spherical harmonics model complete to $n = 180$, obtained by Rapp and associates at OSU from the analysis of a global set of mean 1°x 1° gravity anomalies using quadrature formulas. The anomalies themselves were obtained from gravimetry and altimetry by means of least squares collocation. As a further step the anomalies were adjusted in a combination solution that included the coefficients of GEM-9 as data. For a report on this adjusted data set, see Rapp (1978) and also Rapp (1979a). GEM-9 is described in (Lerch et al., 1977). If one ignores orbit errors, then expression (1.23) is reduced to an identity between the time derivative of the residual line of sight velocity and the residual line of sight inertial acceleration. The residual inertial acceleration corresponds to the difference between the true field and the field model used to compute the orbit. If $\bar{C}^{\alpha}_{nm}$ (Model) is a coefficient of the model, and NM the maximum degree in that model, then the geopotential coefficients corresponding to the residual accelerations are $\Delta C^{\alpha}_{nm} = \bar{C}^{\alpha}_{nm} - \bar{C}^{\alpha}_{nm}(\text{Model})$ for $n \le NM$, and $\bar{C}^{\alpha}_{nm}$ for $NM < n < \infty$. The degree variances of the power spectrum are, therefore, $\delta^2_n = \sum_{m,\alpha} (C^{\alpha}_{nm} - C^{\alpha}_{nm}(\text{Model}))^2$ for $n \le NM$, and $\sigma^2_n = \sum_{m,\alpha} (C^{\alpha}_{nm})^2$ for $NM < n$. The δ^2_n are the degree variances corresponding to the errors in the reference field. As these errors are not known exactly, one must use some "likely numbers" instead, such as the formal error variances of the coefficients obtained during the adjustment of the reference model. This criterion has been adopted here, the error variances being those of the first NM degrees in Rapp's model, with NM = 20 . For $n > NM$, the σ_n^2 as implied by Rapp's coefficients have been used up to degree 100 . For $n > 100$ the σ_n^2 have been calculated with the two-term formula

$$\sigma^2 = \frac{a^2}{G^2M^2}(n-1)^{-1}[A1(n+A)^{-1}S1^{n+2} + A2(n+B)^{-1}(n-2)^{-1}S2^{n+2}] \tag{3.4}$$

where $a = 6371$ km, $\frac{GM}{a} = 982026.41$ mgal $A = 1$, $B = 2$, $A1 = 3.4050$, $A2 = 140.03$, $S1 = 0.998006$, and $S2 = 0.914232$. These parameter values were obtained by Rapp (1979b) by fitting the formula to the empirical variances of his 180, 180 model and other data. In summary:

For $2 \leq n \leq NM$, the error degree variances $\varepsilon^2{}_{nm}$ of Rapp's model;

for $NM < n \leq 100$, the degree variances of Rapp's model;

for $100 < n \leq 2000$, the σ_n^2 according to (3.4). The harmonic content of the geopotential, according to that formula, is negligible for $n > 2000$.

The error in geoid undulation due to the errors in the coefficients up to degree n has been calculated with the formula

$$\sigma^2 \varepsilon N_n = a^2 \sum_{k=2}^{n} \sigma^2 \varepsilon_k \tag{3.5}$$

If no coefficients above n are estimated, the total error must be

$$\sigma^2 \varepsilon N_T = \sigma^2 \varepsilon N_n + a^2 \sum_{k=n+1}^{2000} \sigma^2{}_k \tag{3.6}$$

(Compare to (2.76)).
Expression (3.6) depends on the "tail" of the spectrum containing the high frequency terms. The higher n , the least that is known about σ_n^2 , so this "tail" is the least reliable part of the spectrum model, and the total error calculated with (3.6) is the least credible among the results. Originally, when the least squares collocation results were obtained, the total error was not computed, though its calculation was added in the main program afterwards, when other results were found. For completness' sake, the punched output of that first run was read by an auxiliary program, which then produced the printout shown in Table 3.2, including the total error in the last column, and also the full listing of Table C.2 in Appendix C.A minor mistake in the auxiliary program resulted in values of the total error that are incorrect. To obtain the "true" values e_n , the listed values e'_n should be corrected according to the formulas

$$e_n = (e'^2_n + 0.1074)^{\frac{1}{2}} \tag{3.7}$$

3.2 Results According to Least Squares Adjustment

Table 3.1 shows the accuracies of potential coefficients and geoidal undulations estimated from SST data collected during a mission whose parameters were:

Circular, polar orbit,
satellite height: 160 k. [illegible],
intersatellite separation: 300 km,
accuracy of the data: $\sqrt{2} \times 10^{-6}$ m s^{-1},
averaging interval: 4 s ,
sampling interval: 4 s ,
length of the mission: 179 days ,
maximum degree and order in reference model: 20 .

The error analysis was carried out according to least squares adjustment theory, as put forward in paragraph 2.8. The first column shows the relative percentage error, which is the ratio defined by expression (3.2) multiplied by 100 . Notice that above $n = 270$ the error exceeds 100% consistently. The size of the the errors become so large that the total undulation error, which decreases steadily up to $n = 270$, according to the last column, begins to increase quite perceptibly once more. The last but one column shows the error up to n , according to expression (3.5). The second column contains the values of $\frac{GM}{R^2}^2 (\frac{a}{R})^{2n} \sigma^2 \varepsilon_n$, the variance per degree of the error in the <u>scaled</u> coefficients (see expression (2.70) in par. 2.11). Notice the very low percentage errors for $n < 100$, which are much the same as those shown in the next paragraph for least squares collocation.

As explained in the last part of paragraph 2.10, the error in the recovered coefficients may show local peaks at those degrees satisfying the condition

$$n\psi = 2\pi k \qquad (k = 1, 2, \ldots)$$

Since $\psi = 2\sin^{-1}(\frac{\rho}{2R})$, for a separation of 300 km the maxima should occur within the band $0 \leq n \leq 331$ at degrees $n = 136$ and $n = 273$, approximately, the next peak above the band being at $n = 410$. The listing in table 3.1 is too coarsely spaced in n to show these peaks, which are rather narrow features, each atop a broader rise that surrounds it, but they are quite clear in the detailed listing of Table C.1 in Appendix C.

The values that appear both in Tables 3.1 and in Table C.1 can be modified according to expression (2.79) in paragraph 2.13, to obtain the results corresponding to missions with different parameters. Values corresponding to undulation errors (last two columns) are in meters.

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	.34857D-01	.44945D-23	.13507D-04	2.4628
10	.62618D-02	.21568D-23	.39397D-04	2.4306
20	.82555D-02	.26011D-23	.50898D-04	2.3888
30	.31427D-02	.35568D-23	.62315D-04	1.9152
40	.53409D-02	.51921D-23	.75516D-04	1.6198
50	.73134D-02	.79167D-23	.91745D-04	1.4171
60	.11369D-01	.12498D-22	.11239D-03	1.2560
70	.19457D-01	.20388D-22	.13932D-03	1.1362
80	.27223D-01	.34489D-22	.17520D-03	1.0444
90	.43714D-01	.61148D-22	.22432D-03	.96483
100	.68586D-01	.11636D-21	.29424D-03	.90239
110	.10372	.25104D-21	.40090D-03	.84270
120	.19283	.71472D-21	.58953D-03	.79056
130	.59562	.56954D-20	.11284D-02	.74451
140	2.8004	.10641D-18	.72716D-02	.70351
150	2.8876	.96610D-19	.95861D-02	.66668
160	3.4877	.12142D-18	.11663D-01	.63338
170	4.3906	.16709D-18	.13958D-01	.60310
180	5.6244	.23973D-18	.16669D-01	.57544
190	7.2658	.35197D-18	.19969D-01	.55007
200	9.4254	.52398D-18	.24037D-01	.52674
210	12.249	.78685D-18	.29083D-01	.50525
220	15.928	.11883D-17	.35353D-01	.48546
230	20.715	.18026D-17	.43149D-01	.46731
240	26.962	.27491D-17	.52852D-01	.45079
250	35.325	.42629D-17	.65012D-01	.43604
260	48.103	.71641D-17	.80839D-01	.42344
270	143.98	.58341D-16	.11760	.41793
280	218.07	.12199D-15	.29200	.48503
290	224.30	.11793D-15	.36447	.52241
300	249.58	.13374D-15	.42921	.56145
310	278.35	.15271D-15	.49331	.60502
320	293.89	.15662D-15	.55521	.65062
330	211.29	.74611D-16	.60160	.68551

Table 3.1 Parameters: height 160 km, separation 300 km, $\sigma = \sqrt{2} \times 10^{-6} ms^{-1}$

Procedure : least squares adjustment

3.3 Results According to Least Squares Collocation

Table 3.2 shows the results corresponding to least squares collocation, the principle of which has been explained in paragraph 2.11. The mission parameters are the same as for Table 3.1. Comparing both tables one can see that the collocation accuracies are consistently better than those for least squares adjustment, particularly above degree $n = 210$. The results listed in the last column of this table, and also of Table C.2, Appendix C , have to be corrected according to expression (3.7), at the end of paragraph 3.1. When contrasting tables 3.1 and 3.2 one must bear in mind that the first corresponds purely to propagated noise, while the second contains hybrid variances: noise plus bias. The peaks in the error mentioned in the previous paragraph are much less noticeable, in fact the one at $n = 273$ has disappeared altogether (Table C.2), and only an "acceleration" in the increase of the error with n remains in the neighborhood of the missing peak. Nowhere the error exceeds 100%, also according to the theory.[1]

Table 3.3 lists the accuracies corresponding to a mission with the same parameters as in the preceeding tables, except that the accuracy of the data is now four times worse, i.e., $\sqrt{2} \times 4 \times 10^{-6} ms^{-1}$.

Table 3.4 corresponds to the same parameters as in Table 3.1, but the height has been changed to 220 km above the Earth.

Table 3.5 is for the same parameters as Table 3.4, with the data noise increased to $\sqrt{2} \times 4 \times 10^{-6} m\ s^{-1}$. The results for Tables 3.3 through 3.5 are clearly worse than for Table 3.2, as can be expected, because the data is noisier, the signal weaker (higher altitude), or both, compared to the case of Table 3.2.

Table 3.6 corresponds to the accuracies that would be achieved if all the coefficients in a given degree could be estimated with the same accuracy as the zonal harmonic. The results were computed with a modified version of the main program of Appendix B that sets up and inverts only that part of the normal matrix corresponding to the zonals. In this way, an approximate analysis can be carried out at much less cost than the complete studies of tables 3.1 through 3.5. The mission parameters in 3.6 are the same as in Tables 3.1 and 3.2. In Table 3.7 the separation between satellites has been changed to 150 m , and in Table 3.8, to 600 km. The results in Table 3.7 are clearly much worse than for Table 3.6, showing a particular deterioration at low degrees due to the differential nature of the SST measurements that tends to eliminate the low frequency information, so the percentage of the error in the estimated coefficients increases as the satellites become closer. Table 3.8 shows the smallest band error and the best accuracies, but these are quite irregular because the error peaks in this case are spaced closer than in the other cases, and are much more prominent, so that they can be noticed even with the coarse spacing in n used in the table.

From the results given in this paragraph one can conclude that the orbits should be as low as possible, and the separation between satellites as wide as allowed by natural limitations, the main among which is the need to keep the radar beam from entering too deep into the upper layers of the atmosphere at the mid point. Data noise, of course, should be as low as technically feasible.

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(1)The optimal estimator should not be worse than a null estimator (one that predicts only zeroes), whose error is always 100%.

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	.31895D-01	.37633D-23	.12359D-04	2.4409
10	.62495D-02	.21484D-23	.38899D-04	2.4084
20	.82514D-02	.25985D-23	.50502D-04	2.3662
30	.31419D-02	.35551D-23	.61987D-04	1.8870
40	.53401D-02	.51906D-23	.75243D-04	1.5863
50	.73125D-02	.79147D-23	.91518D-04	1.3787
60	.11368D-01	.12495D-22	.11221D-03	1.2126
70	.19454D-01	.20383D-22	.13916D-03	1.0880
80	.27219D-01	.34478D-22	.17507D-03	.99169
90	.43704D-01	.61120D-22	.22420D-03	.90751
100	.68558D-01	.11627D-21	.29412D-03	.84082
110	.10363	.25060D-21	.40074D-03	.77640
120	.19229	.71071D-21	.58903D-03	.71947
130	.58478	.54900D-20	.11213D-02	.66855
140	2.7821	.10502D-18	.69080D-02	.62256
150	2.8401	.93455D-19	.92654D-02	.58062
160	3.3757	.11375D-18	.11305D-01	.54205
170	4.1536	.14953D-18	.13472D-01	.50634
180	5.1600	.20177D-18	.15915D-01	.47304
190	6.4076	.27374D-18	.18724D-01	.44181
200	7.9146	.36947D-18	.21961D-01	.41234
210	9.7022	.49364D-18	.25671D-01	.38439
220	11.804	.65266D-18	.29896D-01	.35774
230	14.295	.85836D-18	.34685D-01	.33224
240	17.365	.11404D-17	.40145D-01	.30773
250	21.620	.15969D-17	.46571D-01	.28414
260	29.470	.26888D-17	.54997D-01	.26163
270	46.766	.61549D-17	.69239D-01	.24145
280	56.507	.81906D-17	.88813D-01	.22480
290	60.786	.86610D-17	.10640	.20990
300	65.941	.93360D-17	.12241	.19661
310	71.771	.10153D-16	.13770	.18524
320	77.665	.10937D-16	.15253	.17600
330	82.269	.11312D-16	.16684	.16884

Table 3.2 Parameters: height 160 km, separation 300 km, $\sigma = \sqrt{2} \times 10^{-6}$ m s^{-1}

Procedure: least squares collocation

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	.10858	.43616D-22	.22076D-04	2.4628
10	.24968D-01	.34291D-22	.15363D-03	2.4306
20	.32997D-01	.41555D-22	.20002D-03	2.3888
30	.12565D-01	.56860D-22	.24632D-03	1.9152
40	.21356D-01	.83016D-22	.29962D-03	1.6198
50	.29243D-01	.12657D-21	.36494D-03	1.4171
60	.45457D-01	.19980D-21	.44787D-03	1.2560
70	.77785D-01	.32586D-21	.55581D-03	1.1362
80	.10802	.55105D-21	.69954D-03	1.0444
90	.17468	.97638D-21	.89607D-03	.96484
100	.27388	.18556D-20	.11756D-02	.90239
110	.41348	.39899D-20	.16015D-02	.84270
120	.76434	.11229D-19	.23518D-02	.79056
130	2.2636	.82256D-19	.44394D-02	.74452
140	10.437	.14782D-17	.24124D-01	.70389
150	10.111	.11845D-17	.33025D-01	.66743
160	11.307	.12761D-17	.39847D-01	.63452
170	13.041	.14741D-17	.46345D-01	.60472
180	15.181	.17466D-17	.52965D-01	.57763
190	17.718	.20931D-17	.59913D-01	.55296
200	20.703	.25280D-17	.67334D-01	.53048
210	24.249	.30836D-17	.75370D-01	.51001
220	28.574	.38240D-17	.84215D-01	.49144
230	34.091	.48819D-17	.94183D-01	.47475
240	41.591	.65415D-17	.10584	.46002
250	52.191	.93055D-17	.12015	.44759
260	65.333	.13215D-16	.13810	.43799
270	76.705	.16558D-16	.15886	.43136
280	84.246	.18206D-16	.17993	.42704
290	89.282	.18685D-16	.19976	.42424
300	92.782	.18483D-16	.21786	.42245
310	95.214	.17869D-16	.23418	.42134
320	96.843	.17066D-16	.24880	.42066
330	97.660	.15940D-16	.26188	.42023

Table 3.3 Parameters: height 160 km, separation 300 km, $\sigma = \sqrt{2} \times 4 \times 10^{-6}$ m s^{-1}

Procedure: least squares collocation

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	.28512D-01	.30072D-23	.11048D-04	2.4628
10	.69646D-02	.26682D-23	.40892D-04	2.4306
20	.10086D-01	.38825D-23	.55701D-04	2.3888
30	.42086D-02	.63787D-23	.72456D-04	1.9152
40	.78378D-02	.11181D-22	.94257D-04	1.6198
50	.11760D-01	.20469D-22	.12409D-03	1.4171
60	.20029D-01	.38790D-22	.16598D-03	1.2560
70	.37555D-01	.75958D-22	.22586D-03	1.1362
80	.57567D-01	.15422D-21	.31297D-03	1.0444
90	.10126	.32811D-21	.44268D-03	.96484
100	.17398	.74880D-21	.64305D-03	.90239
110	.28786	.19338D-20	.97371D-03	.84270
120	.58331	.65401D-20	.16016D-02	.79056
130	1.8973	.57791D-19	.34638D-02	.74452
140	9.5593	.12399D-17	.21658D-01	.70381
150	9.9570	.11487D-17	.30510D-01	.66731
160	11.905	.14148D-17	.38096D-01	.63442
170	14.585	.18438D-17	.46010D-01	.60469
180	17.932	.24369D-17	.54706D-01	.57779
190	22.025	.32343D-17	.64446D-01	.55347
200	27.057	.43178D-17	.75489D-01	.53158
210	33.372	.58404D-17	.88184D-01	.51206
220	41.529	.80779D-17	.10306	.49502
230	52.233	.11461D-16	.12091	.48077
240	65.497	.16223D-16	.14249	.46981
250	78.797	.21211D-16	.16745	.46254
260	88.562	.24284D-16	.19345	.45846
270	94.191	.24968D-16	.21795	.45644
280	97.096	.24183D-16	.23978	.45549
290	98.470	.22728D-16	.25883	.45505
300	99.103	.21087D-16	.27542	.45482
310	99.563	.19539D-16	.28995	.45471
320	99.766	.18047D-16	.30275	.45465
330	99.628	.16589D-16	.31410	.45461

Table 3.4 Parameters: height 220 km, separation 300 km, $\sigma = \sqrt{2} \times 10^{-6} m\ s^{-1}$

Procedure: least squares collocation

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	.10510	.40858D-22	.40724D-04	2.4628
10	.27839D-01	.42631D-22	.16239D-03	2.4306
20	.40334D-01	.62090D-22	.22192D-03	2.3808
30	.16830D-01	.10201D-21	.28912D-03	1.9152
40	.31341D-01	.17879D-21	.3[illegible]645D-03	1.6198
50	.47018D-01	.32722D-21	.49586D-03	1.4171
60	.80076D-01	.62000D-21	.66343D-03	1.2560
70	.15013	.12138D-20	.90286D-03	1.1362
80	.23010	.24640D-20	.12511D-02	1.0444
90	.40468	.52404D-20	.17695D-02	.96484
100	.69510	.11952D-19	.25702D-02	.90240
110	1.1493	.30826D-19	.38909D-02	.84271
120	2.3236	.10378D-18	.63952D-02	.79058
130	7.3600	.86965D-18	.13693D-01	.74464
140	26.748	.97077D-17	.58429D-01	.70590
150	25.831	.77306D-17	.82246D-01	.67166
160	28.902	.83380D-17	.10006	.64113
170	33.715	.98525D-17	.11713	.61421
180	40.090	.12180D-16	.13498	.59082
190	48.164	.15466D-16	.15452	.57101
200	58.011	.19849D-16	.17638	.55497
210	69.164	.25086D-16	.20084	.54293
220	80.124	.30069D-16	.22733	.53489
230	88.820	.33138D-16	.25429	.53026
240	94.363	.33673D-16	.27986	.52796
250	97.346	.32373D-16	.30288	.52691
260	98.663	.30139D-16	.32310	.52646
270	99.319	.27761D-16	.34074	.52625
280	99.705	.25500D-16	.35617	.52616
290	99.833	.23362D-16	.36975	.52611
300	99.837	.21400D-16	.38178	.52608
310	99.944	.19688D-16	.39250	.52607
320	99.975	.18123D-16	.40210	.52606
330	99.905	.16681D-16	.41075	.52605

Table 3.5 Parameters: height 220 km, separation 300 km, $\sigma = \sqrt{2} \times 4 \times 10^{-6} m\ s^{-1}$
Procedure: least squares collocation

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	.54171D-01	.10855D-22	.20991D-04	2.4628
10	.64022D-02	.22547D-23	.37180D-04	2.4306
20	.74292D-02	.21065D-23	.47124D-04	2.3888
30	.27053D-02	.26356D-23	.56300D-04	1.9152
40	.45779D-02	.38145D-23	.66829D-04	1.6198
50	.64363D-02	.61318D-23	.80361D-04	1.4171
60	.10564D-01	.10790D-22	.99090D-04	1.2560
70	.19621D-01	.20735D-22	.12650D-03	1.1362
80	.30675D-01	.43789D-22	.16865D-03	1.0444
90	.56842D-01	.10339D-21	.23695D-03	.96483
100	.10694	.28290D-21	.35543D-03	.90239
110	.20396	.97078D-21	.58402D-03	.84270
120	.51907	.51788D-20	.11265D-02	.79056
130	2.7408	.12060D-18	.36277D-02	.74452
140	2.9581	.11075D-18	.36479D-01	.70442
150	1.3367	.20704D-19	.36701D-01	.66762
160	1.1748	.13776D-19	.36789D-01	.63434
170	1.2541	.13631D-19	.36863D-01	.60406
180	1.4831	.16669D-19	.36945D-01	.57638
190	1.8796	.23555D-19	.37055D-01	.55096
200	2.5173	.37375D-19	.37221D-01	.52751
210	3.5442	.65873D-19	.37500D-01	.50580
220	5.2505	.12912D-18	.38018D-01	.48567
230	8.2395	.28517D-18	.39085D-01	.46695
240	13.900	.73070D-18	.41549D-01	.44961
250	25.879	.22880D-17	.47978D-01	.43383
260	53.941	.90085D-17	.66372D-01	.42092
270	93.490	.24598D-16	.10766	.41525
280	86.313	.19110D-16	.14537	.41366
290	73.108	.12528D-16	.16482	.40895
300	70.181	.10575D-16	.17813	.40341
310	72.205	.10276D-16	.18953	.39826
320	76.689	.10664D-16	.20042	.39397
330	79.894	.10668D-16	.21113	.39063

<u>Table 3.6</u> Parameters: height 160 km, separation 300 km, $\sigma = \sqrt{2} \times 10^{-6}$ m s^{-1}

Procedure: least squares collocation

(Based on zonals only)

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	50.911	.95881D-17	.19728D-01	2.4629
10	12.197	.81828D-17	.58531D-01	2.4313
20	14.001	.74810D-17	.80666D-01	2.3901
30	4.9094	.86798D-17	.98980D-01	1.9178
40	7.7272	.10868D-16	.11735	1.6241
50	9.8631	.14399D-16	.13753	1.4237
60	14.255	.19649D-16	.16079	1.2663
70	22.340	.26879D-16	.18808	1.1517
80	28.336	.37365D-16	.22018	1.0674
90	39.298	.49417D-16	.25759	.99863
100	50.572	.63265D-16	.29932	.95074
110	58.704	.80422D-16	.34581	.91089
120	69.095	.91766D-16	.39360	.88312
130	77.970	.97599D-16	.44008	.86485
140	84.898	.97801D-16	.48322	.85345
150	89.952	.93749D-16	.52192	.84662
160	93.472	.87213D-16	.55592	.84267
170	95.848	.79627D-16	.58547	.84042
180	97.413	.71912D-16	.61104	.83918
190	98.417	.64578D-16	.63317	.83850
200	99.048	.57864D-16	.65237	.83813
210	99.434	.51849D-16	.66911	.83794
220	99.667	.46526D-16	.68377	.83784
230	99.806	.41843D-16	.69668	.83778
240	99.887	.37731D-16	.70810	.83775
250	99.935	.34118D-16	.71827	.83774
260	99.962	.30938D-16	.72734	.83773
270	99.978	.28130D-16	.73549	.83773
280	99.987	.25645D-16	.74283	.83773
290	99.993	.23437D-16	.74947	.83773
300	99.996	.21469D-16	.75550	.83772
310	99.998	.19709D-16	.76098	.83772
320	99.999	.18132D-16	.76598	.83772
330	99.999	.16713D-16	.77057	.83772

Table 3.7 Parameters: height 160 km, separation 150 meters $\sigma = \sqrt{2} \times 10^{-6} m\ s^{-1}$

Procedure: least squares collocation

(Based on zonals only)

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	.30368D-01	.34114D-23	.11767D-04	2.4628
10	.33459D-02	.61581D-24	.19914D-04	2.4306
20	.42366D-02	.68501D-24	.25397D-04	2.3888
30	.18197D-02	.11925D-23	.31853D-04	1.9152
40	.40211D-02	.29430D-23	.42868D-04	1.6198
50	.88181D-02	.11509D-22	.66534D-04	1.4171
60	.37230D-01	.13402D-21	.15760D-03	1.2560
70	.14423	.11204D-20	.15647D-02	1.1362
80	.49807D-01	.11545D-21	.15976D-02	1.0444
90	.53511D-01	.91628D-22	.16095D-02	.96484
100	.73830D-01	.13484D-21	.16230D-02	.90239
110	.11669	.31774D-21	.16482D-02	.84270
120	.26668	.13670D-20	.17282D-02	.79056
130	1.3372	.28708D-19	.24448D-02	.74451
140	1.4756	.29546D-19	.17653D-01	.70369
150	.70764	.58019D-20	.17772D-01	.66685
160	.69968	.48868D-20	.17829D-01	.63352
170	.91017	.71802D-20	.17895D-01	.60320
180	1.4794	.16586D-19	.18021D-01	.57548
190	3.2642	.71040D-19	.18433D-01	.55002
200	19.503	.22434D-17	.23663D-01	.52672
210	12.322	.79628D-18	.67846D-01	.50895
220	7.1526	.23962D-18	.68937D-01	.48906
230	7.2425	.22034D-18	.69573D-01	.47049
240	9.4383	.33687D-18	.70359D-01	.45318
250	15.256	.79516D-18	.71892D-01	.43712
260	32.677	.33060D-17	.76786D-01	.42268
270	86.509	.21062D-16	.10196	.41381
280	67.191	.11581D-16	.13517	.41019
290	52.194	.63856D-17	.14657	.40194
300	53.844	.62248D-17	.15478	.39365
310	63.954	.80618D-17	.16381	.38669
320	79.075	.11338D-16	.17553	.38192
330	91.945	.14129D-16	.19000	.37962

Table 3.8 Parameters: height 160 km, separation 600 km, $\sigma = \sqrt{2} \times 10^{-6} m\ s^{-1}$

Procedure: least squares collocation

(Based on zonals only)

Finally, the results listed in Table 3.9 below show the sensitivity of the relative accuracies of the estimated coefficients to the choice of power spectrum model. The degree variances ε_n^2 of subroutine NVAR, described in paragraph 3.1 and in appendix B, have been replaced by those obtained according to a two-term model of the form described by expression (3.4), but with the following parameters' values:

$A = 100$, $B = 20$, $A1 = 18.3906$, $A2 = 658.6132$, $S1 = 0.9943667$,

$S2 = 0.908949$.

These values correspond to a model obtained by Jekeli ("2L" model, Report No. 275, Dept. of Geodetic Science, The Ohio State University, Columbus, Ohio, 1978). The values in the first column are the same as in Table 3.6; those in the second column correspond to the "2L" spectrum. As in the case of Tables 3.6 to 3.8, these values correspond to the <u>zonals</u> *only*.

Table 3.9

Relative Accuracies with Two Different Spectral Models

n	As in Table 3.6 (%)	Jekeli's "2L" (%)
2	$.54171 \times 10^{-1}$	$.55870 \times 10^{-1}$
20	$.74292 \times 10^{-2}$	$.74343 \times 10^{-2}$
40	.45779 "	.36486 "
60	$.10564 \times 10^{-1}$	.87091 "
80	.30675 "	$.20738 \times 10^{-1}$
100	.10694	.72841 "
120	.51907	.382197
140	2.9581	2.1529
160	1.1748	.86360
180	1.4831	1.1068
200	2.5173	1.9143
220	5.2505	4.0815
240	13.900	11.106
260	53.941	46.794
280	86.313	82.380
300	70.181	65.245
320	76.689	73.190
330	76.327	79.894

The ε_n^2 up to $n = NM = 20$ are the same for both columns (subroutime MODEL in appendix B). Clearly, while the change in spectrum does make a difference, the changes have little influence on the calculated accuracies.

3.4 Accuracies of Different Harmonics of the Same Degree

Having come across the argument that SST data collected in a polar orbit should be most sensitive to the zonals, as all their variation occurs in the N-S direction, and least sensitive to the sectorials, with the tesserals falling somewhere in between, and the difference from zonals to sectorials being quite large, the author included in the main program statements to list the accuracies for the cosine terms ($\alpha = 0$) of all potential coefficients of degree $2 \le n \le 40$. The accuracies for $n = 30$ are listed below. The mission parameters are those for Table 3.1, and the principle used is that of least squares adjustment. The accuracies correspond to dimensionless, scaled coefficients (par. 2.3).

Table 3.10

Accuracies of potential coefficients of degree $n = 30$

m	$\sigma\varepsilon\hat{C}_{nm}$
0	2.08×10^{-13}
1	1.46 "
2	1.50 "
3	1.51 "
4	1.53 "
5	1.58 "
6	1.61 "
7	1.68 "
8	1.71 "
9	1.78 "
10	1.83 "
11	1.90 "
12	1.97 "
13	2.03 "
14	2.11 "
15	2.17 "
16	2.27 "
17	2.33 "
18	2.44 "
19	2.49 "
20	2.62 "
21	2.67 "
22	2.72 "
23	2.87 "
24	3.05 "
25	3.09 "
26	3.31 "
27	3.35 "
28	3.60 "
29	3.68 "
30	2.82 "

The values above are typical of those listed for other degrees in the interval $2 \le n \le 40$. While there are fluctuations, the sensitivity of the adjustment to the various harmonic terms of degree $n = 30$ does not change very much.

4. Validity of the Results

The numerical results of the previous section have been obtained under the simplifying assumptions of paragraph 2.1. Among those assumptions, numbers (1) to (4) define an orbital geometry more regular than what can be found in reality, and assumption (11) disregards all sources of error other than SST data errors. Of the remaining ones, (9) and (10) have been explained already in the first section of this report, while (5) to (8) merely describe what an ideally successful mission would produce in terms of data. Assumption (3) defines an idealized Earth rotation, where there are no fluctuations in the angular velocity Ω due to such causes as tidal friction, redistribution of atmospheric masses, lack of rigidity of the Earth, etc., and no changes in the inertial orientation of the spin axis due to precession-nutation and polar wandering. Changes in Ω are of two kinds: short term, due to solid Earth-atmosphere interaction, etc., altogether probably too small to matter, and secular, due to tidal effects caused by the Sun and the Moon, also very small and quite predictable from long records of observations. Polar motion is also well modelled from very long series of observations. The main effect of changes in Earth rotation is in the calculation of the satellites' orbits, as orbital errors affect also the accuracy of the results. Because of the existence of good models, this effect is probably negligible in the present context. So this section is going to consider only the consequences of assumptions (1), (2), (4), and (11) on the credibility of the results of the error analysis. The basic argument is that the actual data set, with its complex three-dimensional distribution, can be reduced or transformed into another with the geometry implied by the assumptions and with the same information content as the original. The analysis of this transformed data set can be done, then, in the manner explained in section 2, the accuracy of the estimated coefficients being much the same as that shown in the previous section.

4.1 The Geometry of the Real Orbit

The departure of the gravitational field from that of a central point mass causes the orbit to take a shape that is not exactly circular, however carefully the satellites are manouvered into it. Most of the departure from cicularity is due to the part of the anomalous field that is already known from the study of terrestrial and spacecraft data, and the major portion of this known departure is caused by the second zonal, which is almost three orders of magnitude larger than any other harmonic. This zonal represents most of the effect of the Earth's equatorial bulge on the geopotential. The result is a wavy motion of the satellites, which alternatively run above and below the mean sphere of radius R (average orbital radius), gain or loose ground in the along-track direction with respect to a perfectly uniform circular motion, and also move with respect to each other because of their different positions along the (more or less) common orbit, so the orientation of the line of sight is not always perpendicular to the radial direction, but fluctuates about, and there are also variations in the intersatellite distance. The non-zonal terms of the harmonic expansion introduce further irregularities, particularly in the across-track direction, so the orbit does not lie in any given plane except approximately. The effect of the non-zonals is, however, of the order of meters, while that of the second zonal amounts to several kilometers. The discussion that follows suggests that errors of a few meters have negligible consequence, so only the departures due to oblatness need to be considered. If all measurements had been taken on a sphere of radius R, the line of sight being always perpendicular to the radial direction, and the separation between satellites constant, but the actual positions of the spacecrafts had been displaced in latitude and longitude from the ideal, periodical, regularly spaced pattern implied by the assumptions, one could transform this "semi-perfect" set of observations into a "perfect" set by interpolating horizontally the data from their actual positions to their ideal ones. As far as the signal content is concerned, this interpolation can be exact, because the information is band-limited, in the sense of paragraph 1.2, and the sampling is very dense. That such interpolation, to be exact, would probably require the use of all data values to create each interpolated value is of no consequence here, as this discussion is concerned only with the the existence of valid transformation procedures, regardless of whether they are practical or not. The practical aspects of the matter are left for section 5. The only problem would be that the interpolated noise would not be exactly uncorrelated and of constant variance, its departure from those ideal qualities depending on how much the true groundtrack departs from the ideal one. Here one can only assume that, if this departure is not too large, neither would be the change in the nature of the noise too large. At this stage of the argument it will be assumed that the actual positions of the satellites can be known with negligible errors from orbital calculations, the effect of orbital error being treated later on in paragraph 4.3, where assumption (11) is discussed. As explained there, the model adopted for the line of sight velocity implies that such errors can have a very small effect on the estimated coefficients as long as they do not exceed a few meters, an accuracy achievable with existing orbit determination techniques.

The transformation of the actual set of observations into a set of pseudo-observations lying on the mean sphere along the ideal orbit can be carried out in two steps:

(a) a vertical reduction of the original observations to the mean sphere, where an intermediate set consisting of pseudo-obs. with the following characteristics is created: the "midpoint" between "satellites" lies directly below the true midpoint, the "line of sight" is oriented North-South and perpendicular to the radial direction, the "satellites" are at the same distance from each other in all the pseudo obs; in other words: the intermediate pseudo obs. are identical to ideal observations in everything except their arrangement on the sphere;

(b) a horizontal interpolation using the intermediate data set to form the final set of pseudo-obs. at regularly spaced points along the ideal orbit.

As already explained, the horizontal interpolation can be carried out, in principle, exactly, because of the band-limited nature of the gravitational signal. The main problem, therefore, is the vertical reduction or step (a). This step involves a downward or upward continuation of the signal, depending on where the satellites happen to be with respect to the sphere, and a correction for the change in their relative position, which results in a variable intersatellite distance and line of sight direction. Both the continuation and this correction can be considered *together*, as an overall operation.

4.2 Vertical Reduction to the Mean Sphere

The discussion can be simplified by considering observations of line of sight relative acceleration, $\hat{a}_{12}$, rather than line of sight velocity, v_{12} . The signal being band-limited, it is possible to differentiate the velocity exactly by computing the Fourier coefficients of the data over the whole mission (assuming the orbit to be nearly periodical), and using these coefficients to calculate the acceleration, since the acceleration coefficients are related to the velocity ones by the simple relationship

$$\begin{Bmatrix} a_n \\ b_n \\ (\text{accel.}) \end{Bmatrix} = n\omega \begin{Bmatrix} a_n \\ b_n \\ (\text{veloc}). \end{Bmatrix}$$

The N_p - vector of acceleration values $\underline{\alpha}$ can be obtained, formally, by multiplying the N_p - vector of velocity values by a $N_p x N_p$ "differentiator" matrix S :

$$\underline{\alpha} = S\ \bar{\underline{v}}_{12} \tag{4.1}$$

The model for the accelerations is, therefore,

$$\underline{\alpha} = S\ A\ \underline{c} \tag{4.2}$$

and the "observed" accelerations, which consist of differentiated signal and noise, are

$$\underline{\alpha}_{(obs)} = S\ \bar{\underline{v}}_{12(obs)} = S\ A\ \underline{c} + S\ \underline{n} \tag{4.3}$$

The least squares adjustment estimator is, therefore (paragraph (2.8))

$$\hat{\underline{c}} = (A^T\ S^T(D')^{-1}S\ A)^{-1}A^T\ S^T\ (D')^{-1}S\ \bar{v}_{12(obs)} \tag{4.4}$$

where D' is the variance-covariance matrix of the "differentiated" noise

$$\begin{aligned} D' &= E\ \{(S\underline{n})(S\underline{n})^T\} = S\ E\ \{\underline{n}\underline{n}^T\}\ S^T \\ &= S\ D\ S^T \end{aligned} \tag{4.5}$$

Replacing (4.5) in (4.4)

$$\hat{c} = (A^T S^T (S^t)^T D^{-1} S^t S A)^{-1} A S^T (S^t)^T D^{-1} S^t S \, \bar{\underline{v}}_{12(obs)}$$

$$= (A^T D^{-1} A)^{-1} A D^{-1} \, \bar{\underline{v}}_{12(obs)} \qquad (4.6)$$

where "S^t" indicates "The pseudo inverse of S " ($S^t S A = A$) , so the least squares adjustment estimates of the coefficients based on the differentiated data are identical to those obtained from the data directly. The errors in the estimates being the same, any conclusions derived for accelerations are identical to the corresponding conclusions for velocities. The same principle applies to least squares collocation, as it can be seen by replacing $(A^T S^T (D')^{-1} S A)$ with $(A^T S^T (D')^{-1} S A + C^{-1})$ in the reasoning.

Figure 4.1 below shows the true position of the satellites, S_1 and S_2 , and their "pseudo-positions" on the mean sphere S_1' and S_2' (the "prima" symbols indicate the counterparts of real elements on the mean sphere). The relative positions shown have been chosen for pure convenience of representation: the satellites can be both above, both below, or on either side of the surface of the sphere.

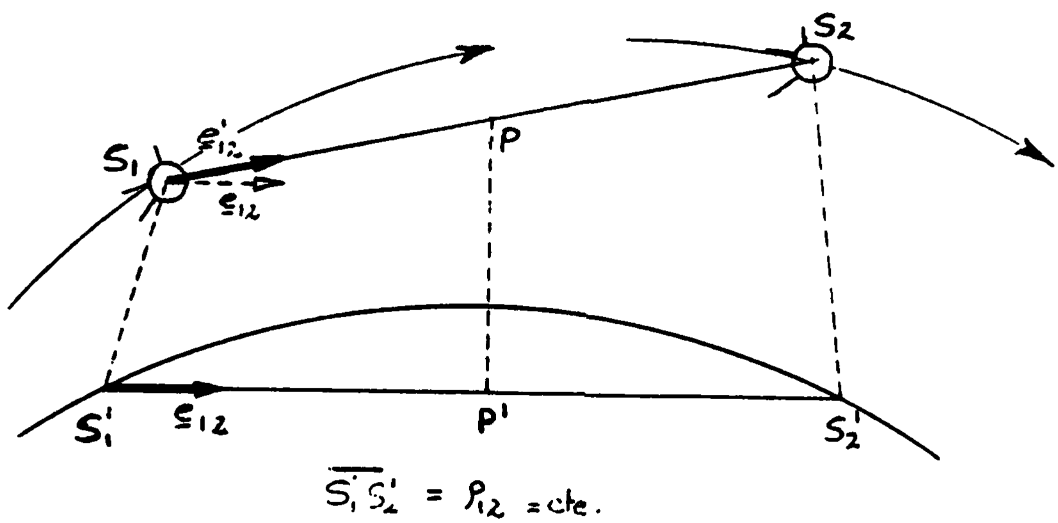

Figure 4.1: True Positions in Orbit and Pseudo-Positions on the Mean Sphere

Let

$$\delta\underline{\hat{a}}(Q',Q) = \underline{\hat{a}}(Q') - \underline{\hat{a}}(Q) = \begin{Bmatrix} \underline{r}_0' \; \hat{a}_r(r',\phi',\lambda') - \underline{r}_0 \; \hat{a}_r(r,\phi,\lambda) \\ \underline{\phi}_0' \; \hat{a}_\phi(r',\phi',\lambda') - \underline{\phi}_0 \; \hat{a}_\phi(r,\phi,\lambda) \\ \underline{\lambda}_0' \; \hat{a}_\lambda(r',\phi',\lambda') - \underline{\lambda}_0 \; \hat{a}_\lambda(r,\phi,\lambda) \end{Bmatrix} \tag{4.7}$$

be the variation in inertial acceleration between points Q and Q' where the r and ϕ components have been modified by the removal of a contribution from the zonals (paragraph 1.3). As it happens, this contribution comes only into a_r and a_ϕ , a_λ is free from this zero frequency problem. Let

$$\underline{\varepsilon} = \underline{e}_{12}' - \underline{e}_{12}$$

be the difference between the unit vector in the direction of the actual line of sight, $\underline{e}_{12}$, and the unit vector $\underline{e}_{12}'$ for the South-North perpendicular to the radial direction at the "pseudo-midpoint" P' . The difference between the true line of sight value and the corresponding "pseudo-observation" on the sphere is then

$$\begin{aligned} \delta\hat{a}_{12} &= \hat{a}_{12}' - \hat{a}_{12} \\ &= \underline{e}_{12}'^T(\underline{\hat{a}}(S_1') - \underline{\hat{a}}(S_2')) - \underline{e}_{12}^T(\underline{\hat{a}}(S_1) - \hat{a}(S_2)) \\ &= \underline{e}_{12}'^T \; \delta\underline{\hat{a}}(S_1',S_2') - (\underline{e}_{12}' - \underline{\varepsilon})^T \; \delta\underline{\hat{a}}(S_1, S_2) \\ &= \underline{e}_{12}'^T(\delta\underline{\hat{a}}(S_1', S_2') - \delta\underline{\hat{a}}(S_1, S_2)) + \underline{\varepsilon} \; \delta\underline{\hat{a}}(S_1, S_2) \end{aligned} \tag{4.8}$$

Let d_{max} be the maximum possible distance between a satellite and the corresponding "pseudo-satellite" position on the sphere. The value of d_{max} will depend on the radial and horizontal components of the separation vector, but one could well argue that a purely radial separation should have much the same effect as the total displacement if both have the same size, because the lateral fluctuations are nearly one order of magnitude smaller than the radial one (appendix A). Consider the first term of (4.8). The contribution of the nth harmonic to this term is

$$\underline{e}_{12}'^T \; (\delta\underline{\hat{a}}_n(S_1', S_2') - \delta\underline{\hat{a}}_n \; (S_1, S_2))$$

where

$$\delta\underline{\hat{a}}_n(S_1, S_2) = \begin{Bmatrix} \hat{a}_{rn}(S_1) \; \underline{r}_0^{(1)} - \hat{a}_{rn}(S_2) \; \underline{r}_0^{(2)} \\ \hat{a}_{\phi n}(S_1) \; \underline{\phi}_0^{(1)} - \hat{a}_{\phi n}(S_2) \; \underline{\phi}_0^{(2)} \\ \hat{a}_{\lambda n}(s_1) \; \underline{\lambda}_0^{(1)} - \hat{a}_{\lambda n}(S_2) \; \underline{\lambda}_0^{(2)} \end{Bmatrix} = \begin{Bmatrix} \delta\hat{a}_{rn} \; (S_1, S_2) \\ \delta\hat{a}_{\phi n} \; (S_1, S_2) \\ \delta\hat{a}_{\lambda n} \; (S_1, S_2) \end{Bmatrix} \tag{4.9}$$

$\hat{a}_{rn}$, $\hat{a}_{\phi n}$, $\hat{a}_{\lambda n}$ being the sum of all terms of degree n in the expansions of $\hat{a}_r$, $\hat{a}_\phi$, $\hat{a}_\lambda$ (expressions (1.8, a-c)). The variation of $\hat{a}_{rn}$ with radial distance is

$$\hat{a}_{rn}(R,\phi,\lambda) - \hat{a}_{rn}(R + h,\phi,\lambda) \cong (1 - (\frac{R}{R+h})^{n+2}) \;\; \hat{a}_{rn}(R,\phi,\lambda) \tag{4.10}$$

and similarly for $\hat{a}_{\phi n}$, and $\hat{a}_{\lambda n}$. As both satellites follow the same orbit with a separation that is small compared to the wavelength of the second zonal and its disturbances, one can regard them, in first approximation, as moving up and down simultaneously, so their heights are the same. In such case

$$\delta\hat{a}_{rn}(S_1', S_2') - \delta\hat{a}_{rn}(S_1, S_2) \cong \left(1 - \left(\frac{R}{R+h}\right)^{n+2}\right) \delta\hat{a}_{rn}(S_1', S_2')$$

and similarly for the other components of $\underline{\hat{a}}_n$, so

$$\delta\underline{\hat{a}}_n(S_1', S_2') - \delta\underline{\hat{a}}_n(S_1, S_2) \cong \left(1 - \left(\frac{R}{R+h}\right)^{n+2}\right) \delta\underline{\hat{a}}_n(S_1', S_2') \tag{4.11}$$

The magnitude of the first term of (4.8) is, therefore,

$$\underline{e}_{12}'^{T}(\delta\underline{\hat{a}}_n(S_1', S_2') - \delta\underline{\hat{a}}_n(S_1, S_2)) \leq ||\underline{e}_{12}'|| \left(1 - \left(\frac{R}{R+h}\right)^{n+2}\right) ||\delta\underline{\hat{a}}_n(S_1', S_2')||$$

As $||\underline{e}_{12}'|| = 1$ and $||\delta\underline{\hat{a}}_n(S_1', S_2')|| \leq ||\delta\underline{\hat{a}}_n||_{max}$ (where $||\delta\underline{\hat{a}}_n||_{max}$ is the maximum size of the magnitude of $\underline{a}_n$ on the mean sphere), and the segment $\overline{S'S'}$ has constant length, it follows that

$$\underline{e}_{12}'^{T}(\delta\underline{\hat{a}}_n(S_1', S_2') - \delta\underline{\hat{a}}_n(S_1, S_2)) \leq \left(1 - \left(\frac{R}{R+h}\right)^{n+2}\right) ||\delta\underline{\hat{a}}_n(S_1', S_2')||_{max} \tag{4.12}$$

The difference between $\delta\underline{\hat{a}}(S_1, S_2)$ and $\delta\underline{\hat{a}}(S_1', S_2')$ is not very large, so the second term in (4.8) is, approximately,

$$\underline{\varepsilon}^T \delta\underline{\hat{a}}(S_1, S_2) \cong \underline{\varepsilon}^T \delta\underline{\hat{a}}(S_1', S_2')$$

The maximum value is

$$\underline{\varepsilon}^T \delta\underline{\hat{a}}(S_1', S_2') \leq ||\underline{\varepsilon}||_{max} ||\delta\underline{\hat{a}}||_{max}$$

The magnitude of $||\underline{\varepsilon}||_{max}$ is, to a first approximation (disregarding the curvature of the Earth), function of the relative displacement of the satellites:

$$||\underline{\varepsilon}||_{max} \cong \frac{\sqrt{\Delta r_{12}^2 + \Delta\rho_{12}^2 + \Delta\tau_{12}^2}}{\rho_{12}} \tag{4.13}$$

where Δr_{12} is radial, $\Delta\rho_{12}$ along the line of sight, and $\Delta\tau_{12}$ across-track. According to Appendix A, the largest displacements due to the anamalous field are: $\Delta\rho_{12max} = 0.7$ km, $\Delta r_{12max} = 0.7$ km, $\Delta\tau_{12max} = 0.4$ km, $\Delta r_{max} = 3.9$ km (vertical displacement of each satellite). Adding 1 km to $\Delta\rho_{12max}$, Δr_{12max} and $\Delta\tau_{12max}$, to introduce an extra error margin, the value of $||\underline{\varepsilon}||_{max}$ according to (4.13) is

$$||\varepsilon||_{max} \cong 0.009$$

The contribution of the nth harmonic to the second term of (4.8), therefore, is

$$\underline{\varepsilon}^T \delta\underline{\hat{a}}_n(S_1', S_2') \leq 0.009 \, ||\delta\underline{\hat{a}}_n||_{max} \tag{4.14}$$

From (4.8), (4.12), and (4.14) one gets, finally,

$$\delta\hat{a}_{12} \leq [(1 - (\frac{R}{R+h})^{n+2}) + 0.009 \quad ||\delta\hat{\underline{a}}_n||_{max} \tag{4.15}$$

as an upper bound for $\delta\hat{a}_{12}$. The maximum value of h is d_{max} , as explained earlier. This maximum absolute displacement of any of the two satellites is

$$d_{max} \cong \sqrt{(\Delta r_{max} + \Delta r_{12\,max})^2 + \Delta\rho_{12}^2 + \Delta\tau_{12\,max}^2} \tag{4.16}$$

Replacing $(\Delta r_{max} + \Delta r_{12\,max})$, $\Delta\rho_{12\,max}$, $\Delta\tau_{12\,max}$ (1 km added to each) in (4.16) with their values according to appendix A :

$$d_{max} \cong 4.7 \text{ km}$$

and the upper bound for $\delta\hat{a}_{12}$ becomes

$$\delta\hat{a}_{12_n} \leq [1.009 - (0.99928087)^{n+2}] \; ||\delta\hat{\underline{a}}_n||_{max}$$

$$= p_n \; ||\delta\hat{\underline{a}}_n||_{max} \tag{4.17}$$

This expression limits the difference between the actual line of sight acceleration along the non-circular orbit, and the corresponding pseudo-observation on the sphere. This is the largest possible value (if the argument behind (4.17) can be accepted) of the correction that has to be introduced in order to transform the original data set into the corresponding set of "accelerations" on the mean sphere. The factor p_n is

$$p_n = [1.009 + (0.99928087)^{n+2}] \tag{4.18}$$

If no correction is applied to the accelerations, and they are analyzed as if they were already on the mean sphere and on the ideal orbit, the error in the estimated coefficients will have two components: (1) that due to the data error, which has been listed in the previous section; (2) the error due to the fact that the uncorrected accelerations are not on the ideal orbit. As both errors have quite independent sources, the total rms error per degree is

$$\sigma\epsilon_n = [\sigma\epsilon_{n(1)}^2 + \sigma\epsilon_{n(2)}^2]^{\frac{1}{2}} \tag{4.19}$$

where $\sigma\epsilon_{n(1)}$ corresponds to the first, and $\sigma\epsilon_{n(2)}$ to the second source of error. The relative error per coefficient is, then

$$\rho_{n(T)} = \frac{\sigma\epsilon_n(2n+1)^{-1}}{\sigma_n(2n+1)^{-1}} = ((\frac{\sigma\epsilon_{n(1)}}{\sigma_n})^2 + (\frac{\sigma_{n(2)}}{\sigma_n})^2)^{\frac{1}{2}} = (\rho_{n(1)}^2 + \rho_{n(2)}^2)^{\frac{1}{2}} \tag{4.20}$$

where $\rho_{n(1)}$ is the propagated noise listed in Tables 3.1 or 3.2 of section 3. If the only difference between the real and the ideal orbits were an increase in radial distance, or expansion, so $R - R' = d_{max}$ were constant, then it would follow from (4.8), (4.10), and (4.17) that

$$\delta\hat{a}_{12_n} = p_n\,\hat{a}_{12_n} \tag{4.21}$$

as $\bar{e}_{12} = \bar{e}'_{12}$ and $\delta\hat{a}_{12n}(S_1, S_2) = (\frac{R}{R+d_{max}})^{n+2}\,\delta\hat{a}_{12n}(S'_1, S'_2)$ at all points along the orbit, so

$$\rho_{n(2)} = \frac{\sigma\varepsilon_{n(2)}}{\sigma_n} = |\delta\hat{a}_{12n}|_{max}\,||\delta\hat{a}_n||^{-1}_{max} = p_n \tag{4.22}$$

In reality $\delta\hat{a}_{12n}$ becomes modulated by the changes in distance between true and ideal orbit, as this distance cannot be constant. Consequently expression (4.21) is no longer true, but merely an approximation to a very complicated relationship. The error should be less than with a constant radial increase d_{max} at almost every point, so perhaps the use of the right hand side of (4.22) as an <u>upper bound</u> for $\rho_{n(2)}$ is a realistic choice: $\rho_{n(2)} \le p_n$.

Let the coefficients estimated from the original set of accelerations $\hat{a}_{12}^{(0)}$ be called $\hat{C}^{\alpha}_{nm}{}^{(0)}$, and let $\hat{a}_{12}^{(1)} = \hat{a}_{12}^{(0)} + \delta\hat{a}_{12}^{(0)}$ be the original accelerations corrected with values $\delta\hat{a}_{12}^{(0)}$ of $\delta\hat{a}_{12}$ computed using the $\hat{C}^{\alpha}_{nm}{}^{(0)}$. Let $\hat{C}^{\alpha}_{nm}{}^{(1)}$ be the coefficients obtained from the analysis of the $\hat{a}_{12}^{(1)}$, and let $\delta\hat{a}_{12}^{(1)}$ be a new estimate of $\delta\hat{a}_{12}$ based on the $\hat{C}^{\alpha}_{nm}{}^{(1)}$. Correcting the original $\hat{a}_{12}^{(0)}$ with the $\delta\hat{a}_{12}^{(1)}$, and then analyzing again, one obtains $\hat{C}^{\alpha}_{nm}{}^{(2)}$. If $\hat{a}_{12}^{(2)} = \hat{a}_{12}^{(0)} + \delta\hat{a}_{12}^{(1)}$ is closer to the transformed set of pseudo-obs $\hat{a}_{12}^{(0)} + \delta\hat{a}_{12}$ (that is the subject of this paragraph) than the $\hat{a}_{12}^{(1)}$, then $\hat{C}_{nm}{}^{(2)}$ should be an improvement on $\hat{C}_{nm}{}^{(1)}$, $\delta\hat{a}_{12}^{(2)}$ on $\delta\hat{a}_{12}^{(1)}$, so $\rho_{n(2)}^{(2)} < \rho_{n(2)}^{(1)}$. Let $\hat{C}^{\alpha}_{nm}{}^{(M)}$, $\delta\hat{a}_{12}^{(M)}$, $\rho_n^{(M)}{}_{(2)}$, etc. be the outcome of $M+1$ iterations of this analysis-correction-analysis procedure. As $M \to \infty$ $\hat{C}^{\alpha}_{nm}{}^{(M)}$ may converge to some limit $\hat{C}^{\alpha\infty}_{nm}$ and $\rho_{n(2)}^{(M)}$ to $\rho^{\infty}_{n(2)}$, where $\rho^{\infty}_{n(2)} < \rho_{n(2)}^{(M)}$ because $\hat{C}^{\alpha\infty}_{nm}$ is better than $\hat{C}_{nm}{}^{(M)}$. As p_n is the ratio of $\delta\hat{a}_{12n}$ to $\hat{a}_{12n}^{(0)}$, if the relative error in the $\hat{C}^{\alpha}_{nm}{}^{(M-1)}$, and thus in the $\delta\hat{a}_{12}^{(M-1)}$, is $\rho_{n(T)}^{(M-1)}$, the error ratio for the corrected $\hat{a}_{12}^{(M)} = \hat{a}_{12}^{(0)} + \delta\hat{a}_{12}^{(M-1)}$ and thus for the $\hat{C}^{\alpha}_{nm}{}^{M}$ must be[1]

$$\rho_n^{(M)} \le (\rho^2_{n(1)} + p^2_n\,\rho^2_{n(T)}{}^{(M-1)})^{\frac{1}{2}}$$

or

$$\rho_n^{2\,(M)} \le \rho^2_{n(1)} + p^2_n(p^2_n\,\rho^2_{n(T)}{}^{(M-2)} + \rho^2_{n(1)}) \tag{4.23}$$

so

$$\rho_n^{2\,(M)} \le q^2_n\,\rho^{2\,(M-2)}_{n(T)} + q_n\,\rho^2_{n(1)} \le q^3_n\,\rho^{2\,(M-3)}_{(T)} + q^2_n\,\rho^2_{n(1)} + q_n\,\rho^2_{n(1)} + \rho^2_{n(1)}$$

where $q_n = p^2_n$. Repeating this reasoning backwards $M-1$ times:

$$\begin{aligned}\rho^{2\,(M)}_{n(T)} &\le q_n{}^{(M-1)}\,(\rho^2_{n(1)} + q_n) + q_n{}^{(M-2)}\,\rho^2_{n(1)} + \dots + q_n\,\rho^2_{n(1)} + \rho^2_{n(1)} \\ &= q_n{}^{M} + \rho^2_{n(1)}\,[q_n{}^{(M-1)} + q_n{}^{(M-2)} + \dots + q_n + 1] \\ &= q_n{}^{M} + \rho^2_{n(1)}\,\frac{1 - q_n^M}{(1 - q_n)}\end{aligned} \tag{4.24}$$

---

[1] The noise S_n in the corrected accelerations $\hat{a}_{12}^{(M)} = \hat{a}_{12}^{(0)} + \delta\hat{a}_{12}^{(M-1)}$ is always the same, so $\rho_{n(1)}$ is independent of M.

so

$$\rho^{\infty}_{n(T)} = \lim_{M\to\infty} \rho_{n(T)}^{(M)} \leq \frac{\rho_{n(1)}}{(1-q_n)^{\frac{1}{2}}} = \frac{\rho_{n(1)}}{(1-\rho_n^2)^{\frac{1}{2}}} \qquad \underline{\text{if } \rho_n < 1}$$

Examining (4.18) one can see that for $n = 331$ (top of the band), is

$$\rho_n \cong 0.22$$

As both ρ_n and $\rho_{n(1)}$ fall with decreasing n, for all n in the band is

$$\rho_{n(T)}^{(M)} < \rho_{331(T)}^{(M)} \quad \text{and} \quad \rho_n < \rho_{331} < 1$$

Therefore it is true that

$$\rho^{\infty}_{n(T)} \leq \frac{\rho_{n(1)}}{(1-\rho_n^2)^{\frac{1}{2}}} \tag{4.25}$$

The values of $\rho^{\infty}_{n(T)}$ and of $\rho_{n(T)}^{(4)}$ have been listed below, together with the corresponding values of the error $\rho_{n(1)}$ given in Table 3.2 and derived under the simplifying assumptions. The closeness between the "true" and the "ideal" relative errors is clear. Moreover, the iterative procedure appears to converge very quickly, as $\rho^{\infty}_{n(T)}$ and $\rho_{n(T)}^{(4)}$ agree to three decimal places.

n	$\rho_{n(1)} \times 100$	$\rho^{\infty}_{n(T)} \times 100$	$\rho_{n(T)}^{(4)} \times 100$
10	0.0063	0.0063	0.0063
100	0.0686	0.0688	0.0689
150	2.8401	2.8582	2.8583
200	7.9146	7.9982	7.9987
250	21.620	21.9581	21.9583
300	65.941	67.3613	67.3614
331	82.269	84.3747	84.3748

It must be remembered that, on the one hand, the reasoning leading to (4.25) is by no means rigorous, while, on the other hand, the assumptions made in it have been mostly on the conservative side. It appears that a series of iterations as described here should transform the original set of observations into a set of "accelerations" on the ideal orbit, in such way that the coefficients estimated from these transformed set would have much the same accuracies as those listed in section 3 and derived under the simplifying assumptions of paragraph 2.1. In this sense, the results of section 3 are supported by those above; they may very well indicate the <u>maximum amount of information</u> on the geopotential that can be extracted by linear estimation techniques from a <u>real</u> set of SST data.

4.3 The Effect of Errors in the Calculated Orbits

To apply the corrections $\delta a_{12}^{(M)}$ to the $\dot{a}_{12}^{(0)}$ one must know the position of the two satellites, S_1 and S_2. In the previous paragraph it was assumed that this knowledge was exact. In reality, however, the gravity field model used to compute the orbits, the coordinates assigned to the tracking stations, and the tracking data used to calculate the orbits, all contain errors, so there is a discrepancy between computed and true orbits. With present tracking, coordinates, and models, the errors in the calculated orbits are not likely to exceed a few meters. The direct effect of these errors is an additional error in the corrections $\delta\hat{a}_{12}^{(M)}$, as the accelerations $a_{12}^{(0)} + \delta\hat{a}_{12}^{(M)}$ are "dropped" from their true positions to the calculated ones. The change in $\delta a_{12_n}^{(M)}$ for a displacement of Δs meters is likely to have a size comparable to the effect of a purely vertical displacement of the same magnitude. From (4.11) follows that such change is

$$\Delta\delta a_{12_n}^{(M)} = \left(1 - \left(\frac{R}{R+\Delta s}\right)^{n+2}\right)\delta\ddot{a}_{12_n}^{(M)} \tag{4.26}$$

or, to a very good first approximation

$$\begin{aligned}\Delta a_{12_n}^{(M)} &= -(n+2)\,\frac{\Delta s}{R}\;\delta\hat{a}_{12_n}^{(M)}\\ &= \beta_n(\Delta s)\;\delta\hat{a}_{12}^{(M)}\end{aligned} \tag{4.27}$$

Even for $n = 331$, at the very top of the band, the relative error β_n caused by an orbital error of 10 m is insignificant: $\beta_{331}(10) = 5 \times 10^{-5}$. This error decreases with n, as indicated by (4.27), so it is smallest at $n = 0$. However, the size of the zero harmonic is so large that its net change over 10 m amounts to some 3 mgal. If no correction term for this and the other even zonals is introduced in the definition of the derivative of the line of sight velocity, as suggested in section 1, then the effect of a vertical displacement is

$$\beta_0(\Delta s)\;\delta\hat{a}_{12_0}^{(m)} = -2\sin\frac{\psi}{2}\;\frac{\partial}{\partial r}\;\frac{GM}{r^2}\;\Delta S \tag{4.28}$$

or 0.138 mgal every 10 m. Because the orbital error is not constant, this change in the zero harmonic constribution is modulated by the complex shape of the error, so the power of the zero harmonic change spills over the whole spectrum corrupting all the estimated coefficients. As the power associated with $n > 100$, in mgal, is well below 0.138 mgal at satellite altitude, this can have serious consequences. In the model adopted here for a_{12} the zero harmonic has been excluded, so only the errors due to the other terms in the expansion, all of them quite negligible, are present.

5. Data Processing

The objective of this section is to include some thoughts on how to process the masses of information collected from a SST experiment into a global geopotential model of very fine resolution. This problem goes beyond the scope of this study, but some further elaboration of concepts introduced in previous sections may help clarify this difficult subject, and perhaps point out directions for future research.

5.1 An Iterative Approach

The argument used in the previous section to substantiate the numerical results of section 3 suggests that a model may be obtained through successive approximations. To be able to use the sort of "analysis-correction-analysis" approach hinted at in that argument, it is necessary to have certain regularities in the data that, though physically possible, may not occur in practice. On practical grounds one can question two assumptions made implicitly in last section: that the departures from perfectly circular orbits were due to the anomalous gravity field alone, and that the data were sampled uninterruptedly at constant intervals during the whole mission, all measurement errors having the same standard deviation.

Even when the compensating mechanism could eliminate all non-gravitational forces, and when gravitational fields (other than the Earth's) and the effect of the body-tide could be calculated exactly from existing models and thus discounted, the relative positions of the satellites are not going to vary as predicted from the action of the anomalous field alone. The reason for this is that it is impossible to determine and control exactly the state of each satellite at "injection" time, to make sure that both move along the same orbit, and that such orbit is as close to circular as the field would allow. As a consequence, besides the relative motion caused by the field there will be a "drift" due to incorrect initial conditions. This drift will follow a more or less arbitrary direction, resulting in the spacecrafts moving towards or away from each other until their separation and the direction of the line of sight change so much that observations taken at different times cannot be described sufficiently well by equations that assume a fixed distance and angle. Since the total relative motion can be calculated from the reference orbits with an accuracy of a few meters, the change in relative configuration with time should soon become apparent. This change should have a more or less periodic component due to the irregular gravitational field, and a trend due to previous orbital manouvers' errors. While these errors cannot be determined accurately, the drift itself increases with time until it becomes easy to estimate. The fact that the satellites are tracking each other continuously, in addition to being tracked from terrestrial stations, should help to obtain a good estimate of the drift and the drift-rate. When the drift reaches a maximum allowed value, the controlling rockets of either spacecraft can be fired briefly to reverse the drifting motion, so the drift begins to decrease (the reversal need not be exact). This scheme is a "dead zone" control policy where correcting action is applied only when a certain limit is about to be exceeded. As the purpose is to reverse the drift rate rather than to eliminate the drift altogether, the correction need not be particulary drastic. Even so, the use of such a scheme must increase the amount of fuel required during the mission, and thus the overall cost. Of course, some correcting maneuvers of the sort described here will have to take place from time to time, to stop the two spacecraft from separating too much (so the radar beam does not cut too deeply inside the atmosphere, or is intercepted by the curvature of the Earth itself, at a separation of about 2700 km) or from becoming too close, as the shorter their distance, the weaker the signal and the less accurate the results, as shown in section 3, Tables 3.6, 3.7, and 3.8. The question is how often such maneuvers will take place, not whether they shall be needed at all. Compensatory changes in the design of the mission could allow the use of "dead zone" control to maintain relative orientation without raising the cost too much. It may be possible to choose a higher orbit, where less fuel for drag compensation is needed (the decrease in fuel requirement with height should be quite fast) so the balance can be used for maneuvering. The length of the mission could be increased, the accuracy of the data improved, or both, to compensate for the weaker signal at greater height. All of these factors, some contradictory, and others, as well, can only be balanced properly in a thorough mission design study incorporating the demands of data processing among the main questions to be considered. Given the magnitude of such demands, some regard for them appears natural.

Another assumption made in section 4 that may not be fulfilled in practice is the existence of uninterrupted series of measurements lasting the whole mission. Breaks will occur in the data, some much too large to be ignored or "patched up" by interpolation from surrounding data. There may be fluctuations, as well, in the quality of those measurements due to problems in the satellites themselves, or in the surrounding medium, as in the case of severe ionospheric perturbations. So the stream of data will not by perfectly uniform in quality (variations in the standard deviation of the errors) or unbroken. These departures from the assumptions, if severe enough, would make a close application of the ideas presented in earlier sections quite impossible. One could begin, as an alternative, by differentiating numerically the data to obtain line of sight accelerations. Supposing that the model of section 1 is correct, these accelerations on a sphere have the form

$$a_{12}(R, \phi', \lambda) = \sum_{\alpha=0}^{1} \sum_{n=0}^{N} \sum_{m=0}^{n} \tilde{C}^{\alpha}_{nm}\, g_{nm}(\phi') \left\{ {\cos \atop \sin} \right\} m\lambda \tag{5.1}$$

where, according to expression (2.9) and (2.15,a-b)

$$g_{nm}(\phi') = (n+1)\left[\bar{P}_{nm}(\phi' - \frac{\psi}{2}) + \bar{P}_{nm}(\phi' + \frac{\psi}{2})\right] \sin\frac{\psi}{2} +$$

$$+ \frac{d}{d\phi}\left[\bar{P}_{nm}(\phi' - \frac{\psi}{2}) - \bar{P}_{nm} + (\phi' + \frac{\psi}{2})\right] \cos\frac{\psi}{2}$$

and

$$\tilde{C}^{\alpha}_{nm} = \frac{GM}{R^2} \left(\frac{a}{r}\right)^n \bar{C}^{\alpha}_{nm}$$

Functions that can be expanded in series of the type of (5.1), which includes the ordinary spherical harmonic expansion, can be analyzed to obtain the values of the C^{α}_{nm} by very efficient methods resembling numerical quadratures. The derivation and implementation of optimal methods that minimize the estimation error in the presence of unhomogeneous and and correlated noise have been discussed by Colombo (1981, paragraphs 2.9

and 2.12). While the procedures presented by this author in that work have been derived for the case

$$g_{nm}(\phi') = \bar{P}_{nm}(\phi')$$

(i.e., spherical harmonics) the extension to the more general case of (5.1) is quite direct. The only requirement is that the data occupy the nodes of a regular grid where the separation between meridians is constant. This can be done by interpolating the differentiated velocities on those nodes from surrounding data points, while also determining the correlations and standard deviations of the interpolated values from those of the observations, as these are needed to set up the optimal estimator. If no reliable interpolation is possible on some node, because of large data breaks in the vicinity, a value of zero with a "standard deviation" equal to the rms of the line of sight acceleration could be used instead.

By setting up an appropriate control mechanism to maintain the relative configuration of the satellites within sufficiently close limits, and by using the interpolated accelerations as pointed out above, the interative scheme could proceed basically along the lines of section 4, except that the data would be analyzed as in Colombo (1981), rather than as in section 2. If such a control scheme proves to be feasible, the next important question is how to calculate the corrections δa_{12} of expression (4.8) from the C^{α}_{nm} estimated in the previous iteration, to refine the pseudo-observations on the mean sphere. Because of the irregular shape of the orbit, calculation of these corrections using exact relationships is too laborious to be practical, even with very powerful computers. The main reason is the number of operations needed to obtain every $\delta\hat{a}_{12}$, which is proportional to the number of coefficients (some 11 x 10^4 if N = 331). This matter needs thorough investigation, but the answer should involve some sort of approximation, to reduce the computer burden. One possibility that may be worth exploring is as follows: consider a spherical shell extending 5 km above and 5 km below the mean orbital sphere. The whole orbit, according to Appendix A, should lie within this shell. Imagine the shell subdivided according to a regular grid, equal angular for instance, where the blocks are y^0 x y^0 in size, so the shell is partitioned into cells each y^0 x y^0 x 10 km in volume (the size of y should be decided by detailed study). The vertices of the cells are arranged in equal angular fashion, so one can compute the three inertial acceleration components at every vertex according to expressions (1.8, a-c) very efficiently using, for example, algorithms like those described by Colombo (1981). To calculate the correction δa_{12} one must know the value of a_{12} in the actual orbit and in the ideal orbit, and for that the three accelerations at points S_1 and S_2, S_1' and S_2' (in Fig. 4.1) are needed. These accelerations can be computed from their values at the vertices of the cell containing those points by some interpolatory procedure, which should be a great deal easier than an exact calculation.

The type of approximate calculation of the acceleration components just described may be used, as well, to obtain the forcing function due to gravitation needed to <u>integrate numerically</u> satellite orbits with a very high degree and order field model, like the one whose determination is being discussed here. Such calculations may be necessary, for example, to reduce the orbital errors at each step of the iterative modelling procedure.

5.2 Other Methods

Instead of converting the data into pseudoobservations on the mean sphere by successive iterations, this could be done directly by estimating the pseudoobs. in one step from neighbouring data points. This means carrying out a series of local reductions that finally covers the whole sphere with a regular grid of estimated values. These values should correspond to a variable that can be described by an expansion of the type of (5.1), and which does not have to be the line of sight acceleration of equation (2.24). These local reductions can be done in many different ways. One possibility is to use least squares collocation in a local manner. A problem with local solutions by collocation is that the signal in SST measurements tends to be too strongly correlated over considerable distances, and this causes the variance-covariance matrix to be too ill-conditioned to be inverted in a dependable way. According to R. Rummel (private communication) a minimum separation of about 100 km between data points is necessary for stable inversion. As the actual points are likely to be spaced some 30 km apart along-track (with a sampling interval of 4 s) and by less than that across-track, (because the separation between adjacent passes should be of only a few kilometers by the end of a six-month's mission) some kind of paring, or decimation, of the data will be needed to achieve the larger spacing. This could be done in the first of two steps: beginning with decimated data, a set of pseudoobservations could be obtained without stability problems, and a first estimate of the harmonic coefficients could be made from this set. In the second step, all the data could be used after substracting from them their nominal values according to the model produced in step one, which could have high degree terms already. Such residuals are going to be less correlated than the original measurents, because much of their low frequency content would have been removed, so the inversion of their variance-covariance matrix may be stable in spite of the close spacing of the data points.

If no control of the relative alignment of the satellites takes place and they are allowed to drift freely with respect to each other, the corresponding observation equations would lack the regular structure assumed in section 2. In such a case the ideas for analysing the data discussed in the previous paragraph are not applicable, and only non-iterative methods like the one just outlined seem to offer any real hope.

Any method that first creates a regularly spaced set of pseudoobservations on the mean sphere and then analyses it efficiently to obtain the potential coefficients requires two main things:

(a) a model for the data that is both practical and accurate. An example of this may be the model adopted in sections 1 and 2, but this idea needs further validation, as pointed out at the end of paragraph 1.3;

(b) an efficient and accurate way of reducing the measurements in the actual orbit to pseudoobservations on the mean sphere.

Both problems require more research to clarify them, and this clarification may be absolutely necessary before developing an effective technique for processing SST data.

5.3 The Use of Local Solutions

As mentioned in the introduction, there have been studies of the SST problem that have concentrated on purely local solutions that use data taken from inside a relatively small neighborhood of the points at the Earth's surface where certain quantities associated with the gravitational field are estimated. The advantage of the global approach is that every value that is estimated is obtained from the processing of all the data available, so it can be more accurate than a local solution based only on part of the data. Global solutions may be free from the numerical instabilities that tend to be associated with local solutions. Global solutions have also some important limitations. A spherical harmonics model, for instance, must be truncated at some finite degree for practical reasons and this limits the size of the finest detail that the model can represent. Even if the number of coefficients is not a problem, some very fine but also quite strong features that may be sensed by the satellite pair, such as anomalies along ocean ridges and trenches, mountain ranges, etc., are likely to be smoothed out by an optimal global estimation procedure because, on a global scale, they are similar to measurement noise. Such small but marked features may be recovered by using local estimation methods, applied in the knowledge that sharp field variations may occur in certain areas. These methods may use residual SST data, with respect to the global solution, to ensure numerical stability and the removal of trends of non-local nature.

Local modelling should be regarded, therefore, as complementary to global analysis, because its careful application in selected areas may push the level of resolution to the very limits allowed by the information contained in SST data. Local solutions may permit also the combination of SST data with terrestrial measurements of gravity, with satellite altimetry over the oceans, and with knowledge of the geological structures that may be responsible for some of the high frequency content of the signal, all of which may be difficult to incorporate into a global solution.

6. Conclusions

According to the theory given in sections 1 and 2 , assuming that the power spectrum of the geopotential is as described in paragraph 3.1, and that the reasoning of section 4 is valid, the results presented in section 3 can be summarized as follows: with two satellites in nearly the same circular, polar orbit, at a height of 160 km, 300 km apart, with a tracking accuracy of $\sqrt{2} \times 10^{-6}$ m s^{-1}, a sampling period of 4 s , an averaging period of 4 s , and a mission length of six months, the coefficients of the spherical harmonic expansion of the potential up to degree $n = 331$, may be estimated to the following entent:

(1) the relative accuracy of the potential coefficients could be better than 1% for $n \leq 130$, than 10% for $n \leq 210$, and than 50% for $n \leq 270$, using least squares collocation. With least squares adjustment, the results may be the same up to $n = 200$, and for higher degrees collocation may work better.

(2) the accuracy of the geoid undulation implied by the coefficients could be better than 0.05 mm rms for wavelengths of between 3000 km and 40030 km, and better than 10 cm rms in the band between 140 km and 3000 km.

The method of analysis of section 2 can be used for studying missions where the orbital plane is oblique to the equator. This would require some changes in the programs listed in Appendix B , subroutine ONEREV in particular, because those are written for the case of polar orbits only. As they are, they can be used to carry out an analysis of the magnitude reported here in any modern computer with some 1.5 megabytes of core. If the highest degree studied is $N = 331$, the central processor unit time required is of the order of one hour. The effect on the results of changes in some of the mission parameters can be studied, in the case of least squares adjustment, using the results of section 3 and the simple formulas of paragraph 2.13. Only potential coefficients and geoid undulations' accuracies have been calculated; the accuracies of other quantities, gravity anomalies, for example, can be obtained in a simple way from the coefficients' accuracies, a complete listing of which, degree by degree, appears in Appendix C .

The method of section 3 allows for a spherical, rotating Earth, and for data consisting of velocity averages. A number of simplifying assumptions listed in paragraph 2.1 are substantiated in sections 1 and 4, in an attempt to show that the results listed in section 3 represent the limit of accuracy for a global model obtainable from real SST data if the mission could be carried out without a single fault.

The accuracies listed being global, the undulation errors are likely to be worse in some areas, perhaps where the field is strongly anomalous, such as mid-ocean ridges, ocean trenches, mountain ranges, etc., and necessarily better than average over the remainder of the planet. In those areas where the model may perform poorly, local solutions (using the model as a reference field to calculate the residuals of satellite and terrestrial data, and these residuals, in turn, as the observed values) may provide the finer detail that a global technique alone cannot reveal.

As argued in section 5 , an iterative solution based on the ideas of sections 2 and 4 may be used for the actual analysis of SST data. Alternatively, non-iterative methods could be developed for that purpose. All such methods should have in common the fact that they rely on some convenient aproximation, as rigorous solutions such as those based on the "numerical" or the "analytical" (i.e., celestial mechanics) approaches are virtually impossible to implement, given the enormous number of harmonic coefficients to be adjusted and of observations. Two essential tasks to be accomplished through further research before a satisfactory technique can be found are, in all likelihood, the following:

(a) validation of the model of the line of sight relative acceleration used in sections 1 and 2 , or its replacement by another that provides a better approximation and is just as tractable mathematically;

(b) development of a satisfactory procedure for reducing the SST measurements to a set of pseudoobservations regularly distributed over the mean orbital sphere, whose analysis can then be carried out with efficient algorithms.

Further work on how to model the SST signal should also help to clarify the question of the influence of orbital errors on the estimated quantities, influence that, according to the model adopted here, may be small.

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Appendix A: Orbital Perturbations

A spacecraft orbiting a planet cannot follow a perfectly circular orbit, as assumed in section 2, but must move in a more irregular course, because of anomalies, or fluctuations, in the actual gravity field compared to that of a central point mass. In the case of the Earth, most of this effect is due to the attraction of the equatorial bulge. In the spherical harmonic expansion of the anomalous field, the influence of this bulge is represented mostly by the second zonal. This term is much larger than all the others in the expansion, so most of the orbital perturbation is due to this term alone. The following calculations will take into account the second zonal in detail, and for the rest a displacement of 200 m in every direction will probably account well enough, provided no strong resonances occur, as assumed in paragraphs 2.1 and 2.6. For relative displacements between satellites, 400 m will be added to those caused by the second zonal alone. Another assumption made here is that the force corresponding to the second zonal is the same along the true orbit as along the ideal, mean circular orbit. As the total displacement of such craft is mostly vertical and of less than 5 km, it is easy to determine that the force of the second zonal cannot change by more than 0.3%, so it may be acceptable to regard it as having the same value along either orbit. As the rotation of the Earth has no effect on the force of a purely zonal field, it will be ignored. The motion in the orbital plane which is perpendicular to the equator, is supposed to be periodical, which is mathematically possible with the right initial conditions.

Consider the system of inertial coordinates with axes $\vec{x}$ and $\vec{y}$, where $\vec{y}$ coincides with the polar figure axis of the Earth, and $\vec{x}$ is in the equator (figure A.1). The polar coordinates r and ϕ' correspond to a point moving along the orbit with an approximately uniform circular motion represented by

$$\phi' = [\omega t]_{\text{MODULE } 2\pi}$$

The relationships between the components of the inertial acceleration in the (x, y) and the (r, ϕ') systems are

$$a_x = a_r \cos\phi' - a_\phi \sin\phi' \qquad \text{(A.1,a)}$$

$$a_y = a_r \sin\phi' + a_\phi \cos\phi' \qquad \text{(A.1,b)}$$

or

$$a_x(t) = a_r(t) \cos\omega t - a_\phi(t) \sin\omega t \qquad \text{(A.2,a)}$$

$$a_y(t) = a_r(t) \sin\omega t + a_\phi(t) \cos\omega t \qquad \text{(A.2,b)}$$

From the definition of the unnormalized Legendre function introduced in paragraph 2.2, the polar components of the acceleration are

$$a_r(t) = -3 \frac{GM}{r^2} \left(\frac{a}{r}\right)^2 J_2 L_{nm}(\omega t) \qquad \text{(A.3,a)}$$

$$= 3 \frac{GM}{r^4} a^2 J_2 \left(\frac{3}{4} \cos 2\omega t - \frac{1}{4}\right)$$

$$a_\phi(t) = \frac{GM}{r^2}\left(\frac{a}{r}\right)^2 J_2 \frac{dL_{nm}(\omega t)}{d(\omega t)} \tag{A.3,b}$$
$$= \frac{3}{2}\frac{GM}{r^4} a^2 J_2 \sin 2\omega t$$

where $J_2 = C_{20}$ (unnormalized)

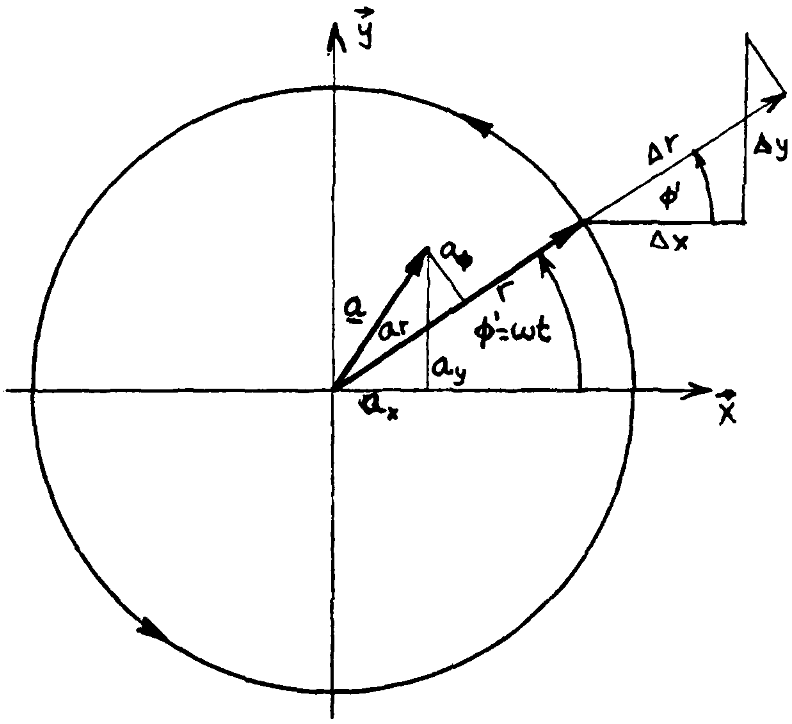


Figure A.1: Acceleration and Position in the (x,y) and (r_1 $\phi^1 = \omega t$) Coordinate Systems.

The radial perturbation in the position of the satellite is

$$\Delta r(t) = \Delta x(t) \cos\omega t + \Delta y(t) \sin\omega t$$
$$= \left[\int_0^t dt' \int_0^{t'} a_x(t'')\, dt'' + \Delta\dot{x}_0 t + \Delta x_0 \right] \cos\omega t$$
$$+ \left[\int_0^t dt' \int_0^{t'} a_y(t'')\, dt'' + \Delta\dot{y}_0 t + \Delta y_0 \right] \sin\omega t \tag{A.4}$$

where $\Delta\dot{x}$, Δx_0 etc., are initial conditions. As $\Delta r(t)$ is supposed to be periodical with the right Δx_0, $\Delta\dot{x}_0$, Δy, Δy_0, and ignoring a constant term that merely changes the size of the orbit by a few kilometers, the radial displacement is

$$\Delta r(t) = \frac{-39}{72} \frac{GM}{r^4} \frac{a^2}{\omega^2} J_2 \cos 2\omega t \tag{A.5}$$

according to (A.2,a-b), (A.3,a-b), and (A.4). Furthermore $\omega = \sqrt{\frac{GM}{r^3}}$, so

$$\Delta r(t) = \frac{-39}{72} \frac{a^2}{r} J_2 \cos 2\omega t \tag{A.6}$$

and using the values

$$a = 6371 \text{ km}$$

$$r = 6531 \text{ km}$$

$$J_2 = 1.1 \times 10^{-3}$$

(A.6) becomes

$$\Delta r(t) = -3.7 \cos 2\omega t \quad [\text{km}] \tag{A.7}$$

So the amplitude of the oscillation due to J_2 alone is

$$\Delta r_{MAX} = 3.7 \text{ km} \tag{A.8}$$

The relative radial displacement between two satellites, assuming that the change in their distance, and thus in ψ, can be ignored, is

$$\Delta r_{12}(t) = -3.7 (\cos 2\omega t - \cos 2(\omega t + \psi))$$

$$= -3.7 \times 2 \sin\psi \sin(2\omega t + \psi)$$

$$= 0.33 \sin(2\omega t + \psi) \tag{A.9}$$

for an intersatellite distance $\rho = 300$ km, so

$$\Delta r_{12MAX} = 0.33 \text{ km} \tag{A.10}$$

The relative motion along the line of sight can be estimated by integrating the relative velocity v_{12}. As this velocity is nearly a sinewave in time, of frequency 2ω, the expression for the variation in ρ is

$$\Delta\rho_{12}(t) = \sqrt{2} \frac{v_{12}(rms)}{2\omega} \cos(2\omega t + \beta) \tag{A.11}$$

where β is some phase angle of no consequence here. Replacing $v_{12}(rms)$ with its total value according to Table 1.1, and adopting $\omega = 1.2 \times 10^{-3}$ rad s^{-1} (as $r \cong 6531$ km),

$$\Delta\rho_{12MAX} = 0.3 \text{ km} \tag{A.12}$$

Because of the force is due to a zonal, the across-track displacement is

$$\Delta\tau_{12}(t) = 0 \quad \text{for all } t. \tag{A.13}$$

The absolute along-track displacement of each satellite can be obtained from an expression similar to (A.7), but it is not needed in section 4. Finally, adding 200 m or 400 m, as the case may be, to account for the rest of the anomalous field, the perturbations amount, approximately, to

$$\Delta r_{MAX} = 3.7 + 0.2 = 3.9 \text{ km}$$

$$\Delta r_{12_{MAX}} = 0.3 + (2 \times 0.2) = 0.7 \text{ km}$$

$$\Delta \rho_{12_{MAX}} = 0.3 + (2 \times 0,2) = 0.7 \text{ km}$$

$$\Delta \tau_{12_{MAX}} = 0.0 + (2 \times 0.2) = 0.4 \text{ km}$$

Appendix B: Computer Programs

This Appendix contains descriptions and listings of the main programs and subroutines used for the error analysis whose theory is contained in sections 1 and 2 and the results of which appear in section 3.

B.1 Main Program

Two main program versions were developed: the first one creates the non-zero diagonal blocks of $(A^T D^{-1} A)$, stores them in unit 10 (a tape file), creates the normal matrix's blocks by adding C^{-1} or not, depending on whether least squares collocation or least squares adjustment is required, inverts the blocks, and uses the diagonal elements of the inverses to calculate the error degree variances, the relative error per degree, and the corresponding accuracy of the geoid undulation; the second version of the main program does not create the normals but reads them from tape (unit 10, once more) and then proceeds as before. The reason for having two versions is that, with the first, one creates the normals and then carries out the error analysis using, say, least squares adjustment theory; as the normals are stored without C^{-1}, one can then use the second program to add C^{-1} to the stored matrix and then carry out the analysis according to collocation without having to recompute unnecessarily $(A^T D^{-1} A)$, which is the most time-consuming part of the analysis.

(a) Full Version

This program both creates and inverts the diagonal blocks of the normal required by the analysis. It call subroutines ONEREV, MATV, MODEL, NVAR, and WRIT, which are listed in this Appendix; it also calls the system-provided subroutines ERRSET, SCLOK1, and RCLOK1, and the subroutines GGNOR (from the single precision library produced by the International Mathematical and Statistical Libraries Inc. (IMSL) company), and DSINV from the double precision library of the IBM System/360 Scientific Subroutine Package.

The program contains a device against a possible underestimation of the total execution time requested in the JOB card. The running time is checked inside the loop where the diagonal blocks are created, so that if this time is a few seconds below the assigned running time before a new block is to be calculated, the run terminates in an orderly way. The blocks created so far are left safely in unit 10, and the remaining ones can be created in a later run where the minimum order of the blocks is set to MMIN = (order of last (even, odd) pair of blocks completed previously + 1). The time for eventual premature exit is the value of parameter YLIM in seconds, and should be smaller than the time declared in the JOB card by at least 10 seconds + compile time. Blocks created in additional runs should be stored in new tapes, which can be combined with the initial one by means of a simple FORTRAN program to produce a tape with all blocks in the required sequence (increasing order). The running time is checked with the help of SCLOK1 (which sets the time counter to zero before the main do loop) and RCLOK1 that "reads" the time counter at every turn in the loop.

All calculations are carried out in double precision, with the exception of those related to subroutine GGNOR, that take place in single precision. For this reason all variables are declared REAL 8 , except array TP which is REAL 4 .

The only COMMON statement is "P", through which the value of the total geoid undulation power POWNMX is communicated to the program from subroutine NVAR, which defines the values of the degree variances according to the model explained in section 3.

The various arrays are somewhat overdimensioned for the needs of the actual analysis, when N = 331 . The maximum degree N is defined by the parameter NMAX which must be an odd number, for reasons given in the description of subroutine ONEREV (if the value declared happens to be even, by mistake, the program adds 1 to make it odd). The small arrays (dimensioned 500) should be at least of dimension NMAXP = NMAX + 1 . As for the large arrays, their minimum dimensions should be

GPNMS : (NMAX - MMIN + 2) x (NMAXP/2+1)

AE : $\frac{1}{8}$ x (NMAX + 2)^2

AO : "

GPNMS contains the Fourier coefficients of the columns of the A matrix, calculated by subroutine ONEREV, and AE, AO contain the elements of the normal matrix corresponding to a given order m and (n-m) <u>even</u> and <u>odd</u>, respectively. Storage is in "upper-diagonal form" as required by the inversion subroutine DSINV.

Subroutine ERRSET is used to suppress unwanted error messages due to underflows in the calculations carried out by ONEREV.

The run parameters, including most of the mission parameters, are assigned values between statements 15 and 31. Two other parameters, the acceleration of gravity at ground level and the mean Earth radius, are assigned values in statements 50 and 51. The meanings of the symbolic names given to the parameters are explained in the comments of this section of the program. Parameter IAC, for example, tells the program whether a least squares adjustment or a least squares collocation error analysis is requested. The main parameter values are listed by the program (statements 36 and 37 if plotter output, as well as printer output, is desired) at the beginning of the run and, together with some other important values, saved at the start of the file created in unit 10, where the diagonal blocks are also stored. The length of day is upposed to be 24 x 3600 seconds exactly.

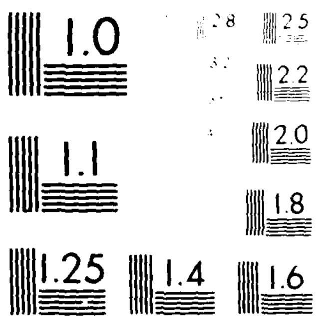
Certain loaders initialize all array areas to zero before execution, as in the case of the loader used to run this program. Others set arrays and registers not declared in the program to "indeterminate", or else the values left in core by a previous job remain unchanged. In such cases the program does not work unless additional DATA statements are written to give all array elements an initial value of zero.

AD-A111 872 OHIO STATE UNIV COLUMBUS DEPT OF GEODETIC SCIENCE A--ETC F/G 8/5
GLOBAL GEOPOTENTIAL MODELLING FROM SATELLITE-TO-SATELLITE TRACK--ETC(U)
OCT 81 O L COLOMBO F19628-79-C-0027
UNCLASSIFIED 317 AFGL-TR-81-0319 NL

2 OF 2

AD A
111872

END
DATE
FILMED
04-82
DTIC



MICROCOPY RESOLUTION TEST CHART

The run begins with the computation of sin $(\frac{\psi}{2})$, $\cos(\frac{\psi}{2})$, and other quantities to be explained later, as well as the quantities $p \sin p \frac{\psi}{2} \cos\frac{\psi}{2}$ and $\cos p \frac{\psi}{2} \sin\frac{\psi}{2}$ stored in arrays SPC and CPS, respectively (statements 68 to 72). This numbers are used later to form the Fourier coefficients of the columns of A according to (2.13). Subroutines NVAR and MODEL are called to set up the array of degree variances of the geopotential, DVAR. The inverse of the variances of the scaled coefficients (equations (2.15,a-b)) are stored in array CM1. The variances produced by NVAR correspond to $n > NMOD$ and are in the form of gravity anomaly variances. They are converted to the inverse of geopotential variances in statement 81. All information relative to C^{-1} is stored in CM1 and in GMR.

The main loop begins at statement 91. Statements 92 through 95 check that the running time allocated in the JOB card is not exceeded. The elapsed time from the beginning of the main loop is printed at the beginning of a new pass through the loop (statement 93). Each pass creates the two diagonal blocks, one for n even, the other for n odd, corresponding to order $m = M$. There are four non-zero blocks for each m , but those corresponding to "sine" terms are identical to those for "cosine" terms (expression (2.51) is independent of α), so only one pair of blocks is needed. Statements 97 through 105 calculate the factors

$$FF(P) = \frac{Np}{4\sigma^2\ \omega_0^4\ \Delta a^2(N+1)^2}\ [(1-\cos((p\omega+m\Omega)\Delta a)(p\frac{\omega}{\omega_0}+m\frac{\Omega}{\omega_0})^{-4} + (1-\cos((p\omega-m\Omega)\Delta a)(p\frac{\omega}{\omega_0}-\frac{\Omega}{\omega_0})^{-4}\ ]$$

(see expression (2.51)) where WO is the fundamental angular frequency for the whole mission (period TO = Ndaysx24x3600). The "a_p^{nm}" calculated with ONEREV are too large by a factor of 2(N+1) , and this fact is accounted for in the denominator above.

After ONEREV has been called, it returns the "Fourier coefficients"

$$a_p^{nm} = \bar{h}_p^{nm}\ [p \sin\ p\frac{\psi}{2}\ \cos\frac{\psi}{2} + (n+1)\ \cos p\ \frac{\psi}{2} \sin\frac{\psi}{2}]$$

where the $\bar{h}_p^{nm}$ correspond to all $\mathcal{L}_{nm}(\phi')$ with $m = M$, $n \geq M$, in array GPNMS. These coefficients are then multiplied by each other and scaled by the factor FF(P) in statements 107 through 137, to form the element $g_{\bar{n}\bar{m}nm}^{\bar{\alpha}\alpha}$ in AIJ according to (2.51). If a collocation analysis is requested and AIJ corresponds to an element on the diagonal, the corresponding term in C^{-1} is added to AIJ in statement 136. Between 138 and 154 the calculated elements are stored in AE and AO, the arrays containing the blocks for (n-m) even, and (n-m) odd, respectively, in upper diagonal form. The vectors PIVE and PIVO are formed with the square roots of the diagonal elements. From 155 to 157 the two blocks and relevant information regarding their size are stored in unit 10, for further possible use. From 158 to 165 the two blocks are "pivoted" according to

$$G^P_{m,\alpha,}\ \left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\} = P^{-1}\quad G_{m,\alpha,}\ \left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}\ P^{-1} \qquad (B.1)$$

where P is a matrix of "pivots", i.e., the square roots of the diagonal elements of $G_{m,\alpha},\{^{even}_{odd}\}$. PIVE and PIVO are used for pivoting AE and AO, respectively.. The "pivoted" blocks are inverted in statements 173 and 174 (there is no "AO" block for M = N). Notice that the program has been written in such a way that AE and AO correspond to even and to odd $n-m$, not n. As m is fixed during each pass of the main loop, the result depends on n, so AE sometimes contains "even". and sometines "odd" elements, and the reciprocal is true of AO, depending on the value of m.

The accuracy of the inverse is tested by the inversion subroutine DSINV according to the tolerance limit defined in the main program (statement 172). If DSINV returns a value of zero in register IER1, the inversion can be regarded as successful (free of serious numerical instability). In addition to this test by DSINV, a further check has been written in the segment from statement 167 to 194. The idea is to calculate

$$\delta\underline{t} = \underline{t} - [(G^{P}_{m,\alpha}, \{^{even}_{odd}\})^{-1}_{(computed)} \; G^{P}_{m,\alpha}, \{^{even}_{odd}\}]\underline{t} = \underline{t} - \underline{t}'$$

and then

$$\varepsilon = [\delta\underline{t}^{T} \; \delta\underline{t} \; (\underline{t}^{T} \; \underline{t})^{-1}]^{\frac{1}{2}}$$

The exponent, in floating point notation, of ε is, approximately, the number of significant figures within which $\underline{t}$ and $\underline{t}'$ agree. Unfortunately, because 188 and 192 are not coded correctly, the values of ε printed at statement 193 are not useful. The elements of $\underline{t}$ are random numbers created by the IMSL subroutine GGNOR.

From 196 to the end of the main loop the inverted normal blocks are obtained by "de-pivoting":

$$G^{-1}_{m,\alpha}, \{^{even}_{odd}\} = P^{-1} \quad (G^{P}_{m,\alpha}, \{^{even}_{odd}\})^{-1} \, P^{-1} \qquad (B.2)$$

and the variance of the error per degree is totalized in array RMS as follows

$$RMS(n) = RMS(n) + \frac{G^2M^2}{a^2}\left(\frac{a}{r}\right)^{2n} \sigma^2\varepsilon^{00}_{nmnm} \times \begin{cases}1 \text{ if } m = 0\\ 2 \text{ otherwise}\end{cases}$$

$(\sigma^2\varepsilon^{00}_{nmnm} = \sigma^2\varepsilon^{11}_{nmnm})$ so, at the end of the main loop, $RMS(n) = \frac{G^2M^2}{a^2}(\frac{a}{r})^{2n}\sigma^2\varepsilon_n$, where $\sigma^2\varepsilon_n$ is the n degree variance of the errors, and the rest is the scaling factor squared (expressions (2.15,a-b)). From statement 219 to the end the error in the undulation up to degree n and this error plus the truncation error above n are both calculated and stored in arrays PPB and PPTR, respectively. Array PERC receives the formal percentage error per degree (expression (3.2) multiplied by 100). While "debugging" the program it was found useful to monitor the values of some of the coefficients' accuracies as they were being obtained. This feature was left in the program, where, for the reason given in paragraph 3.4, all the $\sigma^2\varepsilon^{\alpha\alpha}_{nmnm}$ for $n \le 40$ are still printed out (statement 210). If the inversion of AE or of AO fails. subroutine DSINV returns an explanatory code in IER1 and IER2, which should be different from 0 and is also printed out. A failure to invert is most likely due to a set-up error; perhaps to an improperly dimensioned array; to a loader that does not preset all undefined values to 0 before execution; or to improperly punched cards. An example of all the messages printed during a normal run can be seen in paragraph B.4.

The program prints out all results in unit IU (a plotter, for instance) degree by degree (statement 238) in blocks of fifty lines, and at the end prints a summary in increments of 10 degrees. The results are also punched out (statements 219 and 220).

(b) Reduced Version

This version of the main program does not compute the normals, but reads them from unit 10 (disk or tape), where they have been stored during a previous run of the full version. It also reads the original mission parameters, with which the normals were created, as the first record in file 10 (stat. 22). Some of those parameters can be changed in value by re-scaling the normals (paragraph 2.13). The new parameters' values can be declared by inserting statements between lines 22 and 25 in the listing.

This version does not call subroutine ONEREV, and reads unit 10 from subroutine RED. Subroutine WRIT is not used. All the other subroutines called in the full version are also used here. The fact that there is no AO matrix block when M = NMAX is taken into account (statements 94 and 98). Re-scaling according to (2.78) happens between 99 and 104. If no change in parameters is desired, FNSR is 1. Expression (2.84) for least squares collocation requires knowledge of the singular values and eigenvectors of the normal matrix, instead of the inverse. Lack of time and of familiarity with the decomposition subroutines available resulted in inversion being chosen for both least squares adjustment and for collocation. The fastest way to study the effect of parameter value changes on the results with collocation is, therefore, to run this reduced version with the new parameters. With NMAX = 331, this requires some 15 minutes. As no PIVE or PIVO arrays have been computed so far, this is done now in statements 122-125, and then pivoting is applied. From there on things are done much as in the full version, except that no test for numerical stability of the inversion is done in addition to that of DSINV. Results are printed and punched as in the full version.

B.2 Subroutine ONEREV

This subroutine computes the coefficients a_p^{nm} needed to fin[illegible] elements of the normal matrix according to expression (2.51). The [illegible] are defined by expression (2.13)

$$a_p^{nm} = \bar{h}_p^{nm} \left[(n+1) \cos p \frac{\psi}{2} \sin \frac{\psi}{2} + p \sin p \frac{\psi}{2} \cos \frac{\psi}{2}\right]$$

as proportional to the Fourier coefficients $\bar{h}_p^{nm}$ of the extended Legendre functions $\bar{L}_{nm}(\phi')$. The values of $(n+1) \cos p \frac{\psi}{2} \sin \frac{\psi}{2}$ and of $p \sin p \frac{\psi}{2}$ $\cos \frac{\psi}{2}$ are passed on to the subroutine from the main program in arrays SPC and CPS, respectively. The $\bar{C}_{nm}(\phi')$ are sampled at equal intervals $\Delta\phi' = \frac{2\pi}{2(N+1)}$ (N is the highest degree in the band, so the highest frequency term in any $\bar{C}_{nm}$ is either $h_N^{nm} \cos N\phi'$ or $h_N^{nm} \sin N\phi'$), and then analysed with subroutine FFCSIN, which implements a mixed-radix Fast Fourier Transform (FFT) algorithm. FFCSIN belongs to the International Mathematical and Statistical Libraries (IMSL) Inc.'s double precision library. It returns $2(N+1)\bar{h}_p^{nm}$ instead of $\bar{h}_p^{nm}$, but this is taken into account in the main program. The $\bar{C}_{nm}(\phi')$ are calculated taking advantage of the relationships (2.2,a-d), so only the values in the interval $0 \le \phi < \frac{\pi}{2}$ are absolutely necessary. These values are the same as those of the corresponding $\bar{P}_{nm}$, which are obtained with subroutine LEGFDN. There are (N+1)/2 points[1] where the $\bar{P}_{nm}$ have to be calculated in $0 \le \phi < \frac{\pi}{2}$, so the space required by GPNMS is (N - MMIN + 2) x ((N + 1/2). Here MMIN is the lowest order to be studied (a feature of the main program is that it allows the study of coefficients in the band MMIN $\le m \le$ N ; in the case of the results of section 3, MMIN = 0). The successive values of the $\bar{P}_{nm}$ are put first in array GPNMS. Then all values corresponding to the same $\bar{P}_{nm}$ are moved to REV, where $\bar{C}_{nm}$ is determined from (2.2,a-d), so the dimension of REV is 2(N+1). Then FFCSIN replaces the values of $\bar{C}_{nm}$ with those of the $\bar{h}_p^{nm}$, also in REV, and the latter are moved on to GPNMS, where they replace the original $\bar{P}_{nm}$. IWK is an auxiliary array needed by FFCSIN (see IMSL Handbook). All a_p^{nm} (multiplied by 2(N+1)) are returned to the main program. If the loader used does not preset all undefined registers to zero, a DO loop should be put at the beginning, between statements 5 and 6, setting to zero all arrays with dimensions different from 1 . Alternatively, depending on the compiler, a DATA statement to the same effect could be used.

If an even function is added to an odd function, the Fourier coefficients of the sum are those of the even function in the cosine terms, and those of the odd function in the sine terms. This simple fact is exploited to reduce calculations by half, taking advantage of the even or odd nature of the $\bar{C}_{nm}$ with respect to ϕ' .

The a_p^{nm} with $p = 0$ are handled separately from the rest, and are returned to the main program in array GMN (statement 78). Besides FFCSIN and LEGFDN, no other subroutines are called.

B.3 Subroutines LEGFDN, MODEL, and NVAR

Subroutine LEGFDN can calculate both the values of all normalized Legendre functions at colatitude THETA, for the same order m = M , up to degree n = NMAX , and also their derivatives. The functions are returned in array RLEG, the derivatives in DLEG; after exit, RLMN contains a redundant set of all sectorials up to degree n = m . All arrays, except DRTS and DIRT should have the dimension NMAX1 (statement 5). DRTS and DIRT should have twice that size. IR is a register that should be set to zero before the first call to this subroutine, in the main program. IFLAG tells the subroutine whether only the Legendre functions or these and their derivatives are required (only the $\bar{P}_{nm}$ were needed for the error analysis, so IFLAG was set to 1). The $\bar{P}_{nm}$ and their derivatives are calculated using recursive formulas given in Colombo ((1981), paragraphs (1.10) and (4.4)).

---

(1) (N + 1)/2 must be integer, so N = NMAX must be an odd number.

Subroutine MODEL sets the values of the first NMOD components of arrays DVAR adn DVER to the values of the degree variances of the errors in the potential coefficients of the reference model with respect to which the residual line of sight velocities are determined. The values in the listing corespond to the first 30 degrees in a model obtained by R. H. Rapp at OSU from a global data set of $1°\times 1°$ mean anomalies by numerical quadratures. This model is, in fact, complete up to degree and order 180, though only the accuracies of the first 30 degrees are used here.

Subroutine NVAR initializes array DVAR so that its elements are the same as the degree variances for the gravity anomaly implied by R. Rapp's coefficients mentioned above, up to $n = 100$. Above $n = 100$, the variances are those obtained from a two-term model for the gravity anomaly spectrum, also the work of R. Rapp (1979b). The main program calls NVAR first and MODEL afterwards, so the first NMOD degree variances are, finally, those in MODEL (error variances), which correspond to dimensionless potential coefficients. The remainder comes from NVAR, so they must be converted to dimensionless potential from gravity anomaly variances, a step that occurs in statements 80 and 81 of the main program. In NVAR, statements 120 and 122 add all potential degree variances between n = NMAX and $n = 2000$, and return this sum in POWNMX through COMMON/P/ to the main program.

The same caution regarding the initialization of undeclared arrays and variables to zero that was made for the main program and for ONEREV apply to LEGFDN and to MODEL (array DVAR) as well.

B.4 Sample Output

After the listings of the various routines described previously, the reader will find a sample of the printed output created by either version of the main program (they are the same). This listing should help whoever wants to use these programs to check that his own punched version works properly. The program should also punch out some cards with the results (statements 238 and 239). Because of a minor error in the program, now corrected, the standard deviation of the data is listed as 10^{-6} m s^{-1}, instead of as $\sqrt{2} \times 10^{-6}$ m s^{-1}, which is the actual value corresponding to the results printed in the sample. The parameters used, with the exception of σ , are as listed. To enter them into the main program, see comments at the beginning of either version.

The first page of the output contains a listing of the mission parameter values chosen. The "TIME BEFORE M " statements give the time in seconds at the beginning of a new pass through the main loop, and they are printed just before the time check in statement 95. the "ACCURACY OF INVERSION" lines should indicate the stability of the inversion of the blocks of order M , but they are useless because of the coding error mentioned in paragraph B.1. The other numbers are scaled variances $\frac{G^2M^2}{R^2}(\frac{a}{r})^{2n} \sigma^2\varepsilon_{nmnm}$ of the coefficients up to degree and order 40 (α=0) . If the numerical inversion of either block of order M fails the stability test in subroutine DSINV, a line is printed saying "AT ORDER m IER1 = x IER2 = y", where x and y are two integers whose value should be interpreted according to instructions in the Handbook of the SSP library. At the end of the run, the various accuracies are listed in unit IU, first degree by degree and then, in a final summary page, every 10 degrees.

FULL VERSION OF MAIN PROGRAM

```
LEVEL 2.3.0 (JUNE 78)                    OS/360  FORTRAN H EXTENDED                 DATE 80.273/21.48.08        PAGE   1

REQUESTED OPTIONS: MAP,ID,OPT=2

OPTIONS IN EFFECT: NAME(MAIN) OPTIMIZE(2) LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
                   SOURCE EBCDIC NOLIST NODECK OBJECT MAP NOFORMAT GOSTMT NOXREF NOALC NOANSF NOTERM IBM FLAG(I)
                 C
                 C            PROGRAM FOR THE ERROR ANALYSIS OF THE GLOBAL MODELLING
                 C            OF A GRAVITY FIELD FROM SATELLITE TO SATELLITE TRACKING DATA .
                 C
                 C     **** PROGRAMMED BY OSCAR L. COLOMBO, GEODETIC SCIENCE,
                 C    THE OHIO STATE UNIVERSITY, SEPTEMBER 1980  ******
                 C
ISN 0002               IMPLICIT REAL*8(A-H,O-Z)
ISN 0003               REAL*4 EPS
ISN 0004               REAL*4 Y,Z
ISN 0005               REAL*4 YLIM
ISN 0006               REAL*4 TP(500)
ISN 0007               COMMON/ P / POWNMX
ISN 0008               DIMENSION CMN(500),SPC(500),CPS(500),FF(500),DVAR(500),
                      2 RCPNM(500),IDD(500)
ISN 0009               DIMENSION CPNMS(56000),AE(15000),AO(15000),CNI(500)
ISN 0010               DIMENSION PIVE(500),PIVO(500),RMS(500)
ISN 0011               DIMENSION  CNR(500),T(500),PE(500),PO(500),PE2(500),PO2(500)
ISN 0012               DIMENSION PPB(500),PPTR(500),PERC(500)
ISN 0013               DATA RMS/500*0.D0/
ISN 0014               IX = 1
ISN 0015               CALL ERRSET(208,256,-1,2,IX,0)
                 C
                 C            DEFINITION OF THE PROBLEM. EVERYTHING IS IN METERS AND
                 C            IN SECONDS .
                 C            NE  AND  NS  MUST BE MUTUALLY PRIME, AND  NS GT. 2*NMAX .
                 C            THE SAMPLING INTERVAL  DS  MUST DIVIDE    NE*24*3600
                 C            EXACTLY .
                 C
                 C            NMOD = MAXIMUM DEGREE AND ORDER IN REFERENCE MODEL.
                 C            NMAX = MAX. DECREE AND ORDER IN SIGNAL,
                 C                   (NMAX MUST BE ALWAYS AN ODD NUMBER),
                 C            MMIN = MINIMUM ORDER CONSIDERED IN ANALYSIS,
                 C            NE   = DURATION OF MISSION IN DAYS,
                 C            NS   = NUMBER OF REVOLUTIONS OF SATELLITES PER DAY,
                 C            HS   = HEIGHT OF SATELLITES ABOVE SPHERE OF RADIUS  RE ,
                 C            D12 = SEPARATION BETWEEN THE TWO SATELLITES IN METERS,
                 C            SIGMA= STANDARD DEVIATION OF ERRORS IN DATA (RELATIVE
                 C                   VELOCITY) IN METERS PER SECOND ,
                 C            DS   = SAMPLING INTERVAL,
                 C            DA   = AVERAGING INTERVAL,
                 C            YLIM = TIME IN SECONDS BEFORE EXITING FROM MAIN LOOP,
                 C            IU   = UNIT NUMBER FOR PLOTTER,
                 C            IP   = UNIT NUMBER FOR PUNCH .
                 C
                 C *******************************************************************
ISN 0016               YLIM = 4740.
ISN 0017               NMOD = 20
ISN 0018               NMAX = 331
ISN 0019               NMAX = 11
ISN 0020               MMIN = 0
ISN 0021               NE = 179
ISN 0022               NS = 2933
```

```
LEVEL 2.3.0 (JUNE 78)          MAIN          OS/360  FORTRAN H EXTENDED          DATE 80.273/21.48.08       PAGE   2

ISN 0023          HS = 160.D3
ISN 0024          D12 = 300.D3
ISN 0025          SIGMA = 1.D-6
ISN 0026          DS = 4.D0
ISN 0027          DA = DS
ISN 0028          IU = 9
ISN 0029          IP = 7
ISN 0030          IAC = 0
        C                IAC = 0 FOR L.S. ADJUSTMENT,
        C                IAC = 1 FOR L.S. COLLOCATION .
        C ***********************************************************************
        C
        C                PRELIMINARY CALCULATIONS AND PRINTING OF PROBLEM DEFINITION .
        C
ISN 0031          IF((NMAX/2)*2.EQ.NMAX) NMAX =  NMAX+1
ISN 0033          NMAXP = NMAX+1
ISN 0034          NML = NMAXP/2+1
ISN 0035          NMOD1 = NMOD+1
ISN 0036          WRITE(6 ,2) NMAX,NMIN,NMOD,NE,NS,HS,D12,DA,DS,SIGMA
ISN 0037          WRITE(IU,2) NMAX,NMIN,NMOD,NE,NS,HS,D12,DA,DS,SIGMA
ISN 0038        2 FORMAT('1',' ERROR ANALYSIS OF A SATELLITE TO SATELLITE TRACKING
                 2MISSION'//,' MAXIMUM DEGREE AND ORDER CONSIDERED = ',I5//
                 3,' MINIMUM DEGREE CONSIDERED =  2      MINIMUM ORDER ANALIZED = ',
                 &I5//,
                 #' MAXIMUM DEGREE AND ORDER IN REFERENCE MODEL = ',I5//,
                 4' DURATION OF MISSION IN DAYS = ',I5//,
                 5' NUMBER OF REVOLUTIONS DURING MISSION = ',I5//,
                 6' HEIGHT OF THE SATELLITES ABOVE GROUND = ',G20.10,' IN METERS'//,
                 7' SEPARATION BETWEEN SATELLITES = ',G20.10,' IN METERS'//
                 8,' AVERAGING INTERVAL = ',G20.10,' SECONDS '//,
                 9' SAMPLING INTERVAL  = ',G20.10,' SECONDS '//,
                 #' ACCURACY OF THE DATA = ',G12.4,' METERS PER SECOND '////)
ISN 0039          IF(IAC.EQ.0) WRITE(6,3)
ISN 0041          IF(IAC.EQ.0) WRITE(IU,3)
ISN 0043          IF(IAC.EQ.1) WRITE(6,4)
ISN 0045          IF(IAC.EQ.1) WRITE(IU,4)
ISN 0047        3 FORMAT(' ANALYSIS IS BASED ON LEAST SQUARES ADJUSTMENT *****'////)
ISN 0048        4 FORMAT(' ANALYSIS IS BASED ON LEAST SQUARES COLLOCATION ****'////)
ISN 0049          ISEED = 20000
ISN 0050          GAMMA = 9.820264D0
ISN 0051          RE = 6371.D3
ISN 0052          RS = RE+HS
ISN 0053          GAMS = GAMMA*(RE/RS)**2
ISN 0054          PSI122 = DARSIN(D12*0.5D0/RS)
ISN 0055          PSI12 = 2.D0*PSI122
ISN 0056          SIN122 = DSIN(PSI122)
ISN 0057          COS122 = DCOS(PSI122)
ISN 0058          PI = 4.D0*DATAN(1.D0)
ISN 0059          T0 = NE*24.D0*3600.D0
ISN 0060          W0 = 2.D0*PI/T0
ISN 0061          NSR = 2*NMAXP
ISN 0062          NM = T0/DS+0.0000001D0
ISN 0063          FNS = NM       /(SIGMA**2*W0**4*DA**2*NSR**2)
ISN 0064          WRITE(10) NMOD,NMAX,NMIN,NE,NS,HS,D12,SIGMA,DA,FNS
ISN 0065          W0DA = W0*DA
ISN 0066          XNS = NS
ISN 0067          XNE = NE
ISN 0068          DO 5      KP = 1,NMAX
```

```
ISN 0069            IDD(KP) = ((KP-1)*KP)/2
ISN 0070            SPC(KP) = KP*DSIN(KP*PSI122)*COS122
ISN 0071            CPS(KP) = DCOS(KP*PSI122)*SIN122
ISN 0072          5 CONTINUE
           C
           C              FORM THE INVERSE OF THE DIAGONAL MATRIX  C  IN ARRAY  CMI .
           C
ISN 0073            CALL NVAR(DVAR,NMAX,DVAR0)
ISN 0074            CALL MODEL(DVAR,NMOD)
ISN 0075            DO 8      N = 2,NMOD
ISN 0076          8 CMI(N) = 1.D0/DVAR(N)**2
ISN 0077            NMAXT = NMAX
ISN 0078            IF(NMOD1.GT.NMAX) NMAXT = NMOD1
ISN 0080            DO 9      N = NMOD1,NMAXT
ISN 0081          9 CMI(N) = (2.D0*N+1.D0)*((N-1)*GAMMA*1.D5)**2/DVAR(N)
ISN 0082            CMI(1) = 0.D0
ISN 0083            DO 11     N = 2,NMAX
ISN 0084            CMR(N) = 1.D0/(GAMS*(RE/RS)**N)**2
ISN 0085            CMI(N) = CMI(N)*CMR(N)
ISN 0086         11 CONTINUE
ISN 0087            WRITE(6,17)
ISN 0088         17 FORMAT(//' N   M   COEFFICIENTS VARIANCES (UP TO N,M = 10 )'//)
ISN 0089            CALL SCLOK1
ISN 0090            Z = 1.
           C
           C              BEGINNING OF THE MAIN LOOP.
           C              HERE THE ERRORS PER COEFFICIENT (VARIANCES) ARE FOUND, WITH
           C              INCREASING ORDER  M , AND ADDED UP TO OBTAIN THE RMS
           C              PERCENTAGE ERROR PER DEGREE .
           C
ISN 0091            DO 100        M = MMIN,MMAX
ISN 0092            Y = RCLOK1(Z)
ISN 0093            WRITE(6,13) Y
ISN 0094         13 FORMAT(  ' TIME BEFORE M = ',F10.2,' SECONDS')
ISN 0095            IF(Y.GE.YLIM) GO TO 500
ISN 0097            XM = M
ISN 0098            IF(M.NE.0)FNS0=FNS*(1.D0-DCOS(XM*XNE*W0DA))/(XM*XNE)**4*SIN122**2
ISN 0100            DO 12      KP = 1,NMAX
ISN 0101            XKP = KP
ISN 0102            F1 = XKP*XNS+XM*XNE
ISN 0103            F2 = XKP*XNS-XM*XNE
ISN 0104            FF(KP) = ((1.D0-DCOS(F1*W0DA))/F1**4+(1.D0-DCOS(F2*W0DA))/F2**4)*
                   2 FNS
ISN 0105         12 CONTINUE
           C
           C              OBTAIN THE FOURIER COEFFICIENTS  CPNM  OF THE 'ONE REV.'
           C              ACCELERATIONS .
           C
ISN 0106            CALL ONEREV(CPNMS,M,SPC,CPS,NMAX,CMN)
           C
           C              FORM THE NON-ZERO ELEMENTS OF THE NORMAL MATRIX CORRESPONDING
           C              TO THE ORDER  M  , AND STORE THEM IN UPPER DIAGONAL FORM
           C              IN  AE  ( IF  N-M  IS EVEN), OR IN  AO (IF N-M  IS ODD) .
           C
ISN 0107            NMIN = M
ISN 0108            IF(M.LT.2) NMIN = 2
ISN 0110            ISHI = -NML
ISN 0111            NEV = 0
```

```
ISN 0112          NOD = 0
ISN 0113          DO 20     N1 = NMIN,NMAX
ISN 0114          IKB = 0
ISN 0115          IF(((N1-M)/2)*2.NE.(N1-M)) IKB = 1
ISN 0117          INC = 0
ISN 0118          IF((N1/2)*2.NE.N1) INC = 1
ISN 0120          INCC = 1-INC
ISN 0121          ISH1 = ISH1+NML
ISN 0122          KT = 0
ISN 0123          DO 10      KP = 1,N1,2
ISN 0124          KT = KT+1
ISN 0125          KP1 = KP+INCC
ISN 0126       10 RCPNM(KT) = CPNMS(ISH1+KT)*FF(KP1)
ISN 0127          ISH2 = ISH1
ISN 0128          DO 20      N2 = N1,NMAX,2
ISN 0129          KTT = 0
ISN 0130          AIJ = 0.D0
ISN 0131          DO 15      KP = 1,N1,2
ISN 0132          KTT = KTT+1
ISN 0133       15 AIJ = AIJ+RCPNM(KTT)*CPNMS(ISH2+KTT)
ISN 0134          IF(M.NE.0.AND.IKS.EQ.0.AND.INC.EQ.0) AIJ = AIJ+CMN(N1)*CMN(N2)
                2 *FNS0
ISN 0136          IF(N1.EQ.N2.AND.IAC.EQ.1) AIJ = AIJ+CH1(N1)
ISN 0138          I = (N2-NMIN)/2+1
ISN 0139          J = (N1-NMIN)/2+1
ISN 0140          IF(IKB.EQ.1) GO TO 18
ISN 0142          AE(IDD(I)+J) = AIJ
ISN 0143          IF(I.GT.NEV) NEV = I
ISN 0145          IF(I.EQ.J) PIVE(I) = 1.D0/DSQRT(AIJ)
ISN 0147          GO TO 19
ISN 0148       18 AO(IDD(I)+J) = AIJ
ISN 0149          IF(I.GT.NOD) NOD = I
ISN 0151          IF(I.EQ.J) PIVO(I) = 1.D0/DSQRT(AIJ)
ISN 0153       19 ISH2 = ISH2+2*NML
ISN 0154       20 CONTINUE
             C
             C            SAVE NORMALS IN UNIT 10 .
             C
ISN 0155          NNEV = (NEV**2-NEV)/2+NEV
ISN 0156          NNOD = (NOD**2-NOD)/2+NOD
ISN 0157          CALL WRIT(AE,AO,NEV,NOD,NNEV,NNOD)
             C
             C
             C            APPLY PIVOTING TO NORMALS .
             C
ISN 0158          DO 25      I = 1,NEV
ISN 0159          DO 25      J = 1,I
ISN 0160          AE(IDD(I)+J) = AE(IDD(I)+J)*PIVE(I)*PIVE(J)
ISN 0161       25 CONTINUE
ISN 0162          DO 30      I = 1,NOD
ISN 0163          DO 30      J = 1,I
ISN 0164          AO(IDD(I)+J) = AO(IDD(I)+J)*PIVO(I)*PIVO(J)
ISN 0165       30 CONTINUE
             C
             C            INVERTING THE DIAGONAL BLOCKS OF THE NORMALS CORRESPONDING
             C            TO ORDER  M , AND TESTING FOR NUMERICAL ACCURACY.
             C
ISN 0166          NN2P = (NMAX-M+1)/2+2
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ISN 0167          CALL GGNOR(ISEED,MN2P,TP)
ISN 0168          DO 333       I = 1,MN2P
ISN 0169      333 T(I) = TP(I)
ISN 0170          CALL MATV(PE,AE,T,NEV,IDD)
ISN 0171          CALL MATV(PO,AO,T,NOD,IDD)
ISN 0172          EPS = 1.E-7
ISN 0173          CALL DSINV(AE,NEV,EPS,IER1)
ISN 0174          IF(M.NE.NMAX) CALL DSINV(AO,NOD,EPS,IER2)
ISN 0176          IF(IER1.NE.0.OR.IER2.NE.0) WRITE(6,21) M,IER1,IER2
ISN 0178       21 FORMAT(//' ***** AT ORDER  ',I5,': '/,' ** IER1 = ',I5,' IER2 = ',
                 2 I5,' *************'//)
ISN 0179          IF(IER1.NE.0.OR.IER2.NE.0) GO TO 500
ISN 0181          IF(M.EQ.NMAX) GO TO 334
ISN 0183          CALL MATV(PE2,AE,PE,NEV,IDD)
ISN 0184          CALL MATV(PO2,AO,PO,NOD,IDD)
ISN 0185          ER1 = 0.D0
ISN 0186          DO 31       I = 1,NEV
ISN 0187       31 ER1 = ER1+(PE2(I)-T(I))**2
ISN 0188          ER1 = DSQRT(ER1/NEV)
ISN 0189          ER2 = 0.D0
ISN 0190          DO 32       I = 1,NOD
ISN 0191       32 ER2 = ER2+(PO2(I)-T(I))**2
ISN 0192          ER2 = DSQRT(ER2/NOD)
ISN 0193          WRITE(6,33) M,ER1,ER2
ISN 0194       33 FORMAT(' ACCURACY OF INVERSION, M = ',I5,'= ',2G20.10)
ISN 0195      334 DO 55       I = 1,NEV
ISN 0196          AE(IDD(I)+I) = AE(IDD(I)+I)*PIVE(I)**2
ISN 0197       55 CONTINUE
ISN 0198          DO 65       I = 1,NOD
ISN 0199          AO(IDD(I)+I) = AO(IDD(I)+I)*PIVO(I)**2
ISN 0200       65 CONTINUE
ISN 0201          DO 80       N = NMIN,NMAX
ISN 0202          ITS = 0
ISN 0203          IF(((N-M)/2)*2.NE.N-M) ITS = 1
ISN 0205          I = (N-NMIN)/2+1
ISN 0206          IF(ITS.EQ.0) DIRV = AE(IDD(I)+I)*CMR(N)
ISN 0208          IF(ITS.EQ.1) DIRV = AO(IDD(I)+I)*CMR(N)
ISN 0210          IF(N.LE.40) WRITE(6,104) N,M,DIRV
ISN 0212      104 FORMAT(2I5,G26.10)
ISN 0213          DIRVO = DIRV*2.D0
ISN 0214          IF(M.EQ.0) DIRVO = DIRV
ISN 0216          RMS(N) = RMS(N)+DIRVO
ISN 0217       80 CONTINUE
ISN 0218      100 CONTINUE
            C
            C               END OF MAIN LOOP .
            C
            C               PRINT AND PUNCH ALL THE RESULTS .
            C
ISN 0219      500 WRITE(IP,120) (N,RMS(N),N=2,NMAX)
ISN 0220      120 FORMAT(4(I3,G17.11))
ISN 0221          WRITE(IU,105)
ISN 0222      105 FORMAT('1',20X,'  N  PERCENTAGE ERROR      ERROR VARIANCE(POT.)
                 2BAND ERROR (UNDUL.) (M)     TOTAL UNDUL. ERROR (M) '////)
ISN 0223          PB = 0.D0
ISN 0224          POW = 0.D0
ISN 0225          DO 106      N = 2,NMAX
ISN 0226      106 POW = POW+CMR(N)*(2*N+1)/CMI(N)
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   ISN 0227            PTR = POW*RE**2+POWNMX*1.D-10
   ISN 0228            DO 110      N = 2,NMAX
   ISN 0229            PN = CMR(N)*RE**2*(2*N+1)/CMI(N)
   ISN 0230            PERR = RMS(N)*RE**2
   ISN 0231            PTR = PTR-PN+PERR
   ISN 0232            PB = PB+PERR
   ISN 0233            PERC(N) = DSQRT(RMS(N)*CMI(N)/(CMR(N)*(2*N+1)))*100.D0
   ISN 0234            PPB(N) = DSQRT(PB)
   ISN 0235            PPTR(N) = DSQRT(PTR)
   ISN 0236            IF((N/50)*50.EQ.N) WRITE(IU,105)
   ISN 0238            WRITE(IU,107) N,PERC(N),RMS(N),PPB(N),PPTR(N)
   ISN 0239        107 FORMAT(20X,I5,4(7X,G13.5))
   ISN 0240        110 CONTINUE
   ISN 0241            WRITE(IU,103)
   ISN 0242            DO 115     N = 2,NMAX
   ISN 0243            IF((N/10)*10.EQ.N.OR.N.EQ.2) WRITE(IU,107)N,PERC(N),RMS(N),
                     2 PPB(N),PPTR(N)
   ISN 0245        115 CONTINUE
   ISN 0246            STOP
   ISN 0247            END
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REQUESTED OPTIONS: MAP,ID,OPT=2

OPTIONS IN EFFECT: NAME(MAIN) OPTIMIZE(2) LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
                   SOURCE EBCDIC NOLIST NODECK OBJECT MAP NOFORMAT GOSTMT NOXREF NOALC NOANSF NOTERM IBM FLAG(I)

  ISN 0002            SUBROUTINE MATV(P,A,T,N,IDD)
                C
                C            THIS SUBROUTINE MULTIPLIES VECTOR  T  BY THE SYMMETRICAL
                C            MATRIX  A  , THE RESULT BEING VECTOR  P .
                C
  ISN 0003            IMPLICIT REAL*8(A-H,O-Z)
  ISN 0004            DIMENSION P(1),A(1),T(1),IDD(1)
  ISN 0005            DO 10       I = 1,N
  ISN 0006            P(I) = 0.D0
  ISN 0007            I1 = I
  ISN 0008            DO 10      J = 1,N
  ISN 0009            J1 = J
  ISN 0010            IF(J.LT.I) GO TO 10
  ISN 0012            I1 = J
  ISN 0013            J1 = I
  ISN 0014         10 P(I) = P(I)+A(IDD(I1)+J1)*T(J)
  ISN 0015            RETURN
  ISN 0016            END
```

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LEVEL 2.3.0 (JUNE 78)                    OS/360  FORTRAN H EXTENDED              DATE 80.273/21.48.18        PAGE   1

REQUESTED OPTIONS: MAP,ID,OPT=2

OPTIONS IN EFFECT: NAME(MAIN) OPTIMIZE(2) LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
                   SOURCE EBCDIC NOLIST NODECK OBJECT MAP NOFORMAT COSTMT NOXREF NOALC NOANSF NOTERM IBM FLAG(I)

   ISN 0002              SUBROUTINE WRIT(AE,AO,NEV,NOD,NNEV,NNOD)
                 C
                 C             THIS SUBROUTINE STORES IN 10 THE NORMALS IN LOWER
                 C             TRIANGULAR FORM.
                 C
   ISN 0003              IMPLICIT REAL*8(A-H,O-Z)
   ISN 0004              DIMENSION AE(NNEV),AO(NNOD)
   ISN 0005              WRITE(10) NEV,NOD,NNEV,NNOD
   ISN 0006              WRITE(10) AE,AO
   ISN 0007              RETURN
   ISN 0008              END
```

```
REQUESTED OPTIONS: MAP,ID,OPT=2

OPTIONS IN EFFECT: NAME(MAIN) OPTIMIZE(2) LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
                   SOURCE EBCDIC NOLIST NODECK OBJECT MAP NOFORMAT GOSTMT NOXREF NOALC NOANSF NOTERM IBM FLAG(I)

  ISN 0002            SUBROUTINE ONEREV(CPNMS,M,SPC,CPS,NMAX,CMN)
                C
                C             THIS SUBROUTINE COMPUTES AND SAVES IN ARRAY  CPNMS  THE
                C             'ONE REV.' FOURIER COEFFICIENTS  CPNM  OF THE COLUMNS
                C             OF THE MATRIX OF OBSERVATION EQUATIONS,FOR THE SPHERICAL
                C             HARMONIC COEFFICIENTS OF ORDER  M  .
                C
                C             NMAX =  THE HIGHEST DEGREE IN THE BAND. IT MUST BE
                C             ALWAYS AN  ODD  NUMBER, AND LESS THAN 500 WITH THE ARRAYS
                C             DIMENSIONED AS SHOWN BELOW.
                C             THE DIMENSION OF  CPNMS  SHOULD BE NO LESS THAN
                C             (NMAX-NMIN+2)*((NMAX+1)/2+1), WHERE  NMAX  IS ALWAYS ODD .
                C
                C     PROGRAMMED BY  OSCAR L. COLOMBO, GEODETIC SCIENCE, OHIO STATE,
                C     SEPTEMBER 1980 *********** * * **********
                C
  ISN 0003            IMPLICIT REAL*8(A-H,O-Z)
  ISN 0004            DIMENSION RLEG(500),DLEG(1),RLNN(500),CPNMS(1),SPC(1),CPS(1),
                    2 CMN(1),REV(2000),CP(500),SP(500),IWK(4000)
  ISN 0005            NMAXP1 = NMAX+1
  ISN 0006            PI = 4.D0*DATAN(1.D0)
  ISN 0007            NMIN = M
  ISN 0008            IF(M.LT.2) NMIN = 2
  ISN 0010            IR = 0
  ISN 0011            IFLAG = 1
  ISN 0012            NM2 = NMAXP1/2+1
  ISN 0013            NXZ = NM2*(NMAX-NMIN+2)
  ISN 0014            DO 5    L = 1,NXZ
  ISN 0015          5 CPNMS(L) = 0.D0
  ISN 0016            DPHI = PI/NMAXP1
  ISN 0017            PHI = 0.D0
  ISN 0018            DO 10      NR = 1,NM2
  ISN 0019            THETA = PI*0.5D0-PHI
  ISN 0020            IF(NR.NE.NM2) GO TO 9
  ISN 0022            DO 7       I = 1,NMAXP1
  ISN 0023          7 RLEG(I) = 0.D0
  ISN 0024          9 CONTINUE
  ISN 0025            IF(NR.EQ.NM2.AND.M.NE.0) GO TO 6
  ISN 0027            CALL LEGFDN(M,THETA,RLEG,DLEG,NMAX,IR,RLNN,IFLAG)
  ISN 0028          6 PHI = PHI+DPHI
  ISN 0029            NM22 = NM2*2
  ISN 0030            ISH = -NM2
                C
                C             PUT IN THE ARRAY  CPNMS  THE VALUES OF ALL NORMALIZED
                C             LEGENDRE FUNCTIONS OF ORDER  M  (EXCEPT THOSE OF DEGREES
                C             0  OR  1 ).
                C
  ISN 0031            DO 8          N = NMIN,NMAX
  ISN 0032            ISH = ISH+NM2
  ISN 0033          8 CPNMS(ISH+NR) = RLEG(N+1)
  ISN 0034         10 CONTINUE
                C
                C             FIND THE FOURIER COEFFICIENTS  FPNM  OF THE 'EXTENDED'
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                  C           LEGENDRE FUNCTIONS, TAKING ADVANTAGE OF THE FACT
                  C           THAT CONSECUTIVE FUNCTIONS ARE  EITHER EVEN OR ODD ABOUT THE
                  C           EQUATOR .
                  C
    ISN 0035            NMAXP2 = NMAX+2
    ISN 0036            NMAXP3 = NMAX+3
    ISN 0037            INK = 0
    ISN 0038            IF((M/2)*2.NE.M) INK = 1
    ISN 0040            NMM = NMAXP1*2+1
    ISN 0041            NML = NMAXP1*2
    ISN 0042            NMM1 = NMM+1
    ISN 0043            ISHT = 0
    ISN 0044            IND = 0
    ISN 0045            IF((NMIN/2)*2.NE.NMIN) IND=1
    ISN 0047            INDC = 1-IND
    ISN 0048            INC = 0
    ISN 0049            IF(((NMIN-M)/2)*2.NE.(NMIN-M)) INC = 1
    ISN 0051            DO 80      N = NMIN,NMAX,2
    ISN 0052            ISHT2 = ISHT+NM2
                  C           FORM THE SUM OF TWO CONSECUTIVE 'EXTENDED' LEGENDRE FUNCTIONS
                  C           IN ARRAY  REV  .
                  C
    ISN 0053            DO 20      NR = 1,NM2
    ISN 0054            NM3R = NMAXP3-NR
    ISN 0055            NPPR = NMAXP1+NR
    ISN 0056            NMMR = NMM1-NR
    ISN 0057            CP1 = CPNMS(ISHT+NR)
    ISN 0058            CP2 = CPNMS(ISHT2+NR)
    ISN 0059            REV(NR) = CP1+CP2
    ISN 0060            IF(INK.EQ.1) GO TO 12
    ISN 0062            REV(NM3R) = REV(NR)
    ISN 0063            GO TO 15
    ISN 0064         12 REV(NM3R) = -REV(NR)
    ISN 0065         15 REV(NMMR) = CP1-CP2
    ISN 0066            IF(INC.EQ.1) REV(NMMR) = -REV(NMMR)
    ISN 0068            IF(INK.EQ.1) GO TO 16
    ISN 0070            REV(NPPR) = REV(NMMR)
    ISN 0071            GO TO 20
    ISN 0072         16 REV(NPPR) = -REV(NMMR)
    ISN 0073         20 CONTINUE
                  C
                  C           COMPUTE THE FOURIER COEFFICIENTS OF TWO CONSECUTIVE 'EXTENDED'
                  C           LEGENDRE FUNCTIONS IN  CP   AND IN SP, RESPECTIVELY .
                  C
    ISN 0074            CALL FFCSIN(REV,NML,CP,SP,IWK)
                  C
                  C           FORM THE CPNM COEFFICIENTS, AND STORE THEM IN  CPNMS .
                  C           IN PLACE OF THE VALUES OF THE CORRESPONDING LEGENDRE FUNCTIONS.
                  C
    ISN 0075            NT1 = 0
    ISN 0076            NN1 = N+1
    ISN 0077            NN2 = N+2
    ISN 0078            IF(IND.EQ.0) CMN(N) = CP(1)*NN1
    ISN 0080            DO 30      NR = 1,N,2
    ISN 0081            NT = NR+INDC
    ISN 0082            NTP = NT+1
    ISN 0083            NT1 = NT1+1
    ISN 0084            IF(INC.EQ.1) GO TO 22
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ISN 0086          CPNMS(ISHT+NT1) = CP(NTP)*(SPC(NT)+NN1*CPS(NT))
ISN 0087          GO TO 24
ISN 0088       22 CPNMS(ISHT+NT1) = SP(NTP)*(SPC(NT)+NN1*CPS(NT))
ISN 0089       24 NTA = NN+IND
ISN 0090          NTAP = NTA+1
ISN 0091          IF(ING.EQ.0) GO TO 26
ISN 0093          CPNMS(ISHT2+NT1) = CP(NTAP)*(SPC(NTA)+NN2*CPS(NTA))
ISN 0094          GO TO 28
ISN 0095       26 CPNMS(ISHT2+NT1) = SP(NTAP)*(SPC(NTA)+NN2*CPS(NTA))
ISN 0096       28 IF( IND.NE.0.OR.NN+1.NE.N) GO TO 30
ISN 0098          NTE = NTA+2
ISN 0099          CPNMS(ISHT2+NT1+1) = SP(NTAP+2)*(SPC(NTE)+NN2*CPS(NTE))
ISN 0100       30 CONTINUE
ISN 0101          ISHT = ISHT+NN22
ISN 0102       50 CONTINUE
ISN 0103          RETURN
ISN 0104          END
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LEVEL 2.3.0 (JUNE 78)                OS/360  FORTRAN H EXTENDED          DATE 80.273/21.48.21        PAGE   1

REQUESTED OPTIONS: MAP,ID,OPT=2

OPTIONS IN EFFECT: NAME(MAIN) OPTIMIZE(2) LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
                   SOURCE EBCDIC NOLIST NODECK OBJECT MAP NOFORMAT GOSTMT NOXREF NOALC NOANSF NOTERM IBM FLAG(I)

  ISN 0002          SUBROUTINE LEGFDN(M,THETA,RLEG,DLEG,NMX,IR,RLNN,IFLAG)
              C
              C             THIS SUBROUTINE COMPUTES  ALL NORMALIZED LEGENDRE FUNCTIONS
              C             IN "RLEG" AND THEIR DERIVATIVES IN "DLEG". ORDER IS ALWAYS
              C             M , AND COLATITUDE IS ALWAYS THETA  (RADIANS). MAXIMUM DEGREE
              C             IS  NMX  . ALL CALCULATIONS IN DOUBLE PRECISION.
              C             IR  MUST BE SET TO ZERO BEFORE THE FIRST CALL TO THIS SUB.
              C             THE DIMENSIONS OF ARRAYS  RLEG, DLEG, AND RLNN  MUST BE
              C             AT LEAST EQUAL TO  NMX+1  .
              C             IF THIS SUBROUTINE IS TO BE USED TO COMPUTE FUNCTIONS
              C             AND THEIR DERIVATIVES FOR MORE THAN ONE ORDER M, THEN
              C             THE HIGHEST ORDER SHOULD BE COMPUTED IN THE FIRST CALL.
              C
              C             THIS PROGRAM DOES NOT COMPUTE DERIVATIVES AT THE POLES .
              C
              C             IF    IFLAG = 1  , ONLY THE LEGENDRE FUNCTIONS ARE
              C             COMPUTED.
              C
              C         PROGRAMMER :  OSCAR L. COLOMBO, DEPT. OF GEODETIC SCIENCE,
              C         THE OHIO STATE UNIVERSITY, AUGUST 1980 . ***************************
              C
  ISN 0003          IMPLICIT REAL*8 (A-H,O-Z)
  ISN 0004          DIMENSION RLEG(1),DLEG(1),RLNN(1)
                   2, DRTS(1300),DIRT(1300)
  ISN 0005          NMX1 = NMX+1
  ISN 0006          NMX2P = 2*NMX+1
  ISN 0007          M1 = M+1
  ISN 0008          M2 = M+2
  ISN 0009          M3 = M+3
  ISN 0010          IF(IR.EQ.1) GO TO 10
  ISN 0012          IR = 1
  ISN 0013          DO 5     N = 1,NMX2P
  ISN 0014          DRTS(N) = DSQRT(N*1.D0)
  ISN 0015        5 DIRT(N) = 1.D0/DRTS(N)
  ISN 0016       10 COTHET = DCOS(THETA)
  ISN 0017          SITHET = DSIN(THETA)
  ISN 0018          IF(IFLAG.NE.1.AND.THETA.NE.0.D0)SITHI = 1.D0/SITHET
              C
              C             COMPUTE THE LEGENDRE FUNCTIONS .
              C
  ISN 0020          RLNN(1) = 1.D0
  ISN 0021          RLNN(2) = SITHET*DRTS(3)
  ISN 0022          DO 15    N1 = 3,M1
  ISN 0023          N = N1-1
  ISN 0024          N2 = 2*N
  ISN 0025       15 RLNN(N1) = DRTS(N2+1)*DIRT(N2)*SITHET*RLNN(N1-1)
  ISN 0026          IF(M.GT.1) GO TO 20
  ISN 0028          IF(M.EQ.0) GO TO 16
  ISN 0030          RLEG(2) = RLNN(2)
  ISN 0031          RLEG(3) = DRTS(5)*COTHET*RLEG(2)
  ISN 0032          GO TO 20
  ISN 0033       16 RLEG(1) = 1.D0
  ISN 0034          RLEG(2) = COTHET*DRTS(3)
```

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LEVEL 2.3.0 (JUNE 78)          LEGFDR            OS/360   FORTRAN H EXTENDED               DATE 80.273/21.48.21           PAGE   2

   ISN 0035          20 CONTINUE
   ISN 0036             RLEG(M1) = RLNN(M1)
   ISN 0037             RLEG(M2) = DRTS(M*2+1)*COTHET*RLEG(M1)
   ISN 0038             DO 30      N1 = M3,NMX1
   ISN 0039             N = N1-1
   ISN 0040             IF(M.EQ.0.AND.N.LT.2.OR.M.EQ.1.AND.N.LT.3) GO TO 30
   ISN 0042             N2 = 2*N
   ISN 0043             RLEG(N1) = DRTS(N2+1)*DIRT(N+M)*DIRT(N-M)*(DRTS(N2-1)*COTHET*
                       2 RLEG(N1-1)-DRTS(N+M-1)*DRTS(N-M-1)*DIRT(N2-3)*RLEG(N1-2))
   ISN 0044             GO TO 30
   ISN 0045          30 CONTINUE
   ISN 0046             IF(IFLAG.EQ.1) RETURN
   ISN 0048             IF(SITHET.EQ.0.D0) WRITE(6,99)
   ISN 0050          99 FORMAT(//' *** LEGFDM  DOES NOT COMPUTE DERIVATIVES AT THE POLES
                       2 *****************'//)
   ISN 0051             IF(SITHET.EQ.0.D0) RETURN
                  C
                  C            COMPUTE ALL THE DERIVATIVES OF THE LEGENDRE FUNCTIONS.
                  C
   ISN 0053             RLNN(1) = 0.D0
   ISN 0054             RLN = RLNN(2)
   ISN 0055             RLNN(2) = DRTS(3)*COTHET
   ISN 0056             DO 40      N1 = 3, M1
   ISN 0057             N = N1-1
   ISN 0058             N2 = 2*N
   ISN 0059             RLN1 = RLNN(N1)
   ISN 0060             RLNN(N1) = DRTS(N2+1)*DIRT(N2)*(SITHET*RLNN(N)+COTHET*RLN)
   ISN 0061             RLN = RLN1
   ISN 0062          40 CONTINUE
   ISN 0063             DLEG(M1) = RLNN(M1)
   ISN 0064             DO 60      N1 = M2,NMX1
   ISN 0065             N = N1-1
   ISN 0066             N2 = N*2
   ISN 0067             DLEG(N1) = SITHI*( N    *RLEG(N1)*COTHET-DRTS(N-M)*DRTS(N+M)*
                       2 DRTS(N2+1)*DIRT(N2-1)*RLEG(N))
   ISN 0068          60 CONTINUE
   ISN 0069             RETURN
   ISN 0070             END
```

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LEVEL 2.3.0 (JUNE 78)                    OS/360  FORTRAN H EXTENDED              DATE 80.273/21.48.24        PAGE   1

REQUESTED OPTIONS: MAP,ID,OPT=2

OPTIONS IN EFFECT: NAME(MAIN) OPTIMIZE(2) LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
                   SOURCE EBCDIC NOLIST NODECK OBJECT MAP NOFORMAT GOSTMT NOXREF NOALC NOANSF NOTERM IBM FLAG(I)

   ISN 0002            SUBROUTINE MODEL(DVER,NMOD)
   ISN 0003            IMPLICIT REAL*8(A-H,O-Z)
   ISN 0004            DIMENSION DVER(1),DVAR(50)
   ISN 0005            DVAR(1) = 0.D0
   ISN 0006            DVAR(2) = 2720.D-12
   ISN 0007            DVAR(3) = 6394.D-12
   ISN 0008            DVAR(4) = 4564.D-12
   ISN 0009            DVAR(5) = 7237.D-12
   ISN 0010            DVAR(6) = 5703.D-12
   ISN 0011            DVAR(7) = 6707.D-12
   ISN 0012            DVAR(8) = 5685.D-12
   ISN 0013            DVAR(9) = 5727.D-12
   ISN 0014            DVAR(10) = 5118.D-12
   ISN 0015            DVAR(11) = 4990.D-12
   ISN 0016            DVAR(12) = 4415.D-12
   ISN 0017            DVAR(13) = 4836.D-12
   ISN 0018            DVAR(14) = 4458.D-12
   ISN 0019            DVAR(15) = 4149.D-12
   ISN 0020            DVAR(16) = 3864.D-12
   ISN 0021            DVAR(17) = 3623.D-12
   ISN 0022            DVAR(18) = 3410.D-12
   ISN 0023            DVAR(19) = 3220.D-12
   ISN 0024            DVAR(20) = 3051.D-12
   ISN 0025            DVAR(21) = 2899.D-12
   ISN 0026            DVAR(22) = 2761.D-12
   ISN 0027            DVAR(23) = 2635.D-12
   ISN 0028            DVAR(24) = 2522.D-12
   ISN 0029            DVAR(25) = 2417.D-12
   ISN 0030            DVAR(26) = 2321.D-12
   ISN 0031            DVAR(27) = 2232.D-12
   ISN 0032            DVAR(28) = 2149.D-12
   ISN 0033            DVAR(29) = 2072.D-12
   ISN 0034            DVAR(30) = 2001.D-12
   ISN 0035            DO 10     N=1,NMOD
   ISN 0036         10 DVER(N) = DVAR(N)
   ISN 0037            RETURN
   ISN 0038            END
```

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LEVEL 2.3.0 (JUNE 78)                    OS/360  FORTRAN H EXTENDED                 DATE 80.273/21.48.26        PAGE   1

REQUESTED OPTIONS: MAP,ID,OPT=2

OPTIONS IN EFFECT: NAME(MAIN) OPTIMIZE(2) LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
                   SOURCE EBCDIC NOLIST NODECK OBJECT MAP NOFORMAT COSTMT NOXREF NOALC NOANSF NOTERM IBM FLAG(I)

    ISN 0002             SUBROUTINE NVAR(DVAR,NMAX,DVAR0)                                   00025400
                   C                                                                        00025500
                   C            THIS SUBROUTINE COMPUTES ALL DECREE VARIANCES UP TO        00025600
                   C            DECREE  'NMAX'  , USING SOME CIVEN MODEL .                 00025700
                   C                                                                        00025800
    ISN 0003             IMPLICIT REAL*8 (A-H,O-Z)                                          00025900
    ISN 0004             DIMENSION DVAR(1)                                                  00026000
    ISN 0005             COMMON/ P / POWNMX
    ISN 0006             DVAR0 = 0.D0                                                       00026100
    ISN 0007             DVAR(  1) =     .0                                                 00026200
    ISN 0008             DVAR(  2) =    7.6396518                                           00026300
    ISN 0009             DVAR(  3) =    33.902065                                           00026400
    ISN 0010             DVAR(  4) =    19.743429                                           00026500
    ISN 0011             DVAR(  5) =    19.696103                                           00026600
    ISN 0012             DVAR(  6) =    18.390314                                           00026700
    ISN 0013             DVAR(  7) =    18.455112                                           00026800
    ISN 0014             DVAR(  8) =    10.399362                                           00026900
    ISN 0015             DVAR(  9) =    8.6398739                                           00027000
    ISN 0016             DVAR( 10) =    9.1838027                                           00027100
    ISN 0017             DVAR( 11) =    6.3690710                                           00027200
    ISN 0018             DVAR( 12) =    2.8993199                                           00027300
    ISN 0019             DVAR( 13) =    6.7286206                                           00027400
    ISN 0020             DVAR( 14) =    3.2033050                                           00027500
    ISN 0021             DVAR( 15) =    3.8198187                                           00027600
    ISN 0022             DVAR( 16) =    4.9444726                                           00027700
    ISN 0023             DVAR( 17) =    3.7168750                                           00027800
    ISN 0024             DVAR( 18) =    4.1988269                                           00027900
    ISN 0025             DVAR( 19) =    3.4360349                                           00028000
    ISN 0026             DVAR( 20) =    2.2931356                                           00028100
    ISN 0027             DVAR( 21) =    3.0053072                                           00028200
    ISN 0028             DVAR( 22) =    3.5082009                                           00028300
    ISN 0029             DVAR( 23) =    2.6197612                                           00028400
    ISN 0030             DVAR( 24) =    2.6880078                                           00028500
    ISN 0031             DVAR( 25) =    3.4761637                                           00028600
    ISN 0032             DVAR( 26) =    2.2578367                                           00028700
    ISN 0033             DVAR( 27) =    2.1490026                                           00028800
    ISN 0034             DVAR( 28) =    2.5805299                                           00028900
    ISN 0035             DVAR( 29) =    2.0568812                                           00029000
    ISN 0036             DVAR( 30) =    2.9207785                                           00029100
    ISN 0037             DVAR( 31) =    2.4778694                                           00029200
    ISN 0038             DVAR( 32) =    2.4971826                                           00029300
    ISN 0039             DVAR( 33) =    3.2040108                                           00029400
    ISN 0040             DVAR( 34) =    3.9115457                                           00029500
    ISN 0041             DVAR( 35) =    3.1070939                                           00029600
    ISN 0042             DVAR( 36) =    2.9680243                                           00029700
    ISN 0043             DVAR( 37) =    3.0126138                                           00029800
    ISN 0044             DVAR( 38) =    2.1749143                                           00029900
    ISN 0045             DVAR( 39) =    2.8229174                                           00030000
    ISN 0046             DVAR( 40) =    2.6698616                                           00030100
    ISN 0047             DVAR( 41) =    3.0049545                                           00030200
    ISN 0048             DVAR( 42) =    2.8244301                                           00030300
    ISN 0049             DVAR( 43) =    2.2442772                                           00030400
    ISN 0050             DVAR( 44) =    8.0705790                                           00030500
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LEVEL 2.3.0 (JUNE 78)          NVAR          OS/360  FORTRAN H EXTENDED          DATE 80.273/21.48.26          PAGE   2
      ISN 0051          DVAR( 45) =     2.9169446                                  00030600
      ISN 0052          DVAR( 46) =     2.7205179                                  00030700
      ISN 0053          DVAR( 47) =     3.4346806                                  00030800
      ISN 0054          DVAR( 48) =     2.8180541                                  00030900
      ISN 0055          DVAR( 49) =     2.2040232                                  00031000
      ISN 0056          DVAR( 50) =     3.4272534                                  00031100
      ISN 0057          DVAR( 51) =     2.8176026                                  00031200
      ISN 0058          DVAR( 52) =     2.6870577                                  00031300
      ISN 0059          DVAR( 53) =     3.4527293                                  00031400
      ISN 0060          DVAR( 54) =     3.0583738                                  00031500
      ISN 0061          DVAR( 55) =     3.1636091                                  00031600
      ISN 0062          DVAR( 56) =     2.8039749                                  00031700
      ISN 0063          DVAR( 57) =     3.2159365                                  00031800
      ISN 0064          DVAR( 58) =     2.8370499                                  00031900
      ISN 0065          DVAR( 59) =     2.9285564                                  00032000
      ISN 0066          DVAR( 60) =     3.2489338                                  00032100
      ISN 0067          DVAR( 61) =     2.6989317                                  00032200
      ISN 0068          DVAR( 62) =     3.3195569                                  00032300
      ISN 0069          DVAR( 63) =     2.8978374                                  00032400
      ISN 0070          DVAR( 64) =     2.7474396                                  00032500
      ISN 0071          DVAR( 65) =     2.2005284                                  00032600
      ISN 0072          DVAR( 66) =    -3.4103530                                  00032700
      ISN 0073          DVAR( 67) =     2.6096258                                  00032800
      ISN 0074          DVAR( 68) =     2.5628852                                  00032900
      ISN 0075          DVAR( 69) =     3.1792570                                  00033000
      ISN 0076          DVAR( 70) =     2.4727662                                  00033100
      ISN 0077          DVAR( 71) =     2.3711426                                  00033200
      ISN 0078          DVAR( 72) =     2.7305264                                  00033300
      ISN 0079          DVAR( 73) =     2.8756958                                  00033400
      ISN 0080          DVAR( 74) =     2.9566934                                  00033500
      ISN 0081          DVAR( 75) =     2.5071508                                  00033600
      ISN 0082          DVAR( 76) =     2.5442190                                  00033700
      ISN 0083          DVAR( 77) =     2.3198284                                  00033800
      ISN 0084          DVAR( 78) =     2.6555035                                  00033900
      ISN 0085          DVAR( 79) =     2.515[illegible]9                          00034000
      ISN 0086          DVAR( 80) =     2.80[illegible]285                         00034100
      ISN 0087          DVAR( 81) =     2.8064117                                  00034200
      ISN 0088          DVAR( 82) =     3.1910590                                  00034300
      ISN 0089          DVAR( 83) =     2.7446152                                  00034400
      ISN 0090          DVAR( 84) =     3.0200313                                  00034500
      ISN 0091          DVAR( 85) =     2.3415316                                  00034600
      ISN 0092          DVAR( 86) =     2.6750626                                  00034700
      ISN 0093          DVAR( 87) =     2.6775625                                  00034800
      ISN 0094          DVAR( 88) =     2.5177941                                  00034900
      ISN 0095          DVAR( 89) =     2.4932026                                  00035000
      ISN 0096          DVAR( 90) =     2.4443577                                  00035100
      ISN 0097          DVAR( 91) =     2.7512543                                  00035200
      ISN 0098          DVAR( 92) =     2.3148969                                  00035300
      ISN 0099          DVAR( 93) =     2.5410976                                  00035400
      ISN 0100          DVAR( 94) =     2.5721650                                  00035500
      ISN 0101          DVAR( 95) =     2.2715440                                  00035600
      ISN 0102          DVAR( 96) =     2.5384265                                  00035700
      ISN 0103          DVAR( 97) =     2.3560790                                  00035800
      ISN 0104          DVAR( 98) =     2.6830127                                  00035900
      ISN 0105          DVAR( 99) =     2.2631529                                  00036000
      ISN 0106          DVAR(100) =     2.3381125                                  00036100
      ISN 0107          A = 1.D0                                                   00036300
      ISN 0108          B = 2.D0                                                   00036400
```

```
                                                                                      00036500
  ISN 0109          A1 = 3.4050D0                                                     00036600
  ISN 0110          A2 = 140.03D0                                                     00036700
  ISN 0111          S1 = 0.998006D0                                                   00036800
  ISN 0112          S2 = 0.914232D0                                                   00036200
  ISN 0113          IF(NMAX.LE.100) GO TO 20                                          00036900
  ISN 0115          DO 10      N = 101,NMAX                                           00037000
  ISN 0116          DVAR(N) = (N-1)*(S1**(N+2)*A1/(N+A)+S2**(N+2)*A2/((N+B)*(N-2)))  00037100
  ISN 0117       10 CONTINUE                                                          00037200
  ISN 0118       20 CONTINUE
  ISN 0119          POWNMX = 0.D0
  ISN 0120          NMXP = NMAX+1
  ISN 0121          DO 30      N = NMXP,2000
  ISN 0122       30 POWNMX = POWNMX+4.2089027D11*((N-1)*(S1**(N+2)*A1/(N+A)+S2**(N+2)
                   2 *A2/((N+B)*(N-2))))/(N-1)**2                                     00037300
  ISN 0123          RETURN                                                            00037400
  ISN 0124          END
```

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LEVEL 2.3.0 (JUNE 78)                    OS/360  FORTRAN H EXTENDED            DATE 80.274/00.02.02        PAGE   1

REQUESTED OPTIONS: MAP,ID,OPT=2

OPTIONS IN EFFECT: NAME(MAIN) OPTIMIZE(2) LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
                   SOURCE EBCDIC NOLIST NODECK OBJECT MAP NOFORMAT GOSTMT NOXREF NOALC NOANSF NOTERM IBM FLAG(I)

                 C
                 C          PROGRAM FOR THE ERROR ANALYSIS OF THE GLOBAL MODELLING
                 C          OF A GRAVITY FIELD FROM SATELLITE TO SATELLITE TRACKING DAT
                 C
                 C     **** PROGRAMMED BY OSCAR L. COLOMBO, GEODETIC SCIENCE,
                 C    THE OHIO STATE UNIVERSITY, SEPTEMBER 1980  ******
                 C
ISN 0002               IMPLICIT REAL*8(A-H,O-Z)
ISN 0003               REAL*4 EPS
ISN 0004               REAL*4 Y,Z
ISN 0005               REAL*4 YLIM
ISN 0006               REAL*4 TP(500)
ISN 0007               COMMON/ P / POWNMX
ISN 0008               DIMENSION CMN(500),SPC(500),CPS(500),FF(500),DVAR(500),
                      2 RCPNM(500),IDD(500)
ISN 0009               DIMENSION CPNMS(56000),AE(15000),AO(15000),CNI(500)
ISN 0010               DIMENSION PIVE(500),PIVO(500),RMS(500)
ISN 0011               DIMENSION  CMR(500),T(500),PE(500),PO(500),PE2(500),PO2(500)
ISN 0012               DIMENSION PPB(500),PPTR(500),PERC(500)
ISN 0013               DATA RMS/500*0.D0/
                 C
                 C          DEFINITION OF THE PROBLEM. EVERYTHING IS IN METERS AND
                 C          IN SECONDS .
                 C          NE  AND  NS  MUST BE MUTUALLY PRIME, AND  NS GT. 2*NMAX .
                 C          THE SAMPLING INTERVAL  DS  MUST DIVIDE  NE*24*3600/NS
                 C          EXACTLY .
                 C
ISN 0014               IX = 1
ISN 0015               CALL ERRSET(208,256,-1,2,IX,0)
ISN 0016               YLIM = 1740.
ISN 0017               IU = 6
ISN 0018               IU = 9
ISN 0019               IP = 6
ISN 0020               IP = 7
ISN 0021               HS = 300.D3
ISN 0022               READ(10) NMOD,NMAX,NMIN,NE,NS,BS,D12,SIGMA,DA,EPS0
                 C
                 C          DEFINE MODIFIED PROBLEM HERE
                 C
                 C ******************************************************************
ISN 0023               HS = 300.D3
ISN 0024               IAC = 1
                 C          IAC = 0 FOR L.S. ADJUSTMENT,
                 C          IAC = 1 FOR L.S. COLLOCATION .
                 C ******************************************************************
ISN 0025               DS = DA
ISN 0026               IF((NMAX/2)*2.EQ.NMAX) NMAX =  NMAX+1
ISN 0028               NMAXP = NMAX+1
ISN 0029               NML = NMAXP/2+1
ISN 0030               NMOD1 = NMOD+1
ISN 0031               WRITE(6 ,2) NMAX,NMIN,NMOD,NE,NS,HS,D12,DA,DS,SIGMA
ISN 0032               WRITE(IU,2) NMAX,NMIN,NMOD,NE,NS,HS,D12,DA,DS,SIGMA
ISN 0033             2 FORMAT('1',' ERROR ANALYSIS OF A SATELLITE TO SATELLITE TRACKING
```

REDUCED VERSION

OF THE MAIN

PROGRAM

```
                 2MISSION'//,' MAXIMUM DEGREE AND ORDER CONSIDERED = ',I5//       00007000
                 3,' MINIMUM DEGREE CONSIDERED =  2      MINIMUM OLDER ANALIZED = ', 00007100
                 8I5//,                                                           00007200
                 #' MAXIMUM DEGREE AND ORDER IN REFERENCE MODEL = ',I5//,         00007300
                 4' DURATION OF MISSION IN DAYS = ',I5//,                         00007400
                 5' NUMBER OF REVOLUTIONS DURING MISSION = ',I5//,                00007500
                 6' HEIGHT OF THE SATELLITES ABOVE GROUND = ',G20.10,' IN METERS'//,00007600
                 7' SEPARATION BETWEEN SATELLITES = ',G20.10,' IN METERS'//       00007700
                 8,' AVERAGING INTERVAL = ',G20.10,' SECONDS '//,                 00007800
                 9' SAMPLING INTERVAL  = ',G20.10,' SECONDS '//,                  00007900
                 #' ACCURACY OF THE DATA = ',G12.4,' METERS PER SECOND '////)     00008000
ISN 0034          IF(IAC.EQ.0) WRITE(6,3)                                         00008100
ISN 0036          IF(IAC.EQ.0) WRITE(IU,3)                                        00008200
ISN 0038          IF(IAC.EQ.1) WRITE(6,4)                                         00008300
ISN 0040          IF(IAC.EQ.1) WRITE(IU,4)                                        00008400
ISN 0042        3 FORMAT(' ANALYSIS IS BASED ON LEAST SQUARES ADJUSTMENT *****'////)00008500
ISN 0043        4 FORMAT(' ANALYSIS IS BASED ON LEAST SQUARES COLLOCATION ****'////)00008600
ISN 0044          ISEED = 20000                                                   00008700
ISN 0045          GAMMA = 9.8202641D0                                             00008800
ISN 0046          RE = 6371.D3                                                    00008900
ISN 0047          RS = RE+HS                                                      00009000
ISN 0048          GAMS = GAMMA*(RE/RS)**2                                         00009100
ISN 0049          PSI122 = DARSIN(D12*0.5D0/RS)                                   00009200
ISN 0050          PSI12 = 2.D0*PSI122                                             00009300
ISN 0051          SIN122 = DSIN(PSI122)                                           00009400
ISN 0052          COS122 = DCOS(PSI122)                                           00009500
ISN 0053          PI = 4.D0*DATAN(1.D0)                                           00009600
ISN 0054          T0 = NE*24.D0*3600.D0                                           00009700
ISN 0055          W0 = 2.D0*PI/T0                                                 00009800
ISN 0056          NSR = 2*NMAXP                                                   00009900
ISN 0057          NM = T0/DS+0.0000001D0                                          00010000
ISN 0058          FNS = NM      /(SIGMA**2*W0**4*DA**2*NSR**2)                    00010100
ISN 0059          FNSR = FNS/FNS0
ISN 0060          W0DA = W0*DA                                                    00010300
ISN 0061          XNS = NS                                                        00010400
ISN 0062          XNE = NE                                                        00010500
ISN 0063          DO 5       KP = 1,NMAX                                          00010600
ISN 0064          IDD(KP) = ((KP-1)*KP)/2                                         00010700
ISN 0065        5 CONTINUE                                                        00011000
           C                                                                      00011100
           C          FORM THE INVERSE OF THE DIAGONAL MATRIX  C  IN ARRAY  CMI  .00011200
           C                                                                      00011300
ISN 0066          CALL NVAR(DVAR,NMAX,DVAR0)                                      00011400
ISN 0067          CALL MODEL(DVAR,NMOD)                                           00011500
ISN 0068          DO 8      N = 2,NMOD                                            00011600
ISN 0069        8 CMI(N) = 1.D0/DVAR(N)**2                                        00011700
ISN 0070          NMAXT = NMAX                                                    00011800
ISN 0071          IF(NMOD1.GT.NMAX) NMAXT = NMOD1                                 00011900
ISN 0073          DO 9       N = NMOD1,NMAXT                                      00012000
ISN 0074        9 CMI(N) = (2.D0*N+1.D0)*((N-1)*GAMMA*1.D5)**2/DVAR(N)            00012100
ISN 0075          CMI(1) = 0.D0                                                   00012200
ISN 0076          DO 11      N = 2,NMAX                                           00012300
ISN 0077          CMR(N) = 1.D0/(GAMS*(RE/RS)**N)**2                              00012400
ISN 0078          CMI(N) = CMI(N)*CMR(N)                                          00012500
ISN 0079       11 CONTINUE                                                        00012600
ISN 0080          WRITE(6,17)                                                     00012700
ISN 0081       17 FORMAT(//' N    M    COEFFICIENTS VARIANCES (UP TO N,M = 10 )'//) 00012800
ISN 0082          CALL SCLOK1                                                     00012900
```

```
ISN 0083              Z = 1.                                                                     00013000
                C                                                                                00013100
                C              BEGINNING OF THE MAIN LOOP.                                       00013200
                C              HERE THE ERRORS PER COEFFICIENT (VARIANCES) ARE FOUND, WITH00013300
                C              INCREASING ORDER  M , AND ADDED UP TO OBTAIN THE RMS              00013400
                C              PERCENTAGE ERROR PER DEGREE .                                     00013500
                C                                                                                00013600
ISN 0084                DO 100        M = MMIN,MMAX                                              00013700
ISN 0085              Y = RCLOK((Z)                                                              00013800
ISN 0086              WRITE(6,13)  M,Y
ISN 0087           13 FORMAT(' TIME BEFORE  M = ',I5,5X,F10.2,' SECONDS')
ISN 0088              IF(Y.GE.YLIM) GO TO 500                                                    00014100
ISN 0090              MMIN = M                                                                   00016100
ISN 0091              IF(M.LT.2) MMIN = 2                                                        00016200
ISN 0093              READ(10) NEV,NOD,NNEV,NNOD
ISN 0094              IF(M.EQ.MMAX) GO TO  77
ISN 0096              CALL REO(AE,AO,NEV,NOD,NNEV,NNOD)
ISN 0097              GO TO 78
ISN 0098           77 READ(10) AE(1)
                C
                C              RE-SCALE THE NORMALS ACCORDING TO NEW DEFINITION
                C              OF THE PROBLEM, AND ADD THE INVERSE OF THE  C  MATRIX
                C              IF THE SOLUTION REQUESTED IS BY COLLOCATION .
                C
ISN 0099           78 DO 14     I = 1,NEV
ISN 0100              DO 14          J = 1,I
ISN 0101           14 AE(IDD(I)+J) = AE(IDD(I)+J)*FNSR
ISN 0102              DO  16       I = 1,NOD
ISN 0103              DO 16      J = 1,I
ISN 0104           16 AO(IDD(I)+J) = AO(IDD(I)+J)*FNSR
ISN 0105              DO 15         N = NMIN,NMAX
ISN 0106              IF(IAC.EQ.0) GO TO 15
ISN 0108              IF(((N-M)/2)*2.NE.(N-M)) GO TO 24
ISN 0110              I = (N-M+1)/2+1
ISN 0111              IF(M.EQ.0) I = I-1
ISN 0113              IF(M.EQ.1) I = I-1
ISN 0115              AE(IDD(I)+I) = AE(IDD(I)+I)+CMI(N)
ISN 0116              GO TO 15
ISN 0117           24 I = (N-M-1)/2+1
ISN 0118              IF(M.EQ.0) I = I-1
ISN 0120              AO(IDD(I)+I) = AO(IDD(I)+I)+CMI(N)
ISN 0121           15 CONTINUE
                C
                C              APPLY PIVOTING .
                C
ISN 0122              DO 18          I = 1,NEV
ISN 0123           18 PIVE(I) = 1.D0/DSQRT(AE(IDD(I)+I))
ISN 0124              DO 19          I = 1,NOD
ISN 0125           19  PIVO(I) = 1.D0/DSQRT(AO(IDD(I)+I))
ISN 0126              DO 23     I = 1,NEV
ISN 0127              DO 23      J = 1,I
ISN 0128           23 AE(IDD(I)+J) = AE(IDD(I)+J)*PIVE(I)*PIVE(J)
ISN 0129              DO 29     I = 1,NOD
ISN 0130              DO 29     J = 1,I
ISN 0131           29 AO(IDD(I)+J) = AO(IDD(I)+J)*PIVO(I)*PIVO(J)
                C              INVERTING THE DIAGONAL BLOCKS OF THE NORMALS CORRESPONDING 00021600
                C              TO ORDER  M  .
                C                                                                                00022000
```

```
ISN 0132          EPS = 1.E-7                                                        00022700
ISN 0133          CALL DSINV(AE,NEV,EPS,IER1)                                        00022800
ISN 0134          IF(M.NE.NMAX) CALL DSINV(AO,NOD,EPS,IER2)                          00022900
ISN 0136          IF(IER1.NE.0.OR.IER2.NE.0) WRITE(6,21) M,IER1,IER2                 00023000
ISN 0138       21 FORMAT(//' $$$$$ AT ORDER  ',I5,': '/,' ** IER1 = ',I5,' IER2 = ',00023100
                 2 I5,' *************'//)                                           00023200
ISN 0139          IF(IER1.NE.0.OR.IER2.NE.0) GO TO 500                               00023300
ISN 0141          IF(M.EQ.NMAX) GO TO 334                                            00023400
ISN 0143      334 DO 55     I = 1,NEV                                                00024700
ISN 0144          AE(IDD(I)+I) = AE(IDD(I)+I)*PIVE(I)**2
ISN 0145       55 CONTINUE                                                           00024900
ISN 0146          DO 65      I = 1,NOD                                               00025000
ISN 0147          AO(IDD(I)+I) = AO(IDD(I)+I)*PIVO(I)**2
ISN 0148       65 CONTINUE                                                           00025200
ISN 0149          DO 80        N = NMIN,NMAX                                         00025300
ISN 0150          ITS = 0                                                            00025400
ISN 0151          IF(((N-M)/2)*2.NE.N-M) ITS = 1                                     00025500
ISN 0153          I = (N-NMIN)/2+1                                                   00025600
ISN 0154          IF(ITS.EQ.0) DIRV = AE(IDD(I)+I)*CMR(N)                            00025700
ISN 0156          IF(ITS.EQ.1) DIRV = AO(IDD(I)+I)*CMR(N)                            00025800
ISN 0158          IF(N.LE.40) WRITE(6,104) N,M,DIRV                                  00025900
ISN 0160      104 FORMAT(2I5,G20.10)                                                 00026000
ISN 0161          DIRVO = DIRV*2.D0                                                  00026100
ISN 0162          IF(M.EQ.0) DIRVO = DIRV                                            00026200
ISN 0164          RMS(N) = RMS(N)+DIRVO                                              00026300
ISN 0165       80 CONTINUE                                                           00026400
ISN 0166      100 CONTINUE                                                           00026500
             C                                                                       00026600
             C             END OF MAIN LOOP .                                        00026700
             C                                                                       00026800
             C             PRINT AND PUNCH ALL THE RESULTS .                         00026900
             C                                                                       00027000
ISN 0167      500 WRITE(IP,120) (N,RMS(N),N=2,NMAX)                                  00027100
ISN 0168      120 FORMAT(4(I3,G17.11))                                               00027200
ISN 0169          WRITE(IU,105)                                                      00027300
ISN 0170      105 FORMAT('1',20X,'  N  PERCENTAGE ERROR      ERROI. VARIANCE(POT.)   00027400
                 2BAND ERROR (UNDUL.) (M)    TOTAL UNDUL. ERROR (M) '////)           00027500
ISN 0171          PB = 0.D0                                                          00027600
ISN 0172          POW = 0.D0                                                         00027700
ISN 0173          DO 106     N = 2,NMAX                                              00027800
ISN 0174      106 POW = POW+CMR(N)*(2*N+1)/CM1(N)                                    00027900
ISN 0175          PTR = POW*RE**2+POWNMX*1.D-10                                      00028000
ISN 0176          DO 110       N = 2,NMAX                                            00028100
ISN 0177          PN = GMR(N)*RE**2*(2*N+1)/CM1(N)                                   00028200
ISN 0178          PERR = RMS(N)*RE**2                                                00028300
ISN 0179          PTR = PTR-PN+PERR                                                  00028400
ISN 0180          PB = PB+PERR                                                       00028500
ISN 0181          PERC(N) = DSQRT(RMS(N)*CM1(N)/(GMR(N)*(2*N+1)))*100.D0             00028600
ISN 0182          PPB(N) = DSQRT(PB)                                                 00028700
ISN 0183          PPTR(N) = DSQRT(PTR)                                               00028800
ISN 0184          IF((N/50)*50.EQ.N) WRITE(IU,105)                                   00028900
ISN 0186          WRITE(IU,107) N,PERC(N),RMS(N),PPB(N),PPTR(N)                      00029000
ISN 0187      107 FORMAT(20X,I5,4(7X,G13.5))                                         00029100
ISN 0188      110 CONTINUE                                                           00029200
ISN 0189          WRITE(IU,105)                                                      00029300
ISN 0190          DO 115       N = 2,NMAX                                            00029400
ISN 0191          IF((N/10)*10.EQ.N.OR.N.EQ.2) WRITE(IU,107)N,PERC(N),RMS(N),        00029500
                 2 PPB(N),PPTR(N)                                                    00029600
```

```
ISN 0193       118 CONTINUE                                                        00029700
ISN 0194           STOP                                                            00029800
ISN 0195           END
```

```
LEVEL 2.3.0 (JUNE 78)                    OS/360  FORTRAN H EXTENDED           DATE 80.274/00.02.05       PAGE   1

REQUESTED OPTIONS: MAP,ID,OPT=2

OPTIONS IN EFFECT: NAME(MAIN) OPTIMIZE(2) LINECOUNT(60) SIZE(MAX) AUTODBL(NONE)
                   SOURCE EBCDIC NOLIST NODECK OBJECT MAP NOFORMAT GOSTMT NOXREF NOALC NOANSF NOTERM IBM FLAG(I)

  ISN 0002              SUBROUTINE RED(AE,AO,NEV,NOD,NNEV,NNOD)
                  C
                  C            READS THE NORMALS FROM UNIT  10
                  C
  ISN 0003              IMPLICIT REAL*8(A-H,O-Z)
  ISN 0004              DIMENSION AE(NNEV),AO(NNOD)
  ISN 0005              READ(10) AE,AO
  ISN 0006              RETURN
  ISN 0007              END
```

```
ERROR ANALYSIS OF A SATELLITE TO SATELLITE TRACKING  MISSION
MAXIMUM DEGREE AND ORDER CONSIDERED =    11
MINIMUM DEGREE CONSIDERED =  2      MINIMUM ORDER ANALIZED =     0
MAXIMUM DEGREE AND ORDER IN REFERENCE MODEL =    20
DURATION OF MISSION IN DAYS =   179
NUMBER OF REVOLUTIONS DURING MISSION =  2933
HEIGHT OF THE SATELLITES ABOVE GROUND =       160000.0000     IN METERS
SEPARATION BETWEEN SATELLITES =      300000.0000     IN METERS
AVERAGING INTERVAL =      4.000000000      SECONDS
SAMPLING INTERVAL  =      4.000000000      SECONDS
ACCURACY OF THE DATA =    .1000D-05 METERS PER SECOND


ANALYSIS IS BASED ON LEAST SQUARES ADJUSTMENT *****



N   M   COEFFICIENTS VARIANCES (UP TO N,M = 10 )

TIME BEFORE M =         0.00 SECONDS
ACCURACY OF INVERSION, M =      0=      .1158309042D-14     .1092473701D-14
  2     0     .6486727949D-24
  3     0     .3666464972D-24
  4     0     .3173934560D-24
  5     0     .2387266999D-24
  6     0     .1917693406D-24
  7     0     .1570709735D-24
  8     0     .1320842714D-24
  9     0     .1138015819D-24
 10     0     .8327279525D-25
 11.    0     .7389421803D-23
TIME BEFORE M =         0.01 SECONDS
ACCURACY OF INVERSION, M =      1=      .1583041390D-14     .1104797430D-14
  2     1     .4068172089D-24
  3     1     .5194702956D-23
  4     1     .1789835767D-24
  5     1     .1957257722D-23
  6     1     .1033747934D-24
  7     1     .9387707878D-24
  8     1     .6984804134D-25
  9     1     .5647601539D-24
 10     1     .4579491904D-25
 11     1     .2994807711D-24
TIME BEFORE M =         0.03 SECONDS
ACCURACY OF INVERSION, M =      2=      .4725952219D-14     .1848487996D-14
  2     2     .2685761113D-24
  3     2     .2914233052D-24
  4     2     .1955174717D-24
  5     2     .1513781500D-24
  6     2     .1147107039D-24
  7     2     .9380313350D-25
  8     2     .7674747319D-25
```

```
   9     2      .6580376468D-25
  10     2      .5239386638D-25
  11     2      .4640697615D-25
TIME BEFORE M =        0.04 SECONDS
ACCURACY OF INVERSION, M =      3=       .2943269082D-15      .1867065398D-15
   3     3      .2781031375D-24
   4     3      .2876680203D-24
   5     3      .2016374929D-24
   6     3      .1548998318D-24
   7     3      .1195023648D-24
   8     3      .9679800206D-25
   9     3      .7993143278D-25
  10     3      .6706386718D-25
  11     3      .5776510686D-25
TIME BEFORE M =        0.06 SECONDS
ACCURACY OF INVERSION, M =      4=       .2422220672D-14      .1311579440D-14
   4     4      .1780190476D-24
   5     4      .2201324673D-24
   6     4      .1548791110D-24
   7     4      .1270338618D-24
   8     4      .1009388478D-24
   9     4      .8512541043D-25
  10     4      .7049249303D-25
  11     4      .6140146305D-25
TIME BEFORE M =        0.07 SECONDS
ACCURACY OF INVERSION, M =      5=       .4661986639D-15      .2366016985D-15
   5     5      .1696120657D-24
   6     5      .2247745936D-24
   7     5      .1644859304D-24
   8     5      .1370181370D-24
   9     5      .1096707214D-24
  10     5      .9251954156D-25
  11     5      .7808065664D-25
TIME BEFORE M =        0.08 SECONDS
ACCURACY OF INVERSION, M =      6=       .9222901164D-16      .8278733416D-16
   6     6      .1313771510D-24
   7     6      .1752356965D-24
   8     6      .1321311534D-24
   9     6      .1094862496D-24
  10     6      .9296781911D-25
  11     6      .7834218947D-25
TIME BEFORE M =        0.09 SECONDS
ACCURACY OF INVERSION, M =      7=       .3116585742D-15      .1148591316D-15
   7     7      .1160630345D-24
   8     7      .1873104298D-24
   9     7      .1406444707D-24
  10     7      .1240218092D-24
  11     7      .1021836742D-24
TIME BEFORE M =        0.10 SECONDS
ACCURACY OF INVERSION, M =      8=       .1570092459D-15      .3925231147D-15
   8     8      .9810596401D-25
   9     8      .1370633284D-24
  10     8      .1190304051D-24
  11     8      .9349494144D-25
TIME BEFORE M =        0.11 SECONDS
ACCURACY OF INVERSION, M =      9=       .7021566937D-15      .4336808690D-17
   9     9      .7569826271D-25
  10     9      .1619917818D-24
  11     9      .1243636673D-24
TIME BEFORE M =        0.11 SECONDS
ACCURACY OF INVERSION, M =     10=       .2220446049D-15      .2220446049D-15
  10    10      .6422421250D-25
  11    10      .8879246034D-25
TIME BEFORE M =        0.12 SECONDS
  11    11      .2056732295D-25
```

```
ERROR ANALYSIS OF A SATELLITE TO SATELLITE TRACKING  MISSION
MAXIMUM DEGREE AND ORDER CONSIDERED =    11
MINIMUM DEGREE CONSIDERED =  2     MINIMUM ORDER ANALIZED =     0
MAXIMUM DEGREE AND ORDER IN REFERENCE MODEL =    20
DURATION OF MISSION IN DAYS =   179
NUMBER OF REVOLUTIONS DURING MISSION =  2933
HEIGHT OF THE SATELLITES ABOVE GROUND =       160000.0000     IN METERS
SEPARATION BETWEEN SATELLITES =      300000.0000     IN METERS
AVERAGING INTERVAL =      4.000000000     SECONDS
SAMPLING INTERVAL  =      4.000000000     SECONDS
ACCURACY OF THE DATA =    .1000D-05 METERS PER SECOND

ANALYSIS IS BASED ON LEAST SQUARES ADJUSTMENT *****
```

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	.23249D-01	.19995D-23	.90087D-05	3.9607
3	.19769D-01	.11895D-22	.23748D-04	3.9591
4	.10323D-01	.19978D-23	.25398D-04	3.9582
5	.98932D-02	.56388D-23	.29562D-04	3.9552
6	.68082D-02	.19598D-23	.30879D-04	3.9530
7	.73315D-02	.36269D-23	.33177D-04	3.9496
8	.59267D-02	.19299D-23	.34337D-04	3.9467
9	.67629D-02	.28502D-23	.35982D-04	3.9435
10	.58216D-02	.18643D-23	.37019D-04	3.9407
11	.61638D-02	.21758D-23	.38193D-04	3.9378

B	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	.28249D-01	.19995D-23	.90087D-05	3.9607
10	.58216D-02	.18643D-23	.37019D-04	3.9407

Appendix C : Detailed Listings Degree by Degree

This Appendix contains the full listing of the formal accuracies of the potential coefficients from the error analysis of the SST mission using (a) least squares adjustment theory, and (b) least squares collocation theory. The rms of the total undulation error (last column) for least squares collocation should be corrected as indicated in paragraph 3.1. Notice the fluctuations in percentage error in the neighborhoods of $n = 136$ and $n = 273$.

N	[illegible]	[illegible]	[illegible] (M)	TOTAL UNDUL. ERROR (M)
2	[illegible]	[illegible]	[illegible]	2.4629
3	[illegible]	[illegible]	[illegible]	2.4605
4	[illegible]	[illegible]	[illegible]	2.4518
5	[illegible]	[illegible]	[illegible]	2.4549
6	.7[illegible]	[illegible]	[illegible]	2.4505
7	[illegible]	[illegible]	[illegible]	2.4419
8	.6[illegible]	[illegible]	[illegible]	2.4404
9	.7[illegible]	[illegible]	[illegible]	2.4352
10	[illegible]	[illegible]	[illegible]	2.4366
11	[illegible]	[illegible]	[illegible]	2.4255
12	[illegible]	[illegible]	[illegible]	2.4277
13	[illegible]	[illegible]	[illegible]	2.4164
14	.6[illegible]	[illegible]	[illegible]	[illegible]
15	.7[illegible]	[illegible]	[illegible]	[illegible]
16	[illegible]	[illegible]	[illegible]	[illegible]
17	.7[illegible]	[illegible]	[illegible]	[illegible]
18	.7[illegible]	[illegible]	[illegible]	[illegible]
19	[illegible]	[illegible]	[illegible]	[illegible]
20	[illegible]	[illegible]	[illegible]	[illegible]
21	[illegible]	[illegible]	[illegible]	[illegible]
22	[illegible]	[illegible]	[illegible]	[illegible]
23	[illegible]	[illegible]	[illegible]	[illegible]
24	[illegible]	[illegible]	[illegible]	[illegible]
25	[illegible]	[illegible]	[illegible]	[illegible]
26	[illegible]	[illegible]	[illegible]	[illegible]
27	.8[illegible]	[illegible]	[illegible]	[illegible]
28	[illegible]	[illegible]	[illegible]	[illegible]
29	[illegible]	[illegible]	[illegible]	[illegible]
30	[illegible]	[illegible]	[illegible]	[illegible]
31	[illegible]	[illegible]	[illegible]	[illegible]
32	[illegible]	[illegible]	[illegible]	[illegible]
33	[illegible]	[illegible]	[illegible]	[illegible]
34	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
45	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
47	[illegible]	[illegible]	[illegible]	[illegible]
49	[illegible]	[illegible]	[illegible]	[illegible]

Table C.1 [illegible]

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
50	.73134D-02	.79167D-23	.91745D-04	1.4171
51	.85445D-02	.86323D-23	.93613D-04	1.4002
52	.90369D-02	.86508D-23	.95470D-04	1.3848
53	.83907D-02	.93220D-23	.97432D-04	1.3653
54	.91565D-02	.94656D-23	.99384D-04	1.3484
55	.95216D-02	.10199D-22	.10145D-03	1.3313
56	.10387D-01	.10371D-22	.10350D-03	1.3166
57	.10251D-01	.11175D-22	.10567D-03	1.3001
58	.11209D-01	.11377D-22	.10783D-03	1.2859
59	.11654D-01	.12261D-22	.11011D-03	1.2716
60	.11369D-01	.12498D-22	.11239D-03	1.2560
61	.13164D-01	.13471D-22	.11480D-03	1.2434
62	.12190D-01	.13746D-22	.11721D-03	1.2282
63	.13770D-01	.14822D-22	.11974D-03	1.2152
64	.14823D-01	.15139D-22	.12228D-03	1.2032
65	.17121D-01	.16330D-22	.12496D-03	1.1937
66	.14123D-01	.16696D-22	.12765D-03	1.1794
67	.17030D-01	.18017D-22	.13040D-03	1.1687
68	.17647D-01	.18437D-22	.13332D-03	1.1584
69	.16710D-01	.19907D-22	.13631D-03	1.1458
70	.19457D-01	.20380D-22	.13932D-03	1.1362
71	.20952D-01	.22027D-22	.14249D-03	1.1273
72	.20049D-01	.22578D-22	.14567D-03	1.1171
73	.20600D-01	.24411D-22	.14903D-03	1.1066
74	.20862D-01	.25040D-22	.15240D-03	1.0960
75	.23890D-01	.27096D-22	.15597D-03	1.0871
76	.24352D-01	.27813D-22	.15955D-03	1.0783
77	.26895D-01	.30125D-22	.16333D-03	1.0705
78	.25813D-01	.30945D-22	.16714D-03	1.0616
79	.27977D-01	.33552D-22	.17116D-03	1.0534
80	.27223D-01	.34409D-22	.17520D-03	1.0444
81	.28694D-01	.37438D-22	.17949D-03	1.0355
82	.27633D-01	.38509D-22	.18379D-03	1.0256
83	.31447D-01	.41857D-22	.18835D-03	1.0172
84	.30786D-01	.43083D-22	.19294D-03	1.0081
85	.36916D-01	.46895D-22	.19781D-03	1.0011
86	.35470D-01	.48303D-22	.20271D-03	.99332
87	.37454D-01	.52660D-22	.20791D-03	.98562
88	.39669D-01	.54279D-22	.21315D-03	.97849
89	.42139D-01	.59280D-22	.21872D-03	.97154
90	.43714D-01	.61148D-22	.22432D-03	.96483
91	.43586D-01	.66911D-22	.23029D-03	.95740
92	.48816D-01	.69076D-22	.23630D-03	.95123
93	.49327D-01	.75748D-22	.24272D-03	.94457
94	.50379D-01	.78267D-22	.24918D-03	.93792
95	.56089D-01	.86032D-22	.25609D-03	.93213
96	.55235D-01	.88981D-22	.26305D-03	.92576
97	.60822D-01	.98066D-22	.27051D-03	.91993
98	.58601D-01	.10154D-21	.27802D-03	.91338
99	.67774D-01	.11224D-21	.28610D-03	.90794

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
100	.68886D-01	.11636D-21	.29424D-03	.90239
101	.67665D-01	.12904D-21	.30301D-03	.89603
102	.69702D-01	.13399D-21	.31185D-03	.88976
103	.74331D-01	.14913D-21	.32141D-03	.88358
104	.76623D-01	.15513D-21	.33106D-03	.87750
105	.81863D-01	.17337D-21	.34152D-03	.87149
106	.84464D-01	.18073D-21	.35210D-03	.86557
107	.90432D-01	.20291D-21	.36361D-03	.85973
108	.93414D-01	.21210D-21	.37526D-03	.85398
109	.10026	.23937D-21	.38799D-03	.84830
110	.10372	.25104D-21	.40090D-03	.84270
111	.11162	.28499D-21	.41508D-03	.83717
112	.11570	.30016D-21	.42951D-03	.83172
113	.12492	.34304D-21	.44542D-03	.82634
114	.12982	.36326D-21	.46168D-03	.82103
115	.14068	.41838D-21	.47971D-03	.81578
116	.14669	.44618D-21	.49823D-03	.81061
117	.15965	.51848D-21	.51892D-03	.80550
118	.16723	.55817D-21	.54031D-03	.80046
119	.18294	.65543D-21	.56439D-03	.79547
120	.19283	.71472D-21	.58953D-03	.79056
121	.21222	.84973D-21	.61809D-03	.78570
122	.22569	.94345D-21	.64833D-03	.78090
123	.25023	.11387D-20	.68305D-03	.77616
124	.26959	.12979D-20	.72058D-03	.77147
125	.30170	.15964D-20	.76422D-03	.76685
126	.33162	.18944D-20	.81298D-03	.76227
127	.37578	.23897D-20	.87059D-03	.75775
128	.42695	.30300D-20	.93859D-03	.75329
129	.49297	.39702D-20	.10208D-02	.74887
130	.59562	.56954D-20	.11284D-02	.74451
131	.71323	.80264D-20	.12645D-02	.74020
132	.98982	.15195D-19	.14806D-02	.73593
133	1.3525	.27898D-19	.18297D-02	.73172
134	2.7136	.11038D-18	.27979D-02	.72755
135	3.0583	.13786D-18	.36639D-02	.72343
136	5.3080	.40837D-18	.54772D-02	.71937
137	3.7068	.19587D-18	.61603D-02	.71534
138	3.0320	.12809D-18	.65712D-02	.71135
139	3.0987	.13243D-18	.69683D-02	.70741
140	2.8004	.10641D-18	.72716D-02	.70351
141	2.7930	.10415D-18	.75567D-02	.69965
142	2.7258	.97610D-19	.78144D-02	.69583
143	2.7360	.96776D-19	.80618D-02	.69206
144	2.7194	.94094D-19	.82953D-02	.68832
145	2.7438	.94286D-19	.85229D-02	.68462
146	2.7517	.93343D-19	.87423D-02	.68096
147	2.7867	.94252D-19	.89504D-02	.67733
148	2.8098	.94341D-19	.91697D-02	.67374
149	2.8535	.95801D-19	.93793D-02	.67019

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
150	2.8876	.96610D-19	.95861D-02	.66668
151	2.9385	.98529D-19	.97925D-02	.66320
152	2.9814	.99899D-19	.99973D-02	.65975
153	3.0389	.10223D-18	.10203D-01	.65634
154	3.0893	.10407D-18	.10408D-01	.65297
155	3.1529	.10679D-18	.[illegible]614D-01	.64962
156	3.2101	.10907D-18	.10820D-01	.64631
157	3.2795	.11217D-18	.11029D-01	.64303
158	3.3430	.11485D-18	.11238D-01	.63978
159	3.4182	.11834D-18	.11450D-01	.63657
160	3.4877	.12142D-18	.11663D-01	.63338
161	3.5688	.12531D-18	.11879D-01	.63022
162	3.6442	.12880D-18	.12097D-01	.62710
163	3.7312	.13310D-18	.12319D-01	.62400
164	3.8125	.13700D-18	.12542D-01	.62093
165	3.9055	.14174D-18	.12769D-01	.61789
166	3.9928	.14608D-18	.13000D-01	.61488
167	4.0920	.15129D-18	.13234D-01	.61189
168	4.1854	.15609D-18	.13471D-01	.60894
169	4.2910	.16180D-18	.13713D-01	.60600
170	4.3906	.16709D-18	.13958D-01	.60310
171	4.5029	.17334D-18	.14207D-01	.60022
172	4.6089	.17914D-18	.14461D-01	.59737
173	4.7282	.18599D-18	.14720D-01	.59454
174	4.8409	.19234D-18	.14983D-01	.59174
175	4.9674	.19982D-18	.15251D-01	.58896
176	5.0870	.20677D-18	.15524D-01	.58621
177	5.2212	.21493D-18	.15802D-01	.58348
178	5.3479	.22252D-18	.16085D-01	.58078
179	5.4901	.23144D-18	.16375D-01	.57810
180	5.6244	.23973D-18	.16669D-01	.57544
181	5.7749	.24945D-18	.16970D-01	.57280
182	5.9169	.25849D-18	.17277D-01	.57019
183	6.0764	.26909D-18	.17590D-01	.56760
184	6.2266	.27894D-18	.17909D-01	.56503
185	6.3952	.29050D-18	.18235D-01	.56249
186	6.5540	.30123D-18	.18567D-01	.55996
187	6.7324	.31384D-18	.18907D-01	.55746
188	6.9001	.32552D-18	.19253D-01	.55497
189	7.0888	.33926D-18	.19608D-01	.55251
190	7.2658	.35197D-18	.19969D-01	.55007
191	7.4653	.36695D-18	.20338D-01	.54765
192	7.6521	.38078D-18	.20715D-01	.54525
193	7.8631	.39712D-18	.21100D-01	.54287
194	8.0601	.41215D-18	.21493D-01	.54051
195	8.2830	.42995D-18	.21895D-01	.53816
196	8.4909	.44631D-18	.22305D-01	.53584
197	8.7265	.46572D-18	.22725D-01	.53354
198	8.9454	.48348D-18	.23153D-01	.53125
199	9.1944	.50464D-18	.23591D-01	.52899

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
200	9.4254	.52398D-18	.24037D-01	.52674
201	9.6883	.54704D-18	.24495D-01	.52451
202	9.9314	.56802D-18	.24961D-01	.52230
203	10.209	.59318D-18	.25439D-01	.52011
204	10.466	.61600D-18	.25926D-01	.51793
205	10.759	.64341D-18	.26424D-01	.51577
206	11.029	.66819D-18	.26933D-01	.51363
207	11.339	.69810D-18	.27454D-01	.51151
208	11.623	.72499D-18	.27985D-01	.50941
209	11.950	.75760D-18	.28529D-01	.50732
210	12.249	.78685D-18	.29083D-01	.50525
211	12.595	.82242D-18	.29651D-01	.50320
212	12.910	.85413D-18	.30230D-01	.50116
213	13.275	.89293D-18	.30824D-01	.49914
214	13.606	.92738D-18	.31429D-01	.49714
215	13.992	.96972D-18	.32049D-01	.49515
216	14.340	.10071D-17	.32680D-01	.49318
217	14.748	.10533D-17	.33328D-01	.49123
218	15.113	.10939D-17	.33988D-01	.48929
219	15.544	.11444D-17	.34664D-01	.48737
220	15.928	.11883D-17	.35353D-01	.48546
221	16.384	.12434D-17	.36060D-01	.48358
222	16.788	.12912D-17	.36779D-01	.48170
223	17.269	.13513D-17	.37518D-01	.47985
224	17.694	.14033D-17	.38269D-01	.47801
225	18.203	.14690D-17	.39040D-01	.47618
226	18.649	.15252D-17	.39825D-01	.47438
227	19.187	.15971D-17	.40631D-01	.47259
228	19.655	.16580D-17	.41451D-01	.47081
229	20.224	.17367D-17	.42293D-01	.46905
230	20.715	.18026D-17	.43149D-01	.46731
231	21.318	.18887D-17	.44028D-01	.46558
232	21.833	.19602D-17	.44923D-01	.46387
233	22.471	.20546D-17	.45842D-01	.46218
234	23.013	.21321D-17	.46776D-01	.46050
235	23.689	.22356D-17	.47736D-01	.45884
236	24.257	.23197D-17	.48712D-01	.45720
237	24.975	.24334D-17	.49716D-01	.45557
238	25.572	.25248D-17	.50736D-01	.45396
239	26.335	.26499D-17	.51785D-01	.45237
240	26.962	.27491D-17	.52852D-01	.45079
241	27.773	.28871D-17	.53949D-01	.44923
242	28.435	.29952D-17	.55064D-01	.44769
243	29.300	.31478D-17	.56212D-01	.44617
244	29.995	.32656D-17	.57379D-01	.44467
245	30.920	.34350D-17	.58582D-01	.44318
246	31.656	.35640D-17	.59803D-01	.44171
247	32.648	.37528D-17	.61064D-01	.44027
248	33.427	.38945D-17	.62345D-01	.43884
249	34.498	.41066D-17	.63667D-01	.43743

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
250	35.325	.42629D-17	.65012D-01	.43604
251	36.488	.45031D-17	.66403D-01	.43467
252	37.374	.46777D-17	.67818D-01	.43332
253	38.648	.49528D-17	.69284D-01	.43200
254	39.605	.51498D-17	.70776D-01	.43070
255	41.021	.54705D-17	.72328D-01	.42942
256	42.071	.56979D-17	.73909D-01	.42817
257	43.673	.60003D-17	.75561D-01	.42694
258	44.856	.63517D-17	.77248D-01	.42574
259	46.720	.68238D-17	.79020D-01	.42458
260	48.103	.71641D-17	.80839D-01	.42344
261	50.362	.77772D-17	.82769D-01	.42234
262	52.097	.82422D-17	.84766D-01	.42128
263	55.012	.91026D-17	.86918D-01	.42026
264	57.441	.98294D-17	.89183D-01	.41930
265	61.600	.11197D-16	.91696D-01	.41841
266	65.698	.12615D-16	.94447D-01	.41761
267	72.732	.15315D-16	.97682D-01	.41694
268	82.291	.19420D-16	.10164	.41649
269	99.043	.27868D-16	.10706	.41646
270	143.98	.58341D-16	.11760	.41793
271	197.22	.10844D-15	.13502	.42183
272	276.78	.21159D-15	.16377	.43059
273	323.18	.28579D-15	.19601	.44260
274	268.62	.19562D-15	.21531	.45026
275	325.01	.28372D-15	.24057	.46169
276	237.43	.15002D-15	.25291	.46709
277	233.63	.14392D-15	.26421	.47217
278	223.95	.13103D-15	.27409	.47665
279	222.25	.12787D-15	.28340	.48098
280	218.07	.12199D-15	.29200	.48503
281	217.96	.12075D-15	.30028	.48900
282	216.00	.11752D-15	.30812	.49281
283	216.94	.11747D-15	.31576	.49661
284	216.20	.11562D-15	.32311	.50031
285	217.89	.11638D-15	.33034	.50402
286	217.90	.11535D-15	.33735	.50768
287	220.18	.11672D-15	.34430	.51137
288	220.69	.11622D-15	.35108	.51502
289	223.44	.11807D-15	.35784	.51873
290	224.30	.11793D-15	.36447	.52241
291	227.45	.12019D-15	.37110	.52616
292	228.54	.12028D-15	.37762	.52990
293	232.04	.12290D-15	.38417	.53372
294	233.29	.12313D-15	.39062	.53753
295	237.11	.12609D-15	.39712	.54143
296	238.43	.12639D-15	.40353	.54532
297	242.56	.12966D-15	.41008	.54931
298	243.89	.12995D-15	.41638	.55329
299	248.30	.13353D-15	.42284	.55738

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
300	249.88	.13374D-15	.42921	.56145
301	254.26	.13760D-15	.43567	.56564
302	255.42	.13767D-15	.44203	.56981
303	260.35	.14182D-15	.44850	.57410
304	261.32	.14165D-15	.45486	.57835
305	266.50	.14607D-15	.46133	.58274
306	267.18	.14556D-15	.46769	.58708
307	272.60	.15024D 15	.47417	.59156
308	272.89	.14929D-15	.48051	.59598
309	278.54	.15423D-15	.48699	.60054
310	278.35	.15271D-15	.49331	.60502
311	284.21	.15788D-15	.49976	.60964
312	283.37	.15562D-15	.50604	.61416
313	289.44	.16100D-15	.51246	.61882
314	287.73	.15779D-15	.51867	.62336
315	293.99	.16335D-15	.52502	.62804
316	291.18	.15892D-15	.53113	.63256
317	297.60	.16463D-15	.53738	.63722
318	293.39	.15868D-15	.54334	.64168
319	299.95	.16449D-15	.54945	.64628
320	293.89	.15662D-15	.55521	.65062
321	300.55	.16245D-15	.56111	.65511
322	291.82	.15189D-15	.56658	.65925
323	298.83	.15765D-15	.57220	.66354
324	286.39	.14392D-15	.57728	.66740
325	293.08	.14950D-15	.58251	.67140
326	273.42	.12985D-15	.58699	.67477
327	279.94	.13419D-15	.59161	.67828
328	251.18	.10716D-15	.59528	.68098
329	257.49	.11171D-15	.59907	.68300
330	211.29	.74611D-16	.60160	.68551
331	217.07	.78121D-16	.60423	.68733

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
2	.31095D-01	.37633D-23	.12359D-[illegible]	2.4109
3	.19963D-01	.12132D-22	.25480D-04	2.4384
4	.11138D-01	.23340D-23	.27201D-04	2.4369
5	.99931D-02	.57532D-23	.31200D-04	2.4321
6	.71488D-02	.21600D-23	.32575D-04	2.4203
7	.74792D-02	.37745D-23	.34848D-04	2.4229
8	.62266D-02	.21302D-23	.36067D-04	2.4183
9	.70322D-02	.30817D-23	.37761D-04	2.4130
10	.62495D-02	.21484D-23	.38899D-04	2.4084
11	.69915D-02	.27994D-23	.40333D-04	2.4036
12	.67133D-02	.21962D-23	.41423D-04	2.3995
13	.65339D-02	.26907D-23	.42721D-04	2.3941
14	.62720D-02	.22672D-23	.43785D-04	2.3892
15	.70929D-02	.26731D-23	.45007D-04	2.3847
16	.69188D-02	.23386D-23	.46038D-04	2.3805
17	.76835D-02	.27122D-23	.47238D-04	2.3766
18	.75755D-02	.24691D-23	.48287D-04	2.3729
19	.83091D-02	.27918D-23	.49447D-04	2.3696
20	.82514D-02	.25985D-23	.50502D-04	2.3662
21	.19306D-02	.29039D-23	.51636D-04	2.2984
22	.18249D-02	.27472D-23	.52724D-04	2.2244
23	.23290D-02	.30444D-23	.53883D-04	2.1725
24	.23525D-02	.29159D-23	.54970D-04	2.1228
25	.22633D-02	.32114D-23	.56143D-04	2.0621
26	.28796D-02	.31057D-23	.57255D-04	2.0249
27	.32137D-02	.34046D-23	.58449D-04	1.9915
28	.30067D-02	.33182D-23	.59591D-04	1.9638
29	.36300D-02	.36245D-23	.60812D-04	1.9253
30	.31419D-02	.38551D-23	.61987D-04	1.8879
31	.36829D-02	.38722D-23	.63243D-04	1.8560
32	.37644D-02	.38184D-23	.64456D-04	1.8263
33	.35763D-02	.41497D-23	.65750D-04	1.7899
34	.33221D-02	.41107D-23	.67007D-04	1.7472
35	.39999D-02	.44592D-23	.68344D-04	1.7145
36	.42013D-02	.44346D-23	.69640D-04	1.6845
37	.44640D-02	.48034D-23	.71034D-04	1.6552
38	.53808D-02	.47934D-23	.72391D-04	1.6340
39	.50570D-02	.51837D-23	.73830D-04	1.6024
40	.53401D-02	.51906D-23	.75243D-04	1.5863
41	.53671D-02	.56098D-23	.76742D-04	1.5612
42	.56847D-02	.56302D-23	.78216D-04	1.5384
43	.67887D-02	.60080D-23	.79778D-04	1.5209
44	.39680D-02	.61169D-23	.81320D-04	1.4977
45	.65802D-02	.66013D-23	.82951D-04	1.4764
46	.69020D-02	.66559D-23	.84563D-04	1.4571
47	.65310D-02	.71792D-23	.86269D-04	1.4333
48	.74046D-02	.72330D-23	.87959D-04	1.4146
49	.88790D-02	.78202D-23	.89745D-04	1.4003

Table C.2 Parameters as for table 3.2

Procedure: least squares collocation

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
50	.73125D-02	.79147D-23	.91810D-04	1.3787
51	.85440D-02	.85314D-23	.93390D-04	1.3614
52	.90350D-02	.86487D-23	.95251D-04	1.3455
53	.83903D-02	.93210D-23	.97217D-04	1.3254
54	.91654D-02	.94634D-23	.99173D-04	1.3080
55	.95211D-02	.10198D-22	.10124D-03	1.2904
56	.10306D-01	.10368D-22	.10330D-03	1.2752
57	.10251D-01	.11174D-22	.10547D-03	1.2581
58	.11208D-01	.11375D-22	.10763D-03	1.2434
59	.11654D-01	.12260D-22	.10992D-03	1.2286
60	.11368D-01	.12495D-22	.11221D-03	1.2125
61	.13163D-01	.13470D-22	.11462D-03	1.1995
62	.12189D-01	.13743D-22	.11702D-03	1.1837
63	.13769D-01	.14020D-22	.11957D-03	1.1702
64	.14521D-01	.15136D-22	.12211D-03	1.1577
65	.17120D-01	.16328D-22	.12479D-03	1.1479
66	.14122D-01	.16692D-22	.12748D-03	1.1330
67	.17029D-01	.18015D-22	.13031D-03	1.1218
68	.17645D-01	.18432D-22	.13315D-03	1.1111
69	.16709D-01	.19904D-22	.13615D-03	1.0900
70	.19454D-01	.20383D-22	.13916D-03	1.0880
71	.20951D-01	.22024D-22	.14234D-03	1.0786
72	.20047D-01	.22572D-22	.14552D-03	1.0679
73	.20599D-01	.24408D-22	.14888D-03	1.0570
74	.20859D-01	.25033D-22	.15226D-03	1.0459
75	.23880D-01	.27092D-22	.15583D-03	1.0366
76	.24349D-01	.27805D-22	.15941D-03	1.0274
77	.26893D-01	.30121D-22	.16320D-03	1.0191
78	.25809D-01	.30936D-22	.16700D-03	1.0098
79	.27975D-01	.33548D-22	.17103D-03	1.0012
80	.27219D-01	.34478D-22	.17507D-03	.99169
81	.28692D-01	.37433D-22	.17936D-03	.98234
82	.27628D-01	.38196D-22	.18366D-03	.97187
83	.31445D-01	.41858D-22	.18823D-03	.96299
84	.30780D-01	.43068D-22	.19282D-03	.95336
85	.36913D-01	.46888D-22	.19769D-03	.94601
86	.35463D-01	.48284D-22	.20259D-03	.93773
87	.37451D-01	.52652D-22	.20780D-03	.92957
88	.39661D-01	.54257D-22	.21303D-03	.92201
89	.42135D-01	.59270D-22	.21860D-03	.91464
90	.43704D-01	.61120D-22	.22420D-03	.90751
91	.43582D-01	.66899D-22	.23010D-03	.89960
92	.48804D-01	.69041D-22	.23619D-03	.89303
93	.49323D-01	.75734D-22	.24261D-03	.88593
94	.50365D-01	.78224D-22	.24907D-03	.87884
95	.56894D-01	.86015D-22	.25598D-03	.87266
96	.55218D-01	.88925D-22	.26294D-03	.86583
97	.60816D-01	.98046D-22	.27040D-03	.85961
98	.58580D-01	.10147D-21	.27791D-03	.85260
99	.67766D-01	.11221D-21	.28599D-03	.84677

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
100	.68358D-01	.11627D-21	.29412D-03	.84082
101	.67437D-01	.12901D-21	.30289D-03	.83399
102	.69670D-01	.13387D-21	.31173D-03	.82725
103	.74321D-01	.14909D-21	.32129D-03	.82060
104	.76582D-01	.15496D-21	.33094D-03	.81404
105	.81851D-01	.17332D-21	.34140D-03	.80756
106	.84411D-01	.18851D-21	.35197D-03	.80117
107	.90416D-01	.20284D-21	.36348D-03	.79486
108	.93348D-01	.21179D-21	.37511D-03	.78863
109	.10024	.23927D-21	.38784D-03	.78248
110	.10363	.25060D-21	.40074D-03	.77640
111	.11160	.28486D-21	.41492D-03	.77040
112	.11558	.29951D-21	.42932D-03	.76447
113	.12489	.34285D-21	.44523D-03	.75861
114	.12964	.36229D-21	.46143D-03	.75282
115	.14063	.41880D-21	.47948D-03	.74710
116	.14644	.44468D-21	.49795D-03	.74145
117	.15958	.51881D-21	.51863D-03	.73586
118	.16687	.55577D-21	.53994D-03	.73034
119	.18283	.65466D-21	.56481D-03	.72487
120	.19229	.71071D-21	.58983D-03	.71947
121	.21285	.84840D-21	.61757D-03	.71413
122	.22484	.93637D-21	.64761D-03	.70885
123	.24996	.11362D-20	.68229D-03	.70362
124	.26819	.12845D-20	.71948D-03	.69845
125	.30122	.15914D-20	.76305D-03	.69334
126	.32915	.18664D-20	.81117D-03	.68828
127	.37485	.23779D-20	.86863D-03	.68327
128	.42216	.29631D-20	.93530D-03	.67831
129	.49889	.39367D-20	.10171D-02	.67341
130	.58478	.54900D-20	.11213D-02	.66855
131	.70712	.78895D-20	.12560D-02	.66374
132	.95757	.14221D-19	.14679D-02	.65898
133	1.3178	.26466D-19	.17970D-02	.65427
134	2.4973	.93481D-19	.26502D-02	.64961
135	2.9753	.13047D-18	.35099D-02	.64499
136	4.6895	.31075D-18	.50257D-02	.64043
137	3.6835	.19027D-18	.57428D-02	.63590
138	3.0159	.12753D-18	.61771D-02	.63141
139	3.0774	.13062D-18	.65923D-02	.62697
140	2.7821	.10582D-18	.69080D-02	.62256
141	2.7730	.10266D-18	.72033D-02	.61820
142	2.7038	.96040D-19	.74690D-02	.61387
143	2.7117	.95062D-19	.77230D-02	.60959
144	2.6927	.92231D-19	.79617D-02	.60534
145	2.7142	.92258D-19	.81935D-02	.60113
146	2.7191	.91140D-19	.84163D-02	.59696
147	2.7587	.91827D-19	.86348D-02	.59282
148	2.7704	.91712D-19	.88478D-02	.58872
149	2.8098	.92892D-19	.90583D-02	.58465

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
150	2.8401	.93485D-19	.92654D-02	.58062
151	2.8860	.95039D-19	.94712D-02	.57662
152	2.9244	.96115D-19	.96750D-02	.57265
153	2.9761	.98046D-19	.98785D-02	.56872
154	3.0213	.99544D-19	.10081D-01	.56482
155	3.0781	.10179D-18	.10284D-01	.56095
156	3.1294	.10365D-18	.10486D-01	.55711
157	3.1910	.10620D-18	.10690D-01	.55330
158	3.2477	.10840D-18	.10894D-01	.54952
159	3.313	.11123D-18	.11099D-01	.54577
160	3.3757	.11375D-18	.11305D-01	.54205
161	3.4465	.11687D-18	.11513D-01	.53836
162	3.5131	.11970D-18	.11722D-01	.53470
163	3.5883	.12310D-18	.11933D-01	.53106
164	3.6597	.12624D-18	.12146D-01	.52745
165	3.7393	.12994D-18	.12361D-01	.52387
166	3.8153	.13338D-18	.12578D-01	.52031
167	3.8993	.13730D-18	.12798D-01	.51678
168	3.9799	.14114D-18	.13020D-01	.51328
169	4.0684	.14545D-18	.13245D-01	.50980
170	4.1536	.14953D-18	.13472D-01	.50634
171	4.2465	.15417D-18	.13702D-01	.50291
172	4.3363	.15858D-18	.13935D-01	.49950
173	4.4338	.16355D-18	.14171D-01	.49611
174	4.5282	.16830D-18	.14410D-01	.49275
175	4.6304	.17362D-18	.14653D-01	.48941
176	4.7294	.17872D-18	.14898D-01	.48609
177	4.8362	.18441D-18	.15147D-01	.48280
178	4.9399	.18987D-18	.15400D-01	.47953
179	5.0515	.19594D-18	.15656D-01	.47627
180	5.1600	.20177D-18	.15915D-01	.47304
181	5.2764	.20824D-18	.16179D-01	.46983
182	5.3897	.21447D-18	.16445D-01	.46664
183	5.5111	.22135D-18	.16716D-01	.46347
184	5.6291	.22798D-18	.16991D-01	.46032
185	5.7556	.23530D-18	.17270D-01	.45719
186	5.8785	.24234D-18	.17552D-01	.45407
187	6.0101	.25011D-18	.17839D-01	.45098
188	6.1380	.25768D-18	.18130D-01	.44790
189	6.2747	.26582D-18	.18425D-01	.44485
190	6.4076	.27374D-18	.18724D-01	.44181
191	6.5497	.28246D-18	.19028D-01	.43878
192	6.6877	.29085D-18	.19335D-01	.43578
193	6.8352	.30007D-18	.19648D-01	.43279
194	6.9782	.30894D-18	.19964D-01	.42982
195	7.1312	.31869D-18	.20286D-01	.42687
196	7.2795	.32804D-18	.20611D-01	.42393
197	7.4381	.33835D-18	.20942D-01	.42101
198	7.5915	.34821D-18	.21277D-01	.41810
199	7.7559	.35907D-18	.21616D-01	.41521

#	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
200	7.9146	.36947D-18	.21961D-01	.41234
201	8.0848	.38094D-18	.22310D-01	.40948
202	8.2489	.39186D-18	.22663D-01	.40663
203	8.4251	.40395D-18	.23022D-01	.40380
204	8.5945	.41542D-18	.23386D-01	.40099
205	8.7769	.42817D-18	.23754D-01	.39819
206	8.9518	.44021D-18	.24128D-01	.39540
207	9.1405	.45363D-18	.24506D-01	.39263
208	9.3209	.46627D-18	.24889D-01	.38987
209	9.5161	.48040D-18	.25278D-01	.38712
210	9.7022	.49364D-18	.25671D-01	.38439
211	9.9042	.50853D-18	.26070D-01	.38167
212	10.096	.52239D-18	.26474D-01	.37896
213	10.305	.53807D-18	.26883D-01	.37627
214	10.503	.55259D-18	.27297D-01	.37358
215	10.719	.56911D-18	.27717D-01	.37091
216	10.922	.58431D-18	.28142D-01	.36826
217	11.147	.60173D-18	.28572D-01	.36561
218	11.356	.61763D-18	.29008D-01	.36298
219	11.588	.63601D-18	.29449D-01	.36036
220	11.804	.65266D-18	.29896D-01	.35774
221	12.045	.67288D-18	.30348D-01	.35515
222	12.268	.68958D-18	.30806D-01	.35256
223	12.518	.71006D-18	.31270D-01	.34998
224	12.747	.72831D-18	.31739D-01	.34741
225	13.008	.75012D-18	.32216D-01	.34486
226	13.244	.76925D-18	.32697D-01	.34231
227	13.515	.79243D-18	.33185D-01	.33978
228	13.759	.81251D-18	.33678D-01	.33726
229	14.042	.83725D-18	.34179D-01	.33474
230	14.298	.85836D-18	.34685D-01	.33224
231	14.591	.88484D-18	.35199D-01	.32974
232	14.852	.90710D-18	.35718D-01	.32726
233	15.164	.93558D-18	.36246D-01	.32478
234	15.435	.95911D-18	.36779D-01	.32232
235	15.763	.98991D-18	.37321D-01	.31986
236	16.045	.10149D-17	.37869D-01	.31742
237	16.393	.10484D-17	.38427D-01	.31498
238	16.687	.10750D-17	.38990D-01	.31255
239	17.038	.11118D-17	.39565D-01	.31013
240	17.365	.11404D-17	.40145D-01	.30773
241	17.763	.11810D-17	.40738D-01	.30533
242	18.087	.12119D-17	.41337D-01	.30293
243	18.517	.12572D-17	.41950D-01	.30055
244	18.860	.12911D-17	.42570D-01	.29818
245	19.329	.13423D-17	.43205D-01	.29582
246	19.697	.13798D-17	.43849D-01	.29346
247	20.212	.14383D-17	.44509D-01	.29112
248	20.610	.14805D-17	.45179D-01	.28878
249	21.182	.15483D-17	.45870D-01	.28645

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
250	21.620	.15969D-17	.46571D-01	.28414
251	22.264	.16766D-17	.47296D-01	.28183
252	22.755	.17339D-17	.48034D-01	.27954
253	23.488	.18293D-17	.48801D-01	.27725
254	24.049	.18980D-17	.49584D-01	.27498
255	24.894	.20147D-17	.50402D-01	.27272
256	25.552	.21019D-17	.51241D-01	.27047
257	26.540	.22454D-17	.52123D-01	.26824
258	27.331	.23581D-17	.53033D-01	.26602
259	28.495	.25383D-17	.53996D-01	.26382
260	29.470	.26880D-17	.54997D-01	.26163
261	30.850	.29183D-17	.56064D-01	.25947
262	32.064	.31223D-17	.57183D-01	.25733
263	33.693	.34145D-17	.58382D-01	.25522
264	35.195	.36902D-17	.59651D-01	.25314
265	37.069	.40540D-17	.61015D-01	.25109
266	38.853	.44121D-17	.62466D-01	.24908
267	40.894	.48415D-17	.64019D-01	.24710
268	42.845	.52644D-17	.65667D-01	.24517
269	44.468	.57193D-17	.67411D-01	.24329
270	46.766	.61549D-17	.69239D-01	.24145
271	48.527	.65654D-17	.71130D-01	.23965
272	50.125	.69397D-17	.73091D-01	.23789
273	51.460	.72463D-17	.75076D-01	.23617
274	52.623	.75073D-17	.77079D-01	.23448
275	53.568	.77073D-17	.79082D-01	.23281
276	54.338	.78575D-17	.81073D-01	.23117
277	55.039	.79873D-17	.83049D-01	.22955
278	55.551	.80623D-17	.84996D-01	.22795
279	56.130	.81559D-17	.86922D-01	.22637
280	56.507	.81906D-17	.88813D-01	.22480
281	57.034	.82682D-17	.90683D-01	.22324
282	57.355	.82859D-17	.92519D-01	.22170
283	57.867	.83583D-17	.94335D-01	.22017
284	58.176	.83715D-17	.96119D-01	.21866
285	58.692	.84443D-17	.97885D-01	.21716
286	59.010	.84594D-17	.99624D-01	.21566
287	59.541	.85353D-17	.10135	.21421
288	59.877	.85552D-17	.10303	.21276
289	60.427	.86354D-17	.10473	.21132
290	60.786	.86610D-17	.10640	.20990
291	61.355	.87459D-17	.10805	.20849
292	61.738	.87774D-17	.10969	.20710
293	62.327	.88670D-17	.11132	.20573
294	62.733	.89041D-17	.11293	.20437
295	63.341	.89979D-17	.11454	.20303
296	63.768	.90482D-17	.11613	.20171
297	64.392	.91378D-17	.11771	.20041
298	64.839	.91845D-17	.11929	.19912
299	65.478	.92054D-17	.12086	.19785

N	PERCENTAGE ERROR	ERROR VARIANCE(POT.)	BAND ERROR (UNDUL.) (M)	TOTAL UNDUL. ERROR (M)
300	65.941	.93360D-17	.12241	.19661
301	66.594	.94396D-17	.12397	.19538
302	67.072	.94933D-17	.12551	.19417
303	67.735	.95990D-17	.12706	.19298
304	68.224	.96550D-17	.12859	.19181
305	68.897	.97622D-17	.13012	.19067
306	69.395	.98198D-17	.13164	.18934
307	70.074	.99200D-17	.13316	.18843
308	70.579	.99863D-17	.13468	.18735
309	71.263	.10095D-16	.13619	.18628
310	71.771	.10153D-16	.13770	.18524
311	72.458	.10261D-16	.13920	.18422
312	72.966	.10319D-16	.14070	.18322
313	73.653	.10426D-16	.14219	.18224
314	74.159	.10481D-16	.14368	.18128
315	74.844	.10587D-16	.14517	.18035
316	75.344	.10640D-16	.14665	.17944
317	76.025	.10743D-16	.14813	.17856
318	76.514	.10792D-16	.14960	.17768
319	77.190	.10893D-16	.15107	.17683
320	77.665	.10937D-16	.15253	.17600
321	78.333	.11035D-16	.15399	.17520
322	78.706	.11071D-16	.15544	.17441
323	79.445	.11166D-16	.15690	.17365
324	79.866	.11192D-16	.15834	.17291
325	80.519	.11283D-16	.15978	.17219
326	80.885	.11294D-16	.16120	.17148
327	81.535	.11383D-16	.16263	.17080
328	81.785	.11361D-16	.16404	.17013
329	82.442	.11431D-16	.16545	.16949
330	82.269	.11312D-16	.16684	.16884
331	82.955	.11409D-16	.16822	.16822

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