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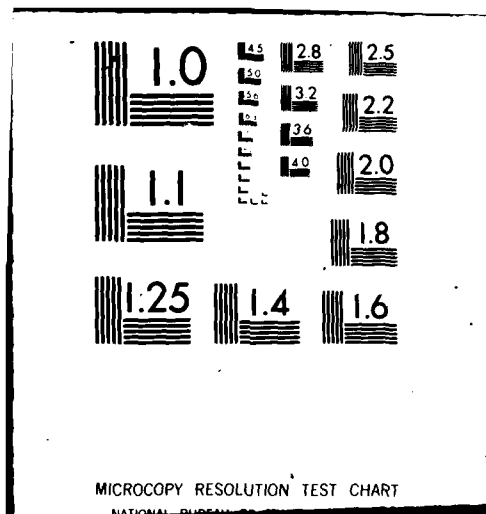
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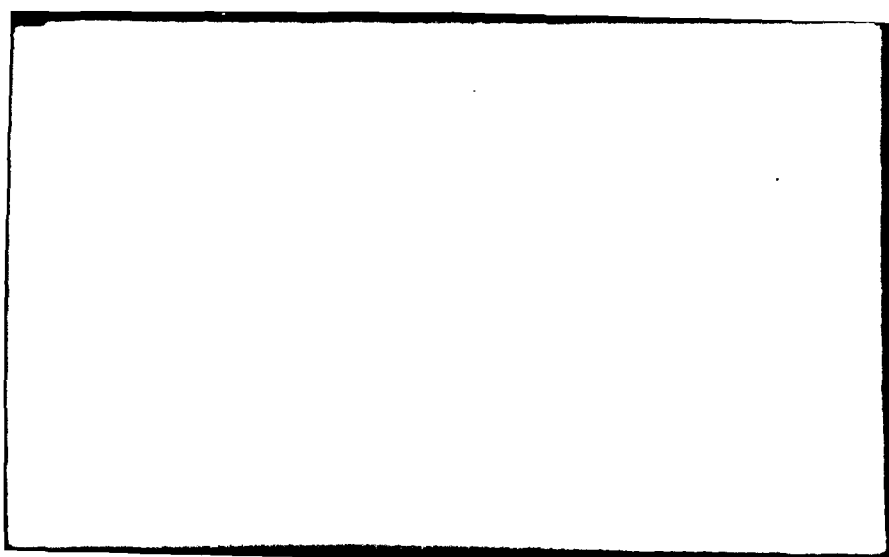
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PERTURBED BIFURCATION OF STATIONARY STRIATIONS
IN A CONTAMINATED, NON-UNIFORM PLASMA

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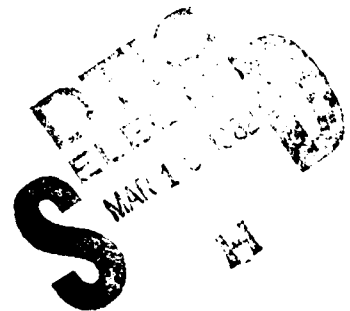
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Perturbed Bifurcation of Stationary Striations
in a Contaminated, Non-uniform Plasma

J. F. Magnan* and R. A. Goldstein**

Abstract

A cylindrical, weakly ionized, and collision dominated neon plasma described by a system of nonlinear, parabolic reaction-diffusion equations for the electron and metastable atom axial densities exhibits a bifurcation from a uniform to a striated state at a critical length of the plasma column. The sharp transition between states predicted by the theory is in contrast with the smooth transition observed in experiments. We apply the theory of singular perturbations of bifurcations to show that small inhomogeneities in the plasma, such as those caused by non-uniform heating and contamination, are sufficient to qualitatively explain the experimental results. We observe that a steady, axial magnetic field in the plasma can also produce a smooth transition.

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1. Introduction

It has been found that the longitudinally uniform and axially symmetric steady state of a cylindrical plasma may bifurcate transcritically to a non-uniform striated state after the onset of an ionization instability [1]. The weakly ionized and collision dominated noble gas plasma which is uniformly heated with radio-frequency electromagnetic waves can be described by a system of non-linear reaction-diffusion equations for the electron N and metastable atom M population densities. The theory in [1] consider the following initial-boundary value problem for N and M :

$$(1.1a) \quad N_t = DN_{zz} + F(N,M) \quad , \quad 0 \leq z \leq L \quad ,$$

$$(1.1b) \quad M_t = \theta DM_{zz} + G(N,M) \quad , \quad 0 \leq z \leq L \quad ,$$

$$(1.1c) \quad N = N_0 \quad , \quad M = M_0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = L \quad ,$$

$$(1.1d) \quad N(z,0) = h(z,\epsilon) \quad , \quad M(z,0) = i(z,\epsilon) \quad .$$

In (1.1) t is the time variable and z the axial coordinate in a cylinder of length L . D is the ambipolar diffusion coefficient of the ions and θ is the ratio of the metastable atom diffusion coefficient to D . N_0 and M_0 are the uniform steady state values of N and M . The initial functions h and i have asymptotic representations $\epsilon h_1(z)$ and $\epsilon i_1(z)$ which have Fourier sine series expansions and vanish at the boundaries. The small parameter ϵ is defined below. The functions F and G are nonlinear particle production and loss rates describing the interactions between electrons, metastables, and neutral gas atoms in the plasma. They are discussed in [1], [2], [3]. Their form is given in [2] as



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$$(1.1e) \quad F(N,M) = -f_1 N + f_2 NM + f_3 N^2 M, \quad ,$$

$$(1.1f) \quad G(N,M) = g_1 - g_2 M - g_3 NM, \quad ,$$

where the positive constants f_1 , f_2 , f_3 , g_1 , g_2 , and g_3 are rate coefficients obtained by properly averaging collision cross sections over the electron energy distribution function in the plasma.

An asymptotic expansion of the solution to problem (1.1) as $\epsilon \rightarrow 0$ is obtained in [1] by a standard two-time method. The analysis of the problem finds that the uniform state may become unstable when a critical value μ_c of the bifurcation parameter $\mu \equiv \frac{L^2}{D\pi^2}$ is exceeded. For μ near μ_c an initial perturbation of the uniform state will develop into a non-uniform steady state with a sinusoidal density variation along the plasma axis. The striated state of the plasma is given by

$$(1.2a) \quad \begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} + \epsilon A_c \begin{bmatrix} 1 \\ b_{n_c} \end{bmatrix} \sin \frac{n_c \pi z}{L} + O(\epsilon^2), \quad ,$$

where the critical mode n_c is odd,

$$(1.2b) \quad A_c = - \frac{q_{n_c}^+}{\alpha} \mu'(0), \quad ,$$

and $q_{n_c}^+$, α , b_{n_c} are constants defined below. The normalization $\mu - \mu_c = \epsilon$ defines ϵ and $\mu'(0)$. The value of n_c in (1.2a) is determined by μ since disconnected regions of this parameter may exist where modes of different n grow in amplitude. For even n_c it is necessary to choose $\mu'(0) = 0$ and a higher order calculation is required.

Experimental investigations of the phenomenon have qualitatively confirmed the results of the analysis [4]. Experiments [5] show however that the transition from the uniform to the striated state ($n_c = 1$) is

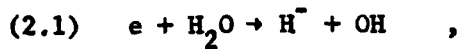
smooth, in contrast with the theoretical prediction that an abrupt change of state occurs at the bifurcation point. The contrast between theory and experiment is shown by Fig. 1. We attribute this discrepancy to the non-uniform heating of the plasma by the radio-frequency electric field and to the presence of impurities in the plasma. In this paper we show that these effects are sufficient to qualitatively explain the smooth transition between states. In addition, we show that the transition can be smoothed by a steady magnetic field in the plasma.

In §2 we modify problem (1.1) to include the effect of impurities and non-uniform heating which are assumed to be of the same small amplitude δ . We specifically consider the case of an argon contaminant in a neon plasma and call the resulting steady state problem the perturbed problem.

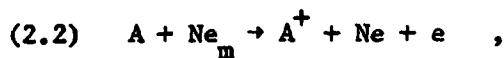
We apply the method of Matkowsky and Reiss [6], [7] in §§3-5 to find asymptotic solutions of the perturbed problem. The technique used is that of matched asymptotic expansions where an outer and inner boundary layer expansion, which are perturbation expansions in δ of the bifurcation branches away from and near to the bifurcation point, are connected by a matching procedure. A uniform asymptotic representation of the solution can then be obtained. The method has been applied to the buckling of columns, forced nonlinear oscillations, Rayleigh-Bénard convection, Poiseuille flow, and panel flutter [7], [8], [9], [10]. In our analysis we have retained much of the notation used in [1].

2. The Perturbed Problem

Impurities in a laboratory plasma are always present. They consist mainly of water vapor and air from leaks and surface desorption. Two sample reactions with impurities are



and



where the two processes are, respectively, dissociative attachment and Penning ionization (of an argon ground state atom A by a neon metastable atom Ne_m). Each impurity has a different effect on the plasma: water molecules destroy electrons; and argon atoms create electrons and destroy metastable atoms.

We shall consider a neon plasma contaminated by an argon impurity which is introduced at a constant rate by an external source. The argon reacts almost immediately with the metastables because of the large argon collision frequency and high probability for Penning ionization to occur. For this reason we can neglect the direct ionization of argon by electrons, the diffusion of argon, and the change of the argon reaction rate with small variations in the metastable concentration. Under these conditions the conservation equation for the argon concentration is decoupled from the N and M equations and gives the result that the steady state concentration has the same spatial dependence of the source.

The time rate of change of the electron and metastable atom populations due to an argon impurity concentration $\delta I(z)$ is proportional to $\delta I(z)M$, where $\delta > 0$ is a small parameter characterizing the magnitude of the impurities.

It may be defined, for example, as

$$(2.3) \quad \delta = \max_{0 \leq z \leq L} \frac{I(z)}{M_0} .$$

In the theory presented in [1] it is assumed that the plasma is uniformly heated by the radio-frequency electric field so that the reaction rates do not vary along z . In experiments, however, the heating can be non-uniform and the rate coefficients of the plasma depend on z . In the experiments described in [4] the heating is concentrated near the ends of the plasma where the field produced by two end electrodes is strongest. The large field at the ends is due to the proximity of the electrodes and the shielding of the central region of the plasma by the plasma at the boundary.

If we assume the non-uniformity of the field to be small and of the same characteristic magnitude δ as the contamination in the plasma we may write the space dependent part of the reaction rates as

$$(2.4a) \quad \delta [-\alpha_1(z)N + \alpha_2(z)NM + \alpha_3(z)N^2M] \equiv \delta X(z, N, M) ,$$

and

$$(2.4b) \quad \delta [\beta_1(z) - \beta_2(z)M - \beta_3(z)NM] \equiv \delta Y(z, N, M) ,$$

where the α 's and β 's are rate coefficients.

The steady state problem for the non-uniformly heated, contaminated plasma is given by

$$(2.5a) \quad N_{\xi\xi} + \mu [F(N, M) + \delta (kMI(\xi) + X(\xi, N, M))] = 0 , \quad 0 \leq \xi \leq \pi ,$$

$$(2.5b) \quad M_{\xi\xi} + \mu [G(N, M) + \delta (-kMI(\xi) + Y(\xi, N, M))] = 0 , \quad 0 \leq \xi \leq \pi ,$$

$$(2.5c) \quad N = N_0 , \quad M = M_0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = \pi ,$$

where $\xi = \frac{\pi z}{L}$ is a dimensionless space variable and k is a rate coefficient.

Equations (2.5) define the perturbed bifurcation problem when $\delta \neq 0$; and the bifurcation problem when $\delta = 0$.

3. The Outer Expansions

The four sub-branches of the bifurcation branches are shown in Fig. 1. The branches include the uniform state sub-branches to the right and left of μ_c and the transcritical striated state sub-branches branching above and below μ_c . Let the vector $\underline{u}$ with components N and M be the outer expansion; and let $\underline{u}_0$ be a bifurcation sub-branch. We seek asymptotic expansions as $\delta \rightarrow 0$ of the solution of the perturbed problem near $\underline{u}_0$ in the form

$$(3.1) \quad \underline{u} = \sum_{j=0}^{\infty} \underline{u}_j(\xi, \mu) \delta^j.$$

The coefficients $\underline{u}_j$ of the outer expansion are obtained by substituting (3.1) into (2.5) and equating to zero coefficients of δ^j . This results in the following linear boundary-value problem for $\underline{u}_1$:

$$(3.2a) \quad (L + \mu Q)\underline{u}_1 = \underline{R}, \quad \underline{u}_1 = 0 \quad \text{for } \xi = 0 \quad \text{and } \xi = \pi,$$

where

$$(3.2b) \quad L = \begin{bmatrix} \frac{\partial^2}{\partial \xi^2} & 0 \\ 0 & \theta \frac{\partial^2}{\partial \xi^2} \end{bmatrix}, \quad Q = \begin{bmatrix} F_N & F_M \\ G_N & G_M \end{bmatrix},$$

and

$$(3.2c) \quad \underline{R} = \mu k M'_0 I(\xi) \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \mu \begin{bmatrix} X(\xi, N'_0, M'_0) \\ Y(\xi, N'_0, M'_0) \end{bmatrix}.$$

In (3.2) the derivatives of F and G are evaluated at $\underline{u}_0$ and N'_0, M'_0 are the components of $\underline{u}_0$.

We solve (3.2) by using eigenfunction expansions. The eigenfunctions v_n and eigenvalues σ_n of the operator $L + \mu Q$ are defined by

$$(3.3) \quad (L + \mu Q)v_n = \sigma_n v_n, \quad v_n = 0 \text{ for } \xi = 0 \text{ and } \xi = \pi, \quad 0 \leq \mu < \infty.$$

To solve the eigenvalue problem (3.3) for the uniform branch where $N'_0 = N_0$ and $M'_0 = M_0$ we rewrite (3.3) as

$$(3.4) \quad [L + \mu_c Q_0 + (\mu - \mu_c) Q_0]v_n = \sigma_n v_n,$$

where Q_0 is Q evaluated at N_0, M_0 . The eigenfunctions $v_n^\pm, v_n^{*\pm}$ and eigenvalues $e_n^\pm, \bar{e}_n^\pm$ of the operator $L + \mu Q_0$ and its adjoint $L^* + \mu Q_0^*$ are defined by

$$(3.5a) \quad (L + \mu Q_0)v_n^\pm = e_n^\pm v_n^\pm, \quad v_n^\pm = 0 \text{ for } \xi = 0 \text{ and } \xi = \pi,$$

and

$$(3.5b) \quad (L^* + \mu Q_0^*)v_n^{*\pm} = \bar{e}_n^\pm v_n^{*\pm}, \quad v_n^{*\pm} = 0 \text{ for } \xi = 0 \text{ and } \xi = \pi,$$

where $\bar{e}_n$ is the complex conjugate of e_n . In (3.5) each operator has two families (+ and -) of eigenfunctions and eigenvalues. They are given in [1] as

$$(3.6a) \quad v_n^+ = \begin{bmatrix} 1 \\ b_n \end{bmatrix} \sin n\xi, \quad b_n = (e_n^+ + n^2 - \mu_{F_N}^+)/\mu_{F_M}^+,$$

$$(3.6b) \quad v_n^- = \begin{bmatrix} 1 \\ c_n \end{bmatrix} \sin n\xi, \quad c_n = (e_n^- + n^2 - \mu_{F_N}^-)/\mu_{F_M}^-,$$

$$(3.6c) \quad v_n^{*+} = E_n \begin{bmatrix} 1 \\ b_n^* \end{bmatrix} \sin n\xi, \quad E_n = \frac{2}{\pi} \frac{1}{1 + b_n b_n^*},$$

$$b_n^* = (\bar{e}_n^+ + n^2 - \mu_{F_N}^+)/\mu_{F_N}^+,$$

$$(3.6d) \quad v_n^{*-} = K_n \begin{bmatrix} 1 \\ c_n^* \end{bmatrix} \sin n\xi, \quad K_n = \frac{2}{\pi} \frac{1}{1 + c_n c_n^*},$$

$$c_n^* = (\bar{e}_n^- + n^2 - \mu_{F_N}^-)/\mu_{F_N}^-,$$

$$(3.6e) \quad e_n^{\pm} = \frac{1}{2} \mu [T_n \pm (T_n^2 - 4P_n)^{\frac{1}{2}}] ,$$

$$(3.6f) \quad T_n = G_M + F_N - \frac{n^2}{\mu} (1 + \theta) , \quad P_n = \frac{\theta n^4}{\mu^2} - (\theta F_N + G_M) \frac{n^2}{\mu} + F_N G_M - F_M G_N .$$

In [1] it was assumed that $T_n < 0$ and $P_n\left(\frac{1}{\mu_c}\right) = 0$.

The eigenfunctions satisfy the orthogonality conditions

$$(3.7a) \quad \langle v_n^{\pm}, v_m^{\pm} \rangle = 0 \quad \text{for } n \neq m ,$$

$$(3.7b) \quad \langle v_n^{\pm}, v_m^{\mp} \rangle = 0 ,$$

$$(3.7c) \quad \langle v_n^{\pm}, v_n^{\pm} \rangle = 1 ,$$

where the inner product of two vectors $\underline{a}, \underline{b}$ is defined by

$$(3.7d) \quad \langle \underline{a}, \underline{b} \rangle = \int_0^{\pi} \underline{a}^{\dagger}(\xi) \underline{b}(\xi) d\xi ,$$

and $\underline{a}^{\dagger}$ is the Hermetian adjoint of $\underline{a}$.

We denote the eigenfunctions of the operator $L + \mu_c Q_0$ and its adjoint $L^* + \mu_c Q_0^*$ by $\underline{v}_n^{\pm}$ and $\underline{v}_n^{*\pm}$ respectively.

It is useful for the matching in §5 to expand the $\underline{v}_n^{\pm}$ and $\underline{v}_n^{*\pm}$ in terms of the $\underline{v}_k^{\pm}$ and $\underline{v}_k^{*\pm}$ as

$$(3.8a) \quad \underline{v}_n^{\pm} = \sum_{k=1}^{\infty} (\alpha_{nk}^{\pm} \underline{v}_k^{+} + \beta_{nk}^{\pm} \underline{v}_k^{-}) ,$$

$$(3.8b) \quad \underline{v}_n^{*\pm} = \sum_{k=1}^{\infty} (\alpha_{nk}^{*\pm} \underline{v}_k^{*+} + \beta_{nk}^{*\pm} \underline{v}_k^{*-}) ,$$

where

$$(3.8c) \quad \alpha_{nk}^{\pm} = \langle \underline{v}_k^{+}, \underline{v}_n^{\pm} \rangle , \quad \beta_{nk}^{\pm} = \langle \underline{v}_k^{-}, \underline{v}_n^{\pm} \rangle ,$$

and

$$(3.8d) \quad \alpha_{nk}^{*\pm} = \langle \tilde{v}_k^{*+}, v_n^{*\pm} \rangle, \quad \beta_{nk}^{*\pm} = \langle \tilde{v}_k^{*-}, v_n^{*\pm} \rangle.$$

The expansion coefficients in (3.8) are functions of μ and are given by

$$(3.9a) \quad \alpha_n^+ \equiv \alpha_{nn}^+ = \frac{1 + b_n(\mu)b_n^*(\mu_c)}{1 + b_n(\mu_c)b_n^*(\mu_c)}, \quad \alpha_n^- \equiv \alpha_{nn}^- = \frac{1 + c_n(\mu)b_n^*(\mu_c)}{1 + b_n(\mu_c)b_n^*(\mu_c)}$$

$$(3.9b) \quad \beta_n^+ \equiv \beta_{nn}^+ = \frac{1 + b_n(\mu)c_n^*(\mu_c)}{1 + c_n(\mu_c)c_n^*(\mu_c)}, \quad \beta_n^- \equiv \beta_{nn}^- = \frac{1 + c_n(\mu)c_n^*(\mu_c)}{1 + c_n(\mu_c)c_n^*(\mu_c)}$$

$$(3.9c) \quad \alpha_n^{*+} \equiv \alpha_{nn}^{*+} = \frac{1 + b_n(\mu_c)b_n^*(\mu)}{1 + b_n(\mu)b_n^*(\mu)}, \quad \alpha_n^{*-} \equiv \alpha_{nn}^{*-} = \frac{1 + b_n(\mu_c)c_n^*(\mu)}{1 + c_n(\mu)c_n^*(\mu)},$$

$$(3.9d) \quad \beta_n^{*+} \equiv \beta_{nn}^{*+} = \frac{1 + c_n(\mu_c)b_n^*(\mu)}{1 + b_n(\mu)b_n^*(\mu)}, \quad \beta_n^{*-} \equiv \beta_{nn}^{*-} = \frac{1 + c_n(\mu_c)c_n^*(\mu)}{1 + c_n(\mu)c_n^*(\mu)},$$

and

$$(3.9e) \quad \alpha_{nk}^{\pm} = \alpha_{nk}^{*\pm} = \beta_{nk}^{\pm} = \beta_{nk}^{*\pm} = 0 \quad \text{for } k \neq n.$$

Thus from (3.8) and (3.9)

$$(3.10a) \quad v_n^{\pm} = \alpha_{nn}^{\pm} v_n^+ + \beta_{nn}^{\pm} v_n^-,$$

and

$$(3.10b) \quad v_n^{*\pm} = \alpha_{nn}^{*\pm} v_n^{*+} + \beta_{nn}^{*\pm} v_n^{*-}.$$

The asymptotic behavior of the coefficients as $\mu \rightarrow \mu_c$ is needed below and is given by

$$(3.11a) \quad \alpha_n^+ = \beta_n^- = \alpha_n^{*+} = \beta_n^{*-} = 1 + O(\mu - \mu_c),$$

and

$$(3.11b) \quad \alpha_n^- = \beta_n^+ = \alpha_n^{*-} = \beta_n^{*+} = 0(\mu - \mu_c) \quad .$$

Now (3.4) may be written as

$$(3.12) \quad \sigma_n^{\pm} v_n^{\pm} = \alpha_n^{\pm} e_n^+(\mu_c) v_n^+ + \beta_n^{\pm} e_n^-(\mu_c) v_n^- + (\mu - \mu_c) (\alpha_n^{\pm} Q_0 v_n^+ + \beta_n^{\pm} Q_0 v_n^-) \quad ,$$

where we have used (3.10a) and (3.5a). Taking the inner product of (3.12) with $v_n^{*\pm}$ and using the orthogonality of the eigenfunctions we arrive at an expression for the $\sigma_n^{\pm}$:

$$(3.13a) \quad \sigma_n^{\pm} = p_n^{\pm} + (\mu - \mu_c) q_n^{\pm} \quad ,$$

where

$$(3.13b) \quad p_n^{\pm} = \alpha_n^{\pm} \alpha_n^{*\pm} e_n^+(\mu_c) + \beta_n^{\pm} \beta_n^{*\pm} e_n^-(\mu_c) \quad ,$$

$$(3.13c) \quad q_n^{\pm} = \alpha_n^{\pm} \alpha_n^{*\pm} q_n^+ + \alpha_n^{\pm} \beta_n^{*\pm} q_n'^- + \alpha_n^{*\pm} \beta_n^{\pm} q_n'^+ + \beta_n^{\pm} \beta_n^{*\pm} q_n^- \quad ,$$

and

$$(3.13d) \quad q_n^{\pm} = \langle v_n^{\pm}, Q_0 v_n^{\pm} \rangle \quad , \quad q_n'^{\pm} = \langle v_n^{\pm}, Q_0 v_n^{\mp} \rangle \quad .$$

We expand the inhomogeneous term R in the $v_n^{\pm}$ as

$$(3.14) \quad R = \sum_{j=1}^{\infty} (r_j^+ v_j^+ + r_j^- v_j^-) \quad .$$

The expansion coefficients $r_j^{\pm}$ are given by

$$(3.15a) \quad r_j^{\pm} = \langle v_j^{\pm}, R \rangle$$

and have the values

$$(3.15b) \quad r_j^{\pm} = \alpha_j^{*\pm} R_j^+ + \beta_j^{*\pm} R_j^- \quad ,$$

where

$$(3.15c) \quad R_j^+ = \frac{2}{\pi} \mu B_j (k M_0 I_j - S_j) \quad ,$$

$$(3.15d) \quad R_j^- = \frac{2}{\pi} \mu C_j (k M_0 I_j - T_j) \quad ,$$

$$(3.15e) \quad B_j = \frac{b_j^*(\mu_c) - 1}{1 + b_j(\mu_c) b_j^*(\mu_c)} \quad , \quad C_j = \frac{c_j^*(\mu_c) - 1}{1 + c_j(\mu_c) c_j^*(\mu_c)} \quad ,$$

$$(3.15f) \quad I_j = \int_0^\pi I(\xi) \sin j\xi \, d\xi \quad ,$$

$$(3.15g) \quad S_j = \frac{1}{b_j^*(\mu_c) - 1} \int_0^\pi [X_0(\xi) + b_j^*(\mu_c) Y_0(\xi)] \sin j\xi \, d\xi \quad ,$$

$$(3.15h) \quad T_j = \frac{1}{c_j^*(\mu_c) - 1} \int_0^\pi [X_0(\xi) + c_j^*(\mu_c) Y_0(\xi)] \sin j\xi \, d\xi \quad ,$$

and

$$(3.15i) \quad X_0(\xi) \equiv X(\xi, N_0, M_0) \quad , \quad Y_0(\xi) \equiv Y(\xi, N_0, M_0) \quad .$$

We assume $R_{jc}^+ \neq 0$.

The solution of (3.2) is given by

$$(3.16) \quad \underline{u}_1 = \sum_{j=1}^{\infty} \left(\frac{r_j^+}{\sigma_j^+} v_j^+ + \frac{r_j^-}{\sigma_j^-} v_j^- \right) \quad ,$$

where the $v_j^\pm$, $\sigma_j^\pm$, and $r_j^\pm$ are given in (3.10a), (3.13a), and (3.15b). The outer expansions corresponding to the uniform sub-branches are

$$(3.17) \quad \underline{u} = \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} + \delta \underline{u}_1 + O(\delta^2) \quad ,$$

where $\underline{u}_1$ is given by (3.16).

The asymptotic behavior of (3.17) as $\mu \rightarrow \mu_c$ is given by

$$(3.18) \quad \underline{u} = \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} + \frac{2}{\pi} \mu \delta \sum_{j=1}^{\infty} \left\{ \frac{B_j (kM_0 I_j - S_j) + O(\mu - \mu_c)}{e_j^+ + (\mu - \mu_c) q_j^+ + e_j^+ O(\mu - \mu_c) + O[(\mu - \mu_c)^2]} \begin{bmatrix} 1 \\ b_j(\mu_c) \end{bmatrix} + \right. \\ \left. \frac{C_j (kM_0 I_j - T_j) + O(\mu - \mu_c)}{e_j^- + (\mu - \mu_c) q_j^- + e_j^- O(\mu - \mu_c) + O[(\mu - \mu_c)^2]} \begin{bmatrix} 1 \\ c_j(\mu_c) \end{bmatrix} \right\} \sin j\xi + \\ O[\delta(\mu - \mu_c)] + O(\delta^2) .$$

For $j = j_c$ the first term inside the brackets in (3.18) becomes

$$(3.19) \quad \frac{B_{j_c} (kM_0 I_{j_c} - S_{j_c})}{(\mu - \mu_c) q_{j_c}^+} \begin{bmatrix} 1 \\ b_{j_c}(\mu_c) \end{bmatrix} \sin j_c \xi + O(1) ,$$

since $e_{j_c}^+ = 0$. It is seen from (3.19) that the outer expansions (3.17) are unbounded at the bifurcation point $\mu = \mu_c$. They are shown in Fig. 2 for $\alpha R_{j_c}^+ > 0$ and $\alpha R_{j_c}^+ < 0$.

The outer expansions for the striation sub-branches (1.2) will depend on ϵ because of the dependence $\underline{u}_0$, μ and Q now have on ϵ . The eigenvalue problem (3.3) thus becomes

$$(3.20) \quad [L + \mu(\epsilon)Q(\epsilon)]v_n(\epsilon) = \sigma_n(\epsilon)v_n(\epsilon) , \quad v_n = 0 \quad \text{for } \xi = 0 \quad \text{and } \xi = \pi .$$

The matrix Q is evaluated at the striation sub-branch and has the form

$$(3.21a) \quad Q(\epsilon) = Q_0 + \epsilon A_c D \sin n_c \xi + O(\epsilon^2) ,$$

where

$$(3.21b) \quad D = \begin{bmatrix} F_{NN} + b_{nc} F_{NM} & F_{NM} \\ G_{NN} + b_{nc} G_{NM} & G_{NM} \end{bmatrix}.$$

We solve (3.20) by seeking perturbation expansions in ϵ of v_n and σ_n of the form

$$(3.22a) \quad v_n(\epsilon) = \sum_{j=0}^{\infty} v_{n,j} \epsilon^j,$$

and

$$(3.22b) \quad \sigma_n(\epsilon) = \sum_{j=0}^{\infty} \sigma_{n,j} \epsilon^j.$$

The coefficients $v_{n,j}$, $\sigma_{n,j}$ in (3.22) are obtained by substituting (3.22) into (3.20) and equating to zero the coefficients of ϵ^j . This leads to the following equations for $v_{n,0}$ and $v_{n,1}$:

$$(3.23) \quad (L + \mu_c Q_0) v_{n,0} = \sigma_{n,0} v_{n,0}$$

and

$$(3.24) \quad (L + \mu_c Q_0 - \sigma_{n,0}) v_{n,1} = (\sigma_{n,1} - \mu'(0) Q_0 - \mu_c A_c D \sin n \xi) v_{n,0}.$$

Equation (3.23) is the same as (3.5a) and therefore $v_{n,0} = v_n^{\pm}$, and $\sigma_{n,0} = e_n^{\pm}$. Thus we can rewrite (3.24) as

$$(3.25a) \quad (L + \mu_c Q_0 - e_n^{\pm}) v_{n,1}^{\pm} = W^{\pm} \sin n \xi + Z \sin n_c \xi \sin n \xi,$$

where

$$(3.25b) \quad W^{\pm} = \begin{bmatrix} w_1^{\pm} \\ w_2^{\pm} \end{bmatrix} = \begin{bmatrix} \sigma_{n,1}^{\pm} - \mu'(0) (F_N + b_n F_M) \\ b_n \sigma_{n,1}^{\pm} - \mu'(0) (G_N + b_n G_M) \end{bmatrix},$$

and

$$(3.25c) \quad Z \equiv \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = -\mu_c A_c \begin{bmatrix} F_{NN} + b_{n_c} F_{NM} + b_n F_{NM} \\ G_{NN} + b_{n_c} G_{NM} + b_n G_{NM} \end{bmatrix} .$$

It will not be necessary to find the $v_{n,1}^{\pm}$ in (3.25a) because they ultimately give terms of $O(\epsilon\delta)$ in the outer expansions and are of higher order than the $O(\epsilon)$ and $O(\delta)$ terms we are keeping. The $\sigma_{n,1}^{\pm}$ are obtained from the solvability requirement

$$(3.26) \quad \langle v_{n,1}^{\pm}, W^{\pm} \sin n\xi + Z \sin n_c \xi \sin n\xi \rangle = 0 .$$

This condition gives us equations which may be solved algebraically for the $\sigma_{n,1}^{\pm}$:

$$(3.27a) \quad \frac{\pi}{2} (w_1^* + b_n^* w_2^*) + \frac{1}{4} \left(\frac{4}{n_c} + \frac{2}{2n - n_c} - \frac{2}{2n + n_c} \right) (z_1 + b_n^* z_2) = 0 ,$$

$$(3.27b) \quad \frac{\pi}{2} (w_1^- + c_n^* w_2^-) + \frac{1}{4} \left(\frac{4}{n_c} + \frac{2}{2n - n_c} - \frac{2}{2n + n_c} \right) (z_1 + c_n^* z_2) = 0 ,$$

for even n and odd $n \neq n_c$,

and

$$(3.27c) \quad \frac{\pi}{2} (w_1^+ + b_n^* w_2^+) + \frac{4}{3n} (z_1 + b_n^* z_2) = 0 ,$$

$$(3.27d) \quad \frac{\pi}{2} (w_1^- + c_n^* w_2^-) + \frac{4}{3n} (z_1 + c_n^* z_2) = 0 ,$$

for $n = n_c$. We solve (3.21c) for $\sigma_{n_c,1}^+$ (it will be needed later) and obtain

$$(3.28) \quad \sigma_{n_c,1}^+ = -\mu'(0) q_{n_c}^+ .$$

The solution of (3.2) for the striation sub-branches is given by

$$(3.29) \quad \underline{u}_1 = \sum_{j=1}^{\infty} \left(\frac{R_j^+}{\sigma_j^+} v_j^+ + \frac{R_j^-}{\sigma_j^-} v_j^- \right),$$

where the $v_j^{\pm}$, $\sigma_j^{\pm}$, and $R_j^{\pm}$ are given by (3.22) and (3.15c,d). Thus it follows from (3.1) and (3.29) that the outer expansions corresponding to the striation sub-branches are

$$(3.30) \quad \underline{u} = \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} + \epsilon A_c \begin{bmatrix} 1 \\ b_{n_c} \end{bmatrix} \sin n_c \xi +$$

$$\frac{2}{\pi} \mu \delta \sum_{j=1}^{\infty} \left\{ \frac{B_j (kM_0 I_j - S_j)}{e_j^+ + \epsilon \sigma_{j,1}^+} \begin{bmatrix} 1 \\ b_j \end{bmatrix} + \frac{C_j (kM_0 I_j - T_j)}{e_j^- + \epsilon \sigma_{j,1}^-} \begin{bmatrix} 1 \\ c_j \end{bmatrix} \right\} \sin j \xi +$$

$$O(\epsilon \delta) + O(\epsilon^2) + O(\delta^2).$$

At the bifurcation point, where $e_{j_c}^+ = 0$ and $\epsilon = 0$, the outer expansions (3.30) are unbounded. They are shown in Fig. 2 for $\alpha R_{j_c}^+ > 0$ and $\alpha R_{j_c}^+ < 0$.

We have used (3.28) to draw the curves.

4. The Inner Expansions

These expansions are valid near μ_c as $\delta \rightarrow 0$ and are obtained by stretching the neighborhoods of μ_c with the transformation

$$(4.1) \quad \mu(v) = \mu_c + \eta v^a + \sum_{j=2}^{\infty} \eta_j (v^a)^j, \quad ,$$

where η is the stretched variable of the method of matched asymptotic expansions and the small parameter v is defined by

$$(4.2) \quad \delta(v) = v^b.$$

The constants a and b are positive integers since we require that $\mu(v) \rightarrow \mu_c$ and $\delta(v) \rightarrow 0$ as $v \rightarrow 0$ and that the derivatives of $\mu(v)$ and $\delta(v)$ be bounded as $v \rightarrow 0$.

Let the vector $\underline{w}$ with components $n(v) = N(\xi, \mu(v), \delta(v))$ and $m(v) = M(\xi, \mu(v), \delta(v))$ be the inner expansion. We seek asymptotic expansions of the solution of the perturbed problem near the singular point defined by

$$(4.3) \quad \mu = \mu_c \quad \text{and} \quad \underline{w}_0 = \begin{bmatrix} N_0 \\ M_0 \end{bmatrix}$$

in the form

$$(4.4) \quad \underline{w} = \sum_{j=0}^{\infty} \underline{w}_j(\xi, \mu(v), \delta(v)) v^j, \quad ,$$

where $\underline{w}_j$ has components n_j, m_j .

In the usual manner we arrive at the following equations for the $\underline{w}_j$:

$$(4.5) \quad (L + \mu_c Q_0) \underline{w}_1 = \mu_c \delta' \left\{ k M_0 I(\xi) \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} X_0(\xi) \\ Y_0(\xi) \end{bmatrix} \right\} = \underline{L}_1, \quad ,$$

$$\underline{w}_1 = 0 \quad \text{for} \quad \xi = 0 \quad \text{and} \quad \xi = \pi, \quad ,$$

$$(4.6) \quad (L + \mu_c Q_0) \omega_2 = -\mu' Q_0 \omega_1 - \mu_c \begin{bmatrix} \frac{n_1^2}{2} F_{NN} + n_1 m_1 F_{NM} \\ \frac{n_1^2}{2} G_{NN} + n_1 m_1 G_{NM} \end{bmatrix} +$$

$$\mu_c \delta' \left\{ km_1 I(\xi) \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} X_1(\xi) \\ Y_1(\xi) \end{bmatrix} \right\} +$$

$$\left(\mu' \delta' + \mu_c \frac{\delta'''}{2} \right) \left\{ kM_0 I(\xi) \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} X_0(\xi) \\ Y_0(\xi) \end{bmatrix} \right\} \equiv \rho_2$$

$$\omega_2 = 0 \quad \text{for } \xi = 0 \quad \text{and } \xi = \pi,$$

and in general for $j > 2$,

$$(4.7) \quad (L + \mu_c Q_0) \omega_j = \rho_j, \quad \omega_j = 0 \quad \text{for } \xi = 0 \quad \text{and } \xi = \pi.$$

The primes on δ and μ denote derivatives with respect to v evaluated at $v = 0$ and come from Taylor expansions of δ and μ . In (4.6) $X_1(\xi) \equiv X(\xi, n_1, m_1)$ and $Y_1(\xi) \equiv Y(\xi, n_1, m_1)$.

The operator $L + \mu_c Q_0$ is not invertible and thus the inhomogeneous problems (4.5)-(4.7) have solutions if and only if ρ_j ($j = 1, 2, \dots$) is orthogonal to the null space of $L^* + \mu_c Q_0^*$ which is spanned by $\tilde{v}_{n_c}^{*+}$. The solvability conditions for these problems are thus

$$(4.8) \quad \langle \tilde{v}_{n_c}, \rho_j \rangle = 0, \quad j = 1, 2, \dots$$

For $j = 1$, (4.5) and (4.8) imply that

$$(4.9) \quad \delta' B_{j_c} (kM_0 I_{j_c} - S_{j_c}) = 0.$$

Since we have assumed that $R_{jc}^{+} \neq 0$, (4.9) implies that

$$(4.10) \quad \delta' = 0$$

and thus $b > 1$. Therefore we conclude from (4.5) and (4.10) that

$$(4.11) \quad \tilde{\omega}_1 = \frac{A\delta}{\tilde{\omega}_{nc}} ,$$

where the amplitude A is to be determined.

We conclude from condition (4.8) with $j = 2$ and (4.10) and (4.11) that A must satisfy the equation

$$(4.12) \quad \alpha \mu_c^2 + q_{nc}^{+} \eta A - 2R_{jc}^{+} = 0 ,$$

where

$$(4.13) \quad \alpha = \frac{4}{3\pi n_c} (F_{NN} + 2b_{nc} F_{NM} + b_{nc}^{*} G_{NN} + 2b_{nc} b_{nc}^{*} G_{NM}) / (1 + b_{nc} b_{nc}^{*}) ,$$

and R_{jc}^{+} has $\mu = \mu_c$. In arriving at (4.12) we have required that $\mu' \neq 0$ and $\delta'' \neq 0$ which by (4.1) imply that $a = 1$ and $b = 2$, or

$$(4.14) \quad \mu = \mu_c + \eta v + O(v^2) , \quad \delta = v^2 .$$

There are two real roots of (4.12) if

$$(4.15) \quad \eta^2 > - \frac{8\mu_c \alpha R_{jc}^{+}}{q_{jc}^{+2}}$$

Thus there are two inner expansions

$$(4.16) \quad \tilde{\omega}^{\pm} = \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} + A^{\pm}(\eta) \tilde{\omega}_{nc}(\xi) \delta^{\frac{1}{2}} + O(\delta) ,$$

with

$$(4.17) \quad \mu = \mu_c + \eta \delta^{\frac{1}{2}} + O(\delta) \quad .$$

If $\alpha R_{jc}^+ > 0$ then these expansions are defined for all η . If $\alpha R_{jc}^+ < 0$ then

there are two expansions for $\eta > (-8\mu_c \alpha R_{jc}^{+'}/q_{jc}^{+2})^{\frac{1}{2}}$ and two expansions for

$$\eta < -(-8\mu_c \alpha R_{jc}^{+'}/q_{jc}^{+2})^{\frac{1}{2}}. \quad \text{At } \eta_0 = (8\mu_c |\alpha R_{jc}^{+'}|/q_{jc}^{+2})^{\frac{1}{2}}, \left| \frac{dA^{\pm}}{d\eta} \right| = \infty \quad .$$

5. The Matching

To obtain a uniform asymptotic representation of the solution it is necessary to connect each of the outer expansions to the appropriate inner expansions by using the matching conditions of the method of matched asymptotic expansions. To apply these conditions we first express the outer expansion in terms of the inner variables by substituting (4.14) into (3.1) and expand the result in a power series in v :

$$(5.1) \quad \tilde{u} = \sum_{j=0}^{\infty} \tilde{u}_j(\xi, \mu(v)) v^{2j} = \sum_{k=0}^{\infty} \tilde{u}_k(\xi) v^k.$$

The matching conditions then are

$$(5.2) \quad \lim_{|\eta| \rightarrow \infty} (\tilde{u}_k - \tilde{u}_k) = 0 \quad \text{for } k = 1, 2, \dots$$

For the outer expansions corresponding to the uniform sub-branches we have

$$(5.3) \quad \tilde{u}_1 = \frac{R_{jc}^{+1}}{q_{jc}^{+1} \eta} \tilde{u}_{jc}$$

and corresponding to the striation sub-branches we have

$$(5.4) \quad \tilde{u}_1 = - \left(\frac{R_{jc}^{+1}}{q_{jc}^{+1} \eta} + \frac{q_{jc}^{+1} \eta}{\alpha} \right) \tilde{u}_{jc}$$

The asymptotic forms of the inner coefficients $\tilde{u}_k$ are found from the asymptotic forms of the amplitudes $A^{\pm}$ according to (4.11). The solution $A^{\pm}$ of (4.12) is

$$(5.5) \quad A^{\pm} = -\frac{q_{nc}^{+} \eta}{2\mu_c \alpha} \pm \frac{\left| \frac{q_{nc}^{+} \eta}{\alpha} \right|}{\left| \frac{q_{nc}^{+} \eta}{\alpha} \right|} \pm \frac{2R_{nc}^{+}}{\left| \frac{q_{nc}^{+} \eta}{\alpha} \right|} + O\left(\frac{1}{|\eta|^3}\right).$$

It then follows from (5.5) that for $\alpha R_{jc}^{+} > 0$

$$(5.6a) \quad \lim_{\eta \rightarrow \infty} A^{+} = \lim_{\eta \rightarrow -\infty} A^{-} = \begin{cases} \frac{1}{\mu_c} \left| \frac{q_{nc}^{+}}{\alpha} \right| \eta, & \alpha > 0, \\ -2 \left| \frac{R_{nc}^{+}}{q_{nc}^{+}} \right| \eta^{-1}, & \alpha < 0, \end{cases}$$

$$(5.6b) \quad \lim_{\eta \rightarrow \infty} A^{-} = \lim_{\eta \rightarrow -\infty} A^{+} = \begin{cases} -2 \left| \frac{R_{nc}^{+}}{q_{nc}^{+}} \right| \eta^{-1}, & \alpha > 0, \\ \frac{1}{\mu_c} \left| \frac{q_{nc}^{+}}{\alpha} \right| \eta, & \alpha < 0 \end{cases}$$

and that for $\alpha R_{jc}^{+} < 0$

$$(5.6c) \quad \lim_{\eta \rightarrow \infty} A^{+} = \lim_{\eta \rightarrow -\infty} A^{-} = \begin{cases} \frac{1}{\mu_c} \left| \frac{q_{nc}^{+}}{\alpha} \right| \eta, & \alpha > 0, \\ 2 \left| \frac{R_{nc}^{+}}{q_{nc}^{+}} \right| \eta^{-1}, & \alpha < 0, \end{cases}$$

$$(5.6d) \quad \lim_{\eta \rightarrow \infty} A^- = \lim_{\eta \rightarrow -\infty} A^+ = \begin{cases} 2 \left| \frac{R_{nc}^+}{q_{nc}^+} \right| \eta^{-1} & , \quad \alpha > 0 \\ \frac{1}{\mu_c} \left| \frac{q_{nc}^+}{\alpha} \right| \eta & , \quad \alpha < 0 \end{cases}$$

In obtaining (5.6) we have assumed $\frac{q_{nc}^+}{\alpha} < 0$ since in experiments this appears to be the case.

The results of the matching are summarized in Tables I and II where we denote the uniform and striation outer expansions for $\mu < \mu_c$ by $u_{(0)}^-$ and $u_{(1)}^-$, respectively, and for $\mu > \mu_c$ by $u_{(0)}^+$ and $u_{(1)}^+$. The resulting composite expansions are shown in Fig. 3.

The qualitative features observed in experiments for the transition from the uniform to the $n = 1$ striated state may be explained by Fig. 3a. The fact that no jump in the state of the plasma is observed as μ is varied rules out the behavior shown by Fig. 3b. The argon contamination and non-uniform heating of the neon plasma are therefore sufficient to qualitatively explain the experimental results.

We note that it may also be possible for a weak, steady magnetic field of strength $B(z)$ to produce smooth transitions between states. The effect of the magnetic field is to reduce the radial diffusion of electrons in the plasma by a factor which depends on $B(z)$. The term in F , proportional to N , which accounts for electron radial diffusion losses to the walls of the cylinder will therefore have a coefficient which varies with z . This effect can be included in the α_1 coefficient in (2.4a) and our results remain valid.

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Table I

$(\alpha > 0)$

$\eta \rightarrow$	∞	$-\infty$
ω^+	$u^+_{(1)}$	$u^-_{(0)}$
ω^-	$u^+_{(0)}$	$u^-_{(1)}$

Table II

$(\alpha < 0)$

$\eta \rightarrow$	∞	$-\infty$
ω^+	$u^+_{(0)}$	$u^-_{(1)}$
ω^-	$u^+_{(1)}$	$u^-_{(0)}$

Figure Captions

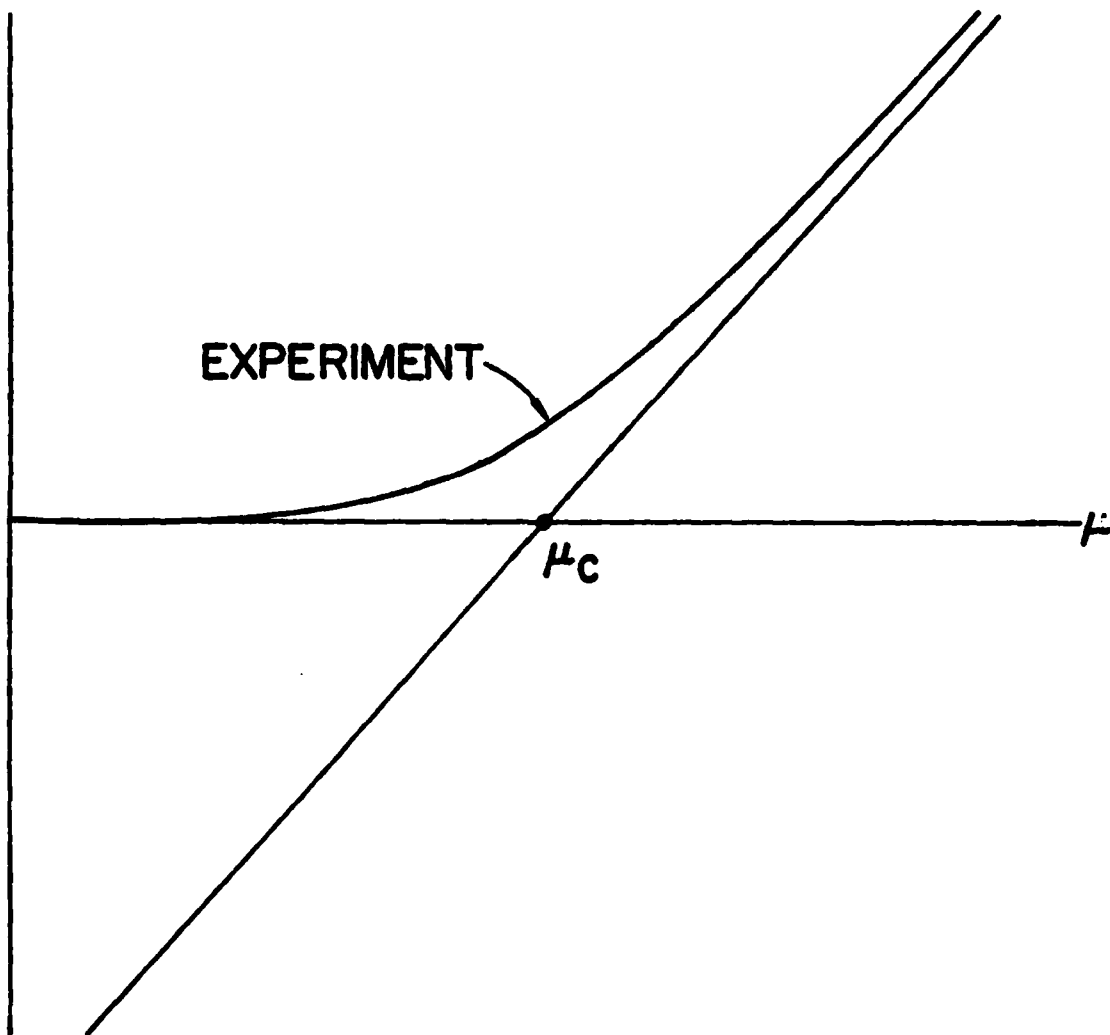
- Fig. 1: The contrast between theory and experiment for stationary striations. The amplitude is the coefficient of $\sin \frac{n \pi z}{L}$ in (1.2a) with $q_n^+ / \alpha < 0$.
- Fig. 2: The curves O_u and O_s are the outer expansions corresponding to the uniform and striation branches of the bifurcation problem, respectively. The dashed curves I are the inner expansions. The amplitude in each case is the coefficient of $\sin n_c \xi$. In (a) $\alpha R_{j_c}^+ > 0$ and in (b) $\alpha R_{j_c}^+ < 0$.
- Fig. 3: The composite solutions of the perturbed problem for (a) $\alpha R_{j_c}^+ > 0$ and (b) $\alpha R_{j_c}^+ < 0$. The noses of the branches in (b) are given by $\mu^\pm = \mu_c \pm \eta_0 \delta^{\frac{1}{2}} + \dots$.

Amplitude

EXPERIMENT

μ_c

Fig. 1



Amplitude

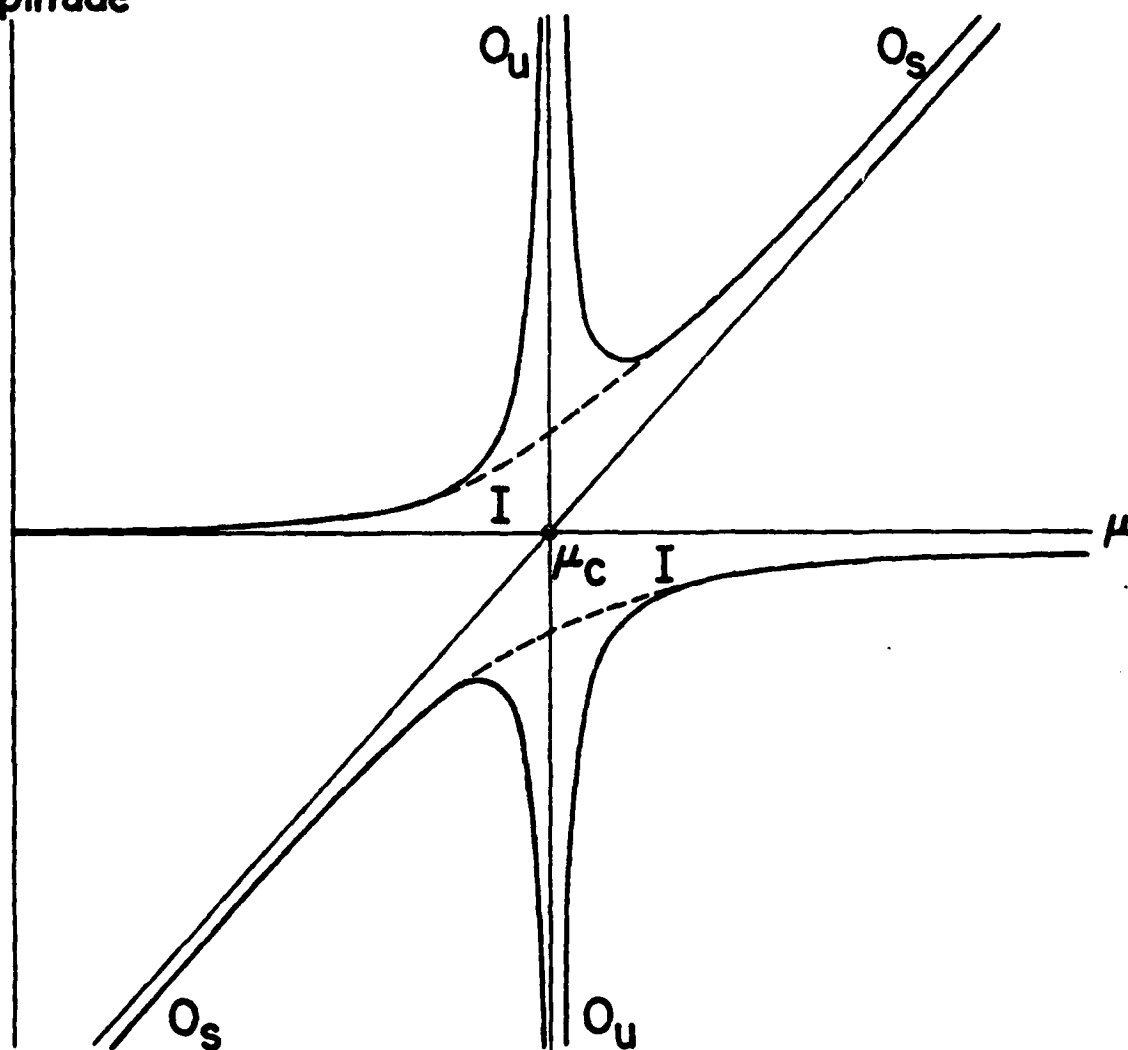


Fig. 2a

Amplitude

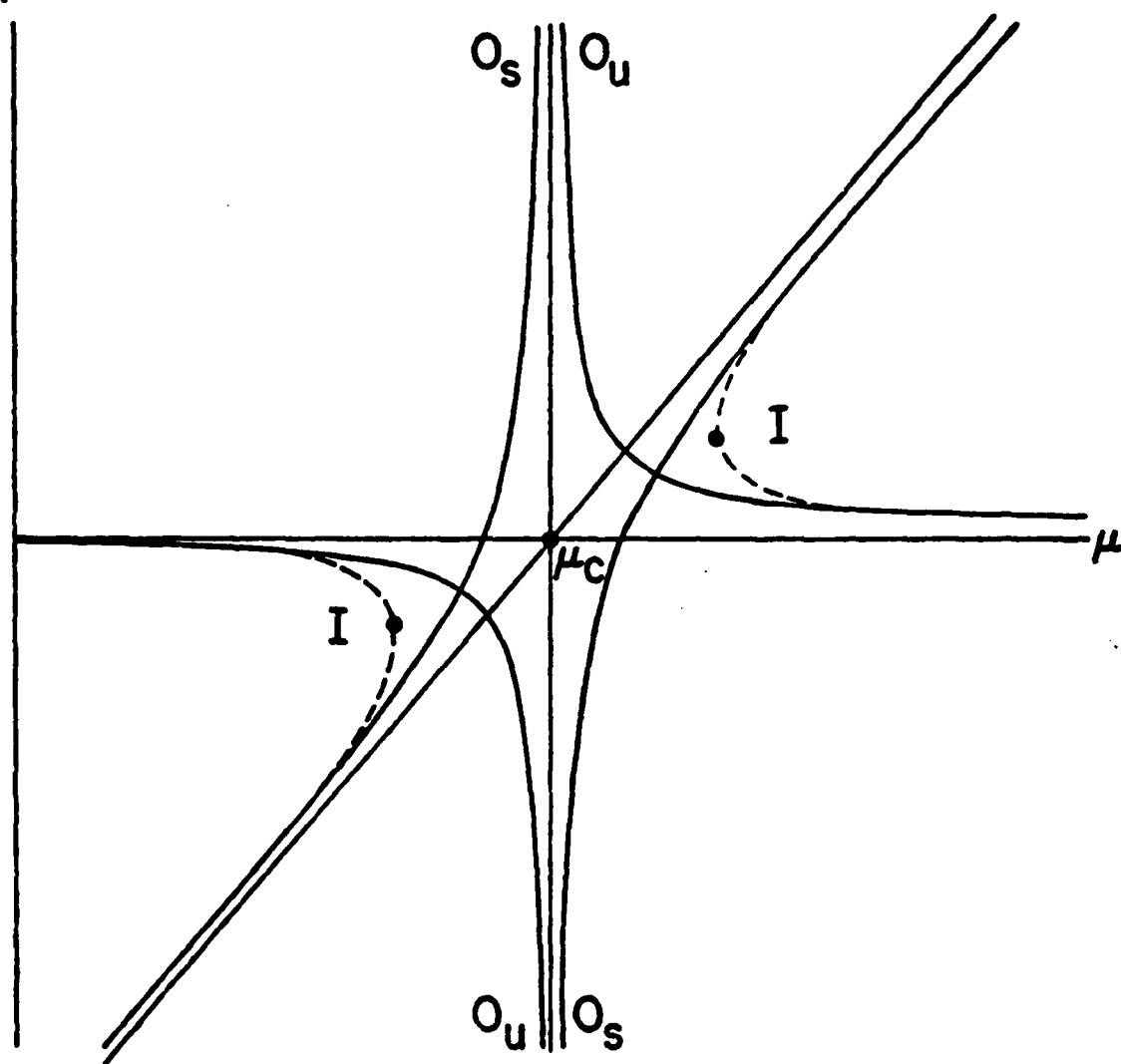


Fig. 2b

Amplitude

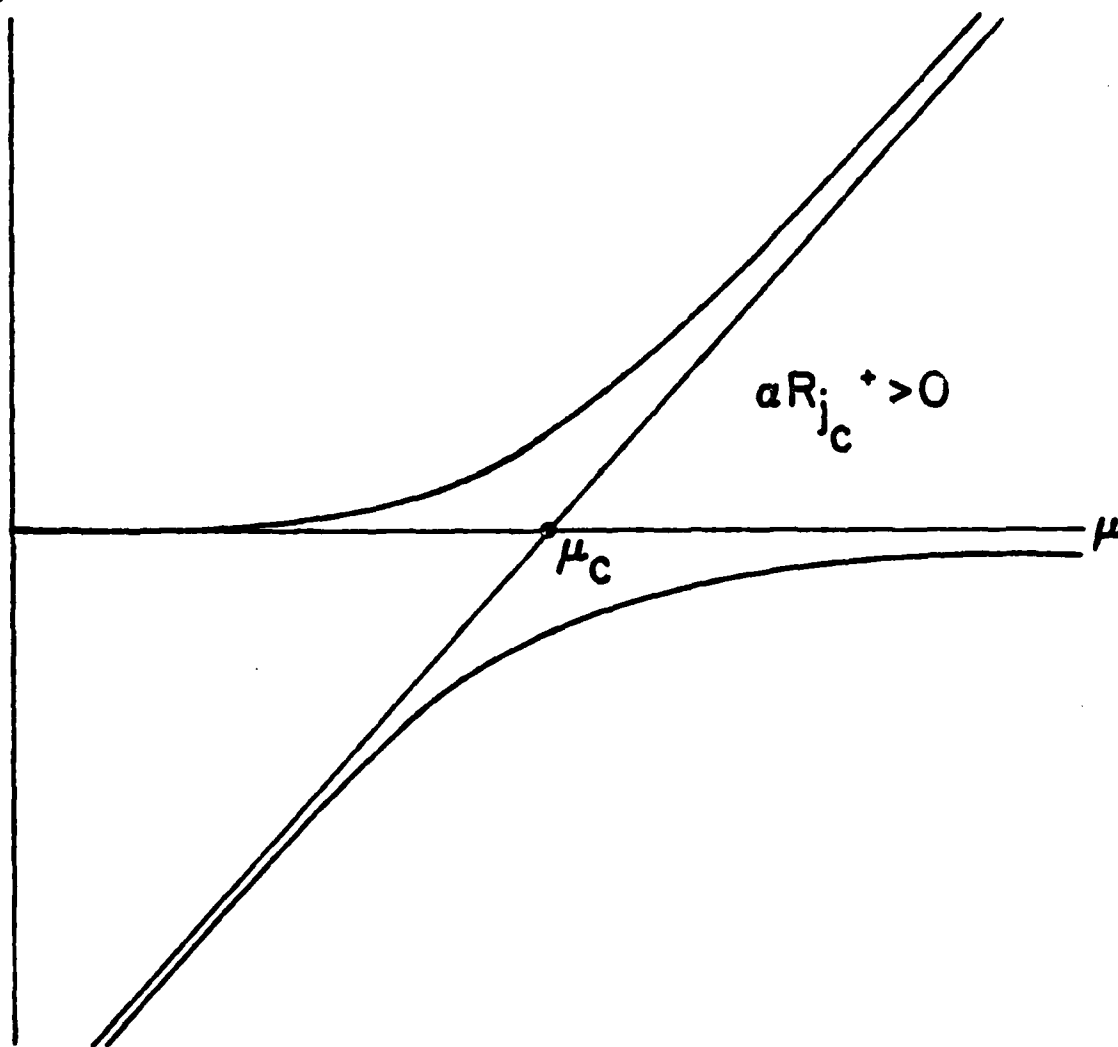


Fig. 3a

Amplitude

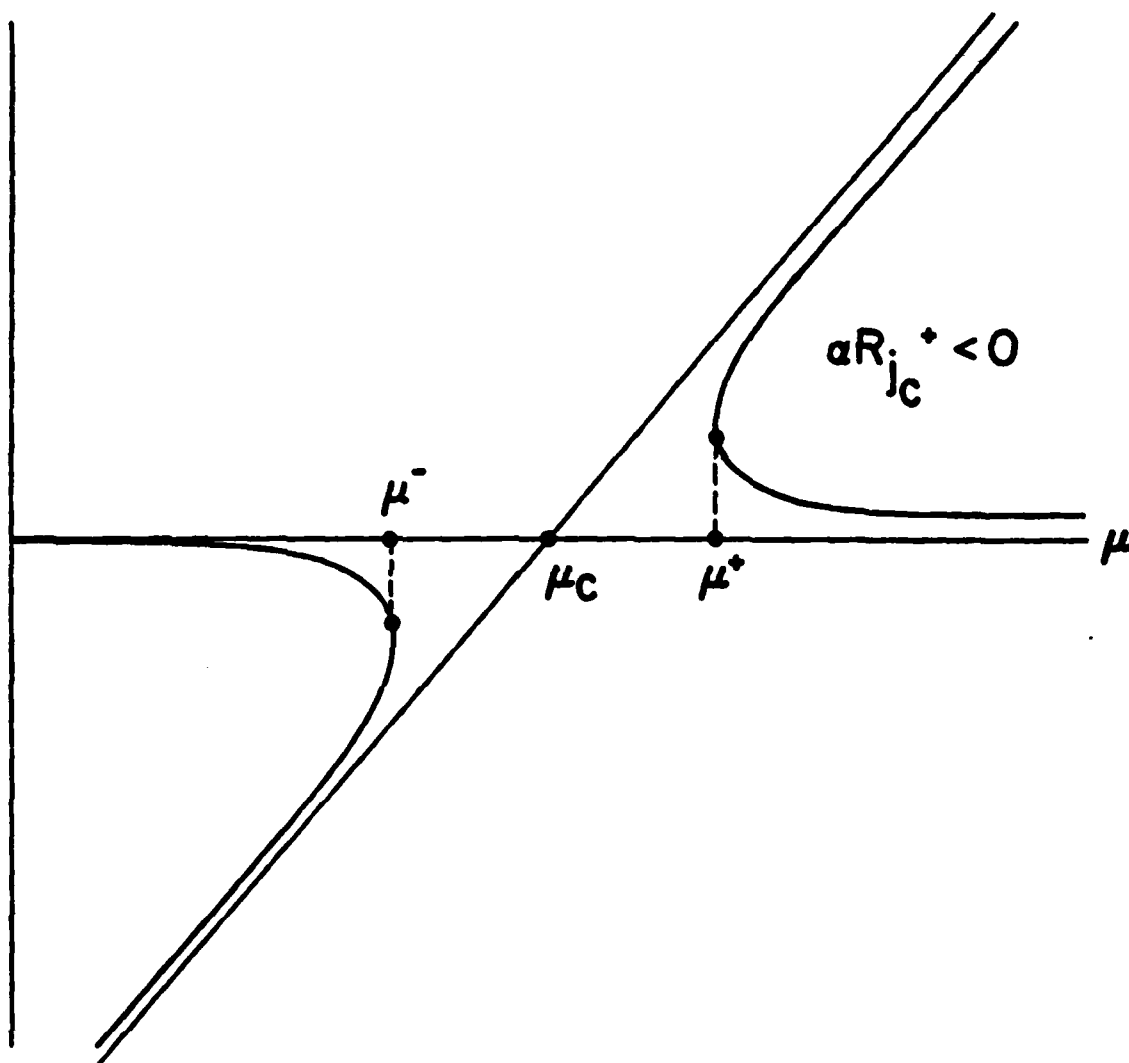


Fig. 3b