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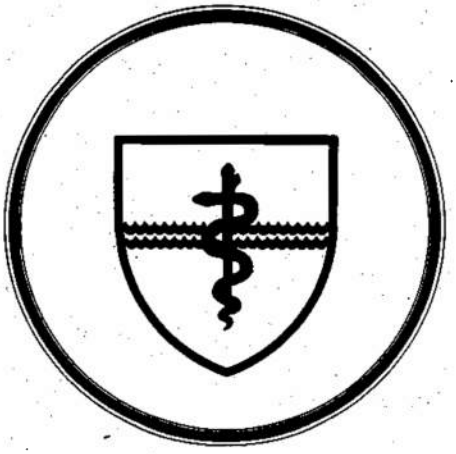

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REFINEMENTS OF FORMULAS FOR RECEPTOR SENSITIVITY

by

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SUMMARY PAGE

PROBLEM

To improve a method for approximating functions of wavelengths which is applicable to computer calculations related to vision.

FINDINGS

Shifting the origin of the wavelength scale to the center of the region of interest resulted in a significant reduction of roundoff errors.

APPLICATION

Use of the method presented enables researchers to use convenient approximation methods for calculations related to vision, without restriction on the computer or calculator employed. A previously published method would fail on some machines due to roundoff error.

ADMINISTRATIVE INFORMATION

This investigation was conducted as part of the Naval Submarine Medical Research Laboratory Work Unit MR000.01.01-5087 -- "Understimulation of the red-green opponent system by fluorescent light." This manuscript was submitted for review on 28 April 1981, approved for publication on 13 May 1981 and designated as NSMRL Report No. 954.

Refinements of formulas for receptor sensitivity

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When it is desired to fit a polynomial to a function of wavelength, roundoff errors may be minimized by shifting the origin of the independent variable (wavelength or frequency) to the center of the region of interest.

As mathematical modeling of visual functions becomes more successful, it is naturally desirable that independent workers should be able to reproduce a given model easily and accurately. Boynton and Wisowaty[1] have taken an important step in this direction by calculating formulas that approximate a certain set of receptor-sensitivity functions put forth by Smith and Pokorny.[2] In this Letter, we present some simple manipulations of the Boynton and Wisowaty results that will make them easier to use. The effect of the transformation that we make is to reduce roundoff errors, but the goal is *not* to make the answers more accurate. Rather, the use of formulas that are more immune to roundoff trouble removes the requirement for a computer that carries 10 significant digits and the need for coefficients carried out to 10 digits.

The step that we take is to shift the origin of the independent variable to the center of the region of interest. This traditional device of applied mathematics has general application when a polynomial is to be fitted to a function of wavelength. If the device were applied at the beginning of a regression calculation, it would reduce or eliminate roundoff error at all stages. Finally, in the same spirit of making calculations more workable and routine, we make explicit the relation of the present receptor sensitivities to Judd's $\bar{x}'$, $\bar{y}'$, and $\bar{z}'$ and to the receptor sensitivities incorporated in the model of Guth *et al.*[3]

In discussing the use of their formulas, Boynton and Wisowaty observe that "... for some machines rounding errors may cause problems with the fourth-order polynomials (it is essential to retain all five decimal places for the coefficients of Table I)."[4] By "five decimal places," Boynton and Wisowaty mean some 8–10 significant digits on most of the coefficients. Roundoff errors occur because the formulas as given depend on the subtraction of nearly equal numbers. For instance, one step in the evaluation of the green receptor function involves the polynomial evaluation

$$\log a = a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0. \tag{1}$$

Using coefficients a_0 through a_4, as provided by Boynton and Wisowaty, and $\lambda = 0.5$ μm leads to the sum

$$\begin{aligned}\log a &= -1747.361113 + 6533.32095 - 9149.3564 \\ &\quad + 5695.5237 - 1332.9406 \\ &= -0.813463.\end{aligned} \tag{2}$$

To shift the origin of wavelength,[5] we define the new variable

$$l = \lambda - \lambda_0. \tag{3}$$

As a practical matter, we put the new origin in the middle of the visible spectrum:

$$\lambda_0 = 0.55\ \mu\text{m}. \tag{4}$$

Equation (1) may be replaced by a polynomial in l:

$$\log a = \alpha_4 l^4 + \alpha_3 l^3 + \alpha_2 l^2 + \alpha_1 l + \alpha_0. \tag{5}$$

In general, to transform the coordinates of a polynomial of degree N, the new coefficients α_k can be calculated from the old a_k by

$$\alpha_k = \sum_{j=0}^{N-k} a_{j+k} \frac{(j+k)!}{j!k!} \lambda_0^j, \tag{6}$$

where $k = 0 \ldots N$. With $\lambda_0 = 0.55$ μm and $\lambda = 0.5$ μm, the evaluation of Eq. (5) involves the sum

$$\begin{aligned}\log a &= -0.1747 + 1.1551 - 2.7525 + 2.0055 - 1.0470 \\ &= -0.8136.\end{aligned} \tag{7}$$

The answer is essentially the same, but roundoff error is less likely to occur since we are no longer adding and subtracting quantities 10^4 times as large as the answer. Any time one wishes to fit a polynomial to data that have a limited domain of the independent variable not centered on the origin, a similar reduction of roundoff problems can be expected from a transformation such as Eq. (3), with λ_0 in the center of the domain.

In the case of the red- and blue-sensitivity functions, Boynton and Wisowaty found it necessary to fit polynomials in frequency to functions expressed as quantum sensitivity. In this case, the origin of frequency must be shifted. We define

$$f = \nu - \nu_0. \tag{8}$$

As a practical matter, we choose

$$\nu_0 = 5.9 \tag{9}$$

in units of 10^{14} Hz—the convenient unit selected by Boynton and Wisowaty.

The working equations now become Eqs. (3), (4), (8), and (9) above, and Eqs. (10)–(22) below. Green-catching sensitivity is given by

$$G = (a^n + b^n)^{1/n}, \tag{10}$$

$$\log a = \alpha_4 l^4 + \alpha_3 l^3 + \alpha_2 l^2 + \alpha_1 l + \alpha_0, \tag{11}$$

$$\log b = \beta_4 l^4 + \beta_3 l^3 + \beta_2 l^2 + \beta_1 l + \beta_0. \tag{12}$$

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Table 1. Values of n for Eq. (10) for G, and for Eq. (14) for R and B, and the coefficients for Eqs. (11), (12), and (15)–(17)

Exponents		
R	G	B
$n_1 = 2.25$	$n = 1.45$	$n_1 = 8.00$
$n_2 = 8.50$	–	$n_2 = 5.00$
Coefficients		
R	G	B
$\alpha_3 = -0.02467$	$\alpha_4 = -27960.$	$\alpha_3 = 0.6877$
$\alpha_2 = -0.9920$	$\alpha_3 = -9241.$	$\alpha_2 = -4.020$
$\alpha_1 = 1.432$	$\alpha_2 = -1101.$	$\alpha_1 = 5.691$
$\alpha_0 = -2.087$	$\alpha_1 = -40.11$	$\alpha_0 = -2.053$
–	$\alpha_0 = -1.047$	–
$\beta_3 = 0.5818$	$\beta_4 = -1706.$	$\beta_3 = -0.3913$
$\beta_2 = -1.042$	$\beta_3 = 1190.$	$\beta_2 = -1.305$
$\beta_1 = -1.403$	$\beta_2 = -319.7$	$\beta_1 = 2.431$
$\beta_0 = -0.5700$	$\beta_1 = 3.979$	$\beta_0 = -0.7289$
–	$\beta_0 = -0.4519$	–
$\gamma_3 = 0.6602$	–	$\gamma_3 = -11.05$
$\gamma_2 = -0.6721$	–	$\gamma_2 = -33.31$
$\gamma_1 = -1.883$	–	$\gamma_1 = -29.86$
$\gamma_0 = -0.9875$	–	$\gamma_0 = -11.16$

Table 2. Values of n and Coefficients for Eqs. (19)–(22) To Be Used in Calculations of Eye Lens and Macular Pigment Density

Exponents	
$n_1 = 15.00$	
$n_2 = 7.50$	
$n_3 = 4.00$	
Coefficients	
$\alpha_3 = -10130.$	$\gamma_3 = -928.0$
$\alpha_2 = -3596.$	$\gamma_2 = -485.1$
$\alpha_1 = -432.8$	$\gamma_1 = -56.18$
$\alpha_0 = -16.91$	$\gamma_0 = -1.295$
$\beta_3 = -6684.$	$\delta_3 = 14.93$
$\beta_2 = -2243.$	$\delta_2 = 2.715$
$\beta_1 = -244.5$	$\delta_1 = -1.119$
$\beta_0 = -8.016$	$\delta_0 = 0.0716$

Red-catching sensitivity R and blue-catching sensitivity B are given by

$$R \text{ or } B = Q(\lambda/555), \tag{13}$$

where λ is in nanometers,

$$Q = [(a^{n_1} + b^{n_1})^{n_2/n_1} + c^{n_2}]^{(1/n_2)}, \tag{14}$$

and

$$\log a = \alpha_3 f^3 + \alpha_2 f^2 + \alpha_1 f + \alpha_0, \tag{15}$$

$$\log b = \beta_3 f^3 + \beta_2 f^2 + \beta_1 f + \beta_0, \tag{16}$$

$$\log c = \gamma_3 f^3 + \gamma_2 f^2 + \gamma_1 f + \gamma_0. \tag{17}$$

Table 1 gives the required constants.

For the density of eye lens plus macular pigment,

$$D = \{[(a^{n_1} + b^{n_1})^{n_2/n_1} + c^{n_2}]^{n_3/n_2} + d^{n_3}\}^{1/n_3}, \tag{18}$$

where the equations for a, b, c, and d are

$$\log a = \alpha_3 l^3 + \alpha_2 l^2 + \alpha_1 l + \alpha_0, \tag{19}$$

$$\log b = \beta_3 l^3 + \beta_2 l^2 + \beta_1 l + \beta_0, \tag{20}$$

$$\log c = \gamma_3 l^3 + \gamma_2 l^2 + \gamma_1 l + \gamma_0, \tag{21}$$

$$\log d = \delta_3 l^3 + \delta_2 l^2 + \delta_1 l + \delta_0. \tag{22}$$

Table 2 gives the required constants.

The polynomial coefficients have been tabulated only to four significant digits. This degree of precision was chosen in a systematic way. We wrote a computer program to tabulate calculated R, G, B, and D with the polynomial coefficients rounded to 2, 3, 4, . . . up to 8 significant digits. Six-digit coefficients give nearly perfect agreement with the Boynton and Wisowaty calculated values; more digits give perfect agreement. Four-significant-digit precision in the coefficients gives discrepancies from those earlier calculated values of a few units in the fourth decimal place—a precision of about 1 in 10^4. This should be tolerable, since Boynton and Wisowaty find their fits to the Smith and Pokorny functions to have precision of the order of 2 in 10^2.

SOME RELATED DATA

Taking the inverse of the transformation matrix derived by Smith and Pokorny[2] gives the expression to recover Judd's modified $\bar{x}, y, \bar{z}$[6]:

$$\begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} 2.9448 & -3.5010 & 0.2118 \\ 1.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{pmatrix} \begin{pmatrix} R \\ G \\ B \end{pmatrix}. \tag{23}$$

This provides smooth versions of the Judd functions for applications where this might be convenient.

The equations above give (R, G, B) with maxima of (0.6442, 0.3912, 1.6431) at wavelengths of (562, 542, 440) nm. Guth *et al.* utilize essentially the same (R, G, B) as a starting point for their vector model for normal and dichromatic color vision,[3] but with the functions adjusted to have maximum values of 1.0. To rescale R, G, and B to have maxima of unity, multiply the expressions given above [Eqs. (10) and (13)] by 1.5523, 2.556, and 0.6086, respectively.

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