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A SPECTRAL MAPPING THEOREM FOR THE EXPONENTIAL FUNCTION, AND SO--ETC(U)
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A SPECTRAL MAPPING THEOREM FOR THE EXPONENTIAL FUNCTION, AND SOME COUNTEREXAMPLES

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ABSTRACT

Elementary proofs are given for the (known) theorems that (1) $e^{\sigma(A)} \subset \sigma(e^A)$ if A is the generator of a C_0-semigroup $\{e^{tA}\}$ of linear operators on a Banach space X, and that (2) $e^{\sigma(A)} = \sigma(e^A)\setminus\{0\}$ if $\{e^{tA}\}$ is a holomorphic semigroup. Also a large class of strongly continuous groups $\{e^{tA}\}$ on a Hilbert space H is given such that $\sigma(A)$ is empty. Note that $\sigma(e^A)$ is not empty, and is away from zero, if $\{e^{tA}\}$ is a group. Some related remarks are given on the relationship between the spectral bound of A and the type of $\{e^{tA}\}$.

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Key Words: C_0-semigroups, holomorphic semigroups, spectral mapping theorem, spectral radius, spectral bound, type.

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SIGNIFICANCE AND EXPLANATION

If one solves a system of linear differential equation $dx/dt = Ax$, where $x = x(t)$ is an n-vector and A is an $n \times n$ matrix, the solution may be written $x(t) = e^{tA}x(0)$. Here the matrix e^{tA} may be given by the Taylor series $\Sigma_{k=0}^{\infty} t^k A^k/k!$, or it may be more easily computed if the Jordan canonical form of A is known. In any case if $\lambda_1,\ldots,\lambda_n$ (some of which may be equal) are the eigenvalues of A, then $e^{\lambda_1 t},\ldots,e^{\lambda_n t}$ are the eigenvalues of e^{tA} (this is a special case of the so-called spectral mapping theorem).

It follows that the growth rate of $x(t)$ in any norm is at most exponential: $\|x(t)\| \le Me^{\omega t}\|x(0)\|$ for $t \ge 0$. The infimum of all possible ω is called the type of the semigroup $\{e^{tA}\}$, and will be denoted by type A in the sequel. The spectral mapping theorem mentioned above implies that $\omega = \max\{\mathrm{Re}\,\lambda_i\}$, which is called the spectral bound of A and will be denoted by spb A. Thus one has the relation spb A = type A.

If the matrix A is replaced with a linear operator A in an infinite-dimensional Banach space X, one can still solve $dx/dt = Ax$ for $x = x(t) \in X$ under certain conditions on A, to obtain a unique solution $x(t) = e^{tA}x(0)$ for $t \ge 0$. A is called the generator of the semigroup $\{e^{tA}\}$. Many problems in linear partial differential equations can be covered by semigroup theory. Here again one can define type A via the optimal growth rate for $\|x(t)\|/\|x(0)\|$, and spb A using the spectrum $\sigma(A)$ of A rather than the eigenvalues. It turns out, however, that the spectrum mapping theorem (which would now take the form $e^{\sigma(A)} = \sigma(e^A)\setminus\{0\}$) need not hold and, consequently, spb A and type A are in general different.

Nevertheless, we show that the previous results are true for a special class of generators A, which are roughly those appearing in parabolic partial differential equations. Also we give a wide class of counterexamples, which are not at all pathological, in which $-\infty$ = spb A < type A < $+\infty$.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A SPECTRAL MAPPING THEOREM FOR THE EXPONENTIAL FUNCTION, AND SOME COUNTEREXAMPLES

Tosio Kato

§0. Introduction

Let A be a linear operator in a complex Banach space X. Consider the mapping $A \to e^A$ and the validity of the spectral mapping theorem

$$\sigma(e^A) = e^{\sigma(A)} , \tag{0.1}$$

where σ denotes the spectrum.

According to the general spectral mapping theorem, (0.1) is true if $A \in B(X)$ (bounded linear operators with domain X). If A is unbounded, (0.1) need note be true even when A is the generator of a strongly continuous group $\{U(t) = e^{tA}; -\infty < t < \infty\}$. A striking counterexample is given in Hille-Phillips [1, p. 665], in which $U(t)$ is the Riemann-Liouville fractional integration of the imaginary order it on $X = L^p(0,1)$. Here $\sigma(A)$ is empty but $\sigma(e^A)$ is nonempty and is away from zero.

On the other hand, Hille-Phillips [1, p. 460] shows that (0.1) is true (except for zero) if A is the generator of a semigroup $\{e^{tA}; t > 0\}$ which is norm-continuous for $t \geq \gamma$ for some constant $\gamma > 0$. The proof in [1] is difficult, however, based on the Gelfand theory of *normed* rings.

The purpose of the present note is twofold. First we give an elementary proof of the Hille-Phillips theorem in the special case when A generates a holomorphic semigroup. We shall then give a wide class of generators A of groups for which $\sigma(A)$ is empty, including the fractional integrals mentioned above as a special case.

§1. A spectral mapping theorem.

We begin with a one-sided inclusion in (0.1) for the generator of a C_0-semigroup.

<u>Theorem</u> 1. Let A be the generator of a semigroup $\{e^{tA}; t \geq 0\}$ of class C_0. Then $e^{\sigma(A)} \subset \sigma(e^A)$.

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Remark. This is a special case of Corollary 2 to Lemma 16.3.2 of [1, p. 457], in which A may be the generator of any semigroup of class A. But our proof is elementary and short.

Proof. For any complex number z_0, set

$$T = e^{z_0} \int_0^1 e^{t(A-z_0)} dt \in B(X) \quad . \tag{1.1}$$

Since $(A-z_0)e^{t(A-z_0)}u = (d/dt)e^{t(A-z_0)}u$ for $u \in D(A)$, it follows on integration that

$$T(A-z_0) \subset (A-z_0)T = e^A - e^{z_0} \quad . \tag{1.2}$$

If $e^{z_0} \in \rho(e^A)$ (ρ denotes the resolvent set), (1.2) gives

$$(A-z_0)^{-1} = (e^A - e^{z_0})^{-1} T \in B(X) \quad , \tag{1.3}$$

so that $z_0 \in \rho(A)$. In other words, $z_0 \in \sigma(A)$ implies $e^{z_0} \in \sigma(e^A)$, q.e.d.

Theorem 2. Let A be the generator of a holomorphic C_0-semigroup. Then $e^{\sigma(A)} = \sigma(e^A) \setminus \{0\}$.

Remarks. (a) By a holomorphic C_0-semigroup we mean a semigroup of class C_0 which has an analytic continuation to a sector containing the positive t-axis. It is known (see [1, Theorem 12.8.1]) that A generates such a semigroup if and only if $\rho(A)$ contains a sector

$$\Sigma = \{z \; ; \; |\arg(z-\gamma)| < \omega\} \quad , \tag{1.4}$$

where γ is a complex number and $\omega > \pi/2$, and

$$\|(z-A)^{-1}\| \le M_\varepsilon |z-\gamma|^{-1} \quad \text{for} \quad |\arg(z-\gamma)| \le \omega - \varepsilon \tag{1.5}$$

for each $\varepsilon \in (0,\omega)$.

(b) Removing 0 from $\sigma(e^A)$ in Theorem 2 is natural since $e^{\sigma(A)}$ never contains 0 but $\sigma(e^A)$ may well do.

(c) Theorem 2 is a special case of Theorem 16.4.1 of [1].

Proof. In view of Theorem 1, it suffices to show that

$$\sigma(e^A) \setminus \{0\} \subset e^{\sigma(A)} \quad . \tag{1.6}$$

In the proof we may assume $\gamma = 0$ in (1.4), (1.5) without loss of generality.

To prove (1.6), it suffices in turn to show that

$$0 \neq \zeta_0 \notin e^{\sigma(A)} \text{ implies } \zeta_0 \in \rho(e^A) . \tag{1.7}$$

To this end we first note that

$$e^A = \frac{1}{2\pi i} \int_{C_0} e^z (z-A)^{-1} dz , \tag{1.8}$$

where C_0 is a curve in Σ running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, where $\pi/2 < \theta < \omega$ (see e.g. Kato [2, p. 489]).

Given a ζ_0 as in (1.7), consider all the complex numbers z_j $(j = 1,\ldots,m)$ lying to the left of C_0 and satisfying $e^{z_j} = \zeta_0$. Obviously there are at most finitely many such z_j; they are on a vertical line and equally spaced. We may assume that there is no $z \in C_0$ with $e^z = \zeta_0$, by deforming C_0 if necessary.

The assumption in (1.7) implies that $z_j \in \rho(A)$ $(j = 1,\ldots,m)$. Hence we can find a small circle C_j about z_j such that C_j and its interior are in $\rho(A)$. We may assume that C_j, including C_0, are separated from one another.

We now construct a Dunford-type integral

$$S = \frac{1}{2\pi i} \int_C \frac{e^z}{e^z - \zeta_0} (z-A)^{-1} dz \in B(X) \tag{1.9}$$

where

$$C = C_0 + C_1 + \cdots + C_m ,$$

the C_j being assumed to be coherently oriented (so that the C_j with $j \geq 1$ are negatively oriented). Note that the integral (1.9) exists because the integrand is analytic for $z \in C$ and decays exponentially at infinity on C_0.

We note that C_0 in (1.8) may be replaced by C, since there is no contribution to the integral from the C_j, the integrand being analytic on each C_j and its interior. Then we can apply the Dunford integral calculus, to obtain (see e.g. [2, p. 44])

$$e^A S = \frac{1}{2\pi i} \int_C \frac{e^{2z}}{e^z - \zeta_0} (z-A)^{-1} dz . \tag{1.10}$$

Here it should be noted that both e^z and $e^z(e^z-\zeta_0)^{-1}$ are analytic in the closed domain bounded by C. The fact that this domain is unbounded causes no difficulty, since these functions decay rapidly at infinity.

Since

$$\frac{e^{2z}}{e^z-\zeta_0} = (1 + \frac{\zeta_0}{e^z-\zeta_0})e^z \quad ,$$

it follows from (1.8) to (1.10) that

$$e^A S = e^A + \zeta_0 S \quad .$$

Hence

$$(\zeta_0-e^A)(1-S) = \zeta_0 \quad .$$

It follows that $\zeta_0 \in \rho(e^A)$, with

$$(\zeta_0 - e^A)^{-1} = \zeta_0^{-1}(1-S) \in B(X) \quad .$$

This proves (1.7), q.e.d.

§2. Counterexamples --- fractional powers of accretive operators.

In this section we show that the counterexample of fractional integrals mentioned in §0 is not an isolated phenomenon.

<u>Theorem</u> 3. Let H be a Hilbert space. Let $B \in B(H)$ be accretive. Then the fractional powers B^α are well defined and form a holomorphic semigroup for $\mathrm{Re}\,\alpha > 0$, with

(2.1) $$\|B^\alpha\| \leq \frac{\sin \pi\xi'}{\pi\xi'(1-\xi')} \|B\|^\xi e^{\pi|\eta|/2} \qquad (\xi = \mathrm{Re}\,\alpha > 0)$$

where $\eta = \mathrm{Im}\,\alpha$ and $\xi' = \xi - [\xi]$. If in particular 0 is not an eigenvalue of B, then B^α is strongly continuous for $\mathrm{Re}\,\alpha \geq 0$, and $\{B^{i\eta};\ -\infty < \eta < \infty\}$ is a strongly continuous group with

(2.2) $$\|B^{i\eta}\| \leq e^{\pi|\eta|/2} \quad .$$

If, in addition, B is quasi-nilpotent, then the generator iA of the group $\{B^{i\eta}\}$ has empty spectrum, while $e^{\pm iA} = B^{\pm i}$ have nonempty spectra away from 0.

<u>Remark</u>. The estimate (2.2) is sharp; equality holds for $k(x,y) = 1$ (see [1, p. 665]).

Proof. Theorem 3 was proved in Kato [3] except for the last assertion regrding the case when B is quasi-nilpotent. In this case the semigroup $\{B^\xi;\ \xi > 0\}$ is of type $-\infty$ (i.e. $\lim_{\xi\to+\infty} \xi^{-1}\log\|B^\xi\| = -\infty$), so that its generator A has empty spectrum. (Note that $\{B^\xi\}$ has generator A if $\{B^{i\eta}\}$ has generator iA.) Since $\{B^{i\eta}\}$ is a group, on the other hand, it is obvious that $e^{\pm iA} = B^{\pm i} \in B(X)$ have nonempty spectra away from 0.

Example. There are abundant examples of operators B satisfying the conditions of Theorem 3. Let $k(x,y)$ be a continuous, hermitian symmetric, nonnegative-definite kernel on $[0,1] \times [0,1]$. $k(x,y)$ defines an integral operator $K \in B(H)$, where $H = L^2(0,1)$, such that $K^* = K \geq 0$. Let B be the associated Volterra operator:

$$Bu(x) = \int_0^x k(x,y)u(y)dy \quad (u \in H) \quad . \tag{2.3}$$

Then B is quasi-nilpoint and accretive, since $2\,\mathrm{Re}(Bu,u) = (Ku,u) \geq 0$. B has no eigenvalue 0 if K is strictly positive, since $Bu = 0$ implies $(Ku,u) = 0$ by the remark above. But B may have no eigenvalue 0 even when K is only semi-definite. The simplest example of B is given by $k(x,y) = 1$. Then B is a simple integration, and $\{B^\alpha\}$ is exactly the fractional integrals considered in §0.

§3. The spectral bound and the type.

If A is a closed linear operator in X, we define the spectral bound of A by

$$\mathrm{spb}\ A = \sup \mathrm{Re}\ \sigma(A) = \sup\{\mathrm{Re}\ \lambda;\ \lambda \in \sigma(A)\} \quad . \tag{3.1}$$

We set $\mathrm{spb}\ A = -\infty$ if $\sigma(A)$ is empty.

If A generates a strongly continuous semigroup $\{e^{tA};\ t > 0\}$, the type of A is defined by

$$\mathrm{type}\ A = \lim_{t\to+\infty} t^{-1} \log\|e^{tA}\| = \log\ \mathrm{spr}\ e^A < +\infty \quad , \tag{3.2}$$

where spr denotes the spectral radius. (type A is usually referred to the semigroup $\{e^{tA}\}$ rather than to the generator A, but we use the notation type A as a convenient abuse.)

Theorem 1 shows that

(3.3) $$\text{spb } A \leq \text{type } A \quad (< +\infty)$$

if A generates a C_0-semigroup. (Actually it is true for more general semigroups.)

Theorem 2 shows that equality holds in (3.3) if A generates a holomorphic semigroup. (Again it is true for more general semigroups.)

Theorem 3 shows that equality in (3.3) need not hold even for the generator A of a strongly continuous group; indeed one has spb $A = -\infty$ while $|\text{type } A| \leq \pi/2$ in Theorem 3.

It may be noted that Greiner-Voigt-Wolff [4] gives examples of positivity-preserving C_0-semigroups $\{e^{tA}\}$ on certain function spaces for which spb $A = -\infty$ and type $A = 0$.

In fact little is known, beyond holomorphic semigroups (or, more generally, norm-continuous semigroups), about the question of when equality holds in (3.3).

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