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FORMATION OF SINGULARITIES FOR A  
CONSERVATION LAW WITH DAMPING

Shao-Shiung Lin and Reza Malek-Madani

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FORMATION OF SINGULARITIES FOR A CONSERVATION  
LAW WITH DAMPING

Shao-Shiung Lin and Reza Malek-Madani

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ABSTRACT

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The equations of gas dynamics in a tube with varying cross section are an example of a nonhomogeneous system of conservation laws. In this work we study the Riemann problem for this system by viewing it as a perturbation of the classical equations of gas dynamics in a uniform tube. We also study the Riemann problem and the formation of singularities for a related, but simpler, problem of a nonhomogeneous Berger's equation.

AMS (MOS) Subject Classifications: 35L65, 35L67, 36L45, 76N15

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Work Unit Number 1 (Applied Analysis)

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#### SIGNIFICANCE AND EXPLANATION

A one-dimensional mathematical model for the flow of gas in a nozzle (a tube with varying cross section) takes the form of a nonhomogeneous system of conservation laws. It is an interesting problem to study how closely this one-dimensional system models the real two- and three-dimensional problem. With this goal in mind we study two problems: a) the Riemann problem (i.e., the solution to the problem whose initial datum is a step function) for the one-dimensional system, b) the formation of shocks for a single first order equation which has most of the structure of the one-dimensional model system.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

## FORMATION OF SINGULARITIES FOR A CONSERVATION LAW WITH DAMPING

Shao-Shiung Lin and Reza Malek-Madani

1. Introduction. The equations of gas dynamics in a uniform tube have been studied quite extensively in recent years. It is well known that, as a hyperbolic conservation law, these equations exhibit discontinuous solutions, while the initial value problem is not mathematically well posed in the class of weak solutions [1]. It is not difficult to envisage the mathematical reason for the nonsmoothness of solutions. These equations enjoy a full set of real characteristics and, if the initial values are chosen properly, the information carried by the characteristics will overlap and shocks develop. The problem under study in this paper has one additional property, namely the variation in the tube's cross section, that will presumably contribute even further to the shock producing mechanisms.

Section 2 concerns with the derivation of the equations studied in this work. The arguments of Hughes [2] have been followed and, as it will become apparent, the system under consideration is an example of nonhomogeneous hyperbolic conservation laws. In Section 3 a simpler but related problem is discussed for the purpose of understanding the shock producing mechanisms that do not exist in the homogeneous problem.

Section 4 concerns the solution of the Riemann problem. It is well known [3], [4] that the solution of the Riemann problem played an essential role in developing a numerical scheme in order to solve the initial value problem for the equations of gas dynamics in a tube with uniform cross section. Motivated by this fact T. P. Liu [5] applied a modified Riemann problem for the general  $n^{\text{th}}$  order nonhomogeneous conservation laws and developed an iterative scheme which converges to the weak solution of the initial value problem. Although the above scheme is quite successful theoretically it is rather difficult to implement it. Since we have in mind a concrete example from the equations of gas dynamics it is our contention to propose a simpler Riemann problem and hope that it would give rise to more manageable computations. We are presently studying this problem.

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2. Derivation of the model equation. Consider an inviscid isentropic gas flow through a two dimensional duct  $D = \{(x,y) | A_1(x) \leq y \leq A_2(x), -\infty < x < \infty\}$ . The motion of the gas is governed by the equations of conservation of mass and linear momentum

$$\begin{aligned} \rho_t + (\rho u)_x + (\rho v)_y &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0 \end{aligned} \quad (2.1)$$

with  $p = f(\rho)$ , where  $\rho = \rho(x,y,t)$  is the density,  $p = p(x,y,t)$  is the pressure and  $\underline{u} = (u,v)$  is the velocity vector, together with the Neumann boundary conditions

$$u(x, A_i(x), t) A'_i(x) = v(x, A_i(x), t), \quad i = 1, 2$$

and the initial conditions

$$\begin{aligned} \rho(x, y, 0) &= \rho_0(x, y), \\ u(x, y, 0) &= u_0(x, y), \\ v(x, y, 0) &= v_0(x, y). \end{aligned}$$

In the remainder of this section we will outline briefly the procedure discussed in [2] which approximates (2.1) by a one-dimensional nonhomogeneous system in the variables  $\rho$  and  $u$ . For a physical quantity  $q(x,y,t)$  defined in the region  $D$  we define the average  $\langle q \rangle$  of  $q$  in the  $y$ -direction

$$\langle q \rangle = \frac{1}{A(x)} \int_{A_1(x)}^{A_2(x)} q(x, y, t) dy$$

where  $A(x) = A_2(x) - A_1(x)$ . Averaging each equation in (2.1) and using the boundary conditions yield

$$\begin{aligned} \langle \rho \rangle_t + \langle \rho u \rangle_x &= - \frac{A'(x)}{A(x)} \langle \rho u \rangle, \\ \langle \rho u \rangle_t + \langle \rho u^2 + p \rangle_x + \langle p \rangle_x &= - \frac{A'(x)}{A(x)} \langle \rho u^2 \rangle, \\ \langle \rho v \rangle_t + \langle \rho uv \rangle_x + \langle p \rangle_y &= - \frac{A'(x)}{A(x)} \langle \rho uv \rangle, \\ \langle p \rangle &= \langle f(\rho) \rangle. \end{aligned} \quad (2.2)$$

In order to further simplify (2.2) we make the following assumptions:

- (A) the total variations of  $A_1(x)$  and  $A_2(x)$  are small,
- (B) the quantity  $|\frac{v}{u}| \ll 1$ , i.e., the flow is predominantly in the x-direction,
- (C)  $\langle f(\rho) \rangle = \hat{f}(\langle \rho \rangle)$  for some  $\hat{f}$ .

Then it is reasonable to assume that

$$(D) \quad \begin{aligned} \langle \rho u \rangle &= \langle \rho \rangle \langle u \rangle, \\ \langle \rho u^2 \rangle &= \langle \rho \rangle \langle u \rangle^2 \end{aligned}$$

etc. An asymptotic analysis with respect to  $|\frac{v}{u}|$  adds more plausibility to the equations (2.3). Thus (2.2) becomes

$$\begin{aligned} \rho_t + (\rho u)_x &= - \frac{A'(x)}{A(x)} \rho u, \\ (\rho u)_t + (\rho u^2 + p)_x &= - \frac{A'(x)}{A(x)} \rho u^2, \\ p &= f(\rho) \end{aligned} \tag{2.3}$$

where we have made the following identifications

$$\langle \rho \rangle \simeq \rho(x, t), \quad \langle u \rangle \simeq u(x, t)$$

etc.

System (2.3) is the one-dimensional approximation of (2.1). It should be pointed out that as far as the authors know there has not been a rigorous analysis of how reasonable the assumptions (A-D) are. Nevertheless, the system (2.3) is a mathematically tractable model of (2.1). It is believed that the study of (2.3) will shed some light to the structure of the solutions of the more difficult, but exact, equations of gas flow in a duct with variable cross section.

3. Formation of singularities for the equation  $u_t + \varphi(u)_x = a(x)u$ . Before proceeding with the solution to the Riemann problem for the system (2.3) it is instructive to study how the spatial dependence of (2.3) enters as an important feature in producing shocks. The nonhomogeneous Burger's equation

$$\begin{cases} u_t + \varphi(u)_x = a(x)u \\ u(x,0) = u \end{cases} \quad (3.1)$$

has most of the essential structure of equations (2.3) with one additional advantage that (3.1) has only one family of characteristics. Our goal in this section is to show that the global smoothness of the solutions of (3.1) depend strongly on the sign of  $a(x)$  and  $a'(x)$ . By way of example, consider the simple case of  $\varphi(u) = 1/2 u^2$ . If  $a(x)$  was a constant  $\alpha$ , then (3.1) can be solved explicitly along characteristics to give us a smooth solution along a family of parallel characteristics. The slope of these characteristics depends on  $\alpha$  and, as easily seen, it is an increasing function of  $\alpha$ . Therefore, it seems plausible that as  $a(x)$  decreases the characteristics will intersect in finite time.

The definition of a shock discussed in the theory of conservation laws lends itself naturally to equation (3.1). Let a family of characteristics  $x(t,\xi)$  be defined by

$$\frac{dx}{dt} = \varphi'(u(x,t)) \quad x(0,\xi) = \xi.$$

We assume throughout this paper that equation (3.1) is hyperbolic and genuinely nonlinear, i.e.,  $\varphi'(u) > 0$  and  $\varphi''(u) > 0$  respectively. Under these conditions, a singularity (shock) develops if two characteristics intersect. Let  $x(t,\xi_1)$  and  $x(t,\xi_2)$  be two characteristics intersecting at  $(T,x^*)$ , that is,  $x^* = x(T,\xi_1) = x(T,\xi_2)$ . Further, suppose that the two characteristics actually cross each other and  $\frac{dx(T,\xi_1)}{dt} \neq \frac{dx(T,\xi_2)}{dt}$ .

It follows from the above definition of characteristic curve that

$\varphi'(u(x(T^-, \xi_1), T^-)) \neq \varphi'(u(x(T^-, \xi_2), T^-))$ . Since  $\varphi'$  is assumed to be monotone the solution  $u(x,t)$  becomes multivalued at  $(T,x^*)$  and a shock develops. It should be noted that a shock formed due to the interaction of two characteristics in forward time already has the entropy inequality embodied in it (cf. [1]).

An indication that the smooth solution  $u(x,t)$  will be discontinuous at a point  $(x,t)$  is that  $u_x(x,t)$  becomes unbounded in finite time [6]. On the other hand,  $u_x$  can be evaluated along the characteristic  $x(t,\xi)$  as

$$u_x(x(t,\xi),t) = \frac{u_\xi(x(t,\xi),t)}{x_\xi(t,\xi)}.$$

Thus  $u_x$  will become unbounded if  $x_\xi(t,\xi)$  approaches zero. This presents an alternative way of establishing the formation of a shock (cf. [7]).

The following lemma states that if  $x_\xi(t,\xi)$  becomes zero in finite time then two characteristics must intersect. Let  $v(t,\xi) \equiv x_\xi(t,\xi)$ . Note that  $v(0,\xi) = 1$ .

Lemma 3.1: Suppose that there exists  $T < \infty$  and  $\xi$  such that

$$v(T,\xi) < 0.$$

Then there are  $\xi_1$  and  $\xi_2$  with  $x(T,\xi_1) = x(T,\xi_2)$ . Moreover, if  $u$  satisfies an equation of the form  $\frac{du}{dt} = g(x,u)$  along the characteristic, it follows that the solution develops a shock in finite time.

Proof: By way of contradiction, suppose that for all  $\xi_1 \neq \xi_2$ ,  $x(T,\xi_1) \neq x(T,\xi_2)$ . This implies that the function  $f(\xi)$  defined by

$$f(\xi) = x(T,\xi)$$

is monotone. Therefore,  $f'(\xi)$  will always be nonnegative which contradicts the hypothesis. Hence there are two characteristics  $\xi_1$  and  $\xi_2$  which meet at  $(\tilde{x}, T)$  for some  $\tilde{x}$ . On the other hand, by the standard uniqueness theorem in ordinary differential equations, the above characteristics viewed in the  $(x,u)$  plane reach the line  $x = \tilde{x}$  at two different values of  $u$  at time  $T$ . This completes the proof of the Lemma.

Theorem (3.1): Suppose  $a \in C^1[0,\infty)$ ,  $u_0$  is a position constant,  $\varphi'(u) > 0$  and  $\varphi''(u) > k > 0$  for some  $k$ :

- (1) If  $a(x) \leq 0$  then (3.1) has global smooth solutions.
- (2) If  $a(x) > 0$  and  $a'(x) > 0$  then (3.1) has global smooth solutions.
- (3) If  $a(x) > 0$  and  $a'(x) < 0$  then a shock develops in finite time.

Proof: We will prove part (3) only. The proofs for (1) and (2) are similar. Let

$v(t, \xi) = x_{\xi}(t, \xi)$ . As before characteristics are defined by

$$\begin{aligned} \frac{dx}{dt} &= \varphi''(u) & x(0, \xi) &= \xi, \\ \frac{du}{dt} &= a(x)u & u(0, \xi) &= u_0, \end{aligned} \quad (3.2)$$

when  $u(t, \xi) = u(x(t, \xi), t)$ . Let  $w(t, \xi) = \frac{\partial}{\partial \xi} u(t, \xi)$ . Differentiating (3.2) with respect to  $\xi$  yields

$$\begin{aligned} \frac{dv}{dt} &= \varphi''(u)w & v(0, \xi) &= \xi \\ \frac{dw}{dt} &= a'(x)uv + a(x)w & w(0, \xi) &= 0. \end{aligned} \quad (3.3)$$

Let

$$G(t, \xi) = \exp - \int_0^t a(x(s, \xi)) ds. \quad (3.4)$$

Then it follows from (3.3b) that

$$w(t, \xi) = G^{-1}(t, \xi) \int_0^t G(s, \xi) a'(x(s, \xi)) u(s, \xi) v(s, \xi) ds. \quad (3.5)$$

From equation (3.2b) we have

$$\frac{d}{dt} (Gu) = 0$$

which implies that

$$u(t, \xi) = u_0 G^{-1}(t, \xi). \quad (3.6)$$

(3.6) combined with (3.5) yields (the dependence on  $\xi$  will be omitted)

$$w(t) = G^{-1}(t) u_0 \int_0^t a'(x(s)) v(s) ds. \quad (3.7)$$

Thus, the analysis of showing the existence of a finite time  $t$  at which  $v$  becomes zero leads to the study of the expression

$$\frac{dv}{dt} = \varphi''(u)G^{-1}(t)u_0 \int_0^t a'(x(s))v(s)ds, \quad (3.8)$$

Let

$$V(t) = \int_0^t a'(x(s))v(s)ds \quad (3.9)$$

so that  $V(0) = 0$  and

$$\dot{V}(t) = a'(x(t))v(t), \quad (3.10)$$

and  $\dot{V}(0) = a'(\xi)$  from (3.2). Hence the problem of finding the time such that

$v(T) = 0$  becomes equivalent to showing  $\dot{V}(T) = 0$ . Differentiating (3.10) again we obtain

$$\ddot{V} = \dot{a}'v + a\dot{v}$$

which combined with (3.9) and (3.10) yields

$$\begin{aligned} \ddot{V} &= \frac{\dot{a}'}{a'} \dot{V} + a'\varphi''G^{-1}u_0V, \\ V(0) &= 0, \\ \dot{V}(0) &= a'(\xi). \end{aligned} \quad (3.11)$$

We note that  $G^{-1}(t) > 1$  since  $a(x) > 0$ . Without loss of generality, we can assume that  $V(t) < 0$  for  $t \in [0, T]$ , otherwise there exists  $T^* > 0$  such that  $V(T^*) = 0$  and by Rolle's theorem there exists a time  $T^{**}$  such that  $\dot{V}(T^{**}) = 0$ . With the same argument as above, we can assume that  $V(t) < -\delta$  for  $t > T_1$ , for some  $\delta > 0$  and  $T_1 > 0$ . Thus we obtain the inequality

$$\ddot{V} - \frac{\dot{a}'}{a'} \dot{V} - a'\varphi''u_0V > 0. \quad (3.12)$$

Multiplying by the integrating factor  $\frac{1}{a'}$  and integrating from 0 to  $t$ , we obtain

$$\frac{1}{a'} \dot{V} - \frac{1}{a'(\xi)} a'(\xi) < u_0 \int_0^t \varphi''(u(s))V(s)ds, \quad (3.13)$$

or

$$\dot{V} > a'(x(t)) \left[ 1 + u_0 \int_0^t \varphi''(u(s))V(s)ds \right]. \quad (3.14)$$

Finally, using the hypothesis on  $\varphi''$  and the bound on  $V(t)$  we arrive at the following estimate for  $\dot{V}$ ,

$$\dot{V}(t) > a'(x(t)) [1 - \delta u_0 kt]. \quad (3.15)$$

We note that the term in brackets becomes negative in finite time which combined with the sign of  $a'(x)$  shows the existence of finite time at which  $v(t)$  becomes zero. This completes the proof of the theorem.

A similar result can be proved for the equation

$$\begin{aligned} u_t + \varphi(u)_x &= a(x)\varphi'(u) \\ u(x_1, 0) &= u_0, \end{aligned} \quad (3.16)$$

where  $u_0$ , as before, is assumed to be a positive constant. On the one hand, the nonlinear dependence of  $a(x)\varphi'(u)$  on  $u$  causes the previous proof to become more cumbersome. On the other hand, since the slope of the characteristics and the nonhomogeneous term have the same dependence on  $u$ , it is possible to find  $u$  explicitly in terms of  $x$  which simplifies the following proof considerably.

Proposition 3.1 Let  $\varphi'(u) > 0$  and  $\varphi''(u) > 0$ .

- 1) If  $a(x) < 0$ , then the solution to (3.16) is globally smooth.
- 2) If  $a(x) > 0$  and  $a'(x) > 0$ , then the solution to (3.16) is globally smooth.
- 3) Assume that  $a(x) > 0$ ,  $a'(x) < 0$  and  $a(x)$  approaches zero as  $x$  approaches infinity. Suppose that there exists a point  $\xi$  such that  $\int_{\xi}^{\infty} a(s)ds < \infty$ . Then a shock develops in finite time.

Proof. Again we confine ourselves to the proof of part 3. Define characteristics by

$$\frac{dx}{dt} = \varphi'(u), \quad x(0, \xi) = \xi \quad (3.17)$$

$$\frac{du}{dt} = a(x)\varphi'(u), \quad u(0, \xi) = u_0.$$

It follows from (3.17) that

$$\frac{du}{dx} = a(x)$$

or

$$u(t, \xi) = u_0 + \int_{\xi}^{x(t, \xi)} a(s)ds. \quad (3.18)$$

Differentiating (3.18) with respect to  $\xi$  yields

$$u_{\xi} = a(x(t, \xi))x_{\xi} - a(\xi). \quad (3.19)$$

On the other hand,  $u_\xi$  can be calculated from (3.17a), i.e.,

$$\frac{\partial}{\partial \xi} \frac{dx}{dt} = \frac{d}{dt} x_\xi = \varphi''(u)u_\xi$$

which combined with (3.19) gives

$$\frac{1}{\varphi''(u)} x_{\xi t} = a(x(t, \xi))x_\xi - a(\xi). \quad (3.20)$$

Now, let  $\xi$  be the point in the hypothesis of part 3. Since  $\varphi'(u) > 0$  it follows that

$x(t, \xi) > \xi$  for  $t > 0$ . (3.18), then, implies that

$$|u(t, \xi)| \leq u_0 + M \quad (3.21)$$

where  $M = \int_{\xi}^{\infty} a(s)ds$ . Thus,

$$\varphi''(u(t, \xi)) > M_1 \quad (3.22)$$

for some  $M_1$ . Since  $a(x)$  is positive and  $x(t, \xi) > \xi$  for  $t > 0$ , it follows that

$u(t, \xi) > u_0$ . On account of  $\varphi''(u)$  being positive, this in turn implies that

$\varphi'(u(t, \xi)) > \varphi'(u_0)$ . By (3.17a)

$$x(t, \xi) > \varphi'(u_0)t + \xi. \quad (3.23)$$

Since by (3.23)  $x(t, \xi)$  approaches infinity with  $t$ , there exists a time  $T$  such that

$a(x(t, \xi)) < \frac{1}{2} a(\xi)$  for  $t > T$ . Next we show that  $x_\xi(t, \xi) < 1$  for  $t > 0$ . This follows

immediately from (3.17) once we show that  $u_\xi(t, \xi) < 0$ . Differentiating (3.17) with

respect to  $\xi$  yields

$$\frac{du_\xi}{dt} = a'(x)x_\xi \varphi'(u) + a(x)\varphi''(u)u_\xi. \quad (3.24)$$

Since  $u(0, \xi) = u_0$  then  $u_\xi(0, \xi) = 0$ . Therefore  $\frac{du_\xi}{dt}(0, \xi) = a'(\xi)\varphi'(u_0)$  which is

negative by hypothesis, i.e.,  $u_\xi(t, \xi)$  becomes negative for a small period of time. On

the other hand, (3.24) shows that once  $u_\xi$  is negative,  $\frac{du_\xi}{dt}$  remains negative due to the

signs of  $a'$ ,  $\varphi'$ , and  $\varphi''$ . The two facts  $a(x(t, \xi)) < \frac{1}{2} a(\xi)$  for  $t > T$  and

$x_\xi(t, \xi) < 1$  enable us to deduce from (3.20) that

$$\frac{1}{\varphi''(u)} x_{\xi t} < -\frac{1}{2} a(\xi), \quad \text{for } t > T \quad (3.25)$$

which reduces to

$$x_{\xi t} < -\frac{1}{2} M_1 a(\xi) \quad \text{for } t > T. \quad (3.26)$$

Integrating (3.26) with respect to  $t$  gives us

$$x_{\xi}(t) \leq x_{\xi}(T) - \frac{1}{2} M_1 \psi(\xi)(t - T). \quad (3.27)$$

Inequality (3.27) shows that for large enough  $t$ ,  $x_{\xi}(t)$  will become negative. We then appeal to Lemma 3.1 to complete the proof of this proposition.

Next, we turn to the question of the Riemann problem for the related equation

$$u_t + \psi(u)_x = q(x, u). \quad (3.28)$$

We assume that  $\psi$  is genuinely nonlinear, i.e.,  $\psi'' > 0$ . Consider (3.28) with the Riemann initial condition

$$u(x, 0) = \begin{cases} u_r, & x > 0 \\ u_l, & x < 0. \end{cases} \quad (3.29)$$

We will give a brief outline of how the local solution to (3.28)-(3.29) is constructed.

Our claim is that the initial discontinuity (3.29) is immediately resolved by the corresponding conservation law

$$u_t + \psi(u)_x = 0. \quad (3.30)$$

Then the term  $q(x, u)$  governs the evolution of the resolved waves. Hence, to solve (3.28)-(3.29) we divide the problem into two cases:

Case A: The solution to (3.29)-(3.30) is a rarefaction. Let

$$u^0(\xi) = \begin{cases} u_l & \text{if } \xi < \xi_l \\ b(\xi) & \text{if } \xi_l < \xi < \xi_r \\ u_r & \text{if } \xi > \xi_r \end{cases} \quad (3.31)$$

be this solution, where

$$\xi_l = \psi'(u_l), \quad \xi_r = \psi'(u_r), \quad \psi'(b(x)) = x, \quad \xi = \frac{x}{t}.$$

Consider

$$\begin{aligned} \frac{dx}{dt} &= \psi'(u), & x(\xi, 0) &= 0 \\ \frac{du}{dt} &= q(x, u), & \tilde{u}(\xi, 0) &= u^0(\xi), \quad \xi_l < \xi < \xi_r. \end{aligned} \quad (3.32)$$

Let  $(\tilde{u}(\xi, t), x(\xi, t))$  be the solution of (3.32) on  $\xi_l \leq \xi \leq \xi_r$ . It is not difficult to show that  $x_r(\xi, t) \neq 0$  for  $\xi_l \leq \xi \leq \xi_r$ . Thus

$$u(x, t) = \tilde{u}(\xi(x, t), t)$$

is a solution of (3.30)-(3.31) within the region  $x_l(t) \leq x \leq x_r(t)$ , with

$$x_i(t) = x(\xi_i, t), \quad i = r, l.$$

Case B: The solution to (3.29)-(3.30) is a shock. Let

$$u^0(\xi) = \begin{cases} u_l & \text{if } \xi > s \\ u_r & \text{if } \xi < s \end{cases} \quad (3.33)$$

be that solution with

$$s = \frac{\varphi(u_r) - \varphi(u_l)}{u_r - u_l}, \quad \xi = \frac{x}{t}.$$

Then, in a similar manner to Case A we construct the solution to (3.28)-(3.29), namely,

$$u(x, t) = \begin{cases} u_l(x, t) & \text{if } x < \tilde{x}(t) \\ u_r(x, t) & \text{if } x > \tilde{x}(t) \end{cases} \quad (3.34)$$

where

$$\begin{cases} \frac{\partial}{\partial t} u_i + \varphi(u_i)_x = \sigma(x, u_i) \\ u_i(x, 0) = u_i \end{cases}, \quad i = r, l, \quad (3.35)$$

and

$$\frac{d\tilde{x}}{dt} = \frac{\varphi(u_r(\tilde{x}, t)) - \varphi(u_l(\tilde{x}, t))}{u_r(\tilde{x}, t) - u_l(\tilde{x}, t)}, \quad \tilde{x}(0) = 0. \quad (3.36)$$

4. Solution to the Riemann problem for the equation (2.3). In [3] Glimm proposed an iterative scheme in order to obtain the solution to the initial value problem for the general conservation law

$$\begin{aligned} u_t + \varphi(u)_x &= 0 \\ u(x, 0) &= u_0(x) \end{aligned} \quad (4.1)$$

where  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is smooth and genuinely nonlinear. The building block of this iterative scheme is the solution to the associated Riemann problems

$$u_c(x, 0) = \begin{cases} u_l, & x < c \\ u_r, & x > c. \end{cases} \quad (4.2)$$

The set of step functions in (4.2) is chosen as a pointwise approximation of the initial data. As shown in Section 2, the original problem of the flow of gas in a duct with varying cross section is a two-dimensional problem. It is the reduction of the latter system of equations to a one-dimensional initial value problem that produces the nonhomogeneous terms in (2.3). In this section we discuss the solution to the Riemann problem for (2.3) which arises from the discretization of the initial data and the boundary of the duct. It is this simultaneous discretization that makes our treatment of the Riemann problem different from the one discussed in [5]. First we note that (2.3) can be written in the form

$$\begin{aligned} (A\rho)_t + (A\rho u)_x &= 0, \\ u_t + \frac{1}{2} (u^2)_x + P(\rho)_x &= 0 \end{aligned} \quad (4.3)$$

where  $P$  is defined by

$$P(\rho) = \int^{\rho} \frac{p'(s)}{s} ds, \quad (4.4)$$

and  $A(x)$  has the form

$$A(x) = \begin{cases} 1, & x < 0 \\ 1 - \epsilon, & x > 0 \end{cases} \quad (4.5)$$

and

$$u_0(x) = \begin{cases} (\rho_-, u_-), & x < 0 \\ (\rho_+, u_+) & x > 0. \end{cases} \quad (4.6)$$

As in [5], we assume that both the initial condition and the boundary of the duct have small bounded variations. When  $\epsilon = 0$  (4.3)-(4.6) reduces to the classical Riemann problem for the equations of gas dynamics in a uniform tube [8]. For the case  $\epsilon$  positive we apply the same ideas as in Section 3, namely, the solution to (4.3)-(4.6) can be viewed as a small perturbation of the solution to the corresponding problem when  $\epsilon = 0$ . The implicit function theorem is the main tool in obtaining the exact solution of (4.3)-(4.6). To illustrate the method we choose a particular solution of the  $\epsilon = 0$  case and carry out the necessary calculations. Let

$$(\rho(x,t), u(x,t)) = \begin{cases} (\rho_-, u_-), & 0 < \frac{x}{t} < s \\ (\rho_m, u_m), & s < \frac{x}{t} < (p'(\rho_m))^{1/2} \\ q(\frac{x}{t}), & (p'(\rho_m))^{1/2} < \frac{x}{t} < (p'(\rho_+))^{1/2} \end{cases}$$

be the physically admissible solution to (4.3)-(4.6) with  $\epsilon = 0$ , i.e., the solution to the Riemann problem consists of a backward shock  $(\rho_-, u_-; \rho_m, u_m; s)$  and a forward rarefaction wave connecting  $(\rho_m, u_m)$  to  $(\rho_+, u_+)$  (cf. [8]). Then the solution to (4.3)-(4.6) with  $\epsilon$  positive consists of a backward shock  $(\rho_-, u_-; \rho_1(\epsilon), u_1(\epsilon); s(\epsilon))$ , a discontinuity  $(\rho_1(\epsilon), u_1(\epsilon); \rho_2(\epsilon), u_2(\epsilon); 0)$  which is due to the geometry of the duct, and a forward rarefaction connecting  $(\rho_2(\epsilon), u_2(\epsilon))$  to  $(\rho_+, u_+)$ . The five formulae relating  $s(\epsilon)$ ,  $\rho_1(\epsilon)$ ,  $u_1(\epsilon)$ ,  $\rho_2(\epsilon)$ , and  $u_2(\epsilon)$  are

$$\begin{aligned} s(\rho_1 - \rho_-) &= \rho_1 u_1 - \rho_- u_- , \\ s(\rho_1 u_1 - \rho_- u_-) &= \rho_1 u_1^2 + p(\rho_1) - \rho_- u_-^2 - p(\rho_-) , \\ (1 - \epsilon)\rho_2 u_2 &= \rho_1 u_1 , \\ \frac{1}{2} u_2^2 + P(\rho_2) &= \frac{1}{2} u_1^2 + P(\rho_1) , \end{aligned} \quad (4.7)$$

$$u_2 = u_+ + \int_{\rho_+}^{\rho_2} \frac{\sqrt{p'(\rho)}}{\rho} d\rho .$$

Equations (4.7) are simplified considerably after eliminating  $s$  and  $u_1$ . In that case we can formulate the above problem in the form

$$F(\rho_1, \rho_2, u_2, \epsilon) = 0 \quad (4.8)$$

where  $F = (F_1, F_2, F_3)$  is

$$\begin{aligned} F_1(\rho_1, \rho_2, u_2, \epsilon) &= (1 - \epsilon)\rho_2 u_2 - \rho_1 u_- + \left[ \frac{\rho_1}{\rho_-} (p(\rho_1) - p(\rho_-)) (\rho_1 - \rho_-) \right]^{1/2}, \\ F_2(\rho_1, \rho_2, u_2, \epsilon) &= u_2 - u_+ - \int_{\rho_+}^{\rho_2} \frac{\sqrt{p'(\rho)}}{\rho} d\rho, \\ F_3(\rho_1, \rho_2, u_2, \epsilon) &= \frac{1}{2} \rho_1^2 u_2^2 + \rho_1^2 p(\rho_2) - \frac{1}{2} (1 - \epsilon)^2 \rho_2^2 u_2^2 - \rho_1^2 p(\rho_1). \end{aligned} \quad (4.9)$$

We point out that the sign of the square root in (4.9) is chosen so that the usual entropy condition is satisfied [1]. (4.8)-(4.9) is now set up for applying the implicit function theorem. The problem is solved if we can uniquely determine  $\rho_1$ ,  $\rho_2$ , and  $u_2$  in terms of  $\epsilon$ . To this end we calculate the Jacobian of  $F$  with respect to  $\rho_1$ ,  $\rho_2$ , and  $u_2$  at  $\epsilon = 0$ . It can be shown that (all partial derivatives are evaluated at  $\rho_1 = \rho_-$ ,  $\rho_2 = \rho_-$ ,  $u_2 = u_-$ , and  $\epsilon = 0$ )

$$\begin{aligned} F_{1,\rho_1} &= -u_- + (p'(\rho_-))^{1/2}, & F_{1,\rho_2} &= u_-, & F_{1,u_2} &= \rho_-, \\ F_{2,\rho_1} &= 0, & F_{2,\rho_2} &= -\frac{(p'(\rho_-))^{1/2}}{\rho_-}, & F_{2,u_2} &= 1, \\ F_{3,\rho_1} &= \rho_- u_-^2 - \rho_- p'(\rho_-), & F_{3,\rho_2} &= -\rho_- u_-^2 + \rho_- p'(\rho_-), & F_{3,u_2} &= 0. \end{aligned}$$

A simple calculation shows that

$$\det \left. \frac{\partial (F_1, F_2, F_3)}{\partial (\rho_1, \rho_2, u_2)} \right|_{\epsilon=0} = 2\rho_- (u_-^2 - p'(\rho_-)) (p'(\rho_-))^{1/2}. \quad (4.10)$$

Thus, if the initial step  $(\rho_-, u_-; \rho_+, u_+)$  is such that

$$u_-^2 - p'(\rho_-) \neq 0 \quad (4.11)$$

i.e., the upstream flow of the gas is either subsonic or supersonic, we see that the

implicit function theorem can be applied to solve uniquely for  $\rho_1$ ,  $\rho_2$ , and  $u_2$  in terms of  $\epsilon$ . This completes the solution to the Riemann problem (4.3)-(4.6).

There are still two interesting problems in connection with (4.3) whose answers would be valuable both to the theory and the application. The first question is whether the above scheme actually converges to the weak solution of the initial value problem. The second question is how easily this scheme can be implemented numerically. We are presently studying these questions.

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