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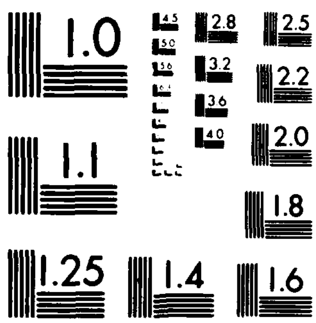

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ON REGULAR GENERALIZED LINE GRAPH DESIGNS

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ON REGULAR GENERALIZED LINE GRAPH DESIGNS

Ching-Shui Cheng†
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Abstract

A class of efficient designs called regular generalized line graph designs are introduced. This class of designs includes many well-known optimum and efficient designs, e.g., balanced incomplete block designs, group-divisible designs with $\lambda_2 = \lambda_1+1$, group-divisible designs with $\lambda_1 = \lambda_2+1$ and group size two, triangular designs with $\lambda_2 = \lambda_1+1$, L_2 designs with $\lambda_2 = \lambda_1+1$, etc. The optimality of regular generalized line graph designs is investigated. This uses graph theory as a tool and unifies much of the previous work in the area.

AMS(MOS) Subject Classifications: 62K05, 05B05

Key Words: generalized line graphs, balanced incomplete block designs, group-divisible designs, triangular designs, L_2 designs, E-optimality, regular graph designs.

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SIGNIFICANCE AND EXPLANATION

Incomplete block designs have been developed as statistical tools for the planning, conduct, analysis, and interpretation of scientific experimentation. Many designs have been shown to be optimum with respect to various criteria. This paper considers a minimax criterion, equivalent to maximum efficiency in the worst situation.

Some connection with graph theory is found. Using available results in graph theory, we introduce a very general class of designs which include many of the well-known optimum or efficient designs. Some important common properties shared by many optimum designs is unveiled after transformation to graphs. Such a property is useful for finding new optimum designs and tends to be obscured without the transformation.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON REGULAR GENERALIZED LINE GRAPH DESIGNS

Ching-Shui Cheng[†]
Gregory M. Constantine[*]

1. Introduction.

The purposes of this paper are two-fold. We shall introduce a class of efficient designs called regular generalized line graph designs (RGLGD), and thereby provide a unified approach to the theory of E-optimum block designs. In recent years, the attention of the researchers on optimum incomplete block designs has been moved from balanced incomplete block designs (BIBD) to the asymmetrical cases. Several classes of designs have been proved to be E-optimum. Many of these designs are special examples of regular generalized line graph designs. Regular generalized line graph designs are very efficient and are often optimum. Thus, besides the unification of earlier results, the optimality of many other designs will also be established. The main tool used involves a transformation of designs into graphs. SOme properties shared by many known optimum designs can be better understood in terms of graphs. These crucial properties tend to be obscured without such a transformation.

Consider the problem of comparing v treatments in b blocks of size k with $k < v$. Note that the assumption $k < v$ is not essential, and is made only for convenience. Any arrangement of the v treatments into the bk experimental units is called a design. Let $\Omega_{v,b,k}$ be the collection of all such designs. The usual additive model assumes the expectation of an

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observation on treatment i in block j to be $\alpha_i + \beta_j$ (treatment effect + block effect), where α_i and β_j are unknown constants, $1 \leq i \leq v$, $1 \leq j \leq b$. Furthermore, the bk observations are assumed to be uncorrelated with common variance. Throughout this paper, we shall assume that $v|bk$ and denote bk/v by r.

Under a design $d \in \Omega_{v,b,k}$, the coefficient matrix of the reduced normal equation for estimating the treatment effects (also called C-matrix) is

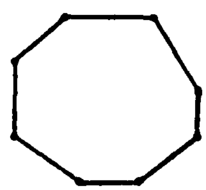
$$\underset{\sim}{C}_d = \text{diag}(r_{d1},\cdots,r_{dv}) - k^{-1}\underset{\sim}{N}_d\underset{\sim}{N}_d',$$

where r_{di} is the number of replications of treatment i, and $\underset{\sim}{N}_d$ is the treatment-block incidence matrix, i.e., $\underset{\sim}{N}_d = [n_{dij}]_{v\times b}$, where n_{dij} is the number of appearances of treatment i in block j. This matrix $\underset{\sim}{C}_d$ is symmetric, nonnegative definite and has zero row sums. Let $\mu_{d,v-1} \geq \mu_{d,v-2} \geq \cdots \geq \mu_{d1} \geq \mu_{d0} = 0$ be the eigenvalues of $\underset{\sim}{C}_d$. A design $d^* \in \Omega_{v,b,k}$ is said to be A-, D-, or E-optimum if it minimizes $\sum_{i=1}^{v-1} \mu_{di}^{-1}$, $\prod_{i=1}^{v-1} \mu_{di}^{-1}$, or μ_{d1}^{-1}, respectively, over $d \in \Omega_{v,b,k}$.

The problem of optimal incomplete block designs is far from being completely solved. Kiefer (1958, 1975) showed that a BIBD, if it exists, is "universally optimum". Takeuchi (1961, 1963) proved the E-optimality of a group-divisible design with $\lambda_2 = \lambda_1+1$. Later, Cheng (1978) proved that a group-divisible design with two groups and $\lambda_2 = \lambda_1+1$ is optimal with respect to a very general class of criteria including the A-, D-, and E-criteria. Extending Takeuchi's result in another direction, Cheng (1980) showed the E-optimality of a group-divisible design with group size two and $\lambda_1 = \lambda_2+1 > 1$ and that of a cyclic design with $\lambda_2 = \lambda_1 \pm 1$ and $v = 5$. All of the designs mentioned above are regular graph designs (RGD) defined by John and Mitchell (1977).

Let $\lambda_{dij} = \sum_{\ell=1}^{b} n_{di\ell}n_{dj\ell}$. Then a regular graph design is defined to be a binary equireplicated design with $|\lambda_{dij} - \lambda_{di'j'}| \leq 1$ for all $i \neq j$, $i' \neq j'$, i.e., there are at most two distinct values of λ_{dij} which differ by one. Let these two values be λ and $\lambda+1$. Then it is straightforward to see that $\lambda = [r(k-1)/(v-1)]$, where $[x]$ is the largest integer $\leq x$.

For convenience of later reference, we now review some terminology in graph theory. In this paper, only simple graphs, i.e., those in which there is at most one edge between any two vertices, are considered. Two vertices are said to be adjacent if there is an edge between them. A graph is called regular if each vertex is adjacent to the same number of other vertices. This number is called the degree of the regular graph. The adjacency matrix of a graph with v vertices is a $v \times v$ (0,1)-matrix with zero diagonal elements such that the (i,j)th entry is one if and only if vertex i and vertex j are adjacent. A *complete graph* K_n is a graph on n vertices in which any two vertices are adjacent. If the vertices can be divided into two disjoint groups of sizes m_1 and m_2, respectively such that two vertices are adjacent if and only if they are in different groups, then we have a complete bipartite graph. Such a graph is denoted by K_{m_1,m_2}. The complementary graph of a graph G is the graph with the same vertices as G in which two vertices are adjacent if and only if they are not adjacent in G. Finally, a circuit of length n is defined to be a regular connected graph with n vertices and degree two. For example, the following is a circuit of length eight:



If d is a regular graph design, then $k\underset{\sim}{C}_d + (\lambda+1)\underset{\sim}{J}_v - \{r(k-1) + \lambda+1\}\underset{\sim}{I}_v$ is a (0,1)-matrix with zero diagonal elements and constant row sum $\beta = (v-1)(\lambda+1) - r(k-1)$, where $\underset{\sim}{I}_v$ is the identity matrix of order v and $\underset{\sim}{J}_v$ is the $v \times v$ matrix of ones. Thus $k\underset{\sim}{C}_d + (\lambda+1)\underset{\sim}{J}_v - \{r(k-1) + \lambda+1\}\underset{\sim}{I}_v$ is the adjacency matrix of a regular graph with v vertices and degree β. Hereafter we shall denote this graph by $G(d)$ and denote the matrix $k\underset{\sim}{C}_d + (\lambda+1)\underset{\sim}{J}_v - \{r(k-1) + \lambda+1\}\underset{\sim}{I}_v$ by $\phi(\underset{\sim}{C}_d)$.

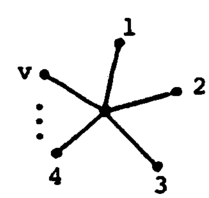
Regular graph designs are expected to be very efficient under the A-, D-, and E-criteria. John and Mitchell even conjectured that for given v, b, and k, if there is a regular graph design, then there must be a regular graph design which is A- (D-, or E-) optimum. Although the conditions imposed on a regular graph design are quite stringent, there can still be many RGD's with the same values of v, b, and k, since usually there are many regular graphs with given numbers of vertices and edges. Thus John and Mitchell (1977) had to search among all the regular graph designs to find the best one. In the present paper, we shall single out a subclass of regular graph designs, the regular generalized line graph designs, which are usually E-optimum and are still so general that all the optimum designs mentioned earlier are included. This provides a clearer picture for the problem of E-optimum designs. Not only are earlier results unified, but also many new optimum designs are obtained. A majority of the optimum regular graph designs listed in the technical report of Mitchell and John (1976) ae regular generalized line graph designs or duals of such designs.

Regular generalized line graph designs are defined in Section 2. Examples are provided. Section 3 is devoted to the E-optimality of regular generalized line graph designs. In particular, the E-optimality of many triangular and Latin square type PBIB designs with $\lambda_2 = \lambda_1+1$ is established.

2. Regular generalized line graph designs.

Generalized line graphs are defined in Hoffman (1970). The line graph $L(G)$ of any given graph G is defined to be the graph with the edges of G as its vertices such that any two edges in G are adjacent in $L(G)$ if and only if they have a common vertex in G. The cocktail party graph $CP(a)$ is the graph on $2a$ vertices in which each vertex is adjacent to every other vertex except one. Let G be a graph with vertices $v_1, v_2, \cdots, v_m$, and let $(a_1, a_2, \cdots, a_m)$ be an m-tuple of nonnegative integers. The generalized line graph $L(G; a_1, a_2, \cdots, a_m)$ is the graph obtained from $L(G)$ by adjoining m disjoint cocktail party graphs $CP(a_i)$, $i = 1, \cdots, m$, where each vertex of the ith cocktail party graph is adjacent to every vertex of $L(G)$ containing the vertex i of G. In particular, a line graph $L(G)$ or a cocktail party graph is a generalized line graph. We call a regular graph design d a regular generalized line graph design if each connected component of $G(d)$ is a generalized line graph.

Example 2a. Let d be a balanced incomplete block design. Then $G(d)$ is a complete graph, which is the line graph of a claw, i.e., a graph of the following kind:



Example 2b. If d is a group-divisible design with $\lambda_2 = \lambda_1 + 1$, then $G(d)$ is a disconnected graph wherein each connected component is a complete graph. Hence d is a regular generalized line graph design.

Example 2c. If d is a group-divisible design with group size two and $\lambda_1 = \lambda_2+1 > 1$, then G(d) is a cocktail party graph. Thus d is also a regular generalized line graph design.

Example 2d. Let d be a cyclic design with $\lambda_2 = \lambda_1 \pm 1$ and $v = 5$. Then G(d) is a circuit of length 5 which is the line graph of itself.

The designs in Examples 2a-2d have been shown to be E-optimal. All of them are regular generalized line graph designs.

Other examples:

Example 2e. Let d be a trinagular type PBIB design with $v = \binom{n}{2}$, and $\lambda_2 = \lambda_1+1$. Then G(d) is the line graph of the complete graph K_n.

Example 2f. Let d be an L_2 type PBIB design with $v = n^2$, and $\lambda_2 = \lambda_1+1$. Then G(d) is the line graph of the regular complete bipartite graph $K_{n,n}$.

The above two types of designs appear frequently in the list of optimum regular graph designs in Mitchell and John (1976).

Example 2g. Let d be design #3 listed in Mitchell and John (1976). It is the (A-, D-, and E-) optimum regular graph design found by them for $v = 8$, $k = 2$ and $b = 20$. The graph G(d) is disconnected and has two components. One component is a circuit of length 3 and the other component is a circuit of length 5. Both components are line graphs. Thus d is a regular generalized line graph design. Note that d is not a PBIB design.

Remark. If d is a regular graph design with $k=2$ and $\lambda=0$, then the graph obtained from d by considering each block of d as an edge connecting the two treatments in the block is not G(d)! Instead, it is the complementary graph of G(d). A fairly straightforward method to construct a regular generalized line graph design of block size two is to pick a regular graph G, then the complementary graph of L(G) provides a regular generalized line graph design with $k=2$ and $\lambda = 0$. Designs with other values of λ can be obtained by adding λ edges to each pair of vertices. Of course, this is not the only way to construct regular generalized line graph designs of block size two. The construction of designs with $k > 2$, however, is much more difficult.

3. <u>E-optimality.</u>

Cheng (1980) (also see Jacroux (1980)) derived some conditions under which there is a regular graph design which is E-optimum over $\Omega_{v,b,k}$. If these conditions are satisfied, then the search for E-optimum designs can be reduced to considering regular graph designs only. So let us tentatively look at the problem of finding the best regular graph design. For a regular graph design d, let $\tilde{\mu}_{d1}$ be the smallest eigenvalue of $\phi(\underset{\sim}{C}_d)$. Then it is clear that $\tilde{\mu}_{d1} = k\mu_{d1} - r(k-1) - \lambda - 1$. Thus to minimize μ_{d1}^{-1} is the same as to maximize $\tilde{\mu}_{d1}$. It is fairly obvious that $\tilde{\mu}_{d1} \leq -1$ for all d. If d is a BIBD or a group-divisible design with $\lambda_2 = \lambda_1 + 1$, then $\tilde{\mu}_{d1} = -1$, achieving the upper bound. This is essentially the idea which Takeuchi used in his 1961 paper. Now the question is: what is next? Obviously, we should keep $\tilde{\mu}_{d1}$ as close to -1 as possible. Examining the optimum regular graph designs found by Mitchell and John (1976), we discovered that a great majority of them satisfy $\tilde{\mu}_{d1} \geq -2$. In fact, all the designs in Examples 2c, 2e, and 2f satisfy $\tilde{\mu}_{d1} = -2$, and the designs in Example 2a, 2b, 2d, and 2g satisfy $\tilde{\mu}_{d1} > -2$. This simple property of $\tilde{\mu}_{d1}$ has been used by Cheng (1981) to establish the optimality (with respect to a very general class of criteria) of the designs in Examples 2b-2f over the PBIB designs with two associate classes. This crucial property is unveiled only after the designs are transformed into graphs. For convenience of later discussion, we define the eigenvalues of a graph as the eigenvalues of its adjacency matrix and denote the smallest eigenvalue of a graph G by $\mu_1(G)$.

In view of the above discussion the regular graph designs with $\tilde{\mu}_{d1} \geq -2$ deserve special attention and it is important to characterize the graphs with the smallest eigenvalues ≥ -2. The latter is a difficult problem and, coincidentally, has interested many graph theorists who tried to characterize the structures of graphs by their eigenvalues, see, e.g., Hoffman (1977), Cameron, Goethals, Seidel, and Shult (1976), and the references cited there. Now the following are well-known:

(i) Any generalized line graph satisfies $\mu_1(G) \geq -2$.

(ii) Other than a finite number of exceptions, the generalized line graphs are the only connected graphs with $\mu_1(G) \geq -2$.

(iii) Other than a finite number of exceptions, regular line graphs and cocktail party graphs are the only connected regular graphs with $\mu_1(G) \geq -2$.

Doob and Cvetković (1979) further showed that the only regular connected graphs with the smallest eigenvalues > -2 are the complete graphs and the odd circuits, i.e., circuits of odd length. Both of these two types of graphs are line graphs.

We have the following corollary of Doob's result:

<u>Theorem 3.1.</u> For given values of v, b, and k, if there exists a regular generalized line graph design d^*, then there must be a regular generalized line graph design which is E-optimum over all the regular graph designs in $\Omega_{v,b,k}$.

<u>Proof.</u> Since $\tilde{\mu}_{d1} \geq -2$ for any regular generalized line graph design, it suffices to show that if d is an RGD and $\tilde{\mu}_{d1} > -2$, then d is also a regular generalized line graph design. Let d be an RGD with $\tilde{\mu}_{d1} \geq -2$. Since $\mu_1(G) = \min\{\mu_1(G') : G'$ is a connected component of $G\}$, all the connected components of $G(d)$ have the least eigenvalues > -2. The assumption that d is an RGD implies that all the connected components of $G(d)$ are regular and hence, by Doob and Cvetković's result, must be complete graphs or odd circuits. We have seen that a complete graph is a line graph. Furthermore, a circuit G is isomorphic to $L(G)$. Therefore d is a regular generalized line graph design. □

So the question is how to search for the best regular generalized line graph design (under the E-criterion). We separate the discussion into three cases: (i) $\beta \neq 2$ and $(\beta+1) \nmid v$, (ii) $\beta \neq 2$ and $(\beta+1)|v$, (iii) $\beta=2$, where $\beta = (v-1)(\lambda+1) - r(k-1)$ as defined in Section 1.

Case (i). $\beta \neq 2$ and $(\beta+1) \nmid v$.

Doob and Cvetković's result tells us that the only regular generalized line graph designs with $\tilde{\mu}_{d1} > -2$ are those with complete graphs or odd circuits as the connected components of $G(d)$. We know that for any RGD d, the degree of $G(d)$ is β (see Section 1). Clearly if $\beta \neq 2$ and $(\beta+1) \nmid v$, then the connected components of $G(d)$ cannot be complete graphs nor odd circuits. Thus when $\beta \neq 2$ and $(\beta+1) \nmid v$, all the regular generalized line graph designs satisfy $\tilde{\mu}_{d1} = -2$; they are equally good under the E-criterion. In view of Theorem 3.1, we have the following

<u>Theorem 3.2.</u> If $\beta \neq 2$ and $(\beta+1) \nmid v$, then any regular generalized line graph design is E-optimum over the regular graph designs in $\Omega_{v,b,k}$.

The following also holds:

<u>Theorem 3.2'.</u> If $\beta \neq 2$ and $(\beta+1) \nmid v$, then any regular graph design with $\tilde{\mu}_{d1} = -2$ is E-optimum over all the regular graph designs in $\Omega_{v,b,k}$.

If $\beta = 2$ or $(\beta+1) | v$, then not all the regular generalized line graph designs satisfy $\tilde{\mu}_{d1} = -2$, and hence not all the regular generalized line graph designs are E-optimum over the RGD's. In both cases, we need to find an ordering of the regular generalized line graph designs.

Case (ii). $\beta \neq 2$ and $(\beta+1) | v$.

In this case, if there exists an RGD d^* such that all the connected components of $G(d^*)$ are complete graphs (i.e., d^* is a group-divisible design with $\lambda_2 = \lambda_1+1$), then of course d^* is E-optimum. If such a design does not exist, then all the regular generalized line graph designs satisfy

$\tilde{\mu}_d = -2$. Again, any regular generalized line graph design is E-optimum over the RGD's.

Case (iii). $\beta = 2$.

If $\beta = 2$, then for any regular generalized line graph design d, all the connected components of G(d) are circuits. It is well-known that for a circuit G with v vertices,

$$\mu_1(G) = \begin{cases} -2\cos(\pi/v), & \text{if } v \text{ is odd,} \\ -2, & \text{if } v \text{ is even.} \end{cases}$$

Note that $-2\cos(\pi/v) > -2$ and is a decreasing function of v. Thus for $\beta = 2$, we should try to find a regular generalized line graph design such that each connected component of G(d) has an odd number of vertices and the number of vertices in the largest connected component is as small as possible. More specifically,

(iiia). If $3|v$, then the best choice is a graph whose connected components are circuits of length three (which are also complete graphs on three vertices). This graph corresponds to a group-divisible design with group size three and $\lambda_2 = \lambda_1+1$. Such a design has $\tilde{\mu}_{d1} = -1$.

(iiib). If $v = 3t+2$ with $t \geq 1$, then the best choice consists of t-1 circuits of length three and one circuit of length five. In this case, $\tilde{\mu}_{d1} = (-1 - \sqrt{5})/2$.

(iiic). If $v = 3t+1$ with $t \geq 3$, then the best choice consists of t-3 circuits of size three and two circuits of size five. In this case, $\tilde{\mu}_{d1} = (-1 - \sqrt{5})/2$.

The only cases not covered by the above discussion are v=4 and 7.

(iiid). If $v = 4$, then there is only one graph with degree 2, i.e., a circuit of length four. This corresponds to a group-divisible design with two groups of size two and $\lambda_1 = \lambda_2+1$, which has $\tilde{\mu}_{d1} = -2$.

(iiie). If $v = 7$, then the best choice clearly is a circuit of length 7 which has $\tilde{\mu}_{d1} > -2$.

For example, when $v = 8$, $k = 2$, and $b = 20$, we have $\beta = 2$. Since $v \equiv 2 (\mathrm{mod}\ 3)$, this falls into Case (iiib). As demonstrated in Example 2g, the (A-, D-, and E-) optimum RGD found by Mitchell and John (1976) indeed is a regular generalized line graph design such that $G(d)$ has two connected components: a circuit of length 3 and a circuit of length 5.

Among the E-optimum regular generalized line graph designs listed in Mitchell and John (1976), 34 of them have $\beta = 2$. They have parameters (v,b,k) = (4,8,2), (4,14,2), (4,20,2), (5,5,2), (5,15,2), (5,25,2), (5,5,3), (5,15,3), (6,9,2), (6,24,2), (6,8,3), (6,18,3), (6,9,4), (7,14,2), (7,35,2) (7,14,5), (8,20,2), (8,16,3), (8,8,4), (8,16,5), (9,27,2), (9,9,3), (9,21,3), (9,9,6), (10,35,2), (10,8,5), (11,44,2), (11,33,3), (12,54,2), (12,40,3), (12,12,5), (12,8,6), (12,12,7), and (12,9,8). The graphs of all but four can be constructed in the manner as we described. The four exceptions are (6,8,3), (8,8,4), (10,8,5), and (12,8,6); all have block sizes bigger than two and it happens that no design corresponding to the optimum graph given by the above discussion exists. In situations like this, a design corresponding to the next best graph, if it exists, will be optimum.

Thus we know how to find the E-optimum regular generalized line graph design which, by Theorem 3.1, is E-optimum over the regular graph designs. If the conditions obtained in Cheng (1980) or Jacroux (1980) are satisfied, then it is also E-optimum over $\Omega_{v,b,k}$. The E-optimality of a BIBD (Kiefer (1958)), a group-divisible design with $\lambda_2 = \lambda_1+1$ (Takeuchi (1961)), a group-divisible design with group size two and $\lambda_1 = \lambda_2+1 > 1$, and a cyclic design with $v = 5$ and $\lambda_2 = \lambda_1 \pm 1$ (Cheng (1980)) can be unified in this way. We now state the following corollaries of Theorems 2.1, 2.2 of Cheng (1980) and Theorems 3.2, 3.2' of the present paper.

Corollary 3.3. If there is a regular generalized line graph design d in $\Omega_{v,b,k}$ such that $\tilde{\mu}_{d1} > -(v+2\beta)/(v-2)$, then there is a regular generalized line graph design which is E-optimum over all the equally replicated designs in $\Omega_{v,b,k}$.

Corollary 3.4. If there is a regular generalized line graph design d in $\Omega_{v,b,k}$ such that $\tilde{\mu}_{d1} > \max[-(v+2\beta)/(v-2), -\{v(k-1)+\beta\}/(v-1), -2]$, then there is a regular generalized line graph design which is E-optimum over $\Omega_{v,b,k}$.

Corollary 3.5. If $k > 3$ and there is a regular generalized line graph design d in $\Omega_{v,b,k}$ such that $\tilde{\mu}_{d1} > -(v+2\beta)/(v-2)$, then there is a regular generalized line graph design which is E-optimum over $\Omega_{v,b,k}$.

Note that if $\beta > \frac{v}{2} - 2$, then the quantity $-(v+2\beta)/(v-2)$ in Corollary 3.3 cannot be bigger than -2, so we have

Corollary 3.6. If $\beta > \frac{v}{2} - 2$, $\beta \neq 2$, and $(\beta+1) \nmid v$, then any regular generalized line graph design in $\Omega_{v,b,k}$ is E-optimum over all the equally replicated designs in $\Omega_{v,b,k}$.

Corollary 3.7. If $\beta > \frac{v}{2} - 2$, $\beta \neq 2$, $(\beta+1) \nmid v$, and $k > 3$, then any regular generalized line graph design in $\Omega_{v,b,k}$ is E-optimum over $\Omega_{v,b,k}$.

The readers can also write down results for other cases ($\beta=2$, $(\beta+1) \mid v$, etc.) which we shall omit. Corollaries 3.3 - 3.7 still hold if we replace the phrase "regular generalized line graph design" by "regular graph design with $\tilde{\mu}_{d1} > -2$" everywhere. All the connected regular graphs with $\mu_1(G) = -2$ which are not generalized line graphs have been found by

Bussemaker, Cvetković, and Seidel (1976). This provides another source of E-optimum designs.

Using Corollary 3.7, we can establish the E-optimality of some triangular and L_2 type PBIB designs.

Theorem 3.8. A triangular PBIB design with $k > 3$, $\lambda_2 = \lambda_1+1$, and $v < 28$ is E-optimum over $\Omega_{v,b,k}$.

Proof. It is clear that $\beta > \frac{v}{2} - 2$, $\beta \neq 2$ and $(\beta+1) \nmid v$ under the assumptions.

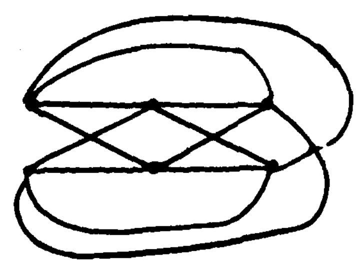
Similarly, one has

Theorem 3.9. An L_2 type PBIB design with $k > 3$, $\lambda_2 = \lambda_1+1$, and $v < 16$ is E-optimum over $\Omega_{v,b,k}$.

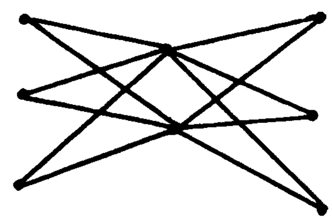
By Corollary 3.6, when $k = 2$, Theorems 3.8 and 3.9 hold if we replace $\Omega_{v,b,k}$ by the collection of equally replicated designs in $\Omega_{v,b,k}$. The E-optimality of the duals of the designs in Theorems 3.8 and 3.9 can also be proved as in Section 4 of Cheng (1980).

Among the triangular type PBIB designs with $\lambda_1 = \lambda_2+1$, only the one with $v = 10$ has $\tilde{\mu}_{d1} = -2$. This design has $\beta = 3$. Since $\beta \neq 2$, $(\beta+1) \nmid v$, and $\beta > \frac{v}{2} -2$, we conclude that such a triangular design is also E-optimum.

One more example. The parameter combination $(v,b,k) = (12,30,2)$ is one case which Mitchell and John (1976) cannot handle. Now let G be the complete tripartite graph on three partite sets of size 2:



Then the complementary graph of L(G) is regular and has 12 vertices and 30 edges. Considering each edge of this graph as a block of size two, we obtain a regular generalized line graph design d^* in $\Omega_{12,30,2}$. In this case $\beta = 5$, so $(\beta+1) \mid v$. But it is fairly easy to see that a group-divisible design with $\lambda_2 = \lambda_1+1$ does not exist. Furthermore, $\beta > \frac{v}{2} - 2$, therefore d^* is at least E-optimum over the equally replicated designs in $\Omega_{12,30,2}$. Another E-optimum design can be obtained by considering the complementary graph of the line graph of the following:



4. Concluding Remarks.

The main purpose of this paper is to define and point out the importance of regular generalized line graph designs. They are often E-optimum and yet so general to cover many of the previously known optimum designs. Among the 209 E-optimum regular graph designs found by Mitchell and John (1976), 176 of them have $\tilde{\mu}_{d1} \geq -2$ or are duals of such designs. The 33 exceptions are exactly those listed in Section 5 of Cheng (1980). Certainly there are more works to be done. We do believe that the best regular generalized line graph designs are always E-optimum over $\Omega_{v,b,k}$. To show this, we need a substantial improvement of Theorem 2.2 of Cheng (1980) on the conditions under which the best regular graph design is E-optimum over $\Omega_{v,b,k}$. More precisely, we need to drop the condition $\beta \geq \frac{v}{2} - 2$ in Corollary 3.6 and Corollary 3.7. Using the method of Cheng (1980) and Jacroux (1980), one can easily show that the competing designs can be reduced to the equally replicated binary designs with $\lambda_{dij} = \lambda$, $\lambda+1$, or $\lambda+2$.

We also expect regular generalized line graph designs to perform very well under other criteria. This is supported by the fact that in Mitchell and John's studies, most of the time the same regular graph design is A-, D-, and E-optimum. When a regular graph design with $\tilde{\mu}_{d1} \geq -2$ does not exist, naturally we should consider regular graph designs with $\tilde{\mu}_{d1} < -2$, while keeping $\tilde{\mu}_{d1}$ as close to -2 as possible. We should point out that all the optimum regular graph designs in Mitchell and John's list either satisfy $\tilde{\mu}_{d1} \geq -3$ or are duals of such designs! To prove the optimality of designs with $\tilde{\mu}_{d1} < -2$, however, would be much more difficult.

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Replacement abstract $\lambda_2 \rightarrow$ lambda 2
$\lambda_1 \rightarrow$ lambda 1

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A class of efficient designs called regular generalized line graph designs are introduced. This class of designs includes many well-known optimum and efficient designs, e.g., balanced incomplete block designs, group-divisible designs with $\lambda_2 = \lambda_1+1$, group-divisible designs with $\lambda_1 = \lambda_2+1$ and group size two, triangular designs with $\lambda_2 = \lambda_1+1$, L_2 designs with $\lambda_2 = \lambda_1+1$, etc. The optimality of regular generalized line graph designs is investigated. This uses graph theory as a tool and unifies much of the previous work in the area.

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