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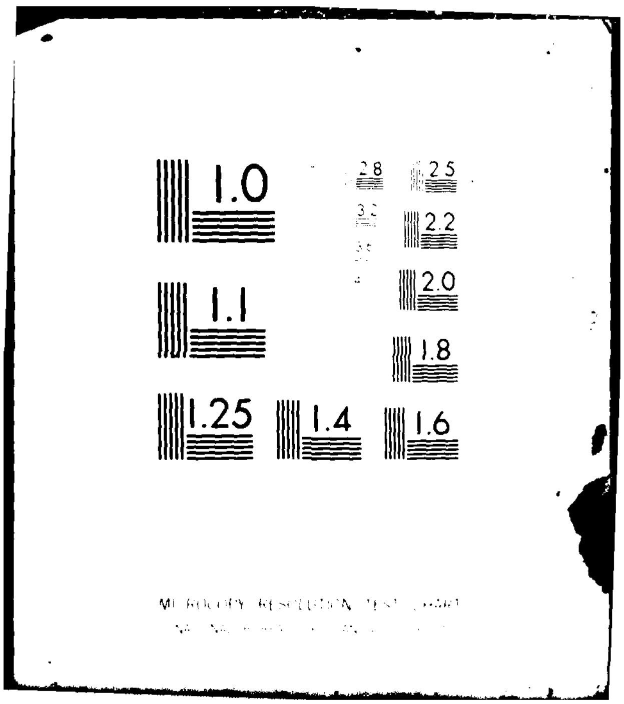
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DIFFERENCE METHODS FOR PARABOLIC HISTORY VALUE PROBLEMS

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ABSTRACT

This paper is concerned with one-step difference methods for parabolic history value problems in one space variable. These problems, which have the feature that the evolution of the solution is influenced by 'all its past' occur in the theory of viscoelastic liquids (materials with 'memory'). The history dependence is represented by a Volterra-integral in the equation of motion. Using recently obtained existence results (see Renardy (1981b)) and smoothness assumptions on the solution we derive a local stability and convergence result for a Crank-Nicolson-type difference scheme by interpreting the linearized scheme as perturbation of a strictly parabolic scheme without memory term. Second order convergence is shown on sufficiently small time intervals. The presented approach carries over to other one-step difference methods like implicit and explicit Euler schemes.

AMS (MOS) Subject Classification: 39A11, 65Q05, 45K05, 35K22

Key Words: Stability of finite difference equations, difference and functional equations, Integro-partial differential equations, Evolution equations.

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SIGNIFICANCE AND EXPLANATION

In this paper we deal with the numerical solution of parabolic history value problems. These problems have the feature that the governing equation depends on the history of the solution such that it is posed as functional differential equation (that means that the equation can involve Volterra type integrals and not only derivatives of the function in question). Problems of this kind occur in the theory of viscoelastic fluids and there the functional term of the equation represents the 'memory' of the material. We devise a finite difference method for the numerical solution of such problems and investigate the convergence properties. It turns out that this method (which is of Crank-Nicolson type) is second order accurate as the grid parameters tend to zero.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

DIFFERENCE METHODS FOR PARABOLIC HISTORY VALUE PROBLEMS

Peter Markowich*

1. Introduction

We are concerned with finite difference methods for scalar functional differential equations with prescribed history data:

(1.1) $u_{tt} = \frac{\partial}{\partial x} g(u_{xt}, u_x^t)$, $x \in [-1,1]$, $t > 0$

(1.2) $u(x,t) = \bar{u}(x,t)$, $x \in [-1,1]$, $t \leq 0$

(1.3) $g(u_{xt}(-1,t), u_x^t(-1,\cdot)) = f_-(t)$, $t > 0$

(1.4) $g(u_{xt}(1,t), u_x^t(1,\cdot)) = f_+(t)$, $t > 0$.

Here we denoted the history of a function $y \in C([-\infty,0])$ (which is the space of all continuous functions on $(-\infty,0]$ with a finite limit at $t = -\infty$) by $y^t(s) = y(t+s)$, $s \leq 0$ and the (possibly) nonlinear functional $g : \mathbb{R} \times \Omega \to \mathbb{R}$ where Ω is an open set in $C([-\infty,0])$. We also assume that the Frechet derivative (with respect to the first argument) $D_1 g \geq \varepsilon > 0$ in $\mathbb{R} \times \Omega$. This assumption makes it possible to interpret (1.1)-(1.4) as a parabolic initial-value problem in a certain Banach space (see Renardy (1981b)). Histories for the boundary data $u_x(-1,t)$ and $u_x(1,t)$ are $\bar{u}_x(-1,t)$ and $\bar{u}_x(1,t)$ resp.

Assuming that (1.3), (1.4) with the corresponding prescribed histories can be solved for $u_x(-1,t)$ and $u_x(1,t)$ resp. we are left with a (parabolic) history value problem with Neumann boundary conditions at $x = \pm 1$.

Problems of this kind occur in the theory of viscoelastic liquids when the constitutive law is expressed as a function of the strain history (see Lodge (1974), Lodge, McLeod and Nohel (1978) and Renardy (1981b)). The functional g is assumed to be of Volterra type:

(1.5) $$g(u_{xt}, u_x^t) = \gamma(u_{xt}, u_x) + \int_{-\infty}^{t} a(t-s) b(u_{xt}(t), u_x(s)) ds \quad .$$

where $\gamma, b : \mathbb{R}^2 \to \mathbb{R}$ and $a(\sigma)$ is an exponentially decaying (as $\sigma \to \infty$) memory kernel. The dependence of u on the space variable x is not stressed explicity.

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In this paper we set up a Crank-Nicolson-type discretization for (1.1) on an interval $[0,T]$ with given history (1.2). The boundary problems (1.3), (1.4) are discretized by the mid-point rule thus giving discrete Neumann boundary data for the Crank-Nicolson scheme. Assuming that T is sufficiently small and that the solution u of (1.1)-(1.4) is sufficiently smooth we show stability of the linearized difference scheme and consistency of the nonlinear scheme at the exact solution u in a discrete Sobolev space norm. From this and from the uniform (in the mesh-sizes) Lipschitz continuity of the linearized scheme we conclude convergence (of order two) from Keller's (1975) theory.

The approach is to interpret the linearized difference scheme as perturbation of a strictly parabolic scheme (without history term) and stability of the scheme for the history value problem will be concluded from the stability of the parabolic difference scheme. Therefore this approach is applicable to other one-step difference schemes like the implicit and explicit Euler schemes. The implicit Euler scheme may be chosen if approximations are needed on a large interval $[0,T]$ (assuming the exact solution exists there) because it is strongly A-α-stable (see Markowich and Renardy (1981a,b)).

The paper is organized as follows. In Section two we define the function spaces which we will need and introduce some notations. Section three deals with the discretization of the boundary problem and Section four is concerned with the Crank-Nicolson scheme for (1.1).

2. Definitions and Notations

We denote

(2.1) $$C^i([-\infty,t_1]) = C^i((-\infty,t_1]) \cap \{f:(-\infty,t_1] \to \mathbb{R} \mid \lim_{t\to-\infty} f(t) \text{ is finite and } \lim_{t\to-\infty} f^{(j)}(t) = 0 \text{ for } j = 1,\dots,i\},\ i \in \mathbb{N}_0 .$$

for some $t_1 \in \mathbb{R}$. For $y \in C^i([-\infty,t_1])$ we define the history $y^t \in C^i([-\infty,0])$ by $y^t(s) = y(t+s)$, $s \in [-\infty,0]$.

Renardy (1981a) investigated a model for the stretching of a filament of polymeric liquid in the form of an initial value problem ((1.1), (1.3), (1.4) are assumed to hold on $[-\infty,T]$, $T \in \mathbb{R} \cup \{\infty\}$ and $u(x,t=-\infty)$ is prescribed). He used a functional g of the form (1.5). A global $(T = \infty)$ existence and uniqueness theorem for sufficiently small data f_+, f_- (in the sense of a certain Sobolev space) and a local (for $T < 0$, $|T|$ large) existence and uniqueness theorem for arbitrarily large boundary data was established. The boundary problem (1.3) (in initial value form) was investigated analytically and numerically in Markowich and Renardy (1981a) using an implicit Euler-type discretization.

The full spatial-temporal problem (in initial value form) was investigated numerically in Markowich and Renardy (1981b). Again, an implicit Euler-type discretization was used in order to get approximate solutions with the same asymptotic behaviour as the exact solution (as $t \to \infty$) and in order to cope with the singular perturbation character of the problem (the Newtonian contribution of the viscosity acts as singular perturbation parameter).

Lodge, McLeod and Nohel (1978) investigated the history value problem for the boundary problem (1.3) assuming the relation (1.5) with γ specified. Under certain assumptions on a and b they proved a global existence theorem and investigated the asymptotic behaviour of the solution as $t \to \infty$.

Nevanlinna (1978) employed an implicit Euler-type discretization for the boundary history value problem and proved uniform convergence on $[0,\infty]$ of the order $O(h^\lambda)$, $0 < \lambda \leq 1$.

Renardy (1981b) proved a local (for $t \in [0,T]$, T sufficiently small) existence and uniqueness theorem for the history value problem (1.1)-(1.4) under mild assumptions on the functional g (not using the special form (1.5)). He transformed (1.1) to a system of equations which can be interpreted as parabolic in the Banach space $C([-\infty,0], H^1([-1,1]))$. This system can be treated following Sobolevskii (1966).

In the sequel g always denotes a functional from $\mathbb{R} \times \Omega$ into $\mathbb{R}$ where Ω is an open set in some space of functions which are defined on $[-\infty,0]$. Frechet derivatives of g are denoted by indices (g_1 denotes the derivative of g with respect to the first argument, g_2 with respect to the second argument, g_{12} for example denotes the second Frechet derivative of g obtained by differentiating first with respect to the first argument and then with respect to the second argument).

$L^2([-1,1])$ denotes the space of square integrable functions on $[-1,1]$, $H^{\ell}([-1,1])$ denotes the space of $L^2([-1,1])$ functions whose (generalized) derivatives of order up to ℓ are square integrable and $C^m([t_1,t_2], H^{\ell}([-1,1])$ denotes the space of C^m-functions $u : [t_1,t_2] \to H^{\ell}([-1,1])$. All these spaces are equipped with their natural norms.

For the difference scheme we define a grid $Gr(h,k)$ on the infinite strip $[-1,1] \times [-\infty,T]$ by setting $h = \frac{1}{M}$, $k = \frac{T}{N}$; $M, N \in \mathbb{N}$ such that

(2.2) $$Gr(h,k) = \{(x_j,t_n) | x_j = jh,\ j = (-M-1)(1)(M+1);\ t_n = nk,\ n \le N\}$$

holds. The exterior gridpoints (x_{-M-1},t_n), $(x_{M+1}t_n)$ will be needed to define a second order approximation to the Neumann boundary conditions. We denote grid functions by $U^{(N)} = ((U_j^n)_{n=-\infty}^{N})_{j=-M-1}^{M+1}$, $U_i^j \in \mathbb{R}$. We also need grid functions on $Gr(h,k)$ - {exterior grid points}. They are defined analogously.

By $U_i^{(N)}$ we denote the grid function on $Gr(h,k) \cap \{(x_j,t_n) | j = i,\ n \le N\}$:

(2.3) $$U_i^{(N)} = (U_i^n)_{n=-\infty}^{N} .$$

For the discretization of the boundary problems we need the $\{t_n = nk | n \le N\}$ grid on the real line and

nd

(2.4) $$y^{(N)} = (y^n)_{n=-\infty}^{N} , \quad y^n \in \mathbb{R}$$

are functions on this grid. Also we set $t_{n+1/2} = (n + \frac{1}{2})k$. We define the following time-difference quotients:

(2.5)(a) $$\delta^+ y^n = \frac{y^{n+1}-y^n}{k}$$

(2.5)(b) $$\delta^- y^n = \frac{y^n-y^{n-1}}{k}$$

(2.5)(c) $$\delta y^n = \frac{y^{n+1}-y^{n-1}}{2k} \quad .$$

Obviously $\delta^+y^n = \delta^-y^{n+1}$, $\delta y^n = \frac{1}{2}(\delta^+y^n + \delta^-y^n)$ holds. We need the spatial differences:

(2.6)(a) $$\Delta_+ z_i = \frac{z_{i+1}-z_i}{h}$$

(2.6)(b) $$\Delta_- z_i = \frac{z_i-z_{i-1}}{h}$$

(2.6)(c) $$\Delta z_i = \frac{z_{i+1}-z_{i-1}}{2h} \quad .$$

These difference quotients will also be applied (component wise) to grid functions, for example $\Delta U_i^{(N)} = (\Delta U_i^n)_{n=-\infty}^N$.

3. The Boundary Problem

In order to solve the boundary problems (1.3), (1.4) for $u_x(1,t)$ and $u_x(-1,t)$ resp. we discretize

(3.1) $$g(y'(t),y^t) = f(t) \quad , \quad t \in [0,T] \quad , \quad T \in \mathbb{R}$$

(3.2) $$y(t) = \bar{y}(t) \quad , \quad t \in [-\infty,0]$$

by the midpoint rule

(3.3) $$g\left(\delta^+y^n, (i_k y^{(N)})^{t_{n+1/2}}\right) - f(t_{n+1/2}) = 0, \; 0 \leq n < N$$

(3.4) $$y^n - \bar{y}(t_n) = 0 \quad , \quad n \leq 0 \quad .$$

Here y^n denotes the approximation to $y(t_n)$, $y^{(N)} = (y^n)_{n=-\infty}^N$ and i_k is the linear interpolation operator defined by

(3.5)(a) $$i_k : A_N = \{u^{(N)} = (u^n)_{n=-\infty}^N | u^i \in \mathbb{R}, \lim_{i\to-\infty} u^i = u^* \in \mathbb{R}\} \longrightarrow C([-\infty,T])$$

(3.5)(b) $$(i_k u^{(N)})(t) = u^i + \delta^+u^i \cdot (t-t_i) \quad \text{for} \quad t_i \leq t \leq t_{i+1}, \; i < N \quad .$$

We denote the right hand side of (3.3), (3.4) by

(3.6) $$F_k(y^{(N)}) = 0$$

where

(3.7) $$F_k : \Omega_N \to B_N \quad , \quad \Omega_N \text{ open, } \Omega_N \subset A_N \quad .$$

Here A_N is equipped with the norm

(3.8) $$\|u^{(N)}\|_{A_N} = \max_{n \le N}|u^n| + \max_{n \le N}|\delta^- u^n| + \max_{1 \le n \le N}|\delta^+\delta^- u^n| \quad .$$

B_N equals A_N (as a set) but as norm we take

(3.9) $$\|w^{(N)}\|_{B_N} = \max_{n \le N}|w^n| + \max_{n \le N}|\delta^- w^n|$$

Ω_N is open in A_N since $g(v,\cdot)$, $v \in \mathbb{R}$ is defined on an open set $\Omega \subset C([-\infty,0])$.

The convergence analysis of (3.6) proceeds along the lines of Keller's (1975) stability-consistency concept. For consistency we need smoothness assumptions on (3.1), (3.2). We assume that there is locally unique solutions y of (3.1), (3.2) and that

(3.10)(a) $g \in C^2(\mathbb{R} \times \Omega, \mathbb{R})$, $\Omega \subset C([-\infty,0])$, $g_1 \ge \varepsilon > 0$ on $\mathbb{R} \times \Omega$

(3.10)(b) $f \in C^3([0,T])$

(3.10)(c) $y \in C^4([0,T])$, $\bar{y} \in C^4([-\infty,0])$;

(3.10)(d) $y \in C^2([-\infty,T])$, $y^t \in \Omega$ for $t \in [0,T]$

holds. A local existence (and uniqueness) theorem is given in Renardy (1981b). (3.10d) holds if the compatibility requirements

(3.11)(a) $g(\bar{y}'(0), \bar{y}) = f(0)$

(3.11)(b) $g_1(\bar{y}'(0), \bar{y})\bar{y}''(0) + g_2(\bar{y}'(0), \bar{y})\bar{y}' = f'(0)$

are fulfilled.

We check consistency of the scheme (3.3), (3.3). We denote

(3.12) $$\hat{y}^{(N)} = (y(t_n))_{n=-\infty}^{N} \quad .$$

(3.10)(c),(d) imply that $(i_k\hat{y}^{(N)})^{t_{n+1/2}} \in \Omega$ for k sufficiently small and so $\hat{y}^{(N)} \in \Omega_N$ holds. Obviously $(F_k(\hat{y}^{(N)}))^n = \delta^-(F_k(\hat{y}^{(N)}))^n = 0$ for $n \le 0$. Since

(3.13) $$\|i_k\hat{y}^{(N)} - y\|_{[-\infty,T]} = O(k^2)$$

(because $\bar{y} \in C^2([-\infty,0])$, $y \in C^2([0,T])$ and $t = 0$ is a grid point) we obtain using (3.10)(a),(b) and the mean value theorem

(3.14) $$|(F_k(\hat{y}^{(N)}))^n| = O(k^2) \quad .$$

The smoothness assumptions (3.10)(c),(d) imply that

(3.15) $$\left\| \frac{(i_k\hat{y}^{(N)})^{t_{n+1/2}} - (i_k\hat{y}^{(N)})^{t_{n-1/2}}}{k} - (y')^{t_n} \right\|_{[-\infty,0]} = O(k^2)$$

holds. Using (3.10), (3.15) and the differentiated equation (3.1)

(3.16) $$g_1(y'(t),y^t)y''(t) + g_2(y'(t),y^t)(y')^t - f'(t) = 0$$

((3.10)(d) implies that $(y')^t \in C^1([-\infty,0])$) we get, after a simple calculation

(3.17) $$|\delta^+(F_k(\hat{y}^{(N)}))^n| = O(k^2) \quad , \quad n < N \quad .$$

Since all estimates hold uniformly for $n \leq N$ we obtain from (3.14), (3.17)

(3.18) $$\|F_k(\hat{y}^{(N)})\|_{B_N} = O(k^2) \quad .$$

Therefore the scheme (3.3), (3.4) is consistent of order 2 at $\hat{y}^{(N)}$ if (3.10) holds.

To check stability we calculate the Frechet-derivative of F_k at $\hat{y}^{(N)}$ and get for $u^{(N)} \in A_N$

(3.19(a) $$(F_k'(\hat{y}^{(N)})u^{(N)})^{n+1} = g_1(\delta^+y(t_n),(i_k\hat{y}^{(N)})^{t_{n+1/2}})\delta^+u^n +$$

$$+ g_2(\delta^+y(t_n),(i_k\hat{y}^{(N)})^{t_{n+1/2}})(i_ku^{(N)})^{t_{n+1/2}}, \; 0 \leq n < N$$

(3.19)(b) $$(F_k'(\hat{y}^{(N)})u^{(N)})^n = u^n \quad , \; n \leq 0 \quad .$$

In order to get an estimate on the norm of $(F_k'(\hat{y}^{(N)}))^{-1} : B_N \to A_N$ we investigate the equation

(3.20) $$F_k'(\hat{y}^{(N)})u^{(N)} = \varphi^{(N)} \quad , \quad \varphi^{(N)} = (\varphi^i)_{i=-\infty}^N \in B_N \quad .$$

For $n \leq 0$ we have $u^n = \varphi^n$, $\delta^-u^n = \delta^-\varphi^n$ and for $0 < n < N$

(3.21) $$\delta^+u^n = \hat{g}_n(k)(i_ku^{(N)})^{t_{n+1/2}} + \varphi^{n+1}$$

where $\hat{g}_n(k)$ is the linear functional

(3.22) $$\hat{g}_n(k) = -\frac{g_2(\delta^+\hat{y}(t_n),(i_k\hat{y}^{(N)})^{t_{n+1/2}})}{g_1(\delta^+\hat{y}(t_n),(i_k\hat{y}^{(N)})^{t_{n+1/2}})}$$

and

$$\hat{\varphi}^{n+1} = \frac{\varphi^{n+1}}{g_1(\delta^+\hat{y}(t_n),(i_k\hat{y}^{(N)})^{t_{n+1/2}})} \tag{3.23}$$

holds. The assumption $g_1 \geq \varepsilon > 0$ on $\mathbb{R} \times \Omega$ implies that $\hat{g}_n(k)$ is uniformly bounded (in k,n) and that

$$|\hat{\varphi}^{n+1}| \leq \text{const}|\varphi^{n+1}| \quad \text{uniformly in } k,\ n < N \tag{3.24}$$

holds. From (3.21) we get:

$$|u^{n+1}| \leq |u^n| + ck \max_{1 \leq j \leq n+1} |u^j| + ck\,|\varphi^{n+1}| \tag{3.25}$$

and setting $v^n = \max\limits_{1 \leq j \leq n} |u^n|$, $z^n = \max\limits_{j \leq n} |\varphi^j|$ we obtain

$$v^{n+1} \leq v^n + ckv^{n+1} + ckz^{n+1} \quad . \tag{3.26}$$

From

$$v^{n+1} \leq \frac{1}{1-ck} v^n + c_1kz^{n+1}$$

we immediately get $|v^n| \leq c_1e^{cT}T \max\limits_{1 \leq j \leq n} |z^j|$ and therefore

$$\max_{n \leq N} |u^n| \leq \text{const.} \max_{n \leq N} |\varphi^n| \tag{3.27}$$

holds. We obtain from (3.21)

$$\max_{n \leq N-1} |\delta^+u^n| \leq \text{const.} \max_{n \leq N} |\varphi^n| \quad . \tag{3.28}$$

In order to prove stability in the norms of A_N, B_N we apply δ^- to (3.21) and obtain

$$\begin{aligned}\delta^-\delta^+u^n = \hat{g}_n(k)\Big(\frac{(i_ku^{(N)})^{t_{n+1/2}} - (i_ku^{(N)})^{t_{n-1/2}}}{k}\Big) + \Big(\frac{\hat{g}_n(k)-\hat{g}_{n-1}(k)}{k}\Big)(i_ku^{(N)})^{t_{n-1/2}} \\ + \delta^-\varphi^n \quad , \quad n > 0 \quad .\end{aligned} \tag{3.29}$$

(3.10) implies that $\dfrac{\hat{g}_n(k)-\hat{g}_{n-1}(k)}{k}$ is a uniformly bounded functional (in k,n) and therefore the second term on the right hand side of (3.29) is bounded by $\text{const.} \max\limits_{n \leq N} |\varphi^n|$. We get by a simple calculation (similar to (3.15))

$$\left\| \frac{(i_ku^{(N)})^{t_{n+1/2}} - (i_ku^{(N)})^{t_{n-1/2}}}{k} \right\|_{[0,\infty]} \leq \max_{n<N} |\delta^+u^n| + \frac{k}{2} \max_{n<N} |\delta^-\delta^+u^n| \quad . \tag{3.30}$$

Repeating the argument following (3.21) we get

$$\max_{1 \leq n < N} |\delta^- \delta^+ u^n| \leq \text{const.}(\max_{n \leq N} |\delta^- \varphi^n| + \max_{n \leq N} |\varphi^n|) \tag{3.31}$$

and stability follows:

$$\|(F_k'(\hat{y}^{(N)}))^{-1}\|_{B_N \to A_N} \leq \text{const.} \tag{3.32}$$

where const. is independent of k. We also have to show uniform Lipschitz continuity of F_k' in a neighborhood $S_\rho(\hat{y}^{(N)}) \subset \Omega_N$. For $a^{(N)}, b^{(N)} \in S_\rho(\hat{y}^{(N)})$ with ρ sufficiently small we get from (3.19) and the mean value theorem:

$$\|(F_k'(a^{(N)}) - F_k'(b^{(N)}))u^{(N)}\|_{B_N} \leq \text{const.}\|a^{(N)} - b^{(N)}\|_{A_N} \|u^{(N)}\|_{A_N} \tag{3.33}$$

for all $u^{(N)} \in A_N$ where const. is independent of k. (3.10)(a) was used for (3.33). Therefore uniform Lipschitz continuity of F_k' holds on $S_\rho(y^{(N)})$.

Now, having proven consistency, stability and Lipschitz continuity we apply Keller's (1975) theory which gives:

<u>Theorem 3.1</u>. Let the assumptions (3.10) on the history value problem (3.1), (3.2) hold. Then for all k sufficiently small there is a locally unique solution $y^{(N)} \in \Omega_N \subset A_N$ of the midpoint rule (3.3), (3.4) and

$$\max_{n \leq N} |y^n - y(t_n)| + \max_{n \leq N} |\delta y^{n-1} - y'(t_{n-1})| + \tag{3.34}$$

$$+ \max_{n \leq N} |\delta^+ \delta^- y^{n-1} - y''(t_{n-1})| = O(k^2) \quad \text{as} \quad k \to 0 \quad .$$

Moreover the (abstract) Newton procedure

$$v_{(0)}^{(N)} = \bar{v}_{(0)}^{(N)} \tag{3.35(a)}$$

$$v_{(i+1)}^{(N)} = v_{(i)}^{(N)} - (F_k'(v_{(i)}^{(N)}))^{-1} F_k(v_{(i)}^{(N)}) \quad , \quad i \geq 0 \tag{3.35(b)}$$

converges quadratically (to $y^{(N)}$) from a sphere of starting values which does not shrink as $k \to 0$.

The abstract Newton method (3.35) immediately translates into the Newton method for determining $y^1, \ldots, y^N$ from (3.3) assuming that $(y^j)_{j=-\infty}^0$ is given.

For the convergence analysis of the boundary problem we do not require T small (except that a smooth solution of (3.1), (3.2) has to exist on $[-\infty,T]$). Therefore Theorem (3.1) holds for any finite T to which the solution y can be smoothly continued. However, the stability constant (3.32) depends on T and so does the estimate (3.34).

4. The Parabolic Problem

We now discretize the full spatial-temporal problem:

$$(4.1) \qquad u_{tt} - g(u_{xt}, u_x^t)_x = 0,\ x \in [-1,1],\ t \in [0,T]$$

$$(4.2) \qquad u(x,t) = \bar{u}(x,t),\ x \in [-1,1],\ t \in [-\infty,0]$$

$$(4.3) \qquad u_x(1,t) = y_+(t),\ t \in [-\infty,T]$$

$$(4.4) \qquad u_x(-1,t) = y_-(t),\ t \in [-\infty,T]$$

where y_+, y_- solve the boundary problems

$$(4.5) \qquad g(y_\pm'(t),\ y_\pm^t) = f_\pm(t),\ t \in [0,T]$$

$$(4.6) \qquad y_\pm(t) = \bar{u}_x(\pm 1,t),\ t \in [-\infty,0]$$

which fulfill the assumptions of Section 3. For convenience we carry out the differentiation in (4.1) (assuming sufficient smoothness)

$$(4.7) \qquad u_{tt} - (g_1(u_{xt}, u_x^t)u_{xxt} + g_2(u_{xt}, u_x^t)u_{xx}^t) = 0,\ x \in [-1,1],\ t \in [0,T] \ .$$

We discretize (4.7), (4.2), (4.3), (4.4) by the Crank-Nicolson method:

$$(4.8) \qquad \begin{aligned} &\delta^+\delta^- U_i^n - (g_1(\Delta\delta U_i^n,\ (i_k \Delta U_i^{(N)})^{t_n})\delta\Delta_+\Delta_- U_i^n + \\ &\quad + g_2(\Delta\delta U_i^n,\ (i_k \Delta U_i^{(N)})^{t_n})(i_k \Delta_+\Delta_- U_i^{(N)})^{t_n}) = 0 \\ &\quad \text{for } i = -M(1)M;\ 0 < n < N \end{aligned}$$

$$(4.9) \qquad U_i^n - \bar{u}(x_i,t_n) = 0,\ i = -M(1)M,\ n \le 0$$

$$(4.10) \qquad \Delta U_M^n - y_+^n = 0,\quad n \le N$$

$$(4.11) \qquad \Delta U_{-M}^n - y_-^n = 0,\quad n \le N \ .$$

U_i^j denotes the approximation to $u(x_i,t_n)$ and $U_i^{(N)} = (U_i^j)_{j=-\infty}^N$. The boundary values y_+^n, y_-^n, $n \le N$ are computed by discretizing (4.5), (4.6) according to Section 3 and therefore are assumed to be known. In (4.10), (4.11) we introduced the exterior grid values U^n_{M+1}, U^n_{-M-1} in order to get second order approximations for the Neumann boundary conditions.

As a device for the analysis of the scheme we substitute

$$(4.12)\qquad W^n_i = U^n_i + \frac{(x_i-1)^2}{4} y^n_- - \frac{(x_i+1)^2}{4} y^n_+,\; i = (-M-1)(1)(M+1)$$

in order to get homogeneous Newmann boundary conditions. In operator form we write

$$(4.13)\qquad G_{k,h}(W^{(N)}) = 0$$

for the left hand sides of (4.8), (4.9) where

$$(4.14)\qquad W^{(N)} = ((W^j_i)^N_{j=-\infty})^{M+1}_{i=-M-1} \quad ,$$

and build the homogeneous boundary conditions into the spaces. We define

$$(4.15)\qquad L^2_h = \{V = (V_{-M-1},\dots,V_0,\dots,V_{M+1}) | V_i \in \mathbb{R},\; V_{-M-1} = V_{-M+1},\; V_{M+1} = V_{M-1}\}$$

$$\|V\|_{L^2_h} = (h \sum_{i=-M}^{M} |V_i|^2)^{\frac{1}{2}} \; .$$

$\bar{L}^2_h$ is defined by skipping the components V_{-M-1}, V_{M+1} of the elements V of L^2_n. Moreover

$$H^1_h = \{V \in L^2_h\},\; H^2_h = \{V \in L^2_h\}$$

$$(4.16)\qquad \|V\|_{H^1_h} = \|V\|_{L^2_h} + \|\Delta V\|_{L^2_h}$$

$$\|V\|_{H^2_h} = \|V\|_{H^1_h} + \|\Delta^+\Delta^- V\|_{L^2_h} \; .$$

L^2_h and H^i_h are the discrete versions of $L^2([-1,1])$ and $H^i([-1,1])$ resp. For arbitrary Banach spaces X,Y we define

$$(4.17)\qquad C_k(X,Y) = \{Z^{(N)} = (Z^j)^N_{j=-\infty} | Z^j \in Y \text{ for } 0 < j \le N,\; Z^j \in X \text{ for } j \le 0 \text{ and } \lim_{j\to-\infty} Z^j \in X\}$$

$$\|Z^{(N)}\|_{C_k(X,Y)} = \max_{0<n\le N} \|Z^n\|_Y + \max_{n\le 0} \|Z^n\|_X + \max_{n\le 0} \|\delta^- Z^n\|_X$$

$$C_k^1(X) = \{z^{(N)} = (z^j)_{j=-\infty}^N \mid z^j \in X,\ \lim_{j\to-\infty} z^j \in X\}$$

(4.18)

$$\|z^{(N)}\|_{C_k^1(X)} = \max_{n\le N} \|z^n\|_X + \max_{n\le N} \|\delta^- z^n\|_X \quad .$$

We regard $G_{k,h}$ as the following mapping:

(4.19) $$G_{k,h} : \Omega_{N,M} \subset C_k^1(H_h^2) \to C_k(H_h^2, \bar{L}_h^2) \quad ,$$

assuming that $f_\pm(t)$ fulfill (3.10)(b) and that the boundary values $y_\pm(t)$ fulfill (3.10)(c),(d). Moreover $g \in C^3(\mathbb{R} \times \Omega, \mathbb{R})$, $\Omega \subset C([-\infty,0])$ open, $g \in C^3(\mathbb{R} \times \bar{\Omega}, L^2([-1,1]))$, $\bar{\Omega} \subset C([-\infty,0], L^2([-1,1])$ and

(4.20)(a) $$\bar{\Omega} \text{ is open, } g_1 > \varepsilon > 0 \text{ on } \mathbb{R} \times \bar{\Omega}$$

shall hold and the parabolic problems (4.1)-(4.4) has a locally unique solution u which fulfills

(4.20)(b) $$u \in C^3([0,T], H^4([-1,1])) \cap C^4([0,T], L_2([-1,1]))$$

(4.20)(c) $$\bar{u} \in C^3([-\infty,0], H^4([-1,1])),\ \bar{u}_x \in C([-\infty,0], H^4([-1,1])) \cap \bar{\Omega} \quad .$$

Assumptions on the history $\bar{u}$ and on (4.1), (4.4) which guarantee the required smoothness of u can be deduced from Renardy (1981b).

A lengthy calculation shows that

(4.21) $$\|G_{k,h}(W^{(N)})\|_{C_k(H_h^2,\bar{L}_h^2)} = O(k^2) + O(h^2)$$

holds, where $\hat{W}^{(N)} = ((\hat{W}_i^j)_{j=-\infty}^N)_{i=-M-1}^{M+1}$ and

(4.22) $$\hat{W}_i^j = u(x_i,t_j) + \frac{(x_i-1)^2}{4} y_-(t_j) - \frac{(x_i+1)^2}{4} y_+(t_j),\ i = -M(1)M, j \le N$$

with $\hat{W}_{M+1}^j = \hat{W}_{M-1}^j$, $\hat{W}_{-M-1}^j = \hat{W}_{-M+1}^j$. For (4.21) we used the boundary convergence result (3.34) (convergence of the second derivative of the boundary grid functions y_-^j, y_+^j is necessary here).

As expected our scheme is consistent of second order at the 'exact' solution.

For the stability analysis we calculate the Frechet derivative of $G_{k,h}$ at $\hat{W}^{(N)}$ getting

$$(4.23)(a) \qquad (G'_{k,h}(\hat{W}^{(N)})V^{(N)})_i^{n+1} = \delta^+\delta^- V_i^n - \hat{g}_1(x_i,t_n,h,k)\delta\Delta_+\Delta_- V_i^n - \hat{g}_2(x_i,t_n,h,k)\Delta\delta V_i^n - \hat{g}^n_{(3)i}(h,k)(i_k\Delta V_i^{(N)})^{t_n} - \hat{g}^n_{(4)i}(h,k)(i_k\Delta_+\Delta_- V_i^{(N)})^{t_n}$$

$$\text{for } i = -M(1)M,\ 0 \le n < N$$

$$(4.23)(b) \qquad (G'_{k,h}(\hat{W}^{(N)})V^{(N)})_i^n = V_i^n,\ i = (-M-1)(1)(M+1),\ n \le 0 \quad .$$

Here $V^{(N)} = ((V_i^j)_{j=-\infty}^N)_{i=-M-1}^{M+1} \in C_k^1(H_h^2)$ such that $V_{M+1}^n = V_{M-1}^n$ and $V_{-M-1}^n = V_{-M+1}^n$ holds for $n \le N$. We obtain

$$(4.24) \qquad \hat{g}_1(x_i,t_n,h,k) = g_1(\Delta\delta u(x_i,t_n),(i_k(\Delta u(x_i,t_j))_{j=-\infty}^N)^{t_n}) \ge \varepsilon > 0$$

$$(4.25) \qquad \hat{g}_2(x_i,t_n,h,k) = g_{11}(\Delta\delta u(x_i,t_n),(i_k(\Delta u(x_i,t_j))_{j=-\infty}^N)^{t_n}) + g_{21}(\Delta\delta u(x_i,t_n),(i_k(\Delta u(x_i,t_j))_{j=-\infty}^N)^{t_n})(i_k(\Delta_+\Delta_- u(x_i,t_j))_{j=-\infty}^N)^{t_n}$$

and $\hat{g}^n_{(3)i}$, $\hat{g}^n_{(4)i}$ are linear functionals *on* $C((-\infty,0])$ involving Frechet derivatives of g of at most order two.

Similarly to the consistency result we get:

$$(4.26) \qquad \hat{g}_1(x_i,t_n,h,k) = g_1(u_{xt}(x_i,t_n),u_x^{t_n}(x_i,\cdot)) + h^2\rho_1(x_i,t_n,h,k) + k^2\rho_2(x_i,t_n,h,k)$$

$$(4.27) \qquad \hat{g}_2(x_i,t_n,h,k) = \frac{\partial}{\partial x} g_1(u_{xt}(x_i,t_n),u_x^{t_n}(x_i,\cdot)) + h^2\sigma_1(x_i,t_n,h,k) + k^2\sigma_2(x_i,t_n,h,k)$$

where the vectors $(\rho_\ell(x_i,t_n,h,k))_{i=-M}^M$, $(\sigma_\ell(x_i,t_n,h,k))_{i=-M}^M$, $\ell = 1,2$ are uniformly bounded in $\bar{L}_h^2$.

We investigate the linear equation

$$(4.28) \qquad G'_{n,k}(\hat{W}^{(N)})V^{(N)} = F^{(N)}$$

for $F^{(N)} = (((F_i^j)_{j=-\infty}^0)_{i=-M-1}^{M+1},((F_i^j)_{j=1}^N)_{i=-M}^M) \in C_k(H_h^2,\bar{L}_h^2)$. $F_{M+1}^n = F_{M-1}^n$, $F_{-M-1}^n = F_{-M+1}^n$ holds for $n \le 0$. As in the continuous case (see Renardy (1981b)) we set

(4.29) $$P_i^n = \Delta V_i^n;\ R_i^n = \delta^- V_i^n;\ Q_i^n = \Delta_+\Delta_- V_i^n$$

and get the parabolic system of difference equations

(4.30)(a) $$\delta^+ P_i^n = \Delta R_i^{n+1}$$

(4.30)(b) $$\delta^+ Q_i^n = \Delta_+\Delta_- R_i^{n+1}$$

(4.30)(c) $$\delta^+ R_i^n = \frac{1}{2}\hat{g}_1(x_i,t_n,h,k)\Delta_+\Delta_-(R_i^{n+1} + R_i^n) + \frac{1}{2}\hat{g}_2(x_i,t_n,h,k)\Delta(R_i^{n+1} + R_i^n) + \hat{g}^n_{3(i)}(h,k)(i_k P_i^{(N)})^{t_n} + \hat{g}^n_{4(i)}(h,k)(i_k Q_i^{(N)})^{t_n} + F_i^{n+1}$$

for $i = -M(1)M$ and $0 \leq n < N$. For $n \leq 0$ we have

(4.31) $$P_i^n = \Delta F_i^n,\ R_i^n = \delta^- F_i^n,\ Q_i^n = \Delta_+\Delta_- F_i^n$$

and $R^n_{M+1} = R^n_{M-1}$, $R^n_{-M-1} = R^n_{-M+1}$ holds. Assuming that T is sufficiently small we regard (4.30)(c) as perturbation of the time-independent scheme

(4.32)(a) $$\delta^+\tilde{R}_i^n = \frac{1}{2}\hat{g}_1(x_i,0,0,0)(\Delta_+\Delta_-\tilde{R}_i^{n+1} + \Delta_+\Delta_-\tilde{R}_i^n) + \frac{1}{2}\hat{g}_2(x_i,0,0,0)(\Delta\tilde{R}_i^{n+1} + \Delta\tilde{R}_i^n) + H_i^{n+1},\ i = -M(1)M,\ 0 \leq n < N$$

(4.32)(b) $$\Delta\tilde{R}_M^n = \Delta\tilde{R}_{-M}^n = 0,\ 0 \leq n \leq N$$

(4.32)(c) $$\tilde{R}_i^0 = \bar{R}_i^0,\ i = (-M-1)(1)(M+1)$$

where $\bar{R}^0_{-M-1} = \bar{R}^0_{-M+1}$, $\bar{R}^0_{M+1} = \bar{R}^0_{M-1}$ holds. Denoting $\tilde{R}^n = (\tilde{R}^n_{-M},\ldots,\tilde{R}^n_0,\ldots,\tilde{R}^n_M)$ we write (4.32)(a),(b),(c) in matrix form

(4.33) $$B_0(h,k)\tilde{R}^{n+1} = B_1(h,k)\tilde{R}^n + kH^{n+1},\ \tilde{R}^0 = \bar{R}^0$$

where $H^{n+1} = (H^{n+1}_{-M},\ldots,H^{n+1}_0,\ldots,H^{n+1}_M)$ has been set. The zero-Neumann boundary conditions (4.32)(b),(c) are incorporated in the $M \times M$ matrices $B_0(h,k)$, $B_1(h,k)$ (see (4.51), (4.52)).

Proceeding similarly to Varah (1971a,b) we derive

(4.34) $$\|B_0^{-1}(h,k)\|_{\bar{L}^2_h} \leq C_1$$

(4.35) $$\|(B_0^{-1}(h,k)B_1(h,k))^i\|_{\bar{L}^2_h} \leq C_2,\ 0 \leq i \leq N$$

as $h \to 0$, $k \to 0$ where C_1, C_2 are independent of h,k. (4.34) allows to rewrite (4.33) as

(4.36) $$\tilde{R}^{n+1} = C(h,k)\tilde{R}^{n} + kB_0^{-1}(h,k)H^{n+1}, \quad \tilde{R}^0 = \bar{R}^0$$

with $C(h,k) = B_0^{-1}(h,k)B_1(h,k)$.

From (4.35), (4.36) and consideration similar to Benderson (1971) we get the stability estimate

(4.37) $$\max_{0\le n\le N} \|\tilde{R}^n\|_{H_h^2} + \max_{0\le n\le N} \|\delta^+\tilde{R}^n\|_{\bar{L}_h^2} \le C_3(\|\bar{R}^0\|_{H^2} + \max_{0<n\le N} \|H^n\|_{\bar{L}_h^2}) \ .$$

Now we assume

(4.38) $$h^2 = \lambda k, \ \lambda = \text{const} \ne 0 \quad \text{as} \quad h,k \to 0 \ .$$

For the difference scheme

$$\left.\begin{aligned}
&(4.39)(a) && \delta^+\tilde{P}_i^n = \Delta\tilde{R}_i^{n+1} \\
&(4.39)(b) && \delta^+\tilde{Q}_i^n = \Delta_+\Delta_-\tilde{R}_i^{n+1} \\
&(4.39)(c) && \delta^+\tilde{R}_i^n = \frac{1}{2}\hat{g}_1(x_i,0,0,0)\Delta_+\Delta_-(\tilde{R}_i^{n+1} + \tilde{R}_i^n) + \\
& && + \frac{1}{2}\hat{g}_2(x_i,0,0,0)\Delta(\tilde{R}_i^{n+1} + \tilde{R}_i^n) + H_i^{n+1}
\end{aligned}\right\} \begin{aligned} &i = -M(1)M \\ &0 \le n < N \end{aligned}$$

and

(4.39)(d) $$\Delta\tilde{R}_{-M}^n = \Delta\tilde{R}_M^n = 0, \quad 0 \le n \le N$$

(4.39)(e) $$\tilde{R}_i^0 = \bar{R}_i^0, \ \tilde{P}_i^0 = \bar{P}_i^0, \ \tilde{Q}_i^0 = \bar{Q}_i^0, \ i = -M(1)M$$

we get immediately

(4.40) $$\max_{0\le n\le N} \|\tilde{P}^n\|_{\bar{L}_h^2} + \max_{0\le n<N} \|\delta^+\tilde{P}^n\|_{\bar{L}_h^2} + \max_{0\le n\le N} \|\tilde{Q}^n\|_{\bar{L}_h^2} +$$
$$+ \max_{0\le n<N} \|\delta^+\tilde{Q}^n\|_{\bar{L}_h^2} + \max_{0\le n\le N} \|\tilde{R}^n\|_{H_h^2} + \max_{0\le n<N} \|\delta^+\tilde{R}^n\|_{\bar{L}_h^2}) \le$$
$$\le \text{const}(\|\bar{P}^0\|_{\bar{L}_h^2} + \|\bar{Q}^0\|_{\bar{L}_h^2} + \|\bar{R}^0\|_{H_h^2} + \max_{0\le n\le N} \|H^n\|_{\bar{L}_h^2}) \ .$$

We now interpret (4.30) *as perturbation of* (4.39). Therefore we write

(4.41) $$\hat{g}^0_{3(i)}(i_k P_i^{(N)})^{t_n} = \hat{g}^0_{3(i)}(i_k \hat{P}_i^{(N)})^{t_n} + \hat{g}^0_{3(i)}(i_k \hat{\hat{P}}_i^{(N)})^{t_n}$$

(4.42) $$\hat{g}^0_{4(i)}(i_k Q_i^{(N)})^{t_n} = \hat{g}^0_{4(i)}(i_k \hat{Q}_i^{(N)})^{t_n} + \hat{g}^0_{4(i)}(i_k \hat{\hat{Q}}_i^{(N)})^{t_n}$$

where $\hat{P}_i^j = \hat{Q}_i^j = 0$ for $j \le 0$; $\hat{P}_i^j = P_i^j$, $\hat{Q}_i^j = Q_i^j$ for $j > 0$ and $\hat{\hat{P}}_i^j = \hat{\hat{Q}}_i^j = 0$ for $j > 0$; $\hat{\hat{P}}_i^j = P_i^j$, $\hat{\hat{Q}}_i^j = Q_i^j$ for $j \le 0$ and incorporate the second terms of the right hand sides of

(4.41), (4.42) *into the inhomogenity* F_i^{n+1} obtaining $\tilde{F}_i^{n+1}$. The perturbation originating from taking $\hat{g}^0_{3(i)}$, $\hat{g}^0_{4(i)}$ instead of $\hat{g}^n_{3(i)}$ and $\hat{g}^n_{4(i)}$ resp. will be investigated later. We denote the sum of the two remaining terms in (4.41), (4.42) by $\Omega_i(P_i^j,Q_i^j)_{j=1}^n$ and from (4.20)(a) we derive the estimate

$$\|(\Omega_i(P_i^j,Q_i^j)_{j=1}^n)_{i=-M}^M\|_{\tilde{L}_h^2} \leq \text{const}(\max_{1\leq j\leq n}\|P^j\|_{\tilde{L}_h^2} + \max_{1\leq j\leq n}\|Q^j\|_{\tilde{L}_h^2}) \tag{4.43}$$

uniformly as $h,k \to 0$.

We denote

$$A^n = (P^n_{-M},\dots,P^n_M,Q^n_{-M},\dots,Q^n_M,R^n_{-M},\dots,R^n_M) \tag{4.44}$$

and rewrite the perturbed system (4.30) (with $t_n = 0$ in $\hat{g}_1$, $\hat{g}_2$ and $\hat{g}^0_{3(i)}$, $\hat{g}^0_{4(i)}$ instead of $g^n_{3(i)}$, $g^n_{4(i)}$) as

$$E_0(h,k)A^{n+1} = E_1(h,k)A^n + kL_0(h,k)A^{n+1} + kL_1(h,k)A^n + k\Omega(A^j)_{j=1}^n + k\tilde{F}^{n+1} \tag{4.45)(a}$$

$$A^0 = \tilde{A}^0 \tag{4.45)(b}$$

where

$$E_0(h,k) = \begin{array}{cc} \begin{bmatrix} I & \theta & -kD_1(h,k) \\ \theta & I & -kD_2(h,k) \\ \theta & \theta & B_0(h,k) \end{bmatrix} & \begin{array}{l} \} \; 2M+1 \\ \} \; 2M+1 \\ \} \; 2M+1 \end{array} \\ \begin{array}{ccc} \underbrace{\quad}_{2M+1} & \underbrace{\quad}_{2M+1} & \underbrace{\qquad}_{2M+1} \end{array} & \end{array} \tag{4.46)(a}$$

$$E_1(h,k) = \begin{bmatrix} I & \theta & \theta \\ \theta & I & \theta \\ \theta & \theta & B_1(h,k) \end{bmatrix} \tag{4.46)(b}$$

$$\Omega(A^j)_{j=1}^n = \begin{pmatrix} \theta \\ \theta \\ (\Omega_i(P_i^j,Q_i^j)_{j=1}^n)_{i=-M}^M \end{pmatrix}, \quad \text{(d)} \quad \tilde{F}^{n+1} = \begin{pmatrix} 0 \\ 0 \\ (\tilde{F}_i^{n+1})_{i=-M}^M \end{pmatrix} \tag{4.46)(c}$$

holds. Here we denoted the difference matrices

(4.47)(a)
$$D_1(h,k) = \frac{1}{2h}\underbrace{\begin{bmatrix} 0 & 0 & 0 & & & \\ 1 & 0 & -1 & & \theta & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & 0 & -1 & \\ & & & \ddots & \ddots & \ddots \\ & \theta & & 1 & 0 & -1 \\ & & & 0 & 0 & 0 \end{bmatrix}}_{2M+1},$$

(4.47)(b)
$$D_2(h,k) = \frac{1}{h^2}\underbrace{\begin{bmatrix} -2 & 2 & 0 & & & \\ 1 & -2 & 1 & & \theta & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & -2 & 1 & \\ & & & \ddots & \ddots & \ddots \\ & \theta & & 1 & -2 & 1 \\ & & & 0 & 2 & -2 \end{bmatrix}}_{2M+1}$$

The first and last rows of $D_1(h,k)$, $D_2(h,k)$ come from incorporating the homogeneous Neumann conditions.

The matrices $L_0(h,k)$, $L_1(h,k)$ are derived by taking $\tilde{g}_1(x_i,0,h,k)$, $\tilde{g}_2(x_i,0,h,k)$ instead of $\tilde{g}_1(x_i,0,0,0)$ and $\tilde{g}_2(x_i,0,0,0)$ resp. Because of (4.26), (4.27) we get

(4.48)
$$\|L_\ell(h,k)\|_{\tilde{L}_h^2} \le \text{const.} \quad , \quad \ell = 0,1$$

uniformly as $h,k \to 0$.

We easily obtain

$$(4.49) \qquad E_0^{-1}(h,k) = \begin{bmatrix} I & \theta & kD_1(h,k)B_0^{-1}(h,k) \\ \theta & I & kD_2(h,k)B_0^{-1}(h,k) \\ \theta & \theta & B_0^{-1}(h,k) \end{bmatrix}$$

and

$$(4.50) \qquad (E_0^{-1}(h,k)E_1\ (h,k))^i = \begin{bmatrix} I & \theta & -kD_1(h,k)\sum_{j=1}^{i} C(h,k)^j \\ 0 & I & -kD_2(h,k)\sum_{j=1}^{i} C(h,k)^j \\ 0 & 0 & C(h,k)^i \end{bmatrix}$$

for $i > 0$.

We set $\alpha_i = \hat{g}_1(x_i,0,0,0)$, $\beta_i = \frac{\partial}{\partial x}\hat{g}_1(x_i,0,0,0)$ and get from (4.32)(a),(b)

$$(4.51) \qquad B_0(h,k) = I-(\frac{k}{2}\Lambda_2 D_2(h,k) + \frac{k}{2}\Lambda_1 D_1(h,k))$$

$$(4.52) \qquad B_1(h,k) = I+\frac{k}{2}\Lambda_2 D_2(h,k) + \frac{k}{2}\Lambda_1 D_1(h,k))$$

where $\Lambda_2 = \mathrm{diag}(\alpha_{-M},\ldots,\alpha_0,\ldots,\alpha_M)$ and $\Lambda_1 = \mathrm{diag}(\beta_{-M},\ldots,\beta_0,\ldots,\beta_M)$ holds. Since $h^2 = \lambda k$ we get

$$(4.53) \qquad \|E_0^{-1}(h,k)\|_{(\bar{L}_h^2)^3} \le \text{const.}$$

Obviously

$$(4.54) \qquad Y^i = k\sum_{j=1}^{i} C(h,k)^j$$

solves the matrix iteration

$$(4.55)(a) \qquad Y^{i+1} = C(h,k)Y^i + kC(h,k) \quad , \quad 0 \le i \le N-1$$

$$(4.55)(b) \qquad Y^0 = \theta \quad .$$

From (4.37) we get

$$(4.56) \qquad \max_{0\le i\le N} \|Y^i\|_{H_h^2} \le \text{const.}\|B_1(h,k)\|_{\bar{L}_h^2} \quad .$$

Because of (4.38) $B_1(h,k)$ is uniformly bounded (in $\bar{L}_h^2$) and therefore the right hand side of (4.56) is uniformly bounded as $h,k \to 0$. We obtain

$$\|D_2 k \sum_{j=1}^{i} C^j\|_{\bar{L}_h^2}, \ \|D_1 k \sum_{j=1}^{i} C^j\|_{\bar{L}_h^2} \le \text{const.}, \ 0 \le i \le N \tag{4.57}$$

uniformly as $h,k \to 0$.

It is interesting to note that the *mesh-size restriction* (4.38) is not necessary for the implicit Euler-scheme since for this fully implicit scheme $B_1 = I$ holds.

(4.57) implies L^2-stability of (4.45):

$$\|(E_0^{-1}(h,k)E_1(h,k)))^i\|_{(\bar{L}_h^2)^3} \le \text{const. for } 0 \le i \le N \tag{4.58}$$

uniformly as $h,k \to 0$.

By proceeding similarly to Richtmeyer and Morton (1965, Chapter 3.9) we get

$$\|A^n\|_{(\bar{L}_h^2)^3} \le \text{const.}(\|\bar{A}^0\|_{(\bar{L}_h^2)^3} + \max_{1\le j\le n} \|(F_i^j)_{i=-M}^M\|_{\bar{L}_h^2}) \tag{4.59}$$

where $(A^n)_{n=0}^N$ solves (4.45). For the (first) time difference quotient of R^n we get

$$\begin{aligned}\delta^+R^n = \frac{1}{k}(C(h,k) - I)R^n + (B_0^{-1}(h,k)\tilde{L}_0(h,k)R^n + \\ + B_0^{-1}(h,k)\tilde{L}_1(h,k)R^{n+1} + B_0^{-1}(h,k)(\Omega_i(P_i^j,Q_i^j)_{j=1}^n)_{i=-M}^M + B_0^{-1}(h,k)(\tilde{F}_i^{n+1})_{i=-M}^M)\end{aligned} \tag{4.60}$$

where $\tilde{L}_\ell(h,k)$, $\ell = 0,1$ stands for the (block) matrix in the (3.3) position of $L_\alpha(h,k)$.

From (4.36), (4.37) we get

$$\|\delta^+R^n\|_{\bar{L}_h^2} \le \text{const.}(\|\bar{R}^0\|_{H_h^2} + \max_{1\le j\le n} \|A^j\|_{(\bar{L}_h^2)^3} + \max_{i\le j\le n+1} \|(\tilde{F}_i^j)_{i=-M}^M\|_{\bar{L}_h^2}) \ . \tag{4.61}$$

Similarly we get bounds for the spatial difference quotients and using (4.59) we obtain the stability estimate

$$\|R^n\|_{H_h^2} \le \text{const.}(\|\bar{R}^0\|_{H_n^2} + \|\bar{P}^0\|_{\bar{L}_h^2} + \|\bar{Q}^0\|_{\bar{L}_h^2} + \max_{1\le j\le n} \|(\tilde{F}_i^j)_{i=-M}^M\|_{\bar{L}_h^2}) \ . \tag{4.62}$$

From (4.29) we get immediately

$$\|(G'_{k,h}(\tilde{W}^{(N)}))^{-1}\|_{C_k(H_h^2,\bar{L}_h^2) + C_k^1(H_h^2)} \le \text{const.} \tag{4.63}$$

uniformly as $h,k \to 0$ where

$$\tilde{W}^{(N)} = ((\tilde{W}_i^j)_{j=-\infty}^N)_{i=-M-1}^{M+1} \tag{4.64}$$

and

$$(4.65)(a) \qquad \tilde{W}_i^j = \hat{W}_i^j \quad , \; i = (-M-1)(1)(M+1), \; j \leq 0$$

$$(4.65)(b) \qquad \tilde{W}_i^j = \tilde{W}_i^0 \quad , \; i = (-M-1)(1)(M+1), \; 0 < j \leq N$$

holds.

As a lengthy (but easy) calculation shows $G'_{k,n}$ is uniformly Lipschitz continuous in a sphere $S_\rho(\hat{W}^{(N)}) \subset C_k^1(H_n^2)$ whose radius ρ is independent of h,k:

$$(4.66) \qquad \| G'_{k,h}(Y^{(N)}) - G'_{k,h}(Z^{(N)}) \|_{C_k^1(H_n^2) \to C_k(H_n^2, L_n^2)} \leq \text{const.} \| Y^{(N)} - Z^{(N)} \|_{C_k^1(H_n^2)}$$

for $Z^{(N)}, Y^{(N)} \in S_\rho(\hat{W}^{(N)})$. Since (4.20) implies that

$$(4.67) \qquad \| \tilde{W}^{(N)} - \hat{W}^{(N)} \|_{C_k^1(H_n^2)} = o(1) \quad \text{as} \quad T \to 0$$

holds, the estimate (4.63) is also fulfilled by $G'_{k,h}(\hat{W}^{(N)})$ if T is sufficiently small. Applying Keller's (1975) theory we obtain

<u>Theorem 4.1</u>. Assume that the assumptions (4.20) and (4.38) hold. Then for h,k sufficiently small the scheme (4.8), (4.9), (4.10), (4.11) has a locally unique solution $((U_i^n)_{i=-M-1}^{M+1})_{n=-\infty}^{N}$ if T is sufficiently small and the convergence estimate

$$(4.68) \qquad \| ((U_i^n - u(x_i,t_n))_{n=-\infty}^{N})_{i=-M}^{M} \|_{C_k^1(H_n^2)} = O(h^2)$$

as $h \to 0$ holds. The (abstract) Newton method for (4.13) converges quadratically from a (sufficiently small) sphere of starting values whose radius is constant as $h,k \to 0$.

Of course, T can be taken independently of h,k.

<u>Remark</u>. Renardy (1981) assumed that $g : \mathbb{R} \times \tilde{\Omega}_1 \to H^1([-1,1])$, $\tilde{\Omega}_1 \subset H^1([-1,1])$ holds instead of (4.20)(a). This is a more realistic assumption (with respect to viscoelastic problems) since $H^1([-1,1])$ is a Banach algebra (elements can be multiplied), but the perturbation approach (4.45) would not go through as presented. However, this is a technicality which can be repaired by incorporating one more x-difference quotient into the spaces.

As an example, we apply our Crank-Nicolson-type scheme to a model for the stretching of a thin filament of a viscoelastic fluid when a force f is applied to its ends (derived by Renardy (1981b)):

$$(4.69)(a) \qquad \rho u_{tt} = \frac{\partial}{\partial x}\left(3\eta \frac{u_{xt}}{u_x^2} + u_x F(u_x^t) - \frac{1}{u_x^2} G(u_x^t)\right), \ x \in [-1,1], \ t \in [0,T]$$

$$(4.69)(b) \qquad u(x,t) = \bar{u}(x,t) \quad , \quad x \in [-1,1], \ t \in [-\infty,0]$$

$$(4.69)(c) \qquad u_x(1,t) = y(t) \quad , \quad t \in [-\infty,T]$$

$$(4.69)(d) \qquad u_x(-1,t) = y(t) \quad , \quad t \in [-\infty,T]$$

where $y(t)$ solves

$$(4.70)(a) \qquad 3\eta \frac{y'(t)}{y^2(t)} + y(t)F(y^t) - \frac{1}{y^2(t)} G(y^t) = f(t), \ t \in [0,T]$$

$$(4.70)(b) \qquad y(t) = \bar{u}_x(1,t) = \bar{u}_x(-1,t) \quad , \quad t \in [-\infty,0] \quad .$$

The history $\bar{u}(x,t)$ is assumed to be an odd function of x for all $t \in [-\infty,0]$, F, G are functionals (of Volterra type). The parameters ρ, η and the physical meaning of u is explained in Renardy (1981b).

At first we apply the midpoint rule (3.3), (3.4) to the boundary problems (4.69) and obtain

$$(4.71)(a) \qquad 12\eta \frac{\delta^+ y^n}{(y^n+y^{n+1})^2} + \frac{y^n+y^{n+1}}{2} F((i_k y^{(N)})^{t_{n+1/2}}) -$$

$$- \frac{4}{(y^n+y^{n+1})^2} G((i_k y^{(N)})^{t_{n+1/2}}) = f(t_{n+1/2}), \ 0 \le n < N$$

$$(4.71)(b) \qquad y^n = \bar{u}_x(1,t_n), \ n \le 0 \quad .$$

(4.8)-(4.11) applied to (4.69) gives

$$(4.72)(a) \qquad \rho\delta^+\delta^- U_i^n = 3\eta \frac{\delta\Delta_+\Delta_- U_i^n(\Delta U_i^n)^2 - 2\Delta\delta U_i^n \Delta U_i^n \Delta_+\Delta_- U_i^n}{(\Delta U_i^n)^4} +$$

$$+ \Delta_+\Delta_- U_i^n F((i_k \Delta U_i^{(N)})^{t_n}) +$$

$$+ \Delta U_i^n F_1((i_k \Delta U_i^{(N)})^{t_n})(i_k \Delta_+ \Delta_- U_i^{(N)})^{t_n} +$$

$$+ 2 \frac{\Delta U_i^n \Delta_+ \Delta_- U_i^n}{(\Delta U_i^n)^4} G((i_k \Delta U_i^{(N)})^{t_n}) -$$

$$- \frac{1}{(\Delta U_i^n)^2} G_1((i_k \Delta U_i^{(N)})^{t_n})(i_k \Delta_+ \Delta_- U_i^{(N)})^{t_n}$$

for $i \leq -M(1)M,\ 0 < n < N$

(4.72)(b) $$U_i^n = \tilde{u}(x_i, t_n),\ i = -M(1)M,\ n \leq 0$$

(4.72)(c) $$\Delta U_M^n = y^n \quad , \quad n \leq N$$

(4.72)(d) $$\Delta U_{-M}^n = y^n \quad , \quad n \leq N \quad .$$

Here F_1, G_1 denote the first Frechet derivatives of F and G resp.

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This paper is concerned with one-step difference methods for parabolic history value problems in one space variable. These problems, which have the feature that the evolution of the solution is influenced by 'all its past' occur in the theory of viscoelastic liquids (materials with 'memory'). The history dependence is represented by a Volterra-integral in the equation of motion. Using recently obtained existence results (see Renardy (1981b)) and smoothness assumptions on the solution we derive a local stability and convergence result for a Crank-Nicolson-type difference scheme by interpreting the linearized scheme as perturbation of a

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strictly parabolic scheme without memory term. Second order convergence is shown on sufficiently small time intervals. The presented approach carries over to other one-step difference methods like implicit and explicit Euler schemes.

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